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# MEASURES AND INDICES OF REFLECTION SYMMETRY FOR CONVEX POLYHEDRA \*

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**Abstract.** This paper discusses measures of reflection symmetry for 3D convex sets which are based on Minkowski addition and Brunn-Minkowski inequalities for volume and mixed volume. These measures are invariant to similitude transformations. It is also shown how these measures can be computed efficiently for convex polyhedra.

**Key words:** Reflection symmetry measure, Minkowski addition, convex polyhedra

## 1. Introduction

For many objects, presence or absence of symmetry is a major feature, and therefore the problem of object symmetry identification is of great interest in image analysis and recognition, computer vision and computational geometry. There exists a vast literature dealing with all kinds of symmetry of shapes and grey-scale images. There exist algorithms for the identification of exact symmetries as well as techniques devoted to the computation of quantitative information about the amount of symmetry in shapes and images. Efficient algorithms for detection of exact symmetries of point sets, polygons and polyhedra can be found e.g. in [9, 19].

Since real images are always disturbed by noise it is useful to have a tool to measure the amount of symmetry in them. Towards this goal Grünbaum [6] introduced the notion of *symmetry measure*. Most of the theoretical results concerning symmetry measures are obtained for convex sets. Some interesting results concerning measures of central symmetry (point reflections) can be found in [6]. Related results for reflection and rotation symmetry of convex sets can be found in [1, 2]. Studies of central symmetry measures for convex sets using morphological transformations

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are reported in [10, 11, 13]. Symmetrization transformations based on Minkowski addition were used in [17] to study rotation and reflection symmetry measures of convex sets.

Most practical algorithms developed for symmetry measurement are applied to the 2D case and only some of them can be extended to 3D (see, for example, the symmetry distance introduced in [20]). Since this paper deals with the 3D case we discuss only some literature concerning the 3D case. The extended Gaussian image representation was used in [15] for finding axes of reflection and rotation symmetries of 3D objects. Octree representations were used in [12] to measure the symmetry degree of 3D objects.

In this paper we extend the approach described in [8] for finding symmetry measures and indices of 2D convex sets to the 3D case. We restrict ourselves to reflection symmetry measures. The measures are introduced based on Minkowski addition and properties of volume and mixed volume functionals.

As was shown in [18] for the case of convex polygons the reflection symmetry measure can be computed in polynomial time. Here it is conjectured that finding reflection symmetry indices for convex polyhedra is reduced to the computation of the symmetry measure for a finite number of reflection planes only. This conjecture is based on a more general result obtained in [16] for comparing convex polyhedra.

## 2. Symmetry measures

Denote by  $\mathcal{K}(\mathbb{R}^3)$ , or briefly  $\mathcal{K}$ , the family of all nonempty compact subsets of  $\mathbb{R}^3$ . The compact convex subsets of  $\mathbb{R}^3$  are denoted by  $\mathcal{C} = \mathcal{C}(\mathbb{R}^3)$ . For two subsets  $A$  and  $B$  we write  $A \equiv B$  if these sets differ only by translation. The group of all linear transformations on  $\mathbb{R}^3$  is denoted by  $G$ . If  $g \in G$  and  $A \subseteq \mathbb{R}^3$ , then  $g(A) = \{g(a) \mid a \in A\}$ . Furthermore, we denote by  $E \subseteq G$  the reflections in the planes through the origin, and by  $R$  the rotations around axes through the origin.

Let  $u$  be a vector on the unit sphere  $S^2$  in  $\mathbb{R}^3$  and let  $\Pi_u$  be the plane in  $\mathbb{R}^3$  orthogonal to the vector  $u$  passing through the origin. The reflection w.r.t. the plane  $\Pi_u$  is denoted by  $e_u$ .

**Definition 1** *A set  $A \subset \mathbb{R}^3$  is called reflection symmetric if there exists a reflection  $e_u \in E$  such that  $e_u(A) \equiv A$ . We say that  $e_u$  is a symmetry of  $A$ , and we call  $\Pi_u$  the plane of reflection symmetry.*

To access the symmetry of sets we need a tool to measure the amount of symmetry. Several years ago, Grünbaum [6] introduced the notion of *symmetry measure*. We adapt his definition of central symmetry measure for the case of reflection symmetry in the following way.

