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Structure Preserving Moment Matching for Port-Hamiltonian Systems: Arnoldi and Lanczos

Rostyslav V. Polyuga and Arjan van der Schaft, *Fellow, IEEE*

Abstract—Structure preserving model reduction of single-input single-output port-Hamiltonian systems is considered by employing the rational Krylov methods. The rational Arnoldi method is shown to preserve (for the reduced order model) not only a specific number of the moments at an arbitrary point in the complex plane but also the port-Hamiltonian structure. Furthermore, it is shown how the rational Lanczos method applied to a subclass of port-Hamiltonian systems, characterized by an algebraic condition, preserves the port-Hamiltonian structure. In fact, for the same subclass of port-Hamiltonian systems the rational Arnoldi method and the rational Lanczos method turn out to be equivalent in the sense of producing reduced order port-Hamiltonian models with the same transfer function.

Index Terms—Model order reduction, port-Hamiltonian systems, rational Krylov methods, structure preservation.

I. INTRODUCTION

The port-Hamiltonian approach to modeling and control of complex physical systems has arisen as a systematic and unifying framework during the last twenty years, see [2], [7], [13]–[15]. The port-Hamiltonian modeling employs the physical properties of the considered system including the energy dissipation, stability and passivity properties as well as the presence of conservation laws. Another important issue the port-Hamiltonian approach deals with is the interconnection of the physical system with other physical systems creating the so-called physical network. In real applications the dimensions of such interconnected port-Hamiltonian state-space systems rapidly grow both for lumped- and (spatially discretized) distributed-parameter models, motivating questions of *structure preserving model reduction*.

The so-called *moment matching* methods, which are of interest in this technical note, are an important class of model reduction methods in which a specific number of moments of the full order system at certain points in the complex plane are preserved by the reduced order system. There is a vast literature on this topic discussing different approaches, drawbacks and advantages, and numerical issues along with the use of the Arnoldi and Lanczos procedures. For an overview of these methods as well as the general model reduction theory we refer to [1], [12].

The goal of this work is to show that the rational Arnoldi and Lanczos methods apart from equalizing a certain amount of moments at an arbitrary point in the complex plane also preserve the *port-Hamiltonian structure*, and, as a consequence, *passivity*. A similar discussion is presented in [18] (see also [8]), where the authors make use of the rational Arnoldi method which results in a reduced order port-Hamiltonian model which is slightly different from the one obtained in this

technical note. Preservation of the port-Hamiltonian structure was also studied in [6], [9], [10], [16] and the references therein.

In Section II we briefly discuss the rational Arnoldi and Lanczos methods as well-known moment matching methods. Basic theory on port-Hamiltonian systems is presented in Section III.

In Section IV we demonstrate how to preserve the port-Hamiltonian structure using the rational Arnoldi method. In Section V we exploit the rational Lanczos method for structure preserving model reduction of a subclass of port-Hamiltonian systems, characterized by an algebraic condition. We will prove that the reduced order port-Hamiltonian models for the given subclass are equivalent to the reduced order models obtained by the rational Arnoldi method, matching $2r$ moments at an arbitrary point in the complex plane. Finally, in Section VI we present a numerical example illustrating that, even though we applied the rational Arnoldi method, which in general preserves only r moments, $2r$ moments are preserved since the considered port-Hamiltonian model belongs to the subclass of port-Hamiltonian systems described above.

II. MOMENT MATCHING FOR LINEAR SYSTEMS AT AN ARBITRARY POINT IN THE COMPLEX PLANE

Consider a linear, single-input, single-output, continuous-time system Σ described by equations of the form

$$\begin{cases} \dot{x} = Ax + bu, \\ y = cx \end{cases} \quad (1)$$

with the state-space vector $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, and constant matrices $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^{1 \times n}$.

Definition 1: [1] The 0-moment of the system (1) at $s_0 \in \mathbb{C}$ is the complex number $\eta_0(s_0) = c(s_0I - A)^{-1}b$. The r -moment of the system (1) at $s_0 \in \mathbb{C}$ is the complex number $\eta_r(s_0) = c(s_0I - A)^{-(r+1)}b$.

