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On second-order consensus in multi-agent dynamical systems with directed topologies and time delays

Wenwu Yu, Guanrong Chen, and Ming Cao

Abstract—This paper establishes some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems with directed topologies and time delays. First, theoretical analysis is carried out for the basic, but fundamentally important case where agents' second-order dynamics are governed by the position and velocity terms. A necessary and sufficient condition is derived to ensure second-order consensus and it is found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix of the corresponding network topology play key roles in reaching consensus. Based on this result, a second-order consensus algorithm is constructed for the multi-agent system with communication delays. A necessary and sufficient condition is then proposed, which shows that consensus can be achieved in a multi-agent system whose topology contains a directed spanning tree if and only if the time delay is less than a critical value. Finally, simulation examples are given to verify the theoretical analysis.

I. INTRODUCTION

Collective behaviors in groups of autonomous mobile agents have attracted increasing attention in recent years due to growing interest in animal group behaviors, such as flocking and swarming, and also due to its applications in biological systems, sensor networks, UAV (Unmanned Air Vehicle) formations, robotic teams, and so on. The study of collective behavior aims to analyze how coordinated group behavior arises as a result of local interactions among mobile individuals who share information with their neighbors locally and simultaneously try to agree on certain global criteria of common interest.

Recently, significant effort has been made to study collective behaviors in multi-agent dynamical systems, such as consensus [5], [8], [10], [12], [13], [14], [15], synchronization [9], [19], [21], [23], [24], [25], and swarming and flocking [11], [16], [18]. It is of great interest to find conditions under which an agreement can be achieved among a group of autonomous agents in a dynamically changing environment. In [18], Vicsek *et al.* proposed a simple discrete-time model to study a group of autonomous agents moving in the plane with the same speed but different headings. Vicsek's model in essence is a simplified version of the model introduced earlier by Reynolds to animate flocking behaviors [16]. Using tools from the algebraic graph theory [4], the study on Vicsek's model and its continuous-time

counterpart has shown that consensus in a network with a dynamically changing topology can be reached if and only if the time-varying network topology contains a spanning tree frequently enough as the network evolves in time [3], [8], [10], [15].

In the literature, most work on the consensus problem considered the case where agents are governed by first-order dynamics [1], [8], [9], [10], [15], [17], [19], [21], [23], [24]. However, there is a growing interest [5], [11], [13], [14] in consensus algorithms where each agent is governed by second-order dynamics. In general, the second-order consensus problem refers to the problem of reaching an agreement among a group of autonomous agents governed by second-order dynamics. A detailed analysis of the second-order consensus algorithms is a key step to bring more complicated dynamics into the model of each individual agent based on the general framework of multi-agent systems, thus can help control engineers implement distributed cooperative control strategies in networked multi-agent systems. It has been shown that, in sharp contrast to first-order consensus problem, consensus may fail to be achieved for agents with second-order dynamics even if the network topology has a directed spanning tree [14]. Some sufficient conditions have been derived for reaching second-order consensus [13], [14], but it is still a challenging unsolved problem to identify the key factors for reaching second-order consensus in a multi-agent system. One contribution of this paper is that a necessary and sufficient condition is obtained for ensuring second-order consensus in a network containing a directed spanning tree. It is found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix of the network play key roles in reaching second-order consensus.

On the other hand, it is well known that time delay is ubiquitous in biological, physical, chemical, and electrical systems [1], [17]. It has been observed from numerical experiments that consensus algorithms without considering time delays may lead to unexpected instability. In [1], [17], some sufficient conditions are derived for first-order consensus in delayed multi-agent systems. This paper also considers explicitly the effect of delays on second-order consensus. In this regard, another contribution of this paper is that we obtain a necessary and sufficient condition, which says that a second-order consensus can be achieved in a delayed multi-agent system with a directed spanning tree if and only if the time delay is less than a certain critical value.

