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On the Canonical Components of Character Varieties of Hyperbolic 2-Bridge Link Complements

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On the Canonical Components of Character Varieties of Hyperbolic 2-Bridge Link Complements

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Dedicated to my grandma Evelyn Landes and good friend and collaborator Dr. David H. Jensen.

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On the Canonical Components of Character Varieties of Hyperbolic 2-Bridge Link Complements

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This dissertation concerns the study of canonical components of the $SL_2(\mathbb{C})$ character varieties of hyperbolic 3-manifolds. Although character varieties have proven to be a useful tool in studying hyperbolic 3-manifolds, very little is known about their structure. Chapter 1 provides background on this subject. Chapter 2 is dedicated to the canonical component of the Whitehead link. We provide a projective model and show that this model is isomorphic to $\mathbb{P}^2$ blown up at 10 points. The Whitehead link can be realized as 1/1 Dehn surgery on one cusp of both the Borromean rings and the 3-chain link. In Chapter 3 we examine the canonical components for the two families of hyperbolic link complements obtained by $1/n$ Dehn filling on one component of both the Borromean rings and the 3-chain link. These examples extend the work of Macasieb, Petersen and van Luijk who have studied the character varieties associated to the twist knot complements. We conjecture that the canonical components for the links obtained by $1/n$ Dehn filling on one component of

the 3-chain link are all rational surfaces isomorphic to $\mathbb{P}^2$ blown up at $9n+1$ points. A major goal is to understand how the algebro-geometric structure of these varieties reflects the topological structure of the associated manifolds. At the end of Chapter 3 we discuss common features of these examples and explain how our results lend insight into the affect Dehn surgery has on the character variety. We conclude, in Chapter 4, with a description of possible directions for future research.

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Chapter 1

Introduction

Character varieties are a point of interest in several areas of mathematics and their study has revealed connections between dynamical systems, topology and algebraic geometry. This dissertation concerns the study of character varieties of discrete groups, particularly fundamental groups of hyperbolic 3-manifolds. These algebraic sets provide an interesting connection between topology and algebraic geometry, and a basic question is to understand which algebraic sets arise as character varieties of hyperbolic 3-manifolds. In my research I have determined topologically the character variety for the Whitehead link complement along with that of other two component 2-bridge hyperbolic link complements. These results expand the work of Macasieb, Petersen and van Luijk who have studied the character varieties associated to the twist knot complements.

Since the seminal work of Culler and Shalen, character varieties have proven to be a powerful tool for studying the topology of hyperbolic 3-manifolds; for example they provide efficient means of detecting essential surfaces in hyperbolic knot complements ([6], [4], [26]). Continuing to determine and study these types of models will help us understand how the algebro-geometric structure of a character variety reflects the topological structure of the associated 3-manifold.

As interesting and useful as character varieties can be, very little is known about their structure. For instance, Dehn surgery on a 3-manifold M naturally gives rise to subvarieties of the character variety for M. Identifying which subvarieties are character varieties associated to manifolds obtained by Dehn filling is a challenging problem we are interested in studying. The canonical components of the character varieties for the two component 2-bridge links we studied are complex surfaces. Although the birational equivalence class of the projective model for these varieties does not depend on the compactification, the isomorphism class does. Hence, for complex surfaces, there is some ambiguity in deciding which smooth model to take as the character variety. What determines the "right" projective completion is a major concern in algebraic geometry. Better understanding the structure of these character varieties will help us determine what we mean by the "right" compactification in this context.

This chapter is devoted to introducing the main results and building the necessary background information. Chapter 2 is devoted to the study of the canonical component for the Whitehead link and Chapter 3 the structure of the canonical components of two families of hyperbolic link complements obtained by Dehn filling a cusp of a fixed 3-manifold. Chapter 4 discusses the implications of these results and presents direction for future research.