**Definition 2** *Let  $\mathcal{J} \subset \mathcal{K}$ . A function  $\mu : \mathcal{J} \times E \rightarrow [0, 1]$  is called a reflection symmetry measure if for every  $e \in E$  the function  $\mu(\cdot, e)$  is continuous on  $\mathcal{J}$  with respect to the Hausdorff topology, and if the following conditions hold:*

1.  $\mu(A, e) = \mu(A', e)$  if  $A \equiv A'$ ;
2.  $\mu(A, e) = \mu(e(A), e)$ ;
3.  $\mu(A, e) = 1$  iff  $e(A) \equiv A$ .

Let  $H \subseteq G$  be a set such that  $heh^{-1}$  is a reflection if  $e$  is a reflection and  $h \in H$ . We say that a reflection symmetry measure  $\mu$  is  $H$ -invariant if

$$\mu(A, e) = \mu(h(A), heh^{-1}) \quad \text{for all } h \in H.$$

Below we introduce two reflection symmetry measures based on Minkowski addition and properties of volume and mixed volume functionals. The *Minkowski addition* of two sets  $A, B \subseteq \mathbb{R}^3$  is defined by

$$A \oplus B = \{a + b \mid a \in A, b \in B\}.$$

It is obvious that for two sets  $A, B \subseteq \mathbb{R}^3$  and  $g \in G$ , we have

$$g(A \oplus B) = g(A) \oplus g(B).$$

Denote by  $V(A)$  the volume (Lebesgue measure) of the set  $A \subset \mathbb{R}^3$ . Given convex sets  $A, B \subset \mathbb{R}^3$  and  $\alpha, \beta \geq 0$  one gets from the Minkowski theorem on mixed volumes [3, p.353] the following relation:

$$V(\alpha A \oplus \beta B) = \alpha^3 V(A) + 3\alpha^2 \beta V(A, A, B) + 3\alpha \beta^2 V(A, B, B) + \beta^3 V(B). \quad (1)$$

Here  $V(A, A, B)$  and  $V(A, B, B)$  are called *mixed volumes*. The following inequalities are used below; see Hadwiger [7] or Schneider [14] for a comprehensive discussion.

*Brunn-Minkowski inequality.* For two arbitrary compact sets  $A, B \subset \mathbb{R}^3$  the following inequality holds:

$$V(A \oplus B)^{\frac{1}{3}} \geq V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}}, \quad (2)$$

with equality if and only if  $A$  and  $B$  are convex and homothetic modulo translation, i.e.,  $B \equiv \alpha A$  for some  $\alpha > 0$ .

*Minkowski inequality.* For convex sets  $A, B \in \mathcal{C}(\mathbb{R}^3)$

$$V(A, A, B)^3 \geq V(A)^2 V(B), \quad (3)$$

and as before equality holds if and only if  $B \equiv \alpha A$  for some  $\alpha > 0$ .

Given a plane reflection  $e_u$  define the transformation  $b_u : \mathcal{K} \rightarrow \mathcal{K}$  by

$$b_u(A) = \frac{1}{2}(A \oplus e_u(A)).$$

It is easy to see that the set  $b_u(A)$  is reflection symmetric with respect to the plane  $\Pi_u$ . In the literature the transformation  $b_u$  is called *Blaschke symmetrization* [14].

**Proposition 1** *Given a set  $A \in \mathcal{C}$  and a plane reflection  $e_u$ , the following inequality holds:  $V(b_u(A)) \geq V(A)$ . Furthermore the following statements are equivalent:*

- (i)  $e_u(A) \equiv A$ , i.e.,  $e_u$  is a symmetry of  $A$ ;
- (ii)  $b_u(A) \equiv A$ ;
- (iii)  $V(b_u(A)) = V(A)$ .

For a proof we refer to [8, Prop.7.2].

