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Abstract. We define and solve a boundary interpolation problem for generalized Schur functions $s(z)$ on the open unit disk $\mathbb{D}$ which have preassigned asymptotics when z from $\mathbb{D}$ tends nontangentially to a boundary point $z_1 \in \mathbb{T}$. The solutions are characterized via a fractional linear parametrization formula. We also prove that a rational J-unitary 2×2-matrix function whose only pole is at z_1 has a unique minimal factorization into elementary factors and we classify these factors. The parametrization formula is then used in an algorithm for obtaining this factorization. In the proofs we use reproducing kernel space methods.

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1. Introduction

Recall that $s(z)$ is a generalized Schur function with κ negative squares (for the latter we write $\mathrm{sq}_-(s) = \kappa$, if it is holomorphic in a nonempty open subset of the open unit disk $\mathbb{D}$ and if the kernel

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \mathcal{D}(s), \tag{1.1}$$

has κ negative squares on $\mathcal{D}(s)$, the domain of holomorphy of $s(z)$. We denote the class of generalized Schur functions $s(z)$ with $\mathrm{sq}_-(s) = \kappa$ by $\mathbf{S}_\kappa$ and set $\mathbf{S} =$

$\cup_{\kappa \geq 0} \mathbf{S}_\kappa$. The function $s(z) \in \mathbf{S}_0$ has a holomorphic and contractive continuation to all of $\mathbb{D}$ and is called a (classical) Schur function. In fact, the following three statements are equivalent:

(a) $s(z) \in \mathbf{S}_0$.
(b) $s(z)$ is holomorphic on $\mathbb{D}$ and bounded by 1 there.
(c) $s(z)$ has the form

$$s(z) = \gamma z^n \prod_j \frac{|\alpha_j|}{\alpha_j} \frac{z - \alpha_j}{1 - \alpha_j^* z} \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right), \tag{1.2}$$

where n is a nonnegative integer, the α_j's are the zeros of $s(z)$ in $\mathbb{D} \setminus \{0\}$ repeated according to multiplicity, γ is a number of modulus one, and $\mu(t)$ is a nondecreasing bounded function on $[0, 2\pi]$. The Blaschke product on the right-hand side of the first equality in (1.2) is finite or infinite and converges on $\mathbb{D}$, because $\sum_j (1 - |\alpha_j|) < \infty$.

By a result of M.G. Krein and H. Langer [24], a function $s(z) \in \mathbf{S}_\kappa$ has a meromorphic extension to $\mathbb{D}$ and can be written as

$$s(z) = \left(\prod_{j=1}^{\kappa} \frac{z - \beta_j}{1 - \beta_j^* z}\right)^{-1} s_0(z), \tag{1.3}$$

where $s_0(z) \in \mathbf{S}_0$, and the zeros β_j of the Blaschke product of order κ belong to $\mathbb{D}$ and satisfy $s_0(\beta_j) \neq 0$, $j = 1, \ldots, \kappa$. Conversely, every function $s(z)$ of the form (1.3) belongs to $\mathbf{S}_\kappa$. It follows from (1.3) that any function $s(z) \in \mathbf{S}$ has nontangential boundary values from $\mathbb{D}$ in almost every point of the unit circle $\mathbb{T}$. In particular, a rational function $s(z) \in \mathbf{S}$ of modulus one on $\mathbb{T}$ is holomorphic on $\mathbb{T}$, and it is the quotient of two finite Blaschke products.

A nonconstant function $s(z) \in \mathbf{S}_0$ has in $z_1 \in \mathbb{T}$ a Carathéodory derivative , if the limits

$$\tau_0 = \lim_{z \hat{\to} z_1} s(z) \text{ with } |\tau_0| = 1, \quad \tau_1 = \lim_{z \hat{\to} z_1} \frac{s(z) - \tau_0}{z - z_1} \tag{1.4}$$

exist, and then

$$\lim_{z \hat{\to} z_1} s'(z) = \tau_1.$$

Here and in the sequel $z \hat{\to} z_1$ means that z tends from $\mathbb{D}$ non-tangentially to z_1. The relation (1.4) is equivalent to the fact that the limit

$$\lim_{z \hat{\to} z_1} \frac{1 - |s(z)|}{1 - |z|}$$

exists and is finite and positive; in this case it equals $\tau_0^* \tau_1 z_1$; see [33, p. 48]. The following basic boundary interpolation problem for Schur functions is a particular case of a multi-point interpolation problem considered by D. Sarason in [34]: Given

$z_1 \in \mathbb{T}$ and numbers $\tau_0, \tau_1, |\tau_0| = 1$, such that $\tau_0^* \tau_1 z_1$ is positive. Find all functions $s(z) \in \mathbf{S}_0$ such that the Carathéodory derivative of $s(z)$ in z_1 exists and

$$\lim_{z \hat{\to} z_1} s(z) = \tau_0, \qquad \lim_{z \hat{\to} z_1} \frac{s(z) - \tau_0}{z - z_1} = \tau_1.$$

The study of the Schur transformation for generalized Schur functions in [14], [1], and [3] motivates the generalization of this basic interpolation problem for generalized Schur functions, which we consider in this note.

Problem 1.1. *Let $z_1 \in \mathbb{T}$, an integer $k \geq 1$, and complex numbers τ_0, τ_k, τ_{k+1}, ..., τ_{2k-1} with $|\tau_0| = 1$, $\tau_k \neq 0$ be given. Find all functions $s(z) \in \mathbf{S}$ such that*

$$s(z) = \tau_0 + \sum_{i=k}^{2k-1} \tau_i (z - z_1)^i + \mathrm{O}((z - z_1)^{2k}), \quad z \hat{\to} z_1. \tag{1.5}$$

We solve this problem under the assumption that the matrix

$$\mathbb{P} := \tau_0^* T B \tag{1.6}$$

is Hermitian, where

$$T = \begin{pmatrix} \tau_k & 0 & \cdots & 0 & 0 \\ \tau_{k+1} & \tau_k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{2k-2} & \tau_{2k-3} & \cdots & \tau_k & 0 \\ \tau_{2k-1} & \tau_{2k-2} & \cdots & \tau_{k+1} & \tau_k \end{pmatrix} \tag{1.7}$$

and

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{k-1}\binom{k-1}{0}z_1^{2k-1} \\ 0 & 0 & \cdots & (-1)^{k-2}\binom{k-2}{0}z_1^{2k-3} & (-1)^{k-1}\binom{k-1}{1}z_1^{2k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\binom{1}{0}z_1^3 & \cdots & (-1)^{k-2}\binom{k-2}{k-3}z_1^k & (-1)^{k-1}\binom{k-1}{k-2}z_1^{k+1} \\ z_1 & -\binom{1}{1}z_1^2 & \cdots & (-1)^{k-2}\binom{k-2}{k-2}z_1^{k-1} & (-1)^{k-1}\binom{k-1}{k-1}z_1^k \end{pmatrix}. \tag{1.8}$$

Evidently, for $k = 1$ the expression in (1.6) reduces to $\tau_0^* \tau_1 z_1$ from above. In Theorem 3.2 we describe all solutions of this problem by a parametrization formula of the form

$$s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)}, \quad \Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \tag{1.9}$$

where the parameter $s_1(z)$ runs through a subclass of $\mathbf{S}$. The matrix function $\Theta(z)$ is rational with a single pole at $z = z_1$ and J-unitary on $\mathbb{T}$ for

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that a rational 2×2-matrix function $\Theta(z)$ is J-unitary on $\mathbb{T}$ if

$$\Theta(z)J\Theta(z)^* = J, \quad z \in \mathbb{T} \setminus \{\text{poles of } \Theta(z)\}.$$

We prove the description (1.9) of the solutions of the Problem 1.1 by making use of the theory of reproducing kernel Pontryagin spaces, see [19], [4], [5], [6] for the positive definite (Hilbert space) case and [2], [3] for the indefinite case. The essential tool is a representation theorem for reproducing kernel Pontryagin spaces which will be formulated at the end of this Introduction.

Boundary interpolation problems for classical Schur functions have been studied by I.V. Kovalishina in [23], [22], by J.A. Ball, I. Gohberg, and L. Rodman in [12, Section 21] and by D. Sarason [34], and for generalized Schur functions which are holomorphic at the interpolation points by J.A. Ball in [11]. In these papers different methods were used: the fundamental matrix inequality, realization theory and extension theory of operators.

Problem 1.1 is similar to the basic interpolation problem for generalized Schur functions at the point $z = 0$ considered in [3]. There, given an arbitrary complex number σ_0, one looks for generalized Schur functions $s(z)$ which are holomorphic in $z = 0$ and satisfy $s(0) = \sigma_0$. In the case that $|\sigma_0| = 1$ a certain number of derivatives has to be preassigned in order to find all solutions. In Problem 1.1 this additional information comes from the preassigned values τ_j, $j = k, k+1, \ldots, 2k-1$, and $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0$.

The Problem 1.1 is equivalent to a basic boundary interpolation problem for generalized Nevanlinna functions at infinity, where one looks for the set of all generalized Nevanlinna functions $N(\zeta)$ with an asymptotics of the form

$$N(\zeta) = -\frac{s_0}{\zeta} - \frac{s_1}{\zeta^2} - \cdots - \frac{s_{2k-2}}{\zeta^{2k-1}} + \mathrm{O}\left(\frac{1}{\zeta^{2k}}\right), \quad \zeta = i\eta, \ \eta \to \infty.$$

In fact, these problems can be transformed into each other via Cayley transformation, and we mention that the cases $\tau_0^* \tau_1 z_1 > 0$, $= 0$, or < 0 correspond to the cases $s_0 > 0$, $= 0$, or < 0, respectively, and the hermiticity of the matrix $\mathbb{P}$ in (1.6) corresponds to the reality of the moments s_j. On the other hand, each of these problems has special features and it seems reasonable to study them also separately. Moreover, the boundary interpolation problem for generalized Nevanlinna functions at infinity is equivalent to the indefinite power moment problem as considered in (see [25], [26], [27], [28] [17], [18]). We shall come back to the basic versions of these problems in another publication.

Basic interpolation problems are closely related to the problem of decomposing a rational J-unitary 2×2-matrix function as a minimal product of elementary factors. For the positive definite case these results go back to V.P. Potapov ([30], [31] and the joint paper [20] with A.V. Efimov); see also L. de Branges [16, Problem 110, p 116]. In the indefinite case, for a J-unitary matrix function on the circle $\mathbb{T}$ with poles in $\mathbb{D}$ this was done in [2], and for the line case in [7]. Here we prove a corresponding factorization result for a rational J-unitary 2×2-matrix function $\Theta(z)$ with a single pole on the boundary $\mathbb{T}$ of $\mathbb{D}$. In fact, with the given

matrix function $\Theta(z)$ a basic boundary interpolation problem can be associated, such that the matrix function which appears in the description of its solutions is an elementary factor of $\Theta(z)$.