1.1 Main Results

The character variety can be defined for any Lie group G and any finitely generated group Γ. Let $Hom(\Gamma, G)$ denote the set of group homomorphisms from Γ to G. Formally, the G-character variety is the GIT quotient $X(\Gamma, G) = Hom(\Gamma, G)//G$ where G acts on $Hom(\Gamma, G)$ by conjugation.

This thesis primarily concerns $SL_2(\mathbb{C})$ character varieties for hyperbolic 3-manifolds. Let M be a hyperbolic 3-manifold. Then the interior of M admits a complete finite volume hyperbolic structure and M is isomorphic to $\mathbb{H}^3/\Gamma$ where Γ is a discrete, torsion free subgroup of $Isom^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. The $(P)SL_2(\mathbb{C})$ character variety for $\Gamma = \pi_1(M)$ reflects topological information about M. In this context, the character variety $X(\Gamma)$ coincides with the set of characters $\{\chi_\rho | \rho \in Hom(\Gamma, G)\}$ where $\chi_\rho : \Gamma \to \mathbb{C}$ is the map $\chi_\rho(\gamma) = tr(\rho(\gamma))$ ([6]). Characters corresponding to discrete faithful representations capture hyperbolic structures on M since $PSL_2(\mathbb{C})$ parameterizes the orientation preserving isometries of 3-dimensional hyperbolic space. Let $Y_0 = Y_0(\Gamma) \subset Y(\Gamma)$ be a component of the $PSL_2(\mathbb{C})$ character variety containing a character y_0 which corresponds to a discrete faithful representation $\tilde{rho}_o$. For hyperbolic manifolds of finite volume, Y_0 is unique (up to orientation) [27]. Fix a lift ρ_0 of $\tilde{\rho_0}$ to $SL_2(\mathbb{C})$. The character $x_0 = x_{\rho_0}$ is contained in a unique component $X_0 = X_0(\Gamma) \subset X(\Gamma)$ of the $SL_2(\mathbb{C})$ character variety. We call X_0 the canonical component. Thurston's Hyperbolic Dehn Surgery Theorem implies that for an orientable, hyperbolic 3-manifold of finite volume, with n-cusps, the canonical component has complex dimension n. Much of the theory developed thus far is particular to case when $n = 1$. For a 1-cusped hyperbolic 3-manifold, the canonical component is a complex curve whose geometric genus coincides with the topological genus. Using the ideal points of this curve, we can detect incompressible surfaces in M ([26]).

Of particular interest is determining explicit models for X_o. This is a difficult problem for even the simplest hyperbolic link complements. Only recently have explicit models for the canonical components of a full family of hyperbolic knots been determined ([17]). Macasieb, Petersen and van Luijk studied a family of 2-bridge knots which include the twist knots. The twist knots can be obtained by $1/n$ Dehn filling on the Whitehead link where Dehn filling on a link complement is a gluing that identifies via homeomorphism the boundary of a solid torus with a boundary torus of the link complement. The Whitehead link is thus a simple two component 2-bridge link whose character variety contains as algebraic sets those associated to the twist knots. Hence, it is a natural example to study and determining the canonical component explicitly extends the work of [17]. Chapter 2 is dedicated to proving the following theorem.

Theorem 1.1.1. *(L [16]) The canonical component of the character variety of the Whitehead link complement is a rational surface isomorphic to* $\mathbb{P}^2$ *blown up at* 10 *points.*

Rational surfaces (surfaces birational to $\mathbb{P}^2$) are well-understood algebro-geometric objects. Smooth rational surfaces are isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$ blown-up at n points. Although we are ultimately interested in the isomorphism class, the birational equivalence class still carries a lot of information. For complex surfaces, although there may be more than one smooth model in a birational equivalence class, there is a notion of a minimal smooth model, i.e. the smooth birational model containing no (-1) curves ([14]). For rational surfaces, there are two possible minimal smooth models: $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. The minimal smooth model for the canonical component of the Whitehead link is $\mathbb{P}^2$. Identifying the birational equivalence class for this surface is not that different than identifying the isomorphism class for the canonical components of the knots studied in [17]. For curves, the birational equivalence class and the isomorphism class coincide.