A. Rational Arnoldi Method

The idea of the rational Arnoldi method is to construct a reduced order model by applying a so-called *Galerkin projection* $V_r V_r^T$, $V_r \in \mathbb{R}^{n \times r}$, to the full order linear system (1). The maps V_r , $r = 1, \dots, n$, satisfy the following properties:

$$\begin{aligned} \text{(i): } & V_r^T V_r = I_r, \text{ i.e., the columns of } V_r \\ & \text{are orthonormal,} \\ \text{(ii): } & \text{span col } V_r = \mathcal{K}_{r, \text{input}}^{\text{shifted}}, \quad r = 1, 2, \dots, n \end{aligned} \quad (2)$$

where $\mathcal{K}_{r, \text{input}}^{\text{shifted}} = \text{span col } \mathcal{R}_r((A - s_0I)^{-1}, (A - s_0I)^{-1}b)$ is a so called *shifted input Krylov subspace*, and $\mathcal{R}_r(A, b) = [b \quad Ab \quad \dots \quad A^{r-1}b] \in \mathbb{R}^{n \times r}$ is the *partial reachability matrix* of the system (1).

Theorem 1: [4], [5] Let V_r be a matrix satisfying (2). Then the r th order system $\hat{\Sigma}$

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}u, \\ \hat{y} = \hat{c}\hat{x} \end{cases}$$

where $\hat{A} = V_r^T A V_r$, $\hat{b} = V_r^T b$, $\hat{c} = c V_r$, defines a reduced order system with the moments $\hat{\eta}_i(s_0)$, $i = 0, \dots, r-1$, at $s_0 \in \mathbb{C}$ equal to the first r moments $\eta_i(s_0)$, $i = 0, \dots, r-1$, of the full order system Σ .

Proof: The idea of the proof is based on the moment matching around $s_0 = 0$ employing the properties of the corresponding input Krylov subspace, with the consequent shift to an arbitrary point s_0

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using the shifted input Krylov subspace. Details of the proof can be also found in [11]. ■

In a similar way we can construct the projection maps $W_r \in \mathbb{R}^{n \times r}$, $r = 1, \dots, n$, based on the *shifted output Krylov subspace* $\mathcal{K}_{r, output}^{shifted} = \text{span col } \mathcal{R}_r((A - s_0 I)^{-T}, (A - s_0 I)^{-T} c^T)$, satisfying the following properties:

- (i): $W_r^T W_r = I_r$, i.e., the columns of W_r are orthonormal,
 (ii): $\text{span col } W_r = \mathcal{K}_{r, output}^{shifted}$, $r = 1, 2, \dots, n$. (3)

Using such a projection map W_r for model reduction establishes an analogous result to that in Theorem 1.

B. Rational Lanczos Method

In order to apply the *rational Lanczos method* one has to construct a reduced order model by applying a so-called *Petrov-Galerkin projection* $V_r W_r^T$, $V_r, W_r \in \mathbb{R}^{n \times r}$, to a full order linear system (1). The maps V_r, W_r satisfy property (ii) of (2), (3). But in this case V_r, W_r are no longer assumed to be orthonormal but instead biorthogonal: $W_r^T V_r = I_r$.

Theorem 2: [4], [5] Let $V_r W_r^T$ be a Petrov-Galerkin projection. Define the reduced order system $\tilde{\Sigma}$

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u, \\ \tilde{y} = \tilde{c}\tilde{x} \end{cases}$$

where $\tilde{A} = W_r^T A V_r$, $\tilde{b} = W_r^T b$, $\tilde{c} = c V_r$. Then the moments $\tilde{\eta}_i(s_0)$, $i = 0, \dots, 2r - 1$, at $s_0 \in \mathbb{C}$ equal to the first $2r$ moments $\eta_i(s_0)$, $i = 0, \dots, 2r - 1$, of the full order system Σ .

Proof: The proof is similar to the proof of Theorem 1 apart from the fact that in this case both the (shifted) input and output Krylov subspaces are used. ■

Thus the rational Lanczos method preserves twice as many moments of the full order model at an arbitrary point $s_0 \in \mathbb{C}$ as the rational Arnoldi method.