The rest of the paper is organized as follows. In Section 2, some preliminaries on graph theory and model formulation are given. Second-order consensus algorithms for multi-agent

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dynamical systems in directed networks and delayed directed networks are proposed in Sections 3 and 4, respectively. In Section 5, numerical examples are simulated to verify the theoretical analysis. Conclusions are drawn in Section 6.

II. PRELIMINARIES

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, G)$ be a weighted directed network of order N , with the set of nodes $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$, the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $G = (G_{ij})_{N \times N}$. A directed edge \mathcal{E}_{ij} in network \mathcal{G} is denoted by the ordered pair of nodes (v_i, v_j) , where v_i and v_j are called the terminal and initial nodes, respectively, which means that node v_i can receive information from node v_j . By the definition of adjacency matrices for weighted graphs [6], when weights are all positive, $G_{ij} > 0$ if and only if there is a directed edge (v_i, v_j) in G . In this paper, only positively weighted networks are considered.

Definition 1: [6] A network \mathcal{G} is *directed* if there is a connection from node v_j to v_i in \mathcal{G} , then $G_{ij} > 0$; otherwise $G_{ij} = 0$ ($i \neq j, i, j = 1, 2, \dots, N$).

Note that undirected networks are special cases of directed networks with $G_{ij} = G_{ji}$ for all $i, j = 1, 2, \dots, N$.

Definition 2: [6] A directed path from node v_j to v_i is a sequence of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_l}, v_j)$ in the directed network with distinct nodes $v_{i_k}, k = 1, 2, \dots, l$. A directed network \mathcal{G} is *strongly connected* if between any pair of distinct nodes v_i and v_j in \mathcal{G} , there is a directed path from v_i to v_j , $i, j = 1, 2, \dots, N$.

Definition 3: [2] A directed network is called a *directed tree* if the underlying network (the direction of the network is ignored as an undirected network) is a tree. A *directed rooted tree* is a directed network with at least a root r having the property that for each node v different from r , there is a unique directed path from r to v . A *directed spanning tree* of a network \mathcal{G} is a directed rooted tree, which contains all the nodes and some edges in \mathcal{G} .

The following notations are used throughout the paper. Let $\lambda_{\max}(F)$ be the largest eigenvalue of the matrix F , I_N (O_N) be the N -dimensional identity (zero) matrix, $\mathbf{1}_N \in \mathbb{R}^N$ ($\mathbf{0}_N \in \mathbb{R}^N$) be a vector with each entry being 1 (0), and $\Re(u)$ and $\Im(u)$ be the real and imaginary parts of a complex number u . For matrices \tilde{A} and \tilde{B} with the same order, $\tilde{A} > \tilde{B}$ means that $\tilde{A} - \tilde{B}$ is positive definite. A vector $x \in \mathbb{R}^N$ is positive if every entry $x_i > 0$ ($1 \leq i \leq N$).

The first-order consensus protocol has been widely studied for networks consisting of N nodes with linearly diffusive coupling [8], [9], [10], [15], [19], [21], [23], [24], [25]:

$$\dot{x}_i(t) = c \sum_{j=1, j \neq i}^N G_{ij}(x_j(t) - x_i(t)), i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the i th node, c is the coupling strength, $G = (G_{ij})_{N \times N}$ is the coupling configuration matrix representing the topological structure of the network and thus is the weighted adjacency matrix of the network. The Laplacian

matrix $L = (L_{ij})_{N \times N}$ is defined by

$$L_{ii} = - \sum_{j=1, j \neq i}^N L_{ij}, L_{ij} = -G_{ij}, i \neq j, \quad (2)$$

which ensures the diffusion property that $\sum_{j=1}^N L_{ij} = 0$. For an undirected network, its Laplacian matrix L is symmetric and positive semi-definite; however, L does not have this property for a directed network in general.