The Whitehead link can be realized as $1/1$ Dehn filling of one cusp of the Magic manifold (i.e. 3-chain link complement). It is natural to ask whether the character varieties of the link complements obtained by $1/n$ Dehn filling one of the cusps of the Magic manifold exhibit any similarities.

Conjecture 1.1.2. *The canonical component of the character variety obtained by $1/n$ Dehn surgery on one cusp of the Magic manifold is a rational surface birational to a conic bundle and isomorphic to $\mathbb{P}^2$ blown up at $1 + 9n$ points.*

Conjecture 1.1.2 is based on our calculations for $n = 1, \dots, 4$ which we discuss in Section 3.1. That these link complement examples exhibit similar yet more complicated recursive relations than the knots studied in [17] motivates the conjecture. Verifying Conjecture 1.1.2 would yield a 2-dimensional generalization of our understanding of twist knot character varieties. These rational surfaces are also all birational to conic bundles. Conic bundles can be realized as smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$, cut out by a polynomial of bidegree $(2, n)$. The twist knot character varieties are all hyperelliptic, meaning they can be realized as smooth hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by a polynomial of bidegree $(2, n)$ ([17]). These surfaces birational to conic bundles are subvarieties of the canonical component of the character variety associated to the Magic manifold, which yields a 3-dimensional analogue. Namely, the canonical component for the Magic manifold is not only rational but also birational to a fiber bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with conic fibers.

All of the character varieties for the examples we have thus far discussed have a single component. We are interested in investigating how the number number and type of components of the character variety reflect the topology of associated manifolds. In particular, we would like to qualify the conditions for which types of varieties arise as components of the character varieties associated to hyperbolic manifolds. In Section we discuss the character varieties for the manifolds which result upon $1/n$ Dehn filling on one cusp of Borromean rings M_{br}. These link complements are hyperbolic two component 2-bridge link complements ([15]). For $n = 2, 3, 4$, we shown that the character variety of $M_{br}(1/n)$ has multiple components, one of which is rational.

Theorem 1.1.3. *For $n = 2, 3, 4$, the character variety of $M_{br}(1/n)$ has a component which is a rational surface isomorphic to $\mathbb{P}^2$ blown up at 7 points.*

1.2 Preliminary Material

Here we provide a description of the $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ representation and character varieties. Standard references for this include [6] and [26].

1.2.1 Representation Varieties

A $(P)SL_2(\mathbb{C})$ *representation* for a group Γ is a homomorphism $\rho : \Gamma \to (P)SL_2(\mathbb{C})$. We say ρ is *irreducible* if there are no nontrivial subspaces of $\mathbb{C}^2$ invariant under the action of $\rho(G)$. Otherwise ρ is *reducible.* Two representations ρ and ρ' are equivalent if they differ by an inner automorphism of $(P)SL_2(\mathbb{C})$. Associated to every representation ρ is a *character* χ_ρ which is the map $\chi_\rho : \Gamma \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = Tr(\rho(\gamma))$. Equivalent representations have the same character since the trace function is invariant under inner automorphisms. Irreducible representations are determined, up to conjugacy, by their character. This is not the case for reducible representations. In fact a reducible representation always shares its character with an abelian representation.