III. LINEAR PORT-HAMILTONIAN SYSTEMS

In the linear case, and in the absence of algebraic constraints and a feed-through term, port-Hamiltonian systems take the following form ([14]):

$$\begin{cases} \dot{x} = (J - R)Qx + bu, \\ y = b^T Qx \end{cases} \quad (4)$$

with $H(x) = 1/2x^T Qx$ the *total energy* (Hamiltonian), $Q = Q^T$ the *energy matrix* and $R = R^T \geq 0$ the *dissipation matrix*. The matrices $J = -J^T$ and b specify the *interconnection structure*. Since J is skew-symmetric and R is positive semi-definite it immediately follows that $(d/dt)(1/2)x^T Qx = u^T y - x^T Q R Q x \leq u^T y$. Thus if $Q \geq 0$ (and the Hamiltonian is non-negative) any port-Hamiltonian system is *passive* (see also [14], [17]). Extended theory on port-Hamiltonian systems is presented in [2], [14] and the references therein. In the sequel we will assume that $Q > 0$.

IV. REDUCTION OF PORT-HAMILTONIAN SYSTEMS BY THE RATIONAL ARNOLDI METHOD

A. Energy Coordinates, Transforming Q to the Identity Matrix

Consider a port-Hamiltonian system (4) with $A = (J - R)Q \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c = b^T Q \in \mathbb{R}^{1 \times n}$, $Q > 0$. Then there exists a coordinate transformation S , $x = Sx_I$, such that in the new coordinates $Q_I = S^T Q S = I$. By defining the transformed system matrices

$$J_I = S^{-1} J S^{-T}, R_I = S^{-1} R S^{-T}, b_I = S^{-1} b \quad (5)$$

we obtain the transformed port-Hamiltonian system with energy $H(x_I) = 1/2\|x_I\|^2$ and input-state-output representation

$$\begin{cases} \dot{x}_I = (J_I - R_I)x_I + b_I u, \\ y = b_I^T x_I. \end{cases} \quad (6)$$

Theorem 3: Consider a full order port-Hamiltonian system (6) and construct V_r satisfying (2) using the Arnoldi procedure. Then the r th order reduced system

$$\begin{cases} \dot{\hat{x}}_I = (\hat{J}_I - \hat{R}_I)\hat{x}_I + \hat{b}_I u, \\ \hat{y} = \hat{c}_I \hat{x}_I \end{cases} \quad (7)$$

is a port-Hamiltonian system with the interconnection matrices $\hat{J}_I = V_r^T J_I V_r$, $\hat{b}_I = V_r^T b_I$, energy matrix $\hat{Q}_I = I$, dissipation matrix $\hat{R}_I = V_r^T R_I V_r$ and output matrix $\hat{c}_I = b_I^T V_r$. Furthermore, the first r moments at $s_0 \in \mathbb{C}$ of the reduced order port-Hamiltonian system (7) and the full order port-Hamiltonian system (6) are equal: $(\hat{\eta}_I(s_0))_i = (\eta_I(s_0))_i = \eta_i(s_0)$, $i = 0, \dots, r - 1$.

Proof: Clearly \hat{J}_I is skew-symmetric and \hat{R}_I is symmetric and positive semi-definite. Moreover $\hat{c}_I = \hat{b}_I^T \hat{Q}_I$. Therefore the reduced order model (7) is port-Hamiltonian. The equality of the first r moments at $s_0 \in \mathbb{C}$, $(\hat{\eta}_I(s_0))_i = (\eta_I(s_0))_i$ follows directly from Theorem 1. The equality $(\eta_I(s_0))_i = \eta_i(s_0)$ is due to the fact that the moments are invariant under state-space coordinate transformations. ■

Remark 1: We do not need to compute coordinate transformation S explicitly. Instead, we compute matrices S^{-1}, S^{-T} , needed in (5), using the Cholesky factorization of the positive definite symmetric banded matrix $Q \in \mathbb{R}^{n \times n}$: $Q = LL^T = S^{-T} S^{-1}$. The algorithm from [3, p. 156], computes the factorization of Q using $n(p^2 + 3p)$ flops and n square roots, which is about $n(p^2 + 3p + 1)$ flops, where p is the lower (upper) bandwidth of Q .

A very conservative (and straightforward) flop count for the matrix product $S^{-1}(J - R)S^{-T}$ (for upper triangular S^{-1} with the upper bandwidth p , lower triangular S^{-T} with the lower bandwidth p and banded $J - R$ with the lower (upper) bandwidth f) reveals that

$$4n \left(f + p + \frac{1}{2} \right) (2p + 1) \quad (8)$$

flops are needed. Combining this count with the previous estimation gives a numerical cost of forming system (6) of

$$n(p^2 + 3p + 1) + 4n \left(f + p + \frac{1}{2} \right) (2p + 1) \quad (9)$$

flops.