A short outline of the paper is as follows. In Section 2 we study the asymptotic behavior of the kernel $K_s(z, w)$ near z_1 for a generalized Schur function $s(z)$ which has an asymptotic behavior (1.5) with not necessarily vanishing coefficients $\tau_1, \ldots, \tau_{k-1}$. It turns out, that an expansion of $s(z)$ up to an order $2k$ implies a corresponding expansion of the kernel up to an order $2k - 1$ only if a certain matrix $\mathbb{P}$ is Hermitian. This matrix $\mathbb{P}$, in some interpolation problems called the Pick or Nevanlinna matrix, is the essential ingredient for the solution of the basic interpolation problem. It satisfies the so-called Stein equation (see (2.17)) which is a basic tool for the definition of the underlying reproducing kernel spaces.

In Section 3 the main result of the paper (Theorem 3.2) is proved, which contains the solution of Problem 1.1. In Section 4 we consider a basic boundary interpolation problem with data given in several points $z_1, z_2, \ldots, z_N$ of the circle $\mathbb{T}$ and describe all its solutions via a parametrization formula. In Section 5 the existence of a minimal factorization of a J-unitary matrix function on $\mathbb{T}$ with a single pole on $\mathbb{T}$ is proved. Finally, in Section 6 we show how by means of the Schur algorithm, based on the parametrization formula of Theorem 3.2, such a minimal factorization can be obtained.

For the convenience of the reader we formulate here a basic representation theorem for reproducing kernel Pontryagin spaces, see [9], which will be essentially used in this paper. Infinite-dimensional versions of this result were proved by L. de Branges [15] and J. Rovnyak [29] for the line case, and by J.A. Ball [10] for the circle case. For a rational J-unitary 2×2-matrix function $\Theta(z)$ on $\mathbb{D}$ we denote by $\mathcal{P}(\Theta)$ the reproducing kernel Pontryagin space with reproducing kernel

$$K_\Theta(z, w) = \frac{J - \Theta(z) J \Theta(w)^*}{1 - zw^*}, \qquad z, w \in \mathcal{D}(\Theta).$$

Theorem 1.2. . *Let $\mathcal{M}$ be a finite-dimensional reproducing kernel Pontryagin space. Then $\mathcal{M} = \mathcal{P}(\Theta)$ for some rational J-unitary 2×2-matrix function $\Theta(z)$ which is holomorphic at $z = 0$ if and only if the following three conditions hold:*

(1) *The elements of $\mathcal{M}$ are 2-vector functions holomorphic at $z = 0$.*
(2) *$\mathcal{M}$ is invariant under the difference quotient operator*

$$(R_0 f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathcal{M}.$$

(3) *The following identity holds:*

$$\langle f, g \rangle_\mathcal{M} - \langle R_0 f, R_0 g \rangle_\mathcal{M} = g(0)^* J f(0), \quad f, g \in \mathcal{M}. \tag{1.10}$$

In this case $\mathcal{M}$ is spanned by the elements of the form $R_0^n \Theta(z) \mathbf{c}$, where n runs through the integers ≥ 1 and $\mathbf{c}$ through $\mathbb{C}^2$.

In the sequel, for $s(z) \in \mathbf{S}$ we denote by $\mathcal{P}(s)$ the reproducing kernel Pontryagin space with reproducing kernel $K_s(z, w)$ given by (1.1). The negative index of this space equals the number of negative squares of $s(z)$.

2. Auxiliary statements

For given numbers $\tau_0, \tau_1, \ldots, \tau_{2k-1}$ we introduce the following $k \times k$-matrices:

$$\widehat{T} = (t_{\ell r})_{\ell, r=0}^{k-1}, \quad t_{\ell r} = \tau_{\ell+r+1}, \tag{2.1}$$

$$\widehat{B} = (b_{rs})_{r,s=0}^{k-1}, \quad b_{rs} = z_1^{k+r-s} \binom{k-1-s}{r} (-1)^{k-1-s}, \tag{2.2}$$

and

$$Q = (c_{sm})_{s,m=0}^{k-1}, \quad c_{sm} = \tau_{s+m-(k-1)}^*. \tag{2.3}$$

Here $\widehat{B}$ is a left upper, Q is a right lower triangular matrix.

Lemma 2.1. *Suppose that the function $s(z) \in \mathbf{S}$ has the asymptotic expansion*

$$s(z) = \tau_0 + \sum_{\ell=1}^{2k-1} \tau_\ell (z - z_1)^\ell + \mathrm{O}\left((z - z_1)^{2k}\right), \quad z \widehat{\rightarrow} z_1, \tag{2.4}$$

with $|\tau_0| = 1$, and that the matrix $\mathbb{P} := \widehat{T}\widehat{B}Q$ is Hermitian. Then the kernel $K_s(z, w)$ has the asymptotic expansion

$$K_s(z, w) = \sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m} (z - z_1)^\ell (w - z_1)^{*m}$$

$$+ \mathrm{O}\left((\max\{|z - z_1|, |w - z_1|\})^{2k-1}\right), \quad z, w \widehat{\rightarrow} z_1, \tag{2.5}$$

where the coefficients $\alpha_{\ell m}$ for $0 \le \ell, m \le k-1$ are the entries of the matrix $\mathbb{P}$: $\mathbb{P} = (\alpha_{\ell m})_{\ell, m=0}^{k-1}$.

Proof. The asymptotic expansion (2.5) will follow if we show that the relation

$$1 - s(z)s(w)^* - \sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m} (z - z_1)^\ell (w - z_1)^{*m} (1 - zw^*)$$

$$= \mathrm{O}\left((\max\{|z - z_1|, |w - z_1|\})^{2k}\right) \tag{2.6}$$

holds, where the symbol O refers again to the non-tangential limit $z, w \widehat{\rightarrow} z_1$. To see this we consider only the radial limits of z and w and observe that then for z and w sufficiently close to z_1 the relation

$$|1 - zw^*| \ge \max\{|z - z_1|, |w - z_1|\}$$

holds. Dividing (2.6) by $1 - zw^*$ we obtain

$$K_s(z, w) - \sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m} (z - z_1)^\ell (w - z_1)^{*m} = \frac{\mathrm{O}\left((\max\{|z - z_1|, |w - z_1|\})^{2k}\right)}{\max\{|z - z_1|, |w - z_1|\}},$$

and this is (2.5).

To prove (2.6) we set $u = z - z_1$, $v = w^* - z_1^*$. Then the expression on the left-hand side of (2.6) becomes

$$1 - \left(\tau_0 + \tau_1 u + \tau_2 u^2 + \cdots + \mathrm{O}(u^{2k})\right)\left(\tau_0^* + \tau_1^* v + \tau_2^* v^2 + \cdots + \mathrm{O}(v^{2k})\right) \tag{2.7}$$
$$- \sum_{0 \le \ell + m \le 2k-2} \alpha_{\ell m} u^\ell v^m (-u z_1^* - v z_1 - uv).$$

Comparing coefficients we find that the following relations are equivalent for (2.6) to hold:

$$u: \quad \tau_0^* \tau_1 = \alpha_{00} z_1^*, \qquad v: \quad \tau_0 \tau_1^* = \alpha_{00} z_1, \tag{2.8}$$

$$u^2: \quad \tau_2 \tau_0^* = \alpha_{10} z_1^*, \quad uv: \quad \tau_1^* \tau_1 = \alpha_{00} + \alpha_{01} z_1^* + \alpha_{10} z_1, \quad v^2: \quad \tau_0 \tau_2^* = \alpha_{01} z_1, \tag{2.9}$$

$$u^3: \quad \tau_3 \tau_0^* = \alpha_{20} z_1^*, \quad u^2 v: \quad \tau_2 \tau_1^* = \alpha_{10} + \alpha_{11} z_1^* + \alpha_{20} z_1,$$

$$uv^2: \quad \tau_1 \tau_2^* = \alpha_{01} + \alpha_{11} z_1 + \alpha_{02} z_1^*, \quad v^3: \quad \tau_0 \tau_3^* = \alpha_{02} z_1,$$

etc. The general relation is

$$\tau_\ell \tau_m^* = \alpha_{\ell-1,m} z_1^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1}, \tag{2.10}$$
$$\ell, m = 0, 1, \ldots, 2k-2, \ 1 \le \ell + m \le 2k-2,$$

where all $\alpha's$ with one index $= -1$ are set equal to zero, and we have to find solutions $\alpha_{\ell m}$ of this system (2.10). The relation (2.10) can be written as

$$\alpha_{\ell m} = -z_1^* \alpha_{\ell-1,m} - z_1^{*2} \alpha_{\ell-1,m+1} + z_1^* \tau_\ell \tau_{m+1}^*, \quad 0 \le \ell + m \le 2k-2, \tag{2.11}$$

and also as

$$\alpha_{\ell m} = -z_1 \alpha_{\ell,m-1} - z_1^2 \alpha_{\ell+1,m-1} + z_1 \tau_{\ell+1} \tau_m^*, \quad 0 \le \ell + m \le 2k-2. \tag{2.12}$$

The numbers $\alpha_{\ell m}$, $0 \le \ell + m \le 2k-2$ in (2.6) or (2.10) can be considered as the entries of a left upper triangular matrix $\widetilde{\mathbb{P}}$, which has the matrix $\mathbb{P}$ as its left upper $k \times k$ diagonal block. According to the assumption, $\mathbb{P}$ is a Hermitian matrix. The elements of the last row of $\mathbb{P}$ determine according to (2.11) the left lower $k \times k$ block of $\widetilde{\mathbb{P}}$, which is a left upper triangular matrix, and, similarly, the last column of $\mathbb{P}$ determines by the relations (2.10) the right upper $k \times k$ block of $\widetilde{\mathbb{P}}$. These relations and the hermiticity of $\mathbb{P}$ imply that also the matrix $\widetilde{\mathbb{P}}$ is Hermitian.