As to the second-order dynamics, what follows is the second-order consensus protocol [5], [13], [14]:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \alpha \sum_{j=1, j \neq i}^N G_{ij}(x_j(t) - x_i(t)) \\ &\quad + \beta \sum_{j=1, j \neq i}^N G_{ij}(v_j(t) - v_i(t)), i = 1, 2, \dots, N, \end{aligned} \quad (3)$$

where $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$ are the position and velocity states of the i th node, respectively, and $\alpha > 0$ and $\beta > 0$ are the coupling strengths.

Equivalently, network (3) can be rewritten in a simpler form as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha \sum_{j=1}^N L_{ij}x_j(t) - \beta \sum_{j=1}^N L_{ij}v_j(t), i = 1, 2, \dots, N \end{aligned} \quad (4)$$

Let $x = (x_1^T, x_2^T, \dots, x_N^T)^T$, $v = (v_1^T, v_2^T, \dots, v_N^T)^T$, and $y = (x^T, v^T)^T$. Then, network (4) can be rewritten in a compact matrix form as

$$\dot{y}(t) = (\tilde{L} \otimes I_n)y, \quad (5)$$

where $\tilde{L} = \begin{pmatrix} O_N & I_N \\ -\alpha L & -\beta L \end{pmatrix}$ and \otimes is the Kronecker product [7].

Lemma 1: [15] The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.

Definition 4: Second-order consensus in the multi-agent system (5) is said to be achieved if for any initial conditions,

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0, \quad \lim_{t \rightarrow \infty} \|v_i(t) - v_j(t)\| = 0, \\ \forall i, j = 1, 2, \dots, N.$$

III. SECOND-ORDER CONSENSUS IN DIRECTED NETWORKS

In this section, some second-order consensus algorithms for the multi-agent system (5) with directed topologies are developed. For the linear model (5), eigenvalues of the Laplacian matrix \tilde{L} are very important in convergence analysis. Suppose that λ_{ij} ($i = 1, 2, \dots, N, j = 1, 2$) and μ_i ($i = 1, 2, \dots, N$) are eigenvalues of \tilde{L} and the Laplacian

matrix L , respectively. First, some relationships between the eigenvalues of \tilde{L} and L are reviewed [13], [14].

Let λ be an eigenvalue of matrix \tilde{L} . Then, one has $\det(\lambda I_{2N} - \tilde{L}) = 0$. Note that

$$\begin{aligned} \det(\lambda I_{2N} - \tilde{L}) &= \det \begin{pmatrix} \lambda I_N & -I_N \\ \alpha L & \lambda I_N + \beta L \end{pmatrix} \\ &= \det(\lambda^2 I_N + (\alpha + \beta \lambda)L) \\ &= \prod_{i=1}^N (\lambda^2 + (\alpha + \beta \lambda)\mu_i) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_{i1} &= \frac{-\beta \mu_i + \sqrt{\beta^2 \mu_i^2 - 4\alpha \mu_i}}{2}, \\ \lambda_{i2} &= \frac{-\beta \mu_i - \sqrt{\beta^2 \mu_i^2 - 4\alpha \mu_i}}{2}, i = 1, 2, \dots, N. \end{aligned} \quad (6)$$

From (6), it is easy to see that L has m zero eigenvalues if and only if \tilde{L} has $2m$ zero eigenvalues.

Lemma 2: Second-order consensus in multi-agent system (5) can be achieved if and only if matrix \tilde{L} has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. In addition, $v_i(t) \rightarrow \sum_{j=1}^N \xi_j v_j(0)$ and $x_i(t) \rightarrow \sum_{j=1}^N \xi_j x_j(0) + \sum_{j=1}^N \xi_j v_j(0)t$ as $t \rightarrow \infty$, where ξ is the nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T \mathbf{1}_N = 1$.

For $n = 1$ and $\alpha = 1$, a proof of this lemma was given in [14]. It is easy to generalize the result for any integer $n \geq 1$ and $\alpha > 0$ by Kronecker product [7].