It is explained in [1, p. 350], that the number of operations needed to compute an r -dimensional reduced system (7) is of order $\mathcal{O}(r^2 n)$ for sparse systems. This flop count, together with the flop count from (9), gives the overall numerical complexity of computing the reduced order model (7).

We would like to underline that the estimation in (8) is very conservative. Furthermore, in practical applications the matrix $J - R$ is very sparse while f is small, and Q is such that p is close to zero ($p = 0$ for diagonal Q). Thus in practice numerical cost of computing system matrices in (6) *does not* dramatically increase the numerical cost of the model reduction procedure.

Using the projection map W_r satisfying (3) instead of V_r in Theorem 3 we obtain a different, but analogous r th order reduced port-Hamiltonian system preserving the first r moments at $s_0 \in \mathbb{C}$:

$$\begin{cases} \dot{\bar{x}}_I = (\bar{J}_I - \bar{R}_I)\bar{x}_I + \bar{b}_I u, \\ \bar{y} = \bar{c}_I \bar{x}_I \end{cases} \quad (10)$$

with the port-Hamiltonian matrices $\bar{J}_I, \bar{R}_I, \bar{Q}_I, \bar{b}_I$ given as in Theorem 3 after substituting W_r for V_r .

In general, the reduced order models (7) and (10) obtained by applying the projection maps V_r , W_r , constructed using the rational Arnoldi method, are *not* equivalent.

Theorem 4: The reduced order port-Hamiltonian model (7) obtained using the projection map V_r based on the shifted input Krylov subspace $\mathcal{K}_{r, input}^{shifted}$ and the reduced order port-Hamiltonian model (10) obtained using the projection map W_r based on the shifted output Krylov subspace $\mathcal{K}_{r, output}^{shifted}$ are equivalent in the sense of sharing the same transfer function if the condition

$$\begin{aligned} & \text{span col} \mathcal{R}_r((FQ - s_0I)^{-1}, (FQ - s_0I)^{-1}b) \\ &= \text{span col} \mathcal{R}_r((F^TQ - s_0I)^{-1}, (F^TQ - s_0I)^{-1}b) \end{aligned} \quad (11)$$

for $F = J - R$ is satisfied.

Proof: Theorem is proven in [9] (see Theorem 6.6). ■

A different yet structure preserving approach to model reduction of port-Hamiltonian systems is considered in [18], where the reduced order moment matching port-Hamiltonian model is defined as $\hat{F} = V_r^T Q F Q V_r$, $\hat{b} = V_r^T Q b$, together with a reduced order energy matrix \hat{Q} which is not the identity matrix: $\hat{Q} = (V_r^T Q V_r)^{-1}$. In general, the projection matrix V_r used in [18] is different from V_r used in Theorem 3. In fact, it is shown in [9] (see Theorem 6.7) that the transfer function of the reduced order port-Hamiltonian model from [18] is equal to the transfer function of (7).

Remark 2: One possible choice to test the condition (11) is to verify that the columns of one of the reachability matrices from (11), attached to the other matrix, do not increase its rank

$$\begin{aligned} & \text{rank}[\mathcal{R}_r((FQ - s_0I)^{-1}, (FQ - s_0I)^{-1}b)] \\ & \mathcal{R}_r((F^TQ - s_0I)^{-1}, (F^TQ - s_0I)^{-1}b)] \\ &= \text{rank}[\mathcal{R}_r((FQ - s_0I)^{-1}, (FQ - s_0I)^{-1}b)]. \end{aligned} \quad (12)$$

The question about the (computationally) efficient test for the condition (11) is currently under investigation.

B. Co-Energy Coordinates

There are various ways to obtain a reduced order port-Hamiltonian model in so-called co-energy coordinates ([9])

$$\begin{cases} \dot{e} = Q(J - R)e + Qbu, \\ y = b^T e \end{cases} \quad (13)$$

either scaling the energy matrix Q , or taking it to the left side of the differential equation in (13). The reduced order port-Hamiltonian models in that case turn out to be equivalent to those in (7) and (10), in the sense of sharing the same transfer function. For the details and proofs see [9].

V. REDUCTION OF PORT-HAMILTONIAN SYSTEMS BY THE RATIONAL LANCZOS METHOD

In this section we show how the rational Lanczos algorithm preserves not only $2r$ moments at $s_0 \in \mathbb{C}$ but also the port-Hamiltonian structure for a subclass of port-Hamiltonian systems.