From (2.12) we find successively

$$\alpha_{\ell 0} = \tau_0^* z_1 \tau_{\ell+1}, \qquad\qquad\qquad \ell = 0, \ldots, 2k-2,$$

$$\alpha_{\ell 1} = \tau_1^* z_1 \tau_{\ell+1} - \tau_0^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}), \qquad\qquad \ell = 0, \ldots, 2k-3,$$

$$\alpha_{\ell 2} = \tau_2^* z_1 \tau_{\ell+1} - \tau_1^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}) + \tau_0^* (z_1^3 \tau_{\ell+1} + 2z_1^4 \tau_{\ell+2} + z_1^5 \tau_{\ell+3}),$$
$$\ell = 0, \ldots, 2k-4,$$

$$\alpha_{\ell 3} = \tau_3^* z_1 \tau_{\ell+1} - \tau_2^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}) - \tau_1^* (z_1^3 \tau_{\ell+1} + 2z_1^4 \tau_{\ell+2} + z_1^5 \tau_{\ell+3})$$
$$- \tau_0^* (z_1^3 \tau_{\ell+1} + 3z_1^4 \tau_{\ell+2} + 3z_1^5 \tau_{\ell+3} + z_1^6 \tau_{\ell+4}), \qquad \ell = 0, \ldots, 2k-5, \tag{2.13}$$

and so for $m = 0, \ldots, 2k - 2$, we have

$$\alpha_{\ell m} = \sum_{s=0}^{m} \tau_{m-s}^* \sum_{r=0}^{s} (-1)^s \binom{s}{r} z_1^{s+r+1} \tau_{\ell+r+1}, \quad \ell = 0, \ldots, 2k - 2 - m.$$

With the convention that $\tau_\ell = 0$ for $\ell < 0$, observing that $\binom{s}{r} = 0$ if $r > s$, and substituting s by $k - 1 - s$ we find for $0 \leq \ell, m \leq k - 1$

$$\alpha_{\ell m} = \sum_{r,s=0}^{k-1} \tau_{\ell+r+1} (-1)^{k-1-s} \binom{k-1-s}{r} z_1^{k-s+r} \tau_{m+s-(k-1)}^* = \sum_{r,s=0}^{k-1} t_{\ell r} b_{rs} c_{sm}$$

and hence (see (2.1)–(2.3))

$$(\alpha_{\ell m})_{\ell,m=0}^{k-1} = \widehat{T}\widehat{B}Q.$$

These considerations also imply that if a solution of the equations (2.10) exists, it is unique.

As to the existence of a solution, the first relation in (2.13) determines the elements of the first column of $\widetilde{\mathbb{P}}$, and the following columns are successively determined by the other relations of (2.13) or by (2.12). Because of the symmetry of $\widetilde{\mathbb{P}}$, the resulting elements $\alpha_{0\ell}$ are the complex conjugates of $\alpha_{\ell 0}$, $\ell = 1, 2, \ldots, 2k - 2$, and α_{00} is real. Thus, these $\alpha's$ satisfy all the relations of the system (2.10) and hence are its unique solution. □

The relation (2.10) implies that

$$\alpha_{\ell-1,m} z_1^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1} = \tau_\ell \tau_m^*, \quad 1 \leq \ell, m \leq k - 1. \quad (2.14)$$

If we introduce the $k \times k$-matrices

$$S_k = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad A = z_1^* I_k + S_k, \quad (2.15)$$

and the $2 \times k$-matrix

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_0^* & \tau_1^* & \cdots & \tau_{k-1}^* \end{pmatrix}, \quad (2.16)$$

then the relation (2.14) is equivalent to the relation (2.17) below, and hence we have:

Corollary 2.2. *Under the assumptions of Lemma* 2.1 *the matrix* $\mathbb{P}$ *satisfies the Stein equation*

$$\mathbb{P} - A^* \mathbb{P} A = C^* J C. \quad (2.17)$$

Remark 2.3. 1) Formula (2.8) implies a condition on τ_0 and τ_1: the number $\tau_0^* \tau_1 z_1$ has to be real. As was mentioned in the Introduction, for Schur functions this number must be nonnegative if it is finite. In (2.9) the first and the last equation determine α_{10} and α_{01}, the second equation is an additional condition. Similarly in the relations following (2.9): the first and last equation determine α_{20} and α_{02}, then there are 2 equations left for to determine α_{11}. These additional conditions are automatically satisfied since the matrix $\mathbb{P}$ is Hermitian.

2) If the equations (2.10) have a solution $\alpha_{\ell m}$, $0 \leq \ell + m \leq 2k - 2$, then these numbers must be symmetric in the sense that $\alpha_{\ell m} = \alpha_{m\ell}^*$, $0 \leq \ell + m \leq 2k - 2$, since they are the coefficients of the expansion of the Hermitian kernel $K_s(z, w)$.

3) For a function $s(z) \in \mathbf{S}$ with an expansion (2.4), such that the corresponding matrix $\mathbb{P}$ is not Hermitian, the kernel $K_s(z, w)$ does in general not have an expansion (2.5). An example is the function

$$s(z) = 1 + \frac{1}{2}(z - 1),$$

which has at $z = 1$ an expansion (2.4) with any $k \geq 1$ but for the corresponding kernel we obtain, for example, for real z, w,

$$K_s(z, w) = \frac{1}{2} + \frac{1}{4} \frac{(z - 1)(w^* - 1)}{1 - zw^*} = \frac{1}{2} + \mathrm{O}\big(\max\{|1 - z|, |1 - w|\}\big),$$

and the order of the last term cannot be improved. For this example it holds

$$\mathbb{P} = \begin{cases} \quad 1/2 & k = 1, \\ \begin{pmatrix} 1/2 & -1/4 \\ 0 & 0 \end{pmatrix} & k = 2. \end{cases}$$

4) For a function $s(z)$ which is analytic on an arc around z_1 and has values of modulus one on this arc the matrices $\mathbb{P}$ are Hermitian for all k and the kernel $K_s(z, w)$ is analytic in z and w^* near $z = w = z_1$. To see this we observe that the function $s(z)$ satisfies in some neighborhood of this arc the relation $s(1/z^*) = 1/s(z)^*$. Now it follows that in this neighborhood, for each fixed w the function $K_s(\,\cdot\,, w)$ and for each fixed z the function $K_s(z, \,\cdot\,)^*$ is holomorphic. According to a theorem of Hartogs [32, Theorem 16.3.1] the kernel $K_s(z, w)$ is holomorphic in z and w and the claim follows. We mention, that a function $s(z) \in \mathbf{S}_\kappa$ has the above properties if and only if in its representation (see (1.2) and (1.3))

$$s(z) = \left(\prod_{j=1}^{\kappa} \frac{z - \beta_j}{1 - \beta_j^* z} \right)^{-1} \gamma z^n \prod_j \frac{|\alpha_j|}{\alpha_j} \frac{z - \alpha_j}{1 - \alpha_j^* z} \, \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right)$$

the nondecreasing function $\mu(t)$ is constant at t_1 where $z_1 = \exp(it_1)$. In particular, all rational functions in $\mathbf{S}$, which are of modulus one on $\mathbb{T}$, have these properties.

Lemma 2.4. *Under the assumptions of Lemma 2.1 the functions*

$$f_0(z) = \frac{1 - s(z)\tau_0^*}{1 - zz_1^*}$$

and

$$f_\ell(z) = \frac{zf_{\ell-1}(z) - s(z)\tau_\ell^*}{1 - zz_1^*}, \quad \ell = 1, 2, \ldots, k - 1,$$

are elements of $\mathcal{P}(s)$ and $\langle f_\ell, f_m \rangle_{\mathcal{P}(s)} = \alpha_{m\ell}$.

Proof. First we note that for $z \in \mathbb{D}$ and $\ell = 0, 1, \ldots, k - 1$,

$$f_\ell(z) = \lim_{w \hat{\to} z_1} \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(z, w).$$

This implies that for all $w' \in \mathbb{D}$

$$\lim_{w \hat{\to} z_1} \left\langle \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(\,\cdot\,, w), K_s(\,\cdot\,, w') \right\rangle_{\mathcal{P}(s)} = \lim_{w \hat{\to} z_1} \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(w', w) = f_\ell(w'), \tag{2.18}$$

and for $\ell, m = 0, 1, \ldots, k - 1$

$$\lim_{w \hat{\to} z_1, w' \hat{\to} z_1} \left\langle \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(\,\cdot\,, w), \frac{1}{m!} \frac{\partial^m}{\partial w'^{*m}} K_s(\,\cdot\,, w') \right\rangle_{\mathcal{P}(s)} \tag{2.19}$$

$$= \lim_{w \hat{\to} z_1, w' \hat{\to} z_1} \frac{1}{\ell!m!} \frac{\partial^{\ell+m}}{\partial w^{*\ell} \partial w'^m} K_s(w', w) = \alpha_{m\ell}. \tag{2.20}$$

The claim follows now from [21, Theorem 2.4] and [8, Theorem 1.1.2]. In fact, (2.18) and (2.19) imply $f_\ell \in \mathcal{P}(s)$, $\ell = 1, 2, \ldots, k - 1$, and (2.19) also yields the formula for the inner product between the f_ℓ's.   □

In Section 4 below we also need the following generalization of Lemma 2.1. To formulate it, we suppose that at two points $z_1, z_2 \in \mathbb{T}$, $z_1 \neq z_2$, the function $s(z) \in \mathbf{S}$ has the asymptotic expansions

$$s(z) = \tau_{1;0} + \sum_{\ell=1}^{2k_1-1} \tau_{1;\ell}(z - z_1)^\ell + O\left((z - z_1)^{2k_1}\right), \quad z \hat{\to} z_1, \tag{2.21}$$

$$s(z) = \tau_{2;0} + \sum_{m=1}^{2k_2-1} \tau_{2;m}(z - z_2)^m + O\left((z - z_2)^{2k_2}\right), \quad z \hat{\to} z_2, \tag{2.22}$$

and we introduce for $i = 1, 2$ the $k_i \times k_i$-matrices

$$A_i = z_i^* I_{k_i} + S_{k_i}$$

and the $2 \times k_i$-matrices

$$C_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_{i;0}^* & \tau_{i;1}^* & \cdots & \tau_{i;k_i-1}^* \end{pmatrix}.$$

Lemma 2.5. *Suppose that at two points $z_1, z_2 \in \mathbb{T}$, $z_1 \neq z_2$, the function $s(z) \in \mathbf{S}$ has the asymptotic expansions (2.21) and (2.22). Then the kernel $K_s(z, w)$ has the asymptotic expansion*

$$K_s(z, w) = \sum_{\substack{0 \leq \ell \leq k_1 - 1, \\ 0 \leq m \leq k_2 - 1}} \alpha_{\ell m}(z - z_1)^\ell (w - z_2)^{*m}$$

$$+ O\left((\max\{|z - z_1|^{k_1}, |w - z_2|^{k_2}\})\right), \quad z \hat{\to} z_1, \ w \hat{\to} z_2,$$

where

$$\alpha_{\ell m} = \lim_{z \hat{\to} z_1, w \hat{\to} z_2} \frac{1}{\ell! \, m!} \frac{\partial^\ell}{\partial z^\ell} \frac{\partial^m}{\partial w^{*m}} K_s(z, w).$$

Moreover, the $k_1 \times k_2$-matrix $\mathbb{P}_{12} = (\alpha_{\ell m})$, $0 \leq \ell \leq k_1 - 1, 0 \leq m \leq k_2 - 1$, satisfies the relation

$$\mathbb{P}_{12} - A_1^* \mathbb{P}_{12} A_2 = C_1^* J C_2. \tag{2.23}$$

Proof. Similar to the proof of Lemma 2.1 we set now $u = z - z_1$, $v = w^* - z_2^*$, and equate the coefficients of their powers in the analog of the expression in (2.7):

$$1 - \left(\tau_{1;0} + \tau_{1;1}u + \tau_{1;2}u^2 + \cdots + O(u^{2k_1})\right)\left(\tau_{2;0}^* + \tau_{2;1}^* v + \tau_{2;2}^* v^2 + \cdots + O(v^{2k_2})\right)$$

$$- \sum_{0 \leq \ell \leq k_1 - 1, 0 \leq m \leq k_2 - 1} \alpha_{\ell m} u^\ell v^m (-u z_2^* - v z_1 - uv + 1 - z_1 z_2^*).$$