Let $R(G)$ and $\bar{R}(G)$ denote the set or $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ representations respectively. For any finitely generated group $\Gamma = \langle g_1, \ldots, g_n | r_1, \ldots \rangle$, $R(\Gamma)$ and $\bar{R}(\Gamma)$ have the structure of an affine algebraic set [6]. Consider $R(\Gamma)$. We can write $R(\Gamma) = \{(x_1, \ldots, x_n) \in (SL_2(\mathbb{C}))^n | r_j(x_1, \ldots, x_n) = I\}$. By appealing to the Hilbert basis theorem, we can assume $\{r_j\}$ is finite, say $\{r_1, \ldots, r_m\}$. Notice then that $R(\Gamma)$ can be identified with $r^{-1}(I, \ldots, I)$ where

$r : (SL_2(\mathbb{C}))^n \to (SL_2(\mathbb{C}))^m$ is the map $r(x) = (r_1(x), \ldots, r_m(x))$. That $R(\Gamma)$ is an algebraic set follows from the fact that r is a regular map. Identifying $SL_2(\mathbb{C})^n$ with a subset of $\mathbb{C}^{4n}$, we can view $R(\Gamma)$ as an algebraic set over $\mathbb{C}$. We can argue similarly for $\bar{R}(\Gamma)$ by replacing $SL_2(\mathbb{C})$ with $PSL_2(\mathbb{C})$ above. We should note that the isomorphism class of $R(\Gamma)$ does not depend on the group presentation and in general, is not irreducible. In fact the abelian representations (i.e. representations with abelian image) comprise a component of this algebraic set [6].

1.2.2 Character Varieties

Let $X(\Gamma)$ and $Y(\Gamma)$ denote the space of $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ characters. Since this thesis primarily concerns $X(\Gamma)$ for hyperbolic 3-manifolds, we will focus on $X(\Gamma)$ here. For each $g \in \Gamma$ there is a regular map $\tau_g : R(\Gamma) \to \mathbb{C}$ defined by $\tau_g(\rho) = \chi_\rho(g)$. Let T be the subring of the coordinate ring on $R(\Gamma)$ generated by 1 and τ_g, $g \in \Gamma$. In [6] it is shown that the ring T is finitely generated, for example by $\{\tau_{g_{i_1} g_{i_2} \cdots g_{i_k}} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. In particular any character $\chi \in X(\Gamma)$ is determined by its value on finitely many elements of Γ. As a result, for $t_1, \ldots, t_s$ generators of T, the map $t = (t_1, \ldots, t_s) : R(\Gamma) \to \mathbb{C}^s$ defined by $\rho \mapsto (t_1(\rho), \ldots, t_s(\rho))$ induces a map $X(\Gamma) \to \mathbb{C}^s$. Culler and Shalen use the fact that this map is injective to show that $X(\Gamma)$ inherits the structure of a closed algebraic subset of $\mathbb{C}^s$ ([6]). From this it follows that $X(\Gamma)$ is an affine algebraic variety with coordinate ring $T_X = T \otimes \mathbb{C}$.

From here onward we will view the map t as a map from $R(\Gamma)$ to $X(\Gamma)$, identifying $(t_1(\rho), \ldots, t_s(\rho))$ with χ_ρ. Let R_0 be an irreducible component of $R(\Gamma)$. Since t is a regular map, $t(R_0)$ is an affine algebraic set. We are particularly interested in the case where $\Gamma = \pi_1(M)$ for M a hyperbolic 3-manifold. Mostow-Prasad rigidity guarantees the existence of a discrete faithful representation, $\tilde{\rho_0} : \pi_1(M) \to Isom^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. It it well known that $\tilde{\rho_0}$ lifts to a discrete faithful representation $\rho_0 : \pi_1(M) \to SL_2(\mathbb{C})$. Let x_0 be the character $t(\rho_0)$. The *canonical component* of $X(\pi_1(M))$ is the irreducible component containing x_0 and is denoted by $X_0(\pi_1(M))$. For hyperbolic knot and link complements, as x_0 is a smooth point of $X_0(\pi_1(M))$, $X_0(\pi_1(M))$ is unique up to orientation ([27]). In this context, Thurston's Hyperbolic Dehn Surgery theorem ([27] Theorem 4.5.1) can be stated as follows.