Theorem 5: Consider a full order port-Hamiltonian system (4) and construct V_r satisfying property (ii) of (2) such that $V_r^T Q V_r = I_r$. Then the r th order reduced system

$$\begin{cases} \dot{\hat{x}} = (\tilde{J} - \tilde{R})\hat{x} + \tilde{b}u, \\ \hat{y} = \tilde{c}\hat{x} \end{cases} \quad (14)$$

is a port-Hamiltonian system reduced by the rational Lanczos method with the interconnection matrices $\tilde{J}_I = V_r^T Q J Q V_r$, $\tilde{b}_I = V_r^T Q b$, energy matrix $\tilde{Q}_I = I$, dissipation matrix $\tilde{R}_I = V_r^T Q R Q V_r$, output matrix $\tilde{c}_I = b^T Q V_r$, and the projection map $W_r = Q V_r$, if condition (11) holds true. Furthermore the first $2r$ moments at $s_0 \in \mathbb{C}$ of the

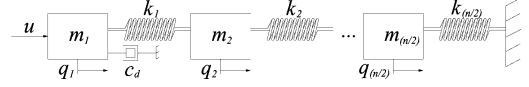


Fig. 1. n-dimensional mass-spring-damper system.

reduced order port-Hamiltonian system (14) and the full order port-Hamiltonian system (4) are equal: $\hat{\eta}_i(s_0) = \eta_i(s_0)$, $i = 0, \dots, 2r - 1$.

Proof: Theorem is proven in [9] (see Theorem 6.8). ■

This scheme of model reduction using the rational Lanczos method works as well in co-energy coordinates resulting in the reduced order port-Hamiltonian model which is equivalent to (14).

The next result establishes a relation between the reduced order port-Hamiltonian models obtained by both the rational Arnoldi and the rational Lanczos methods.

Theorem 6: The reduced order port-Hamiltonian model (7) in energy coordinates obtained by the rational Arnoldi method and the reduced order port-Hamiltonian model (14) in energy coordinates obtained by the rational Lanczos method share the same transfer function if condition (11) is satisfied.

Proof: The proof is similar to the proof of Theorem 6.6 in [9], hence omitted. ■

Corollary 1: A natural conclusion of Theorem 6 is that for a subclass of port-Hamiltonian systems, characterized by the condition (11), the rational Arnoldi method matches twice as many moments of the original system at $s_0 \in \mathbb{C}$ as it does for a general linear system.

Note that for an important point $s_0 = 0$ condition (11) specializes to

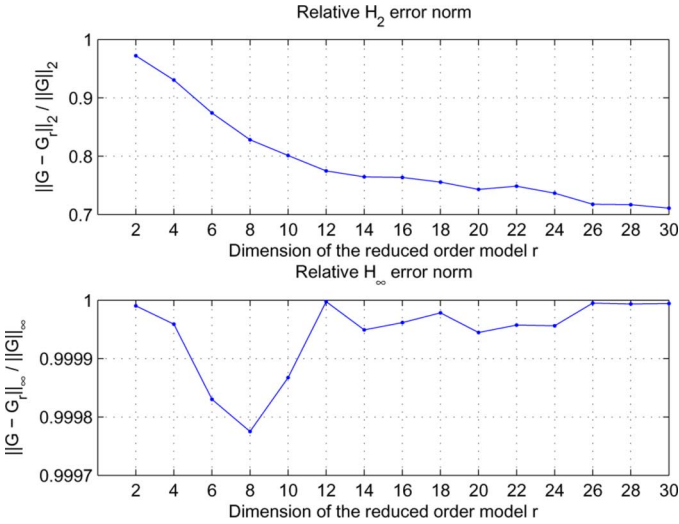
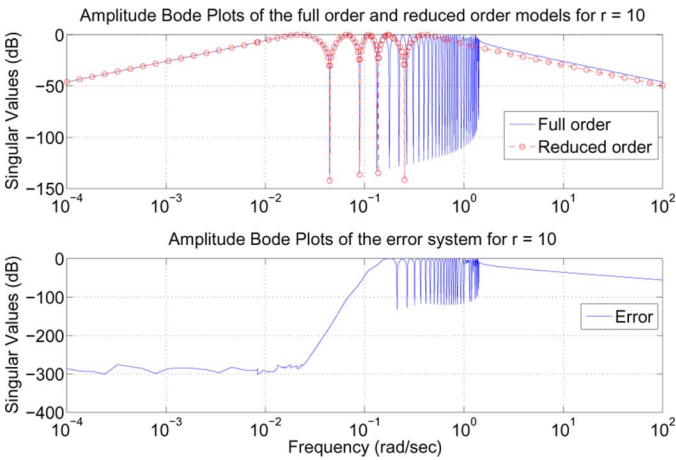
$$\begin{aligned} & \text{span col} \mathcal{R}_r(F^{-1}Q^{-1}, F^{-1}b) \\ &= \text{span col} \mathcal{R}_r(F^{-T}Q^{-1}, F^{-T}b). \end{aligned} \quad (15)$$