This gives

$$1 - \tau_{1;0} \tau_{2;0}^* = \alpha_{0,0}(1 - z_1 z_2^*),$$

and for $0 \leq \ell \leq k_1 - 1$, $0 \leq m \leq k_2 - 1$, and $\ell + m > 0$,

$$\tau_{1;\ell} \tau_{2;m}^* = \alpha_{\ell-1,m} z_2^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1} + \alpha_{\ell m}(1 - z_1 z_2^*),$$

which is easily seen to be equivalent to (2.23). □

3. The basic interpolation problem at one boundary point

With the data of the Problem 1.1 the $k \times k$-matrix T was defined in (1.7), and we recall the definition of B in (1.8). Then the matrix $\mathbb{P}$ from Lemma 2.1 can be written in the form

$$\mathbb{P} = \tau_0^* T B. \tag{3.1}$$

Observe that $\mathbb{P}$ is a right lower triangular matrix, which is invertible because of $\tau_0, \tau_k, z_1 \neq 0$. We define the vector function

$$R(z) = \left(\frac{1}{1 - zz_1^*} \quad \frac{z}{(1 - zz_1^*)^2} \quad \cdots \quad \frac{z^{k-1}}{(1 - zz_1^*)^k} \right),$$

fix some $z_0 \in \mathbb{T}$, $z_0 \neq z_1$ and introduce the polynomial $p(z)$ by

$$p(z) = (1 - zz_1^*)^k \, R(z)\mathbb{P}^{-1}R(z_0)^*. \tag{3.2}$$

It has degree at most $k - 1$ and $p(z_1) \neq 0$.

Lemma 3.1. *With $p(z)$ from* (3.2) *we have that*

$$\tau_0 \frac{(1 - zz_1^*)^k}{(1 - zz_0^*)p(z)} = -\sum_{i=k}^{2k-1} \tau_i(z - z_1)^i + \mathrm{O}\left((z - z_1)^{2k}\right), \quad z \hat{\to} z_1.$$

Proof. Since $1 - zz_1^* = -z_1^*(z - z_1)$, it suffices to show that if

$$\tau_0 \frac{(-1)^{k-1} z_1^{*k}}{(1 - zz_0^*)p(z)} = \sigma_k + \sigma_{k+1}(z - z_1) + \cdots + \sigma_{2k-1}(z - z_1)^{k-1} + \mathrm{O}\left((z - z_1)^k\right), \quad (3.3)$$

then $\sigma_j = \tau_j$, $j = k, k+1, \ldots, 2k-1$. An expansion of the form (3.3) exists because the quotient on the left-hand side is rational and the denominator does not vanish at $z = z_1$. Write

$$1 - zz_0^* = -z_0^*[(z - z_1) + (z_1 - z_0)],$$

$$p(z) = \sum_{j=0}^{k-1} p_j(z - z_1)^j = \begin{pmatrix} 1 & z - z_1 & \cdots & (z - z_1)^{k-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix},$$

and define

$$T' = \begin{pmatrix} \sigma_k & 0 & \cdots & 0 \\ \sigma_{k+1} & \sigma_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2k-1} & \sigma_{2k-2} & \cdots & \sigma_k \end{pmatrix}.$$

From

$$\left(\sigma_k + \sigma_{k+1}(z - z_1) + \cdots + \sigma_{2k-1}(z - z_1)^{k-1}\right) \begin{pmatrix} 1 & z - z_1 & \cdots & (z - z_1)^{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & z - z_1 & \cdots & (z - z_1)^{k-1} \end{pmatrix} T' + \mathrm{O}\left((z - z_1)^k\right),$$

the definition of the shift matrix S_k from (2.15), and (3.3) we obtain

$$\tau_0(-1)^{k-1} z_0 z_1^{*k}$$

$$= \begin{pmatrix} 1 & (z - z_1) & \cdots & (z - z_1)^{k-1} \end{pmatrix} \left((z_1 - z_0)I_k + S_k^*\right) T' \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix},$$

and it follows that

$$T' \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \tau_0 \frac{(-1)^{k-1} z_0 z_1^{*k}}{z_0 - z_1} \begin{pmatrix} 1 \\ \frac{1}{z_0 - z_1} \\ \vdots \\ \frac{1}{(z_0 - z_1)^{k-1}} \end{pmatrix}. \qquad (3.4)$$

On the other hand, from the definition of $p(z)$ it follows that

$$p(z) = \tau_0 \left((1 - zz_1^*)^{k-1} \quad z(1 - zz_1^*)^{k-2} \quad \cdots \quad z^{k-1} \right)$$

$$\times B^{-1} T^{-1} \frac{z_0}{z_0 - z_1} \begin{pmatrix} 1 \\ 1 \\ \frac{1}{z_0 - z_1} \\ \vdots \\ \frac{1}{(z_0 - z_1)^{k-1}} \end{pmatrix}.$$

A straightforward calculation shows that

$$\left((1 - zz_1^*)^{k-1} \quad z(1 - zz_1^*)^{k-2} \quad \cdots \quad z^{k-1} \right)$$
$$= \left(1 \quad z - z_1 \quad \cdots \quad (z - z_1)^{k-1} \right) B(-1)^{k-1} z_1^{*k}$$

and hence

$$T \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \tau_0 \frac{(-1)^{k-1} z_0 z_1^{*k}}{z_0 - z_1} \begin{pmatrix} 1 \\ 1 \\ \frac{1}{z_0 - z_1} \\ \vdots \\ \frac{1}{(z_0 - z_1)^{k-1}} \end{pmatrix}.$$

This equality and (3.4) imply

$$\begin{pmatrix} \sigma_k & 0 & \cdots & 0 \\ \sigma_{k+1} & \sigma_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2k-1} & \sigma_{2k-2} & \cdots & \sigma_k \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \begin{pmatrix} \tau_k & 0 & \cdots & 0 \\ \tau_{k+1} & \tau_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{2k-1} & \tau_{2k-2} & \cdots & \tau_k \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix}.$$

From this relation, because of $p_0 = p(z_1) \neq 0$, it readily follows that $\sigma_j = \tau_j$, $j = k, k+1, \ldots, 2k-1$. $\qquad\square$

For a Hermitian matrix $\mathbb{P}$, by $\mathrm{ev}_-(\mathbb{P})$ we denote the number of negative eigenvalues of $\mathbb{P}$.

Theorem 3.2. *Given $z_1 \in \mathbb{T}$ and $\tau_0, \tau_k, \ldots, \tau_{2k-1}$ as in Problem 1.1 such that the matrix $\mathbb{P}$ in (1.6) is Hermitian, and let $\Theta(z)$ be the J-unitary rational matrix function*

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I_2 - \frac{(1 - zz_0^*)p(z)}{(1 - zz_1^*)^k} \mathbf{u}\mathbf{u}^* J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \tau_0^* \end{pmatrix},$$
(3.5)

with $p(z)$ from (3.2) and fixed $z_0 \in \mathbb{T}$, $z_0 \neq z_1$. Then the fractional linear transformation

$$s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)},$$
(3.6)

establishes a bijective correspondence between all solutions $s(z)$ *of Problem* 1.1 *and all* $s_1(z) \in \mathbf{S}$ *with the property*

$$\liminf_{z \hat{\rightarrow} z_1} |s_1(z) - \tau_0| > 0. \tag{3.7}$$

Moreover, if $s(z)$ *and* $s_1(z)$ *are related by* (3.6) *then*

$$\mathrm{sq}_-(s) = \mathrm{sq}_-(s_1) + \mathrm{ev}_-(\mathbb{P}). \tag{3.8}$$

Proof. With the given numbers $\tau_0, \tau_k, \ldots, \tau_{2k-1}$ we define the space $\mathcal{M}$ as the span of the functions

$$\mathbf{f}_\ell(z) = \frac{z^\ell}{(1 - zz_1^*)^{\ell+1}}\mathbf{u}, \quad \ell = 0, 1, \ldots, k-1. \tag{3.9}$$

Then

$$\begin{pmatrix} \mathbf{f}_0(z) & \mathbf{f}_1(z) & \cdots & \mathbf{f}_{k-1}(z) \end{pmatrix} = C(I_k - zA)^{-1}, \tag{3.10}$$

where the matrix C from (2.16) specializes now to

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_0^* & 0 & \cdots & 0 \end{pmatrix}, \tag{3.11}$$

and $A = z_1^* I_k + S_k$ as in (2.15) with S_k being the $k \times k$ shift matrix. Endowing the space $\mathcal{M}$ with the inner product

$$\langle \mathbf{f}_m, \mathbf{f}_\ell \rangle_{\mathcal{M}} = (\mathbb{P})_{\ell,m} = \alpha_{\ell m} \tag{3.12}$$

we have that $\mathcal{M}$ is a reproducing kernel Pontryagin space with reproducing kernel equal to

$$C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - wA)^{-*}C^*. \tag{3.13}$$

Evidently, the negative index of this space is equal to $\mathrm{ev}_-(\mathbb{P})$.

On the other hand, according to (2.17) the matrix $\mathbb{P}$ satisfies the Stein equation

$$\mathbb{P} - A^*\mathbb{P}A = C^*JC,$$

where now the expressions on both sides are equal to zero. Therefore for $\mathcal{M}$ all the conditions of Theorem 1.2 are satisfied, and hence there exists a J-unitary rational 2×2-matrix function

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

such that $\mathcal{M} = \mathcal{P}(\Theta)$, the reproducing kernel Pontryagin space with reproducing kernel $\dfrac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}$. By the uniqueness of the reproducing kernel it must coincide with the kernel from (3.13):

$$C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - wA)^{-*}C^* = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}.$$

Thus if we normalize $\Theta(z)$ by $\Theta(z_0) = I_2$ we obtain

$$\Theta(z) = I_2 - (1 - zz_0^*)C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - z_0A)^{-*}C^*J.$$

By (3.9) and (3.10) this matrix function can be written as

$$\Theta(z) = I_2 - (1 - zz_0^*)\, \mathbf{u}\, R(z)\, \mathbb{P}^{-1} R(z_0)^*\, \mathbf{u}^* J,$$

and this coincides with the formula for $\Theta(z)$ in the theorem.