Theorem 1.2.1. *Let M be an orientable hyperbolic 3-manifold of finite volume with n-cusps. Then $dim_{\mathbb{C}}(X_0(\pi_1(M))) = n$.*

We have thus far been working with affine algebraic sets. What we are most interested in are smooth projective models. In this work, we obtain projective models, which may or may not be smooth, by compactifying in $\mathbb{P}^2 \times \mathbb{P}^1$. If the projective model is not smooth, we obtain a nonsingular model by resolving the singular points. Throughout this thesis we will refer to the affine algebraic sets $X(\pi_1(M))$ and $X_0(\pi_1(M))$ as the affine $SL_2(\mathbb{C})$-character variety and affine canonical component of M. We use $\tilde{X}(\pi_1(M))$ to denote a projective model for the $SL_2(\mathbb{C})$-character variety and $\tilde{X}_0(\pi_1(M))$ to mean a smooth projective model for canonical component. The canonical component for the character varieties associated to hyperbolic 1-cusped manifolds (e.g knot complements) are complex curves. For these varieties, the birational equivalence class contains a unique smooth model (up to isomorphism). In this sense there no ambiguity in choosing a projective model. The birational equivalence class for varieties of higher dimension do not admit unique smooth models. For complex surfaces, however, there is still a notion of a minimal smooth model, i.e. the smooth birational model containing no (-1) curves. To some extent, identifying the birational class for a complex surface is consistent with identifying the isomorphism class for a complex curve because for curves, the birational equivalence class coincides with the isomorphism class. In Chapters 2 and 3 we determine the birational equivalence class for the canonical component of the Whitehead link and that for other two component 2-bridge link complements. For our examples of hyperbolic two component link complements we identify the minimal model and describe topologically the isomorphism class for a specific projective model.

1.3 Some Basic Algebraic Geometry

In this section we discuss the algebro-geometric concepts relevant to the main proof of this paper. For more details see [14] or [24] .

1.3.1 Definitions and Notations

Let $X \subset \mathbb{A}^n$ be an algebraic set over the ground field k. We denote the coordinate ring of X by $k[X]$. When X is irreducible we can consider the function field $k(X)$ of X. Let Θ_x denote the Zariski tangent space of X at x and let $dim_x X$ denote maximum dimension ranging over the irreducible components of X containing x. A point $x \in X$ is *nonsingular* or *smooth* if $dim\Theta_x = dim_x X$. Otherwise the point is *singular*. When X is a variety (i.e. an irreducible algebraic set) defined by the ideal generated by $f_1, \ldots, f_s \in k[x_1, \ldots, k_n]$, this is equivalent to the condition that the partial derivatives $\frac{\partial f_j}{\partial x_i}$ do not all simultaneously vanish at x. If x is not a smooth point we say that x is a *singularity* or a *singular point* of X.

In algebraic geometry we often work up to birational equivalence. Two algebraic sets, X and Y are *birational* if for some dense open sets $U \subset X$ and $V \subset Y$ there is a polynomial map $\psi : U \to V$ whose inverse is also polynomial. Birational varieties are not generally isomorphic. However, the birational equivalence does carry a lot of information about a variety. For instance, birational varieties have isomorphic function fields. In the case of curves, there is a unique (up to isomorphism) smooth projective model for each equivalence class. The canonical component of the character variety for hyperbolic 1-cusped 3-manifolds is a complex curve. The model we use to identify this component is the unique smooth projective model of the birational equivalence class. For hyperbolic 3-manifolds with 2-cusps, the canonical component of the character variety is a complex surface. In this case, the model with which to identify this component is not as obvious. Unlike curves, smooth projective models do not determine the birational equivalence class for varieties of higher dimension. For surfaces there is, however, still a notion of a minimal smooth model i.e. the smooth birational model containing no (-1) curves ([14]). Aside from rational surfaces (those birational to $\mathbb{P}^2$) and ruled surfaces ($\mathbb{P}^1$ bundles over a curve), the minimal model is unique. Although we are ultimately interested in the isomorphism class, determining the birational equivalence class for the canonical component of the character varieties for 2-cusped hyperbolic 3-manifolds is not that different than identifying the isomorphism class for the canonical components associated to hyperbolic 1-cusped manifolds because for curves, the birational equivalence class and the isomorphism class coincide.