VI. NUMERICAL EXAMPLE

Consider an n-dimensional mass-spring-damper system as shown in Fig. 1 with masses m_i and spring constants k_i , for $i = 1, \dots, n/2$. A damper with a damping constant $c_d \geq 0$ is attached only to the first mass m_1 . p_i and q_i are the momentum and displacement of the mass m_i , respectively. The input u is the external force acting on the first mass m_1 , while the output y is the velocity of the mass m_1 . State variables are defined in the following way: for $i = 1, \dots, n/2$, $x_{2i-1} = q_i$ and $x_{2i} = p_i$.

A minimal realization of this system for order $n = 6$ (corresponding to three masses with one damper and three springs) is

$$\begin{aligned} J &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \\ b &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ c &= \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \end{aligned}$$


 Fig. 2. Evolution of the relative \mathcal{H}_2 and \mathcal{H}_∞ norms.

 Fig. 3. Amplitude Bode plots for $r = 10$.

$$Q = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2 + k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \end{bmatrix}$$

with $R = \text{diag}(0 \ c_d \ 0 \ 0 \ 0 \ 0)$.

We considered a 100-dimensional mass-spring-damper system with $m_i = 2$, $k_i = 1$, and $c_d = 1$. We applied the rational Arnoldi method as shown in Theorem 3 with the approximation point $s_0 = 0$. The reduced order systems are constructed for the orders from $r = 2$ to $r = 30$ with increments of 2. Evolution of the relative \mathcal{H}_2 and \mathcal{H}_∞ norms is shown in Fig. 2. The \mathcal{H}_2 relative norm decays as the dimension r of the reduced order systems increases, whereas the \mathcal{H}_∞ relative norm is almost constant. This effect can be explained by the lack of damping in the system (the \mathcal{H}_∞ relative norm is known to have weakly decreasing behavior for poorly damped systems). Reduced order systems inherit the port-Hamiltonian structure, are asymptotically stable and passive.

The magnitude Bode plots of the full, reduced order with $r = 10$, and error systems are shown in Fig. 3. The figure exhibits that the approximation is very accurate for small frequencies and the error is accumulated for high frequencies. This is to be expected since the moments are matched at $s_0 = 0$. The magnitude plot of the reduced order system

captures the first peaks and zeros of that of the full order system. The model reduction scheme preserves at least the first r moments of the full order transfer function at zero.

In fact, for the mass-spring-damper system considered here condition (15), which is a special case of the condition (11) at $s_0 = 0$, is satisfied. Therefore even though the reduced order port-Hamiltonian model is obtained using the rational Arnoldi method from Theorem 3, it is equivalent to that of the rational Lanczos method, as explained in Theorem 6. Moreover, due to Corollary 1 the reduced order port-Hamiltonian model preserves $2r$ moments at zero, which can be readily checked for the particular case when r , for instance, is equal to 2: $(\eta_1(0) \ \dots \ \eta_{2r}(0)) = (0 \ -50 \ 2500 \ -39150) = (\hat{\eta}_1(0) \ \dots \ \hat{\eta}_{2r}(0))$.

VII. CONCLUSION

In this technical note we used the rational Krylov methods to produce reduced order models which are port-Hamiltonian. We showed how the rational Arnoldi method can be employed for this purpose in energy and co-energy coordinates using the projection maps constructed both on the shifted input and output Krylov subspaces.

The rational Lanczos method can be applied in a structure preserving way only to a subclass of port-Hamiltonian systems, characterized by an algebraic condition. For this subclass of systems all the reduced order models in this technical note share the same transfer function. Consequently, the rational Lanczos method is proven to produce a reduced order port-Hamiltonian model which is equivalent to that of the rational Arnoldi method. Therefore the rational Arnoldi method applied to a port-Hamiltonian system from the subclass preserves twice as many moments at an arbitrary point in the complex plane as it does for a general linear system.