Although a necessary and sufficient condition is given in Lemma 2 to ensure the second-order consensus in multi-agent system (5), it does not show any relationship between the eigenvalues of matrix \tilde{L} and the Laplacian matrix L . A natural question is: in what kind of networks can the second-order consensus be reached? In [14], an example is given where the second-order consensus can be achieved in a directed spanning tree but cannot be achieved after adding only one extra edge into the directed spanning tree. This is a bit surprising as it is inconsistent with the intuition that connections are helpful for reaching consensus.

Theorem 1: Second-order consensus in multi-agent system (5) can be achieved if and only if the network contains a directed spanning tree, and moreover

$$\frac{\beta^2}{\alpha} > \max_{2 \leq i \leq N} \frac{\mathcal{I}^2(\mu_i)}{\mathcal{R}(\mu_i)[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i)]}. \quad (7)$$

In addition, $v_i(t) \rightarrow \sum_{j=1}^N \xi_j v_j(0)$ and $x_i(t) \rightarrow \sum_{j=1}^N \xi_j x_j(0) + \sum_{j=1}^N \xi_j v_j(0)t$ as $t \rightarrow \infty$, where ξ is a nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T \mathbf{1}_N = 1$.

Proof. From Lemma 1, one knows that the Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree. By Lemma 2, one only needs to prove that both $\mathcal{R}(\mu_i) > 0$ ($i = 2, 3, \dots, N$) and (7) hold if and only if $\mathcal{R}(\lambda_{ij}) < 0$ ($i = 2, 3, \dots, N; j = 1, 2$).

Let $\sqrt{\beta^2 \mu_i^2 - 4\alpha \mu_i} = c + id$, where c and d are real. From (6), $\mathcal{R}(\lambda_{ij}) < 0$ ($i = 2, 3, \dots, N; j = 1, 2$) if and only if $-\beta \mathcal{R}(\mu_i) < c < \beta \mathcal{R}(\mu_i)$, which is equivalent to $\mathcal{R}(\mu_i) > 0$ and $c^2 < \beta^2 \mathcal{R}^2(\mu_i)$ ($i = 2, 3, \dots, N$). Then, it suffices to prove that (7) holds if and only if $c^2 < \beta^2 \mathcal{R}^2(\mu_i)$ ($i = 2, 3, \dots, N$). It is easy to see that

$$\beta^2 \mu_i^2 - 4\alpha \mu_i = (c + id)^2.$$

Separating the real and imaginary parts, one has

$$\begin{aligned} c^2 - d^2 &= \beta^2 [\mathcal{R}^2(\mu_i) - \mathcal{I}^2(\mu_i)] - 4\alpha \mathcal{R}(\mu_i), \\ cd &= \beta^2 \mathcal{R}(\mu_i) \mathcal{I}(\mu_i) - 2\alpha \mathcal{I}(\mu_i). \end{aligned}$$

By simple calculations, the following equation is obtained

$$\begin{aligned} c^4 - \{\beta^2 [\mathcal{R}^2(\mu_i) - \mathcal{I}^2(\mu_i)] - 4\alpha \mathcal{R}(\mu_i)\} c^2 \\ - \mathcal{I}^2(\mu_i) [\beta^2 \mathcal{R}(\mu_i) - 2\alpha]^2 = 0. \end{aligned} \quad (8)$$

It is easy to check that $c^2 < \beta^2 \mathcal{R}^2(\mu_i)$ if and only if (7) holds. \square

Remark 1: In Theorem 1, in addition to the condition that the network has a directed spanning tree, (7) should also be satisfied. It is easy to verify that if all the other eigenvalues of the Laplacian matrix L are real, then (7) holds, which indicates that the condition derived in Theorem 1 is more general than the result in [14]. From (7), it is easy to see that both real and imaginary parts of the eigenvalues of the Laplacian matrix play important roles in reaching second-order consensus. Let $\frac{\mathcal{I}^2(\mu_k)}{\mathcal{R}(\mu_k)[\mathcal{R}^2(\mu_k) + \mathcal{I}^2(\mu_k)]} = \max_{2 \leq i \leq N} \frac{\mathcal{I}^2(\mu_i)}{\mathcal{R}(\mu_i)[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i)]}$, where $2 \leq k \leq N$. Then, one can see that in order to reach consensus, the critical value β^2/α increases as $|\mathcal{I}(\mu_k)|$ increases and decreases as $\mathcal{R}(\mu_k)$ increases.