Let us introduce the functionals  $\mu_1, \mu_2 : \mathcal{C} \times E \rightarrow \mathbb{R}_+$  defined for compact convex sets as follows

$$\mu_1(A, e) = \frac{8V(A)}{V(A \oplus e(A))}, \quad (4)$$

$$\mu_2(A, e) = \frac{V(A)}{V(A, A, e(A))}. \quad (5)$$

It is known from properties of mixed volumes that for every linear transformation  $g$ , the following identity holds:  $V(g(A), g(B), g(C)) = |\det g|V(A, B, C)$ . Here  $|\det g|$  denotes the determinant of  $g$ . Now, using relation (1) it follows that

$$\mu_2 = \frac{3\mu_1}{4 - \mu_1}. \quad (6)$$

Moreover the following proposition is true.

**Proposition 2** *The functionals  $\mu_1$  and  $\mu_2$  are reflection symmetry measures which are invariant under rotations and scalings.*

Sometimes, one is not so much interested in the symmetry measure for a specific reflection plane  $\Pi_u$ , but rather in the maximum of these values over all planes. We call this maximum the *index of reflection symmetry*. Thus we define:

$$\iota_1(A) = \sup_{e \in E} \frac{8V(A)}{V(A \oplus e(A))}, \quad (7)$$

$$\iota_2(A) = \sup_{e \in E} \frac{V(A)}{V(A, A, e(A))}. \quad (8)$$

Evidently, both indices are related to each other by a formula analogous to (6). It is obvious that both indices  $\iota_1$  and  $\iota_2$  are invariant under similitude transformations, i.e.,  $\iota_1(g(A)) = \iota_1(A)$  and  $\iota_2(g(A)) = \iota_2(A)$  for every similitude transformation  $g$ .

If the supremum in (7) is achieved for  $e = e_u$ , then we call  $\Pi_u$  the  *$\iota_1$ -optimal plane of reflection symmetry* (note that because of relation (6), the  $\iota_1$ -optimal and  $\iota_2$ -optimal planes coincide).

To find the  $\iota_1$ -optimal plane of reflection symmetry it is necessary to maximize  $\mu_1(A, e_u)$  over all possible positions of reflection planes passing through the coordinate origin. In general this is a time consuming problem. Therefore, one often restricts oneself to a finite number of reflection planes. Following ideas from classical mechanics [5] one can associate with every body its ellipsoid of inertia. It is known that planes of symmetry of reflection symmetrical bodies are orthogonal to the principal axes of this ellipsoid. Therefore to reduce the computation complexity of the optimization, one can define approximate measures by considering only planes orthogonal to the principal axes.

If one restricts attention to the class of convex polyhedra then it is possible to get additional results. In Section 3 we show that it is possible to reduce the time complexity of the given optimization problem.

### 3. Convex polyhedra

To compute the indices of reflection symmetry  $\iota_1, \iota_2$  for a convex polyhedron  $P$  it is necessary to find minima of the following functionals

$$V(P \oplus e_u(P)) \quad \text{and} \quad V(P, P, e_u(P)) \quad (9)$$

for  $u \in S_+^2$ , the hemisphere containing unit vectors with non-negative  $z$  coordinate.

Denote by  $r_{u,\alpha}$  the rotation in  $\mathbb{R}^3$  about the oriented axis directed along vector  $u$  over an angle  $\alpha$  (counter-clockwise direction). Given two unit vectors  $u_1, u_2 \in S_+^2$ , denote by  $\alpha(u_1, u_2)$  the angle smaller than  $180^\circ$  between them. The composition of plane reflections can be expressed as a rotation in the following way

$$e_{u_2} e_{u_1} = r_{u_1 \times u_2, 2\alpha(u_1, u_2)}, \quad (10)$$

where  $u_1 \times u_2$  denotes the outer product of vectors  $u_1$  and  $u_2$ .

Given any vector  $v \in S^2$  denote by  $S_+(v)$  the upper part of the great circle in  $S^2$  which is orthogonal to  $v$ . Let us investigate the functional  $V(P \oplus e_u(P))$  for  $u \in S_+(v)$  (see Fig. 1).