Now we consider a solution $s(z)$ of Problem 1.1:

$$s(z) = \tau_0 + \sum_{\ell=k}^{2k-1} \tau_\ell (z - z_1)^\ell + O((z - z_1)^{2k}), \quad z \hat{\to} z_1.$$

According to Lemma 2.1 the corresponding kernel $K_s(z, w)$ admits the representation (2.5):

$$K_s(z, w) = \sum_{0 \leq \ell + m \leq 2k-2} \alpha_{\ell m} (z - z_1)^\ell (w - z_1)^{*m}$$
$$+ O\left((\max\{|z - z_1|, |w - z_1|\})^{2k-1}\right), \quad z, w \hat{\to} z_1,$$

with

$$\alpha_{\ell m} = \lim_{z, w \hat{\to} z_1} \frac{1}{\ell! m!} \frac{\partial^{\ell+m}}{\partial w^{*m} \partial z^\ell} K_s(z, w) = \alpha_{m\ell}^*. \tag{3.14}$$

From

$$K_s(z, w) = \frac{1 - s(z)s(w)^*}{1 - zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{\begin{pmatrix} 1 \\ s(w)^* \end{pmatrix}}{1 - zw^*}$$

we see that

$$\lim_{w \hat{\to} z_1} \frac{1}{m!} \frac{\partial^m}{\partial w^{*m}} K_s(z, w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}_m(z), \quad m = 0, \ldots, k-1.$$

On the other hand, according to Lemma 2.4 the elements

$$f_m(z) = \lim_{w \hat{\to} z_1} \frac{1}{m!} \frac{\partial^m}{\partial w^{*m}} K_s(z, w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}_m(z), \quad m = 0, 1, \ldots, k-1,$$

belong to the reproducing kernel Pontryagin space $\mathcal{P}(s)$ with reproducing kernel $K_s(z, w)$ and

$$\left\langle \begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f}_m, \begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f}_\ell \right\rangle_{\mathcal{P}(s)} = \lim_{z, w \hat{\to} z_1} \frac{1}{\ell! m!} \frac{\partial^{m+\ell}}{\partial w^{*m} \partial z^\ell} K_S(z, w). \tag{3.15}$$

By (3.15), (3.12), and (3.14) the map $\mathcal{T}$ of multiplication by $\begin{pmatrix} 1 & -s(z) \end{pmatrix}$ is an isometry from $\mathcal{M}$ into $\mathcal{P}(s)$. Setting

$$s_1(z) = \frac{b(z) - d(z)s(z)}{c(z)s(z) - a(z)}$$

we have that $s(z)$ is of the desired form:

$$s(z) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)}. \tag{3.16}$$

From

$$K_s(z,w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \begin{pmatrix} 1 & -s(w) \end{pmatrix}^* \qquad (3.17)$$

$$+ (a(z) - c(z)s(z))K_{s_1}(z,w)(a(w) - c(w)s(w))^*,$$

and since $\mathcal{T}$ is an isometry, it follows that $s_1(z)$ is a generalized Schur function and

$$\mathcal{P}(s) = \mathcal{T}\mathcal{M} \oplus (a - cs)\mathcal{P}(s_1).$$

By the observations at the end of the Introduction and after formula (3.12) this implies the equality (3.8).

From the definition (3.5) of $\Theta(z)$:

$$\Theta(z) = \begin{pmatrix} 1 - \theta(z) & \tau_0\theta(z) \\ -\tau_0^*\theta(z) & 1 + \theta(z) \end{pmatrix}, \quad \theta(z) = \frac{(1 - zz_0^*)p(z)}{(1 - zz_1^*)^k} = (1 - zz_0^*)R(z)\mathbb{P}^{-1}R(z_0)^*,$$
$$(3.18)$$

and (3.16) we obtain

$$s(z) - \tau_0 \left(1 - \frac{(1 - zz_1^*)^k}{(1 - zz_0^*)p(z)} \right)$$

$$= \frac{\tau_0(1 - zz_1^*)^{2k}}{(1 - zz_0^*)p(z)\left\{(1 - zz_1^*)^k - \tau_0^*(1 - zz_0^*)p(z)(s_1(z) - \tau_0)\right\}}. \qquad (3.19)$$

By Lemma 3.1 the expression on the left is $O((z - z_1)^{2k})$, $z\hat\to z_1$, and this can only be the case if (3.7) holds. Thus, every solution of the Problem 1.1 is of the form given in the theorem.

As to the existence of solutions, the equality (3.19) readily implies that any function $s(z)$ of the form (3.6) has the desired asymptotics and since $\Theta(z)$ is J-unitary and rational, the formula (3.17) implies that if $s_1(z)$ belongs to the class **S** then also $s(z)$ belongs to this class. □

Remark 3.3. 1) The J-unitarity of $\Theta(z)$ implies that

$$p(z) = z_0(-z_1^*)^k z^{k-1} p\left(\frac{1}{z^*}\right)^*. \qquad (3.20)$$

2) Note that the matrix function $\Theta(z)$ in Theorem 3.2 is normalized such that $\Theta(z_0) = I_2$. Replacing z_0 by another point $\hat z_0 \in \mathbb{T}$, $\hat z_0 \neq z_1$, amounts to multiplying $\Theta(z)$ from the right by a J-unitary constant matrix. This follows from the fact that the fractional linear transformations with the corresponding matrix function $\widehat\Theta(z)$ and with $\Theta(z)$ have the same range. It can also be shown directly using the equality (3.22) below.

3) For $\theta(z)$ as in (3.18) we have

$$\theta(z) = (1 - zz_0^*)R(z)\mathbb{P}^{-1}R(z_0)^*, \quad R(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}(I - zA)^{-1}, \qquad (3.21)$$

where $A = S_k + z_1^* I_k$. If the point z_0 is replaced by another point $\hat z_0 \in \mathbb{T}$, $\hat z_0 \neq z_0, z_1$, then for the corresponding function $\widehat\theta(z)$ the difference $\theta(z) - \widehat\theta(z)$ is independent

of z. In fact, a direct calculation using (3.21) and (2.17) with $C^*JC = 0$ shows that

$$\theta(z) - \widehat{\theta}(z) = -\widehat{\theta}(z_0). \tag{3.22}$$

4) For rational parameters $s_1(z)$ the condition (3.7) is equivalent to the fact that the denominator in (3.6):

$$c(z)s_1(z) + d(z) = -\tau_0^*(s_1(z) - \tau_0)\theta(z) + 1$$

has a pole of order k (see (3.18)).

5) The matrix $\mathbb{P}$ in (1.6) is right lower triangular and the entries on the second main diagonal are given by

$$(\mathbb{P})_{i,k-1-i} = (-1)^{k-1-i}z_1^{2k-1-2i}\tau_0^*\tau_k, \quad i = 0, 1, \ldots, k-1. \tag{3.23}$$

If $\mathbb{P}$ is Hermitian, then by (3.23), $z_1^k\tau_0^*\tau_k$ is purely imaginary if k is even and real if k is odd, and we have

$$\mathrm{ev}_-(\mathbb{P}) = \begin{cases} k/2, & k \text{ even}, \\ (k-1)/2, & k \text{ odd}, \ (-1)^{(k-1)/2}z_1^k\tau_0^*\tau_k > 0, \\ (k+1)/2, & k \text{ odd}, \ (-1)^{(k-1)/2}z_1^k\tau_0^*\tau_k < 0. \end{cases}$$

Recall that the Schur algorithm is originally defined for a Schur function $s(z)$. Theorem 3.2 allows us to define an analog for functions $s(z)$ in the class $\mathbf{S}$ which have an asymptotics (1.5) at z_1 with a Hermitian matrix $\mathbb{P}_k$ and $\tau_k \neq 0$. The Schur transform of $s(z)$ is the function $\widehat{s}(z) := s_1(z) = T_{\Theta(z)^{-1}}(s(z))$ with $\Theta(z)$ as in Theorem 3.2. By this Schur transformation the set of functions in $\mathbf{S}$ with the above mentioned properties is mapped into $\mathbf{S}$. The Schur algorithm consists in iterating the Schur transformation. It will be considered in Sections 5 and 6.

4. Multipoint boundary interpolation

We generalize Problem 1.1 to an interpolation problem with N distinct points $z_1, \ldots, z_N$ on the unit circle.

Problem 4.1. *Let $N \geq 1$ be an integer, let $z_1, \ldots, z_N$ be N distinct points on $\mathbb{T}$, let $k_1, \ldots, k_N$ be integers ≥ 1, and let $\tau_{i;0}, \tau_{i;k_i}, \tau_{i;k_i+1}, \ldots, \tau_{i;2k_i-1}$ be complex numbers such that $|\tau_{i;0}| = 1$ and $\tau_{i;k_i} \neq 0$, $i = 1, \ldots, N$. Find all generalized Schur functions $s(z) \in \mathbf{S}$ such that*

$$s(z) = \tau_{i;0} + \sum_{\ell=k_i}^{2k_i-1} \tau_{i;\ell}(z - z_i)^\ell + \mathrm{O}((z - z_i)^{2k_i}), \quad z\hat{\to}z_i, \quad i = 1, \ldots, N.$$

Let $\mathbb{P}_i$, C_i, A_i, and $\Theta_i(z)$ be related to z_i as in Section 3 the matrices $\mathbb{P}$, C, A, and $\Theta(z)$ in formulas (3.1), (3.11), (2.15) and (3.5) are related to z_1. Set

$$C = \begin{pmatrix} C_1 & C_2 & \cdots & C_N \end{pmatrix}, \quad A = \mathrm{diag}\,(A_1, A_2, \ldots, A_N),$$

and denote by $\mathbb{P} = (\mathbb{P}_{ij})_{i,j=1}^N$ the $N \times N$ block matrix with $\mathbb{P}_{ii} = \mathbb{P}_i$ and $\mathbb{P}_{ij} \in \mathbb{C}^{k_i \times k_j}$ being the matrix given by (2.23) for $z_1 = z_i$ and $z_2 = z_j$, $i, j = 1, 2, \ldots, N$. Then, according to (2.17) and (2.23) the matrix $\mathbb{P}$ satisfies the Stein equation

$$\mathbb{P} - A^* \mathbb{P} A = C^* J C. \tag{4.1}$$

We note that the relation (2.23) in the situation of this section reads as

$$\mathbb{P}_{ij} - A_i^* \mathbb{P}_{ij} A_j = C_i^* J C_j = \begin{pmatrix} 1 - \tau_{i;0} \tau_{j;0}^* & 0 & 0 & \cdots & 0 \\ 0 & & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If no derivatives are involved, $\mathbb{P}_{ij}$ is a complex number and equal to $\dfrac{1 - \tau_{i;0} \tau_{j;0}^*}{1 - z_i^* z_j}$.