IV. SECOND-ORDER CONSENSUS IN DELAYED DIRECTED NETWORKS

In this section, the following second-order consensus protocol with time delays is considered

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha \sum_{j=1}^N L_{ij} x_j(t - \tau) - \beta \sum_{j=1}^N L_{ij} v_j(t - \tau), \\ & i = 1, 2, \dots, N, \end{aligned} \quad (9)$$

where $\tau > 0$ is the time-delay constant.

Let $x = (x_1^T, x_2^T, \dots, x_N^T)^T$, $v = (v_1^T, v_2^T, \dots, v_N^T)^T$, and $y = (x^T, v^T)^T$. Then, network (9) can be rewritten in a compact matrix form as follows:

$$\dot{y}(t) = (\tilde{L}_1 \otimes I_n) y + (\tilde{L}_2 \otimes I_n) y(t - \tau), \quad (10)$$

where $\tilde{L}_1 = \begin{pmatrix} O_N & I_N \\ O_N & O_N \end{pmatrix}$ and $\tilde{L}_2 = \begin{pmatrix} O_N & O_N \\ -\alpha L & -\beta L \end{pmatrix}$.

In [20], [22], stability and Hopf bifurcation of delayed networks were studied, where the time delays are regarded as bifurcation parameters. It was found that Hopf bifurcation occurs when time delays pass through some critical values

where the conditions for local asymptotical stability of the equilibrium are not satisfied. Similarly, this section aims to find the maximum time delay with which the consensus can be achieved in the multi-agent system (10).

The characteristic equation of system (10) is $\det(\lambda I_{2N} - \tilde{L}_1 - e^{-\lambda\tau}\tilde{L}_2) = 0$, i.e.,

$$\begin{aligned} & \det(\lambda I_{2N} - \tilde{L}_1 - e^{-\lambda\tau}\tilde{L}_2) \\ &= \det \begin{pmatrix} \lambda I_N & -I_N \\ \alpha e^{-\lambda\tau}L & \lambda I_N + \beta e^{-\lambda\tau}L \end{pmatrix} \\ &= \det(\lambda^2 I_N + (\alpha + \beta\lambda)e^{-\lambda\tau}L) \\ &= \prod_{i=1}^N (\lambda^2 + (\alpha + \beta\lambda)e^{-\lambda\tau}\mu_i) = 0. \end{aligned} \quad (11)$$

Let $g_i(\lambda) = \lambda^2 + (\alpha + \beta\lambda)e^{-\lambda\tau}\mu_i$ and $g(\lambda) = \prod_{i=1}^N g_i(\lambda)$. From (11), it is easy to see that L has m zero eigenvalues if and only if $g(\lambda) = 0$ has $2m$ zero roots.

Lemma 3: Suppose that the network contains a directed spanning tree. For a given integer i , $2 \leq i \leq N$, $g_i(\lambda) = 0$ has a purely imaginary root when

$$\tau \in \Psi = \left\{ \frac{1}{\omega_{i1}} (2k\pi + \theta_{i1}) \mid i = 2, \dots, N; k = 0, 1, \dots \right\}, \quad (12)$$

where $0 \leq \theta_{i1} < 2\pi$, which satisfies $\cos \theta_{i1} = \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{i1}\beta]}{\omega_{i1}^2}$ and

$$\sin \theta_{i1} = \frac{[\mathcal{R}(\mu_i)\omega_{i1}\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_{i1}^2}, \quad \text{and} \quad \omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}, \quad i = 1, 2, \dots, N.$$