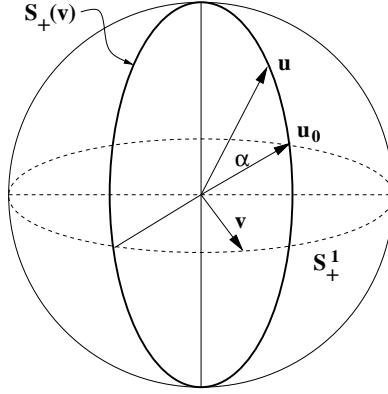


Fig. 1. Vector  $v$  is orthogonal to (upper part of) the great circle  $S_+(v)$  and  $u_0, u \in S_+(v)$ .

**Proposition 3** For a vector  $v \in S^2$  the functionals  $V(P \oplus e_u(P))$  and  $V(P, P, e_u(P))$  are piecewise concave in  $u$ , when  $u$  runs over  $S_+(v)$ .

*Proof.* To prove this result, fix  $u_0 \in S_+(v)$  as in Fig. 1. For  $u \in S_+(v)$  we have  $u \times u_0 = v$ , hence

$$e_u e_{u_0} = r_{v, 2\alpha},$$

where  $\alpha = \alpha(u_0, u)$ . Applying  $e_{u_0}$  at the right of both expressions and using that  $e_{u_0}^2 = \text{id}$ , we get

$$e_u = r_{v, 2\alpha} e_{u_0}.$$

Therefore, putting  $Q = e_{u_0}(P)$ , we find

$$V(P \oplus e_u(P)) = V(P \oplus r_{v, 2\alpha}(Q)).$$

When  $u$  runs over  $S_+(v)$ , the angle  $\alpha$  runs over  $(0, \pi]$ . The concavity of the functional at the right hand-side has been established in [16], and the proof follows. Q.E.D.

Observe that  $u_0$  is the  $90^\circ$ -rotation of  $v$  around the  $z$ -axis. Thus  $Q = e_{u_0}(P)$  depends on  $v$ :  $Q = Q(v)$ . We have shown the following relation:

$$\min_{u \in S_+^2} V(P \oplus e_u(P)) = \min_{v \in S_+^1} \min_{0 < \alpha \leq \pi} V(P \oplus r_{v, 2\alpha}(Q(v))).$$

A similar relation holds for the functional  $V(P, P, e_u(P))$ .

Therefore the optimization problem on the hemisphere  $S_+^2$  is reduced to the optimization problem on the semi-equator  $S_+^1$ .

In our report [16] we have dealt with the computation of

$$\begin{aligned} \min_{u \in S^2, 0 \leq \alpha < 2\pi} V(P \oplus r_{u, \alpha}(Q)), \\ \min_{u \in S^2, 0 \leq \alpha < 2\pi} V(P, P, r_{u, \alpha}(Q)), \end{aligned}$$

for arbitrary convex polyhedra  $P$  and  $Q$ . It was proven that only finitely many vectors  $u \in S^2$  have to be checked to compute the minimum. Based on these results we formulate the following conjectures:

**Conjecture 1** *There exist a finite number of vectors  $v_1, \dots, v_k \in S_+^1$  such that*

$$\begin{aligned} \min_{u \in S_+^2} V(P \oplus e_u(P)) &= \min_{i=1, \dots, k} \min_{0 \leq \alpha < \pi} V(P \oplus r_{v_i, 2\alpha}(Q(v_i))), \\ \min_{u \in S_+^2} V(P, P, e_u(P)) &= \min_{i=1, \dots, k} \min_{0 \leq \alpha < \pi} V(P, P, r_{v_i, 2\alpha}(Q(v_i))). \end{aligned}$$

#### 4. Conclusion

The 3D case is essentially more difficult than the 2D case: for the latter it has been shown [8] that only finitely many cases have to be checked there. Basically, the reason for this difference is that in the 2D case, the composition of a reflection and a rotation is a reflection; this no longer true in the 3D case. The conjecture formulated above is still to be proven. Our present work concerns the implementation of algorithms for computing indices of reflection symmetry for convex polyhedra. Such algorithms are based on the slope diagram representation of convex polyhedra [4].