1.3.2 Divisors

Algebro-geometric invariants such as the geometric genus and the canonical divisor are helpful tools in determining the birational equivalence class and understanding the topological structure of a variety. For divisors, there are two commonly used flavors: Weil divisors and Cartier divisors. These notions agree on non-singular varieties over algebraically closed fields.

A *Weil divisor* D on a variety X is a formal sum of codimension 1 subvarieties $D_1, \ldots, D_n$. Namely, $D = k_1 D_1 + \cdots + k_n D_n$ where k_i are integer multiplicities. We write $D = 0$ if all the $k_i = 0$ and we call D *effective* if all $k_i \geq 0$ and for some i, $k_i > 0$. If D_i is an irreducible codimension 1 subvariety with multiplicity 1, we say D_i is a *prime* divisor. Since any two divisors, D and D' can be expressed as a formal sum of the same prime divisors, we add divisors by adding corresponding multiplicities and the set of divisors on X form a group

$DivX$ isomorphic a free $\mathbb{Z}$ module. For any prime divisor C of X, let v_C be the valuation on $k(X)$ that takes $f \in k[X]$ to its order of vanishing on C and takes $\frac{1}{f}$ to $-v_C(f)$. The divisor of f is $div f = \Sigma v_C(f)$ taken over all prime divisors of X. We say a Weil divisor D is a *principal* divisor if $D = div f$ for some $f \in k(X)$. The set of principal divisors, $P(X)$, forms a subgroup of $Div(X)$ and the quotient group $Cl(X) = Div(X)/P(X)$ is the divisor class group. For projective varieties $Cl(\mathbb{P}^n) = \mathbb{Z}$ and $Cl(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) = \mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_k}$. We say two divisors are *linearly equivalent* if they are in the same divisor class. To each divisor D we associate the vector space $\mathscr{L}(D)$ of all $f \in K(X)$ such that $div f + D$ is effective. When X is a projective variety $\mathscr{L}(D)$ is finite with dimension (D).

Principal divisors are, in some sense, a fundamental class in the group of Weil divisors. Cartier divisors are constructed so that every divisor can be realized as locally principal. For a variety X, a *compatible system of functions* is a collection of functions $\{f_i\}$ together with corresponding open set U_i of a finite cover $X = \cup U_i$ such that f_i are not identically 0 and $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are regular on the overlap $U_i \cap U_j$. Two compatible systems $\{f_i\}$ corresponding to U_i and $\{g_j\}$ corresponding to V_j are equivalent if $\frac{f_i}{g_j}$ and $\frac{g_j}{f_i}$ are regular and nonzero on $U_i \cup V_j$. A *Cartier divisor* on a variety X is a compatible system of rational functions. A Cartier divisor naturally gives rise to a Weil divisor. Let $\{f_i\}$ corresponding to $\{U_i\}$ be a Cartier divisor on X. For each prime Weil divisor C let $k_C = V_C(f_i)$ if $U_i \cap C \neq \emptyset$ where C and f_i are considered as a divisor and rational function on U_i. The compatibility condition guarantees that the values k_C are independent of the choice $\{f_i\}$ and $\{U_i\}$. Hence the system defines a divisor $D = \Sigma k_C C$ and on each U_i, D can be realized as $div f_i$. Conversely, every Weil divisor can be realized as a Cartier divisor. For any prime divisor C, near any point $x \in X$ there is a neighborhood U_x in which C is defined by the local equation ϕ. Consider any divisor $D = \Sigma k_i C_i$ where C_i are defined by locally by ϕ_i in U_x. Then, in U_x, $D = div(g_i)$ where $g_i = \Pi \phi_i^{k_i}$. In this way, choosing a finite covering $\{U_i\}$ for X gives rise to a Cartier divisor D. The product of two compatible systems $\{f_i\}$ associated to $\{U_i\}$ and $\{g_j\}$ associated to $\{V_j\}$ is the compatible system $\{f_i g_j\}$ associated to $\{U_i \cap V_j\}$. Under this multiplication, Cartier divisors form a group and the principal divisors form a subgroup. We often work with the divisor classes in this quotient group which is called the *Picard* group of X and denoted $Pic(X)$.