Proof. Let $\lambda = i\omega_i$ ($\omega_i \neq 0$). From $g_i(\lambda) = 0$, one has

$$\omega_i^2 = (\alpha + i\beta\omega_i)e^{-i\omega_i\tau}\mu_i. \quad (13)$$

Taking modulus on both sides of (13), one obtains

$$\omega_i^4 - [\mathcal{R}(\mu_i)^2 + \mathcal{I}(\mu_i)^2]\beta^2\omega_i^2 - [\mathcal{R}(\mu_i)^2 + \mathcal{I}(\mu_i)^2]\alpha^2 = 0. \quad (14)$$

Then,

$$\omega_i^2 = \frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}. \quad (15)$$

Separating the real and imaginary parts of (13) yields

$$\begin{aligned} \omega_i^2 &= [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta] \cos(\omega_i\tau) \\ &\quad + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha] \sin(\omega_i\tau), \\ 0 &= [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha] \cos(\omega_i\tau) \\ &\quad - [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta] \sin(\omega_i\tau). \end{aligned} \quad (16)$$

It is easy to see that $\omega_i^4 = [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]^2 + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]^2$. By simple calculations, one obtains

$$\begin{aligned} \cos(\omega_i\tau) &= \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]}{\omega_i^2}, \\ \sin(\omega_i\tau) &= \frac{[\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_i^2}. \end{aligned} \quad (17)$$

$$\text{Let } \omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}, \quad \omega_{i2} = -\sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}, \quad \text{and } 0 \leq \theta_{ij} < 2\pi,$$

which satisfies $\cos \theta_{ij} = \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{ij}\beta]}{\omega_{ij}^2}$ and

$$\sin \theta_{ij} = \frac{[\mathcal{R}(\mu_i)\omega_{ij}\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_{ij}^2}, \quad j = 1, 2. \quad \text{Since}$$

$\mathcal{R}(\mu_i) \pm i\mathcal{I}(\mu_i)$ are two eigenvalues of L , if $\mathcal{I}(\mu_i) \neq 0$, then there exists an integer j , $2 \leq j \leq N$, such that

$$\begin{aligned} \cos(\omega_{i1}\tau) &= \cos(\omega_{s2}\tau), \quad \sin(\omega_{i1}\tau) = -\sin(\omega_{s2}\tau), \\ \theta_{i1} &= 2\pi - \theta_{s2}. \end{aligned}$$

If $\mathcal{I}(\mu_i) = 0$, then

$$\begin{aligned} \cos(\omega_{i1}\tau) &= \cos(\omega_{i2}\tau), \quad \sin(\omega_{i1}\tau) = -\sin(\omega_{i2}\tau), \\ \theta_{i1} &= 2\pi - \theta_{i2}. \end{aligned}$$

It follows that

$$\begin{aligned} \tau \in \Psi &= \left\{ \frac{1}{\omega_{i1}} (2k\pi + \theta_{i1}) \mid i = 2, \dots, N; k = 0, 1, \dots \right\} \\ &\cup \left\{ \frac{1}{\omega_{i2}} (2k\pi + \theta_{i2}) \mid i = 2, \dots, N; k = -1, -2, \dots \right\} \\ &= \left\{ \frac{1}{\omega_{i1}} (2k\pi + \theta_{i1}) \mid i = 2, \dots, N; k = 0, 1, \dots \right\}. \end{aligned}$$

The proof is completed. \square

Lemma 4: [20] Consider the exponential polynomial

$$\begin{aligned} & P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}) \\ &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \dots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + [p_1^{(1)}\lambda^{n-1} + \dots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} \\ &\quad + \dots + [p_1^{(m)}\lambda^{n-1} + \dots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}, \end{aligned}$$

where $\tau_i \geq 0$ ($i = 1, 2, \dots, m$) and $p_j^{(i)}$ ($i = 0, 1, \dots, m; j = 1, 2, \dots, n$) are constants. As $(\tau_1, \tau_2, \dots, \tau_m)$ vary, the sum of the orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ on the open right-half plane can change only if a zero appears on or crosses the imaginary axis.