The problem of computation of indices of reflection symmetry is more difficult than computation of central symmetry measures which indicate the amount of central symmetry. Write  $\tilde{A} = \{-x, x \in A\}$  and introduce the functionals  $\mu_3, \mu_4 : \mathcal{C} \rightarrow \mathbb{R}_+$  defined for compact convex sets as follows

$$\begin{aligned} \mu_3(A) &= \frac{8V(A)}{V(A \oplus \tilde{A})}, \\ \mu_4(A) &= \frac{V(A)}{V(A, A, \tilde{A})}. \end{aligned}$$

One can show that the functionals  $\mu_3$  and  $\mu_4$  define affine invariant central symmetry measures for convex sets in  $\mathbb{R}^3$ .

## References

1. C. K. Chui and M. N. Parnes. Measures of  $N$ -fold symmetry for convex sets. *Proceedings of the American Mathematical Society*, 26:480–486, 1970.
2. B. A. deValcourt. Measures of axial symmetry for ovals. *Israel Journal of Mathematics*, 4:65–82, 1966.
3. R. Gardner. *Geometric Tomography*. Cambridge University Press, 1995.
4. P. K. Ghosh and R. M. Haralick. Mathematical morphological operations of boundary-represented geometric objects. *Journal of Mathematical Imaging and Vision*, 6(2/3):199–222, 1996.
5. H. Goldstein. *Classical Mechanics*. Addison-Wesley, Reading, MA, 1950.
6. B. Grünbaum. Measures of symmetry for convex sets. In *Proc. Sympos. Pure Math.*, volume 7, pages 233–270. Providence, USA, 1963.
7. H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer, Berlin, 1957.
8. H. J. A. M. Heijmans and A. Tuzikov. Similarity and symmetry measures for convex sets based on Minkowski addition. Research report BS-R9610, CWI, Amsterdam, 1996.
9. X. Y. Jiang and H. Bunke. Detection of rotational and involutorial symmetries and congruity of polyhedra. *The Visual Computer*, 12(4):193–201, 1996.
10. M. Jourlin and B. Laget. Convexity and symmetry: Part 1. In J. Serra, editor, *Image Analysis and Mathematical Morphology, vol. 2: Theoretical Advances*, London, 1988. Academic Press.
11. G. Matheron and J. Serra. Convexity and symmetry: Part 2. In J. Serra, editor, *Image Analysis and Mathematical Morphology, vol. 2: Theoretical Advances*, pages 359–375, London, 1988. Academic Press.
12. P. Minovic, S. Ishikawa, and K. Kato. Symmetry identification of a 3d object represented by octree. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 15(5):507–514, 1993.
13. R. Schneider. On a morphological transformation for convex domains. *Journal of Geometry*, 34:172–180, 1989.
14. R. Schneider. *Convex Bodies. The Brunn-Minkowski Theory*. Cambridge University Press, Cambridge, 1993.
15. C. Sun and J. Sherrah. 3d symmetry detection using the extended Gaussian image. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 19(2):164–168, 1997.
16. A. Tuzikov, J. B. T. M. Roerdink, and H. J. A. M. Heijmans. Similarity measures for convex polyhedra based on Minkowski addition. Technical Report CS-R9708, University of Groningen, Department of Computer Science, 1997.
17. A. V. Tuzikov, G. L. Margolin, and A. I. Grenov. Convex set symmetry measurement via Minkowski addition. *Journal of Mathematical Imaging and Vision*, 7(1):53–68, 1997.
18. A. V. Tuzikov, G. L. Margolin, and H. J. A. M. Heijmans. Efficient computation of a reflection symmetry measure for convex polygons based Minkowski addition. In *Proceedings of 13th International Conference on Pattern Recognition*, volume B, pages 236–240, 1996.
19. J. D. Wolter, T. C. Woo, and R. A. Volz. Optimal algorithms for symmetry detection in two and three dimensions. *Visual Computer*, 1:37–48, 1985.
20. H. Zabrodsky, S. Peleg, and D. Avnir. Symmetry as a continuous feature. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17:1154–1166, 1995.