A particular divisor class, called the canonical class, carries information about the variety X. This class corresponds to top dimensional rational differential forms on X and can be described as follows. Suppose X is an n-dimensional variety and $\omega \in \Omega^n[X]$ be any n-form. For any point $x \in X$, there is a neighborhood U_x in which $\omega = g du_1 \bigwedge \cdots \bigwedge du_n$. We can take a finite cover $\{U_i\}$ for X such that on each U_i, $\omega = g^{(i)} du_1^{(i)} \bigwedge \cdots \bigwedge du_n^{(i)}$. On the overlap $U_i \cap U_j$, $g^{(i)} = g^{(j)} J(\frac{u_1^{(i)}, \dots u_n^{(i)}}{u_1^{(j)}, \dots u_n^{(j)}})$ where J is the determinant of the Jacobian. These rational functions $\{g_i\}$ associated to $\{U_i\}$ form a compatible system and hence define a divisor, denoted $div\omega$, on X. Each n-form can be written $\omega = f\omega_1$ for some fixed n-from ω_1 since $\Omega^n[X]$ is a 1-dimensional vector space over $k(X)$. When X is smooth all n-forms are linearly equivalent because, in this case, $div(f\omega) = div(f)div(\omega)$ for all $f \in k(X)$. This linear equivalence class is the *canonical class* K_X. We will use ω_k to denote a representative divisor in the canonical class. We can and often do view K_X as the class of global sections of line bundles $\Gamma(X, \omega_k)$ over X. For points y in a neighborhood U_x of $x \in X$, the fiber corresponds to the set of forms $\{f du_1 \bigwedge \cdots \bigwedge du_n\}$ where $f \in k(X)$. Global sections correspond to the compatible systems and hence divisors in the canonical class.

The vector space $L(K_X)$ corresponds precisely to $\Omega^n[X]$ since $div(\omega)$ is effective whenever $\omega \in \Omega^n[X]$. The dimension $l(K_X) = dim_k(\Omega^n[X])$ is a birational invariant called the *geometric genus*, $p_g(X)$. In the case of complex curves, the geometric genus coincides with the topological genus and thus identifies the birational equivalence class.

1.3.3 Geometric genus

The geometric genus, p_g, of a projective variety, S, is the dimension of the vector space of global sections $\Gamma(X, \omega_k)$ of the canonical divisor w_k. For a complex curve, the geometric genus coincides with the topological genus and can thus be used to topologically determine the character varieties of hyperbolic knot complements. Unfortunately for complex surfaces, the geometric genus does not carry as direct topological information (for instance it appears as $h^{2,0}$ in the Hodge decomposition [12]). However, as it may still be helpful in determining which varieties can arise as the character varieties of hyperbolic two component link complements, it it worth keeping track of this value. For a hypersuface, Z, in $\mathbb{P}^2 \times \mathbb{P}^1$ defined by a polynomial f of bidegree (a, b) the geometric genus is $p_g(Z) = \frac{(a-1)(a-2)(b-1)}{2}$.