Both considered methods preserve the port-Hamiltonian structure, implying, among others, the passivity property, and, therefore, stability.

Important questions concerning general error bounds for the structure preserving port-Hamiltonian model reduction methods, numerical efficiency and the physical realization of the obtained port-Hamiltonian reduced order models, as well as systematic characterization of the subclasses of port-Hamiltonian systems, are currently under investigation.

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A Quasi-Newton Interior Point Method for Low Order H-Infinity Controller Synthesis

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Abstract—This technical note proposes a method for low order H-infinity synthesis where the constraint on the order of the controller is formulated as a rational equation. The resulting nonconvex optimization problem is then solved by applying a quasi-Newton primal-dual interior point method.

The proposed method is evaluated together with a well-known method from the literature. The results indicate that the proposed method has comparable performance and speed.

Index Terms—H-infinity synthesis, interior point methods, linear matrix inequalities (LMIs), quasi-Newton methods, rank constraints, rational constraints.

I. INTRODUCTION

The development of robust control theory emerged during the 80s and a contributory factor certainly was the fact that the robustness of

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linear quadratic Gaussian (LQG) controllers can be arbitrarily bad as reported in [1]. A few years later, in [2], an important step in the development towards a robust control theory was taken, where the concept of H_∞ theory was introduced. The H_∞ synthesis, which is an important tool when solving robust control problems, was a cumbersome problem to solve until a technique was presented in [3], which is based on solving two Riccati equations. Using this method, the robust design tools became much easier to use and gained popularity. Quite soon thereafter, linear matrix inequalities (LMIs) were found to be a suitable tool for solving these kinds of problems by using reformulations of the Riccati equations. Also related problems, such as gain scheduling synthesis, fit into the LMI framework, see, e.g., [4]. In parallel to the theory for solving problems using LMIs, numerical methods for this purpose were being developed and made available.

Typical applications for robust control include systems that have high requirements for robustness to parameter variations and for disturbance rejection. The controllers that result from these algorithms are typically of very high order, which complicates implementation. However, if a constraint on the maximum order of the controller is set, that is lower than the order of the augmented plant, the problem is no longer convex and it is then relatively hard to solve. These problems become very complex, even when the order of the system to be controlled is low. This motivates the development of efficient algorithms that can solve these kinds of problems.

When evaluating the proposed method, we iterate through orders and performance. Even if we can use the method to find a controller of a given order directly, evaluating the trade-off between controller order and performance is natural from an engineering perspective. In this way we can find a controller which gives a good balance between complexity and performance, even if the computational effort during the design becomes higher.

This article is based on the work in [5], but the difference is that the proposed algorithm in this work uses an approximate quasi-Newton update of the Hessian instead of a regularized, exact Hessian. New numerical results are also included to demonstrate the improved performance of the new algorithm.

Denote with \mathcal{S}^n the set of symmetric $n \times n$ matrices and with $\mathbb{R}^{m \times n}$ the set of real $m \times n$ matrices. The notation $A \succ 0$ ($A \succeq 0$) and $A \prec 0$ ($A \preceq 0$) means that A is a positive (semi) definite matrix and negative (semi)definite matrix, respectively. Also, denote the symmetric vectorization operator by svec and the symmetric Kronecker product by \otimes_s , as defined in [6].

II. PRELIMINARIES

We begin by describing a linear system, G , with state vector, $x \in \mathbb{R}^n$. The input vector contains the disturbance signal, $w \in \mathbb{R}^w$, and the control signal, $u \in \mathbb{R}^u$. The output vector contains the measurement, $y \in \mathbb{R}^y$, and the performance measure, $z \in \mathbb{R}^z$. In terms of its system matrices, we can represent the linear system as

$$G : \begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix} \quad (1)$$

where D_{22} is assumed to be zero, i.e., the system is strictly proper from u to y . If this is not the case, we can find a controller \tilde{K} for the system where D_{22} is set to zero, and then construct the controller as $K = \tilde{K}(I + D_{22}\tilde{K})^{-1}$. Hence, there is no loss of generality in making this assumption. For simplicity, it is also assumed that the whole system is on minimal form, i.e., it is both observable and controllable. However, in order to find a controller, it is enough to assume detectability and stabilizability (non observable and non controllable modes are stable).