Lemma 5: Suppose that the network contains a directed spanning tree. For each $g_i(\lambda) = 0$, $2 \leq i \leq N$,

$$\mathcal{R} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau \in \Psi} > 0, \quad j = 1, 2. \quad (18)$$

Proof. Taking the derivative of λ with respect to τ in $g_i(\lambda) = 0$, one obtains

$$2\lambda \frac{d\lambda}{d\tau} + e^{-\lambda\tau}\mu_i \left[\beta \frac{d\lambda}{d\tau} - (\alpha + \beta\lambda)(\lambda + \tau \frac{d\lambda}{d\tau}) \right] = 0. \quad (19)$$

If $\tau \in \Psi$, then $\lambda = i\omega_{ij}$ for some i and j , $2 \leq i \leq N$, $1 \leq j \leq 2$.

Let

$$\begin{aligned} q &= [-2\omega_{ij} \sin(\omega_{ij}\tau) + \mathcal{R}(\mu_i)(\beta - \alpha\tau) + \mathcal{I}(\mu_i)\beta\omega_{ij}\tau]^2 \\ &\quad + [2\omega_{ij} \cos(\omega_{ij}\tau) - \mathcal{R}(\mu_i)\beta\omega_{ij}\tau + \mathcal{I}(\mu_i)(\beta - \alpha\tau)]^2. \end{aligned}$$

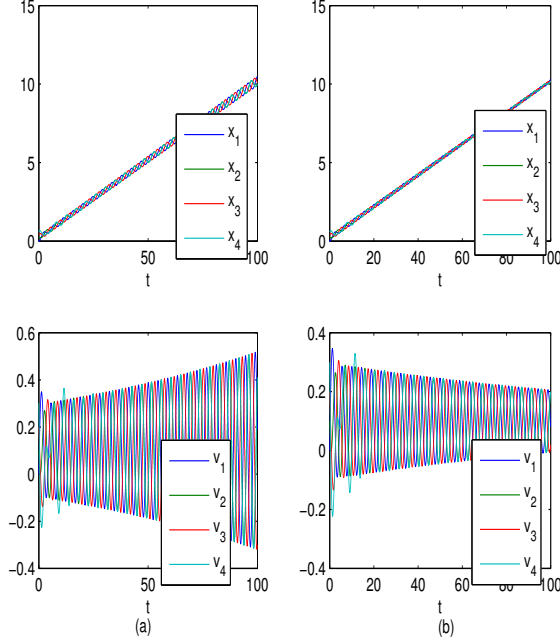


Fig. 1. Position and velocity states of agents in a network, where $\beta = 0.4$ (a) and $\beta = 0.415$ (b).

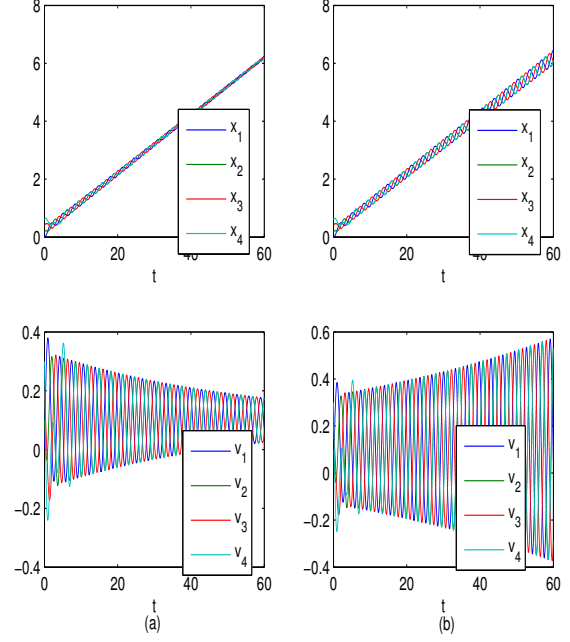


Fig. 2. Position and velocity states of agents in a delayed network, where $\tau = 0.29$ (a) and $\tau = 0.30$ (b).