Theorem 4.2. *Assume that the matrix $\mathbb{P}$ is invertible and Hermitian and define the J-unitary matrix function $\Theta(z)$ by*

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I_2 - (1 - zz_0^*) C(I - zA)^{-1} \mathbb{P}^{-1} (I - z_0 A)^{-*} C^* J,$$

where z_0 is any point in $\mathbb{T}$ different from the interpolation points. Then the fractional linear transformation

$$s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z) s_1(z) + b(z)}{c(z) s_1(z) + d(z)} \tag{4.2}$$

establishes a bijective correspondence between all solutions $s(z)$ of Problem 4.1 and all $s_1(z) \in \mathbf{S}$ with the properties

$$\liminf_{z \hat{\to} z_1} \left| \frac{\hat{a}_i(z) s_1(z) + \hat{b}_i(z)}{\hat{c}_i(z) s_1(z) + \hat{d}_i(z)} - \tau_{i,0} \right| > 0, \quad i = 1, \ldots, N, \tag{4.3}$$

where

$$\begin{pmatrix} \hat{a}_i(z) & \hat{b}_i(z) \\ \hat{c}_i(z) & \hat{d}_i(z) \end{pmatrix} = \widehat{\Theta}_i(z) := \Theta_i^{-1}(z) \Theta(z).$$

In the correspondence (4.2),

$$\mathrm{sq}_-(s) = \mathrm{ev}_-(\mathbb{P}) + \mathrm{sq}_-(s_1). \tag{4.4}$$

Proof. As in the proof of Theorem 3.2, to each of the interpolation points z_i is associated the finite-dimensional resolvent invariant space $\mathcal{M}_i$ of $\mathbb{C}^2$-valued rational functions spanned by the columns of the matrix function $C_i(I - zA_i)^{-1}$. Then the space $\mathcal{M} = \oplus_{i=1}^N \mathcal{M}_i$ is spanned by the columns of the matrix function $C(I - zA)^{-1}$. We endow $\mathcal{M}$ with the inner product defined by $\mathbb{P}$. It follows from Theorem 1.2 that $\mathcal{M} = \mathcal{P}(\Theta)$ with $\Theta(z)$ as in the theorem.

Assume that $s(z)$ is a solution of the interpolation problem. We claim that the map $\mathcal{T} : f(z) \mapsto \begin{pmatrix} 1 & -s(z) \end{pmatrix} f(z)$ is an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{P}(s)$. Indeed,

because of the Stein equation (4.1) and the relations

$$\mathcal{T}C_i(I - zA_i)^{-1} =$$
$$\left(\lim_{w \hat{\to} z_i} K_s(z, w) \quad \lim_{w \hat{\to} z_i} \frac{\partial}{\partial w^*} K_s(z, w) \quad \cdots \quad \lim_{w \hat{\to} z_i} \frac{1}{(k_i - 1)!} \frac{\partial^{k_i - 1}}{\partial w^{*(k_i-1)}} K_s(z, w) \right),$$

where $i = 1, 2, \ldots, N$, the entries of the Gram matrix associated with the basis of the space $\mathcal{M}$, which is the union of the bases of the spaces $\mathcal{M}_i$, coincides with the Gram matrix of the images under $\mathcal{T}$. Hence

$$\mathcal{P}(s) = \mathcal{T}\mathcal{P}(\Theta) \oplus (a - cs)\mathcal{P}(s_1)$$

and $s(z) = T_{\Theta(z)}(s_1(z))$ for some generalized Schur function $s_1(z)$ satisfying (4.4). Since $\mathcal{M}_i$ is a non-degenerate R_0-invariant subspace of $\mathcal{M}$, $\Theta(z)$ admits the factorization $\Theta(z) = \Theta_i(z)\widehat{\Theta}_i(z)$, see [9]. Hence $s(z) = T_{\Theta(z)}(s_1(z)) = T_{\Theta_i(z)}(\widehat{s}_1(z))$ with

$$\widehat{s}_1(z) = T_{\widehat{\Theta}_i(z)}(s_1(z)) = \frac{\widehat{a}_i(z)s_1(z) + \widehat{b}_i(z)}{\widehat{c}_i(z)s_1(z) + \widehat{d}_i(z)}.$$

This shows that $s(z)$ is a solution of the interpolation problem at z_i with parameter $\widehat{s}_1(z)$, therefore, according to (3.7), $\widehat{s}_1(z)$ satisfies (4.3).

Conversely, let $s(z) = T_{\Theta(z)}(s_1(z))$ be given with a function $s_1(z)$ as in the theorem. If we write $s(z) = T_{\Theta_i(z)}(\widehat{s}_1(z))$, then, since $\widehat{\Theta}_i(z) = \Theta_i^{-1}(z)\Theta(z)$ is J-unitary, $\widehat{s}_1(z)$ is a generalized Schur function and by (3.7) it has all the properties of the parameters in Theorem 3.2 and hence $s(z)$ is a solution of Problem 4.1 $\quad\square$

Remark 4.3. 1) There exist rational parameters $s_1(z)$ satisfying the conditions (4.3) for $i = 1, \ldots, N$. Indeed for each i there is a unique constant $s_i = T_{\Theta(z_i)^{-1}}(\tau_{i;0})$ such that in (4.3) there is equality rather than inequality. It suffices to take for $s_1(z)$ any constant of modulus 1 which is different from these s_i, $i = 1, 2, \ldots, N$.
2) If $k_i = 1$, $i = 1, 2, \ldots, N$, a description of all rational Schur functions which satisfy the given interpolation conditions was given by J.A. Ball, I. Gohberg, and L. Rodman [12, Theorem 21.1.2]: in this case the conditions (4.3) reduce to the fact that $c(z)s_1(z) + d(z)$ has poles of order 1 at $z = z_i$, $i = 1, 2, \ldots, N$. Indeed, with

$$\Theta_i(z) = \begin{pmatrix} a_i(z) & b_i(z) \\ c_i(z) & d_i(z) \end{pmatrix}$$

and the relations in the proof of the theorem we have

$$c(z)s_1(z) + d(z) = (c_i(z)\widehat{s}_1(z) + d_i(z))(\widehat{c}_i(z)s_1(z) + \widehat{d}_i(z)).$$

According to Remark 3.3, 4) the first factor on the right-hand side has a pole of order 1 at z_i and the second factor is rational and nonzero at z_i.
3) We give an example where $\mathbb{P}$ is not invertible while its diagonal entries are invertible. For such matrices the assumptions of Theorem 4.2 are not satisfied. Take $N = 2$, two distinct points z_1 and z_2 on $\mathbb{T}$, $k_1 = k_2 = 1$, $\tau_{1;0} = 1$, $\tau_{2;0} = -1$, and numbers $\tau_{1;1}, \tau_{2;1}$ such that $z_1\tau_{1;1}, z_2\tau_{2;1} \in \mathbb{R}$ and $z_1 z_2 \tau_{1;1}\tau_{2;1} = 4/|1 - z_1 z_2^*|^2$. Then $\mathbb{P}_1$ and $\mathbb{P}_2$ are invertible, $\mathbb{P}$ satisfies the Stein equation (4.1) but is not invertible.

5. J-unitary factorization

In this section z_0 and z_1 are two distinct points in $\mathbb{T}$. By $\mathcal{U}_{z_1}$ we denote the set of all rational J-unitary 2×2-matrix functions $\Theta(z)$ with a pole only at $z = z_1$, and by $\mathcal{U}_{z_1}^{z_0}$ the set of all matrix functions $\Theta(z) \in \mathcal{U}_{z_1}$ which are normalized such that $\Theta(z_0) = I_2$. In particular, the matrix functions of $\mathcal{U}_{z_1}$ are bounded at ∞.

Lemma 5.1. *If* $\Theta(z) \in \mathcal{U}_{z_1}$ *then* $\det \Theta(z) \equiv c$ *for some* $c \in \mathbb{T}$, *and* $\Theta(z)^{-1} \in \mathcal{U}_{z_1}$.

Proof. The J-unitarity of $\Theta(z)$ on $\mathbb{T}$ and the analyticity outside $z = z_1$ imply the identity
$$\Theta(z) J \Theta(1/z^*)^* = J, \quad z \in \mathbb{C} \setminus \{0, z_1\}.$$
For the rational function $f(z) = \det\Theta(z)$ it follows that $|f(z)| = 1$, $z \in \mathbb{T}$. Therefore f cannot have a pole at z_1, and since it is also bounded at ∞ it must be constant. □

By the degree of a rational J-unitary matrix function $\Theta(z)$ we mean the McMillan degree (see [13]) and we write it as $\deg \Theta(z)$. If $\Theta(z) \in \mathcal{U}_{z_1}$ and
$$\Theta(z) = \sum_{i=0}^{n} T_i (z - z_1)^{-i},$$
where the T_i's are constant 2×2-matrices and $T_n \neq 0$, then
$$\deg \Theta = \operatorname{rank} \begin{pmatrix} T_n & 0 & \cdots & 0 \\ T_{n-1} & T_n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_1 & T_2 & \cdots & T_n \end{pmatrix}.$$

A product $\Theta_1(z)\Theta_2(z) \cdots \Theta_n(z) = \Theta(z)$ of rational J-unitary matrix functions is called minimal if the degrees add up, that is,
$$\deg \Theta_1(z) + \deg \Theta_2(z) + \cdots + \deg \Theta_n(z) = \deg \Theta(z).$$
In this case the product on the left-hand side is also called a minimal factorization of $\Theta(z)$. An example of a nonminimal product is given by the equality $\Theta(z)\Theta(z)^{-1} = I_2$ for any nonconstant $\Theta(z) \in \mathcal{U}_{z_1}$, since, because of Lemma 5.1, the inverse $\Theta(z)^{-1}$ also belongs to $\mathcal{U}_{z_1}$.

A matrix function $\Theta(z) \in \mathcal{U}_{z_1}$ is called elementary if in any minimal factorization $\Theta(z) = \Theta_1(z)\Theta_2(z)$ at least one of the factors is a J-unitary constant.

Theorem 5.2. *Assume* $z_0, z_1 \in \mathbb{T}$ *and* $z_0 \neq z_1$. *Then:*

(i) *The matrix function* $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$ *is elementary if and only if it is of the form*
$$\Theta(z) = I_2 - \frac{(1 - zz_0^*)p(z)}{(1 - zz_1^*)^k} \mathbf{u}\mathbf{u}^* J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}, \tag{5.1}$$

where k *is an integer* ≥ 1, $\zeta \in \mathbb{T}$, $p(z)$ *is a polynomial of degree* $\leq k - 1$ *satisfying* (3.20) *and* $p(z_1) \neq 0$.

(ii) *Every $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$ admits a unique minimal factorization*

$$\Theta(z) = \Theta_1(z) \cdots \Theta_n(z), \tag{5.2}$$

in which each $\Theta_j(z)$ is an elementary normalized factor of the form (5.1).

The theorem implies that the matrix function $\Theta(z)$ in (3.5) belongs to the class $\mathcal{U}_{z_1}^{z_0}$ and is elementary. The proof of Theorem 5.2 hinges on the fact that the reproducing kernel space $\mathcal{P}(\Theta)$ consists of one Jordan chain for the difference quotient operator R_0, which makes the elementary factors unique. In case of higher dimensions this uniqueness does not hold.

Proof of Theorem 5.2. Let $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$. We claim that $\mathcal{P}(\Theta)$ is spanned by a single chain for R_0 at the eigenvalue $\lambda = z_1^*$. To see this, let λ be an eigenvalue of R_0 with eigenelement $\mathbf{f}_0(z)$: $R_0 \mathbf{f}_0(z) = \lambda \mathbf{f}_0(z)$. Then

$$\mathbf{f}_0(z) = \frac{\mathbf{c}_0}{1 - \lambda z}, \quad \mathbf{c}_0 = \mathbf{f}_0(0) \neq 0,$$

and since the elements of $\mathcal{P}(\Theta)$ have a pole only at $z = z_1$, we conclude that $\lambda = z_1^*$. The identity (1.10) and $|z_1| = 1$ imply that $\mathbf{c}_0$ is J-neutral:

$$\mathbf{c}_0^* J \mathbf{c}_0 = \langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} - \langle z_1^* \mathbf{f}_0, z_1^* \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} = 0.$$

If

$$\mathbf{g}_0(z) = \frac{\mathbf{d}_0}{1 - z z_1^*}, \quad \mathbf{d}_0 \in \mathbb{C}^2,$$

is another eigenfunction, then also $\mathbf{d}_0$ is J-neutral and (1.10) yields $\mathbf{c}_0^* J \mathbf{d}_0 = 0$. Since J is invertible, this implies that $\mathbf{d}_0$ is a multiple of $\mathbf{c}_0$ and hence the geometric multiplicity of the eigenvalue $\lambda = z_1^*$ is 1. This proves the claim. It follows that there are vectors $\mathbf{c}_j \in \mathbb{C}^2$, $\mathbf{c}_0$ being J-neutral, such that $\mathcal{P}(\Theta)$ is spanned by

$$\mathbf{f}_j(z) = \frac{z \mathbf{f}_{j-1}(z) + \mathbf{c}_j}{1 - z z_1^*}, \quad j = 0, \ldots, N-1, \quad \mathbf{f}_{-1}(z) \equiv 0.$$

Since $\mathbf{c}_0$ is nonzero and J-neutral, its components have the same nonzero absolute value and hence we may suppose without loss of generality that for some unimodular number ζ_0,

$$\mathbf{c}_0 = \begin{pmatrix} 1 \\ \zeta_0 \end{pmatrix}.$$

Let k be the smallest integer ≥ 1 such that $\langle \mathbf{f}_0, \mathbf{f}_{k-1} \rangle_{\mathcal{P}(\Theta)} \neq 0$, hence, if $k \geq 2$,

$$\langle \mathbf{f}_0, \mathbf{f}_j \rangle_{\mathcal{P}(\Theta)} = 0, \quad j = 0, \ldots, k-2.$$

Then the subspace

$$\mathcal{M} = \text{span}\{\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}\}$$

is the smallest R_0-invariant subspace of $\mathcal{P}(\Theta)$ which is non-degenerate and hence, by Theorem 1.2, it is a $\mathcal{P}(\Theta_1)$-space for some rational J-unitary 2×2-matrix function $\Theta_1(z)$. We prove that $\Theta_1(z)$ is of the form described by (5.1).

To this end we first show that without loss of generality we may assume that

$$\mathbf{c}_1 = \cdots = \mathbf{c}_{k-1} = 0. \tag{5.3}$$

By the identity (1.10) we have

$$\mathbf{c}_0^* J \mathbf{c}_j = \langle \mathbf{f}_j, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} - \langle z_1^* \mathbf{f}_j + \mathbf{f}_{j-1}, z_1^* \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} = 0$$

and, since $\mathbf{c}_0^* J \mathbf{c}_0 = 0$ and J is invertible, $\mathbf{c}_j$ is a multiple of $\mathbf{c}_0$. Successively for $j = 1, \ldots, k-1$, we may replace $\mathbf{c}_j$ in $\mathbf{f}_j(z)$ by zero by subtracting from $\mathbf{f}_j(z)$ a suitable multiple of the eigenfunction $\mathbf{f}_0(z)$. Thus we obtain a chain which satisfies (5.3) and still spans $\mathcal{M}$. By (5.3), this new chain coincides with the columns of the matrix $C(I_k - zA)^{-1}$ with C and A as in (3.11) and $\tau_0^* = \zeta$. Denote by $\mathbb{P}$ the corresponding Gram matrix:

$$\mathbb{P} = (p_{ij})_{i,j=0}^{k-1}, \quad p_{ij} = \langle \mathbf{f}_j, \mathbf{f}_i \rangle_{\mathcal{P}(\Theta)}, \ i, j = 0, 1, \ldots, k-1.$$

For the reproducing kernel $\Theta_1(z)$ of the space $\mathcal{M}$ we obtain

$$\frac{J - \Theta_1(z) J \Theta_1(w)^*}{1 - zw^*} = C(I_k - zA)^{-1} \mathbb{P}^{-1} (I_k - wA)^{-*} C^*,$$

and hence

$$\Theta_1(z) = I_2 - (1 - z_0^* z) C(I_k - zA)^{-1} \mathbb{P}^{-1} (I_k - z_0 A)^{-*} C^* J.$$

As in the proof of Theorem 3.2 one can show that $\Theta_1(z)$ is of the form (5.1). From its construction it follows that $\Theta_1(z)$ is elementary: Assume on the contrary, that $\Theta_1(z) = \Theta'(z)\Theta''(z)$ is a minimal factorization with nonconstant factors. Then $\mathcal{P}(\Theta_1) = \mathcal{P}(\Theta') \oplus \Theta' \mathcal{P}(\Theta'')$ and $\mathcal{P}(\Theta')$ is a proper non-degenerate R_0-invariant subspace of $\mathcal{P}(\Theta_1)$ and hence also a subspace of $\mathcal{P}(\Theta)$. The construction above and the minimality of k imply that $\mathcal{P}(\Theta')$ is spanned by the same chain as $\mathcal{P}(\Theta_1)$, that is, $\mathcal{P}(\Theta') = \mathcal{P}(\Theta_1)$. The normalization implies $\Theta'(z) = \Theta_1(z)$ and $\Theta''(z) = I_2$.

Now we prove (i) and (ii).

(i) The arguments above imply that if $\Theta(z)$ is elementary, then $\Theta(z) = \Theta_1(z)$. We now prove that if $\Theta(z)$ is given by (5.1), then it is elementary. The formula (5.1) implies that $\Theta(z)$ is J-unitary, rational with only one pole of order k at $z = z_1$ and normalized by $\Theta(z_0) = I_2$. The space $\mathcal{P}(\Theta)$ is spanned by the elements $R_0^n \Theta(z)\mathbf{c}$, $n = 0, 1, \ldots$, and these are 2-vector functions of the form $x(z)\mathbf{u}$, where $x(z)$ is a rational function with at most one pole at $z = z_1$. The chain argument above shows that the space $\mathcal{P}(\Theta)$ is spanned by the following chain of R_0 at z_1

$$\mathbf{g}_0(z) = \frac{1}{(1 - zz_1^*)^1}\mathbf{u}, \quad \mathbf{g}_1(z) = \frac{z}{(1 - zz_1^*)^2}\mathbf{u}, \quad \ldots \quad , \quad \mathbf{g}_{k-1}(z) = \frac{z^{k-1}}{(1 - zz_1^*)^k}\mathbf{u}.$$

We claim that the Gram matrix G associated with this chain is right lower triangular. Then, since the space $\mathcal{P}(\Theta)$ is non-degenerate, the entries on the second diagonal of G are nonzero. The triangular form of G implies that the span of any sub-chain of the given chain is degenerate and hence $\Theta(z)$ is elementary.

It remains to prove the claim. For this we use the matrix representation of the operator R_0 relative to the basis $\mathbf{g}_j(z)$: it is the matrix $A = z_1^* I_k + S_k$ from (2.15). From (1.10) and since $\mathbf{u}$ is J-neutral, we have that

$$G - (z_1^* I_k + S_k)^* G(z_1^* I_k + S_k) = 0,$$

and hence
$$S_k^* G = G \left(-z_1^2 S_k + z_1^3 S_k + \cdots (-1)^{k-1} z_1^k S_k^{k-1} \right).$$

The triangular form of G can be deduced from this equality by comparing the entries of the matrices on both sides.

(ii) If $\Theta(z)$ and $\Theta_1(z)$ are as in the beginning of this proof, then by Lemma 5.1, $\Theta_2(z) = \Theta_1(z)^{-1}\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$. From the orthogonal decomposition
$$\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1) \oplus \Theta_1 \mathcal{P}(\Theta_2)$$

it follows that $\deg \Theta_2 = \deg \Theta - k$. The minimal factorization mentioned in part (ii) of the theorem now follows by repeating the foregoing arguments. $\qquad \square$

Since rank $\mathbf{uu}^* J = 1$, the elementary factor $\Theta(z)$ in Theorem 5.2 (i) has McMillan degree k, which, evidently, is the order of the pole of $\Theta(z)$ at $z = z_1$. The function $\Theta(z)$ in (5.1) is a generalization of a Brune section in the positive definite case where it is of the form
$$(I + \frac{1}{\gamma} \frac{z+a}{z-a} \mathbf{uu}^* J)V,$$

with a normalizing constant J-unitary factor V, $a \in \mathbb{T}$, $\mathbf{u} \in \mathbb{C}^2$ with $\mathbf{u}^* J \mathbf{u} = 0$, and $\gamma > 0$.

6. A factorization algorithm

In this section we show how the factorization of a matrix function
$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathcal{U}_{z_1}^{z_0}$$

with $z_1, z_0 \in \mathbb{T}$, $z_0 \neq z_1$, can be derived from the Schur algorithm described at the end of Section 3. Similar arguments were presented in our previous papers [2] and [7] for polynomial matrix functions which are J-unitary on the unit circle or on the real line. We proceed in a number of steps.

Step 1: *Choose a number $\tau \in \mathbb{T}$ such that* (i)
$$s(z) = s_\tau(z) = \frac{a(z)\tau + b(z)}{c(z)\tau + d(z)} \tag{6.1}$$

is not a constant, (ii) $c(0)\tau + d(0) \neq 0$, *and* (iii)
$$O_{a\tau+b} = \max\{O_a, O_b\}, \quad O_{c\tau+d} = \max\{O_c, O_d\},$$

where, for example, O_a stands for the order of the pole of the function $a(z)$ at $z = z_1$. Then $s(z) \in \mathbf{S}$, it is a rational function holomorphic and of modulus one on $\mathbb{T}$ and hence the quotient of two Blaschke factors.