By simple calculations, one obtains

$$q\mathcal{R}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau \in \Psi} = [\mathcal{R}(\mu_i)^2 + \mathcal{I}(\mu_i)^2](\beta^2\omega_i^2 + 2\alpha^2) > 0. \quad (20)$$

This completes the proof. \square

Theorem 2: Suppose that the network contains a directed spanning tree and (7) is satisfied. Then, second-order consensus in system (10) is achieved if and only if

$$\tau < \tau_0 = \min_{2 \leq i \leq N} \left\{ \frac{\theta_{i1}}{\omega_{i1}} \right\}, \quad (21)$$

where $0 < \theta_{i1} < 2\pi$, which satisfies $\cos \theta_{i1} = \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{i1}\beta]}{\omega_{i1}^2}$ and $\sin \theta_{i1} = \frac{[\mathcal{R}(\mu_i)\omega_{i1}\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_{i1}^2}$, and

$$\omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}, \quad i = 1, 2, \dots, N.$$

Proof. Since the network contains a directed spanning tree and (7) is satisfied, from Theorem 1 it follows that the second-order consensus can be achieved in system (10) when $\tau = 0$, where $g(\lambda) = 0$ has exactly two zero roots and all the other roots have negative real parts. When τ varies from 0 to τ_0 , by Lemma 3, a purely imaginary root emerges. From Lemmas 4 and 5, one knows that $g(\lambda) = 0$ has exactly two zero roots and all the other roots have negative real parts when $0 \leq \tau < \tau_0$, and there is at least one root with positive real part $\tau > \tau_0$. Therefore, the second-order consensus can not be achieved when $\tau \geq \tau_0$. The proof is completed. \square

V. SIMULATION EXAMPLES

In this section, several simulation examples are given to verify the theoretical analysis.

A. Second-order Consensus in Directed Networks

Consider the network (4) where the Laplacian matrix L is $\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ and its four eigenvalues are $0, 1, 1.5 + 0.866i, 1.5 - 0.866i$. Let $\alpha = 1$ and apply Theorem 1. Then, second-order consensus in the multi-agent system (4) can be achieved if and only if $\beta > 0.4082$. The position and velocity states of all the agents are shown in Fig. 1(a) and Fig. 1(b), where consensus cannot be achieved when $\beta = 0.4$ but it can be reached if $\beta = 0.415$. It is easy to see that by appropriately choosing some $\alpha > 0$ and $\beta > 0$, consensus can be achieved but then may fail if a connection between two agents is added.

B. Second-order Consensus in Delayed Directed Networks

Consider the network (9) with a structure shown above where $\alpha = \beta = 1$. When $\tau = 0$, from Theorem 1, one knows that second-order consensus can be achieved in the network. By simple calculations with Theorem 2, the second-order consensus can be reached if and only if $\tau < 0.29415$. The position and velocity states of all the agents are shown in Fig. 2(a) and Fig. 2(b), where consensus is achieved when $\tau = 0.29$ but not achievable if $\tau = 0.30$.

VI. CONCLUSIONS

In this paper, some second-order consensus algorithms for multi-agent dynamical systems with directed topologies and time delays have been studied. Detailed analysis has been performed on the case where the second-order dynamics of each agent are determined by the position and velocity terms. A necessary and sufficient condition has been derived to ensure second-order consensus in multi-agent systems where the network has a directed spanning tree. It was found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix play key roles in reaching consensus. Moreover, the scenario when communication delays are present in the network has been investigated. A necessary and sufficient condition has also been established, and it was shown that, in this case, the second-order consensus can be achieved in the multi-agent systems with a directed spanning tree if and only if the time delay is less than a critical value.

The study in this paper on second-order consensus algorithms can serve as a stepping stone toward more complicated and realistic agent dynamics. Moreover, the effects of more complicated inter-agent connections on group behaviors are being investigated. For example, it is of great interest to generalize the results of this paper to the case when the network topology evolves with time, or has certain hierarchical features.

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