There are at most five distinct points $\tau \in \mathbb{T}$ for which (i)–(iii) do not hold: Assume that for three distinct points $\tau_1, \tau_2, \tau_3 \in \mathbb{T}$ the function $s(z)$ is a constant. Then, since $\Theta(z_0) = I_2$,

$$\frac{a(z)\tau_j + b(z)}{c(z)\tau_j + d(z)} = \tau_j, \quad j = 1, 2, 3, \ z \in \mathbb{C},$$

and we obtain that $c(z) \equiv 0$, $b(z) \equiv 0$, $a(z) \equiv d(z)$. Hence $\Theta(z) = a(z)I_2$. Since $\det \Theta(z)$ is a constant, we have that $a(z)$ is a constant, and so that $\Theta(z)$ is a constant matrix, which is a contradiction. Hence (i) holds with the exception of at most two different values of $\tau \in \mathbb{T}$. The condition in (ii) holds with the exception of at most one $\tau \in \mathbb{T}$, since $|\det \Theta(0)| = 1$. Finally, the conditions in (iii) hold, each with the exception at most one point $\tau \in \mathbb{T}$.

Step 2: *Let $s_1(z) = \widehat{s}(z)$ be the Schur transform of $s(z)$ (see the end of Section* 3*). Then $s_1(z) = T_{\Theta_1(z)^{-1}}(s(z))$ and $\Theta_1(z)$ is an elementary factor of $\Theta(z)$.*

From the proof of Theorem 3.2 we know that the map $\mathcal{T} : \mathbf{f}(z) \mapsto \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}(z)$ is an isometry from $\mathcal{P}(\Theta_1)$ into $\mathcal{P}(s)$. We first show that $\mathcal{T}$ is a unitary mapping from $\mathcal{P}(\Theta)$ onto $\mathcal{P}(s)$. The fact that τ in (6.1) is a constant of modulus one implies the identity

$$\frac{1 - s(z)s(w)^*}{1 - zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \begin{pmatrix} 1 \\ -s(w)^* \end{pmatrix}. \qquad (6.2)$$

This in turn implies that $\mathcal{T}$ is a partial isometry from $\mathcal{P}(\Theta)$ onto $\mathcal{P}(s)$, which is unitary if its kernel $\ker \mathcal{T}$ is trivial, see [8, Theorem 1.5.7]. Suppose

$$0 \neq \mathbf{f} = \begin{pmatrix} f \\ g \end{pmatrix} \in \ker \mathcal{T},$$

that is, $\begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f} = 0$, then

$$\mathbf{f} = \begin{pmatrix} s \\ 1 \end{pmatrix} g = \Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} x \in \mathcal{P}(\Theta), \quad x = \frac{g}{c\tau + d}.$$

Note that since $\det \Theta \neq 0$, we have that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \neq 0$. Apply R_0 to $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} x$ to obtain

$$(R_0\Theta) \begin{pmatrix} \tau \\ 1 \end{pmatrix} x(0) + \Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} R_0 x \in \mathcal{P}(\Theta).$$

The first summand belongs to $\mathcal{P}(\Theta)$ and hence the second summand also belongs to $\mathcal{P}(\Theta)$. By repeatedly applying R_0, we find that

$$\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} R_0^j x \in \mathcal{P}(\Theta), \quad j = 0, 1, 2, \ldots.$$

Since x is a rational function there is an integer $n \geq 0$ such that the span of the functions $R_0^j x$, $j = 0, 1, \ldots, n$, is finite-dimensional and R_0-invariant. It follows that R_0 has an eigenvector v which has one of three possible forms: either $v \equiv 1$

or $v(z) = 1/(z - z_2)$ with $z_2 \neq z_1$ or $v(z) = 1/(1 - zz_1^*)$. All three possibilities lead to a contradiction:

$v \equiv 1$: This implies that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathcal{P}(\Theta)$, and hence, since the elements in $\mathcal{P}(\Theta)$ all tend to 0 as $z \to \infty$, we see that $\Theta(\infty) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0$, but this cannot hold since $\det \Theta(\infty) \neq 0$.

$v(z) = 1/(z - z_2)$: This implies that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \dfrac{1}{z - z_2} \in \mathcal{P}(\Theta)$, and hence, since the elements in $\mathcal{P}(\Theta)$ are all holomorphic at $z = z_2$, we see that $\Theta(z_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0$, and again this cannot hold since $\det \Theta(z_2) \neq 0$.

$v(z) = 1/(1 - zz_1^*)$: This implies that

$$\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \frac{1}{1 - zz_1^*} = \begin{pmatrix} \dfrac{a(z)\tau + b(z)}{1 - zz_1^*} \\ \dfrac{c(z)\tau + d(z)}{1 - zz_1^*} \end{pmatrix} \in \mathcal{P}(\Theta),$$

but this cannot hold because of conditions (iii) in Step 1 and because, according to the last statement in Theorem 1.2, if $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{P}(\Theta)$ then $O_f \leq \max \{O_a, O_b\}$ and $O_g \leq \max \{O_c, O_d\}$.

These contradictions imply that $\mathcal{T}$ has a trivial kernel and hence $\mathcal{T}$ is unitary.

We now claim that $\mathcal{P}(\Theta_1) \subset \mathcal{P}(\Theta)$ and that the inclusion map is isometric. Let $N_1 = \dim \mathcal{P}(\Theta_1)$ and $\mathbf{g}_0, \ldots, \mathbf{g}_{N_1-1}$ be a basis of $\mathcal{P}(\Theta_1)$ such that $R_0 \mathbf{g}_j = z_1 \mathbf{g}_j + \mathbf{g}_{j-1}$. One can choose $\mathbf{g}_j = \mathbf{f}_j$ for $j = 1, \ldots, N_1 - 1$. Indeed, let

$$\mathbf{g}_0(z) = \frac{1}{1 - zz_1^*} \begin{pmatrix} 1 \\ \eta \end{pmatrix},$$

then the function

$$\begin{pmatrix} 1 & -s(z) \end{pmatrix} (\mathbf{f}_0(z) - \mathbf{g}_0(z)) = -\frac{s(z)(\zeta_0 - \eta)}{1 - zz_1^*}$$

belongs to $\mathcal{P}(s)$, and thus $\zeta_0 = \eta$ since the elements of $\mathcal{P}(s)$ are holomorphic in z_1. Hence $\mathbf{f}_0(z) = \mathbf{g}_0(z)$. Moreover,

$$\langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} = \langle \mathcal{T} \mathbf{f}_0, \mathcal{T} \mathbf{f}_0 \rangle_{\mathcal{P}(s)} = \langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta_1)}.$$

In the same way it follows that $\mathbf{f}_\ell(z) = \mathbf{g}_\ell(z)$, $\ell = 1, \ldots, N_1 - 1$, and that for $i, j = 0, \ldots, N_1 - 1$ the inner products satisfy

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle_{\mathcal{P}(\Theta)} = \langle \mathcal{T} \mathbf{f}_i, \mathcal{T} \mathbf{f}_j \rangle_{\mathcal{P}(s)} = \langle \mathbf{f}_i, \mathbf{f}_j \rangle_{\mathcal{P}(\Theta_1)}.$$

We conclude that $\mathcal{P}(\Theta_1)$ is isometrically included in $\mathcal{P}(\Theta)$, and the claim is proved. According to [9], $\Theta_1(z)$ is an elementary factor of $\Theta(z)$.

Step 3: *If $s_1(z)$ is a constant, then $\Theta(z) = \Theta_1(z)$. If $s_1(z)$ is not a constant, let $s_2(z) = \widehat{s}_1(z)$ be the Schur transform of $s_1(z)$ and denote the corresponding coefficient matrix by $\Theta_2(z)$. Then $\Theta_2(z)$ is an elementary factor of $\Theta_1(z)^{-1}\Theta(z)$. We iterate n times until $s_n(z) = \widehat{s}_{n-1}(z)$ is a unitary constant and conclude that $\Theta(z) = \Theta_1(z) \cdots \Theta_n(z)$.*

Because of (6.2) and the relation

$$\frac{1 - s(z)s(w)^*}{1 - zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \begin{pmatrix} 1 \\ -s(w)^* \end{pmatrix}$$
$$+ (a_1(z) - c_1(z)s(z)) \frac{1 - s_1(z)s(w)^*}{1 - zw^*} (a_1(w) - c_1(w)s(w))^*$$

we have the following equalities:

$$\begin{aligned} \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta), \\ \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1) \oplus (a_1 - c_1 s)\mathcal{P}(s_1). \end{aligned} \tag{6.3}$$

In particular, the map

$$\mathbf{f} \mapsto (a_1 - c_1 s)\, \mathbf{f} \tag{6.4}$$

is an isometry from $\mathcal{P}(s_1)$ into $\mathcal{P}(s)$.

If $s_1(z)$ is a constant then $\mathcal{P}(s_1) = \{0\}$ and (6.3) implies that $\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1)$. Since $\Theta(z)$ and $\Theta_1(z)$ are normalized they must be equal.

If $s_1(z)$ is not a constant, we define $\Theta_2(z)$ via $s_1(z) = T_{\Theta_2(z)}(s_2(z))$. Then $\Theta_2(z) \in \mathcal{U}_{z_1}^{z_0}$ and we have the decomposition

$$\mathcal{P}(s_1) = \begin{pmatrix} 1 & -s_1 \end{pmatrix} \mathcal{P}(\Theta_2) \oplus (a_2 - c_2 s_1)\mathcal{P}(s_2).$$

Since (6.4) is an isometry and

$$(a_1(z) - c_1(z)s(z)) \begin{pmatrix} 1 & -s_1(z) \end{pmatrix} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \Theta_1(z)$$

we obtain that

$$(a_1 - c_1 s)\mathcal{P}(s_1) = \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1 \mathcal{P}(\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2).$$

Thus

$$\begin{aligned} \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1) \oplus \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1 \mathcal{P}(\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &= \begin{pmatrix} 1 & -s \end{pmatrix} (\mathcal{P}(\Theta_1) \oplus \Theta_1\mathcal{P}(\Theta_2)) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2). \end{aligned}$$

It follows as above that $\mathcal{P}(\Theta_1\Theta_2)$ is isometrically included in $\mathcal{P}(\Theta)$, and, if $s_2(z)$ is constant, that $\Theta(z) = \Theta_1(z)\Theta_2(z)$. If $s_2(z)$ is not constant, we observe that

$$(a_1 - c_1 s)(a_2 - c_2 s_1) \begin{pmatrix} 1 & -s_2 \end{pmatrix} = (a_1 - c_1 s) \begin{pmatrix} 1 & -s_1 \end{pmatrix} \Theta_2 = \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1\Theta_2.$$

and define $\Theta_3(z)$ via $s_2(z) = T_{\Theta_3(z)}(s_3(z))$. Then we have

$$\begin{aligned} &(a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &\quad = \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1\Theta_2\mathcal{P}(\Theta_3) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)(a_3 - c_3 s_2)\mathcal{P}(s_3). \end{aligned}$$

and the factorization (5.2) follows by repeating the arguments.

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