We give a brief description of this here. As the group of linear equivalence classes of divisors for $\mathbb{P}^2 \times \mathbb{P}^1$ is $Pic(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$, we think of the divisors of $\mathbb{P}^2 \times \mathbb{P}^1$ as elements of $\mathbb{Z} \times \mathbb{Z}$. For a linear class with representative divisor D on $\mathbb{P}^2 \times \mathbb{P}^1$, there is an associated vector space, $L(D)$ of principal divisors E such that D+E is effective. The vector space $L(D)$ is in one-to-one correspondence with the vector space of global sections of the line bundle $\mathbb{L}(D)$ on $\mathbb{P}^2 \times \mathbb{P}^1$. As the vector space of global sections of $\mathbb{L}(D)$ corresponds to the space of polynomials over $\mathbb{P}^2 \times \mathbb{P}^1$ with the same bidegree as that which cuts out D, the restrictions of these polynomials to S which are nonzero on S, correspond to the vector space of global sections of D on S. That is to say the kernel of the surjective map $L(D) \twoheadrightarrow L(D)|_S$ is those polynomials which vanish on S. When D is the canonical divisor K_S of the surface S, assuming all the restricted polynomials are nonzero on S, the geometric genus of the surface $g_g(S)$ is then just the dimension of the vector space of these polynomials.

For the hypersurface S defined by f, we can use the adjunction formula to determine K_S. Namely, $K_S = [K_{\mathbb{P}^2 \times \mathbb{P}^1} \otimes \mathbb{O}(S)]|_S$ where $\mathbb{O}(S)$ is the divisor class of S in $\mathbb{P}^2 \times \mathbb{P}^1$. The canonical divisor $K_{\mathbb{P}^2 \times \mathbb{P}^1}$ of $\mathbb{P}^2 \times \mathbb{P}^1$ is $(-3, -2) \in \mathbb{Z} \times \mathbb{Z}$ and the divisor class $\mathbb{O}(S) = (a, b) \in Pic(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$ where (a, b) is the bidegree of f. Hence, $K_S = (a - 3, b - 2)$. Since the linear class of divisors of $K_S = (a - 3, b - 2)$ corresponds to a polynomials of bidegree $(a - 3, b - 2)$, the global sections of line bundle associated to $K_S = (a - 3, b - 2)|_S$ correspond to polynomials of bidegree $(a-3, b-2)$. Since S is a hypersurface defined by the irreducible polynomial f, no polynomial of bidegree $(a-3, b-2)$ can vanish on all of S. Hence, the geometric genus of the surface $g_g(S)$ is then just the dimension of the vector space of polynomials over $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree (a, b). Determining this dimension is a matter of counting monomials of bidegree $(a-3, b-2)$ for which there are $\frac{(a-1)(a-2)(b-1)}{2}$.

1.3.4 Projective models for character varieties

The affine varieties with which we are concerned are all hypersurfaces in $\mathbb{C}^3$ i.e. they are zero sets $Z(\tilde{f})$ of a single smooth polynomial $\tilde{f} \in \mathbb{C}[x, y, z]$. Finding the right projective completion is tricky, especially with complex surfaces since different projective completions may result in non-isomorphic models. It might seem natural to take projective closures in $\mathbb{P}^3$. One problem with compactifying in $\mathbb{P}^3$ is that, generally, this projective model has singularities which take more than one blow up to resolve. Following the work of [17] it is more natural to consider the compactification in $\mathbb{P}^2 \times \mathbb{P}^1$. This compactification does result in a singular surface. However, the singularities are manageable and away from the singularities this model has the nice structure of a conic bundle. Hence, for these reasons, this is the projective model we choose to use for our examples.

Given an affine variety $Z(\tilde{f})$ defined by a polynomial $\tilde{f} \in \mathbb{C}[x, y, z]$, we construct the projective closure by homogenizing $\tilde{f}$. Let a be the degree of $\tilde{f}$ when viewed as a polynomial in variables x and y. Let b be the degree of

$\tilde{f}$ when viewed as a polynomial in the variable z. The projective model in $\mathbb{P}^2 \times \mathbb{P}^1$ of the affine variety $Z(\tilde{f})$ is cut out by the homogeneous polynomial $f = u^a w^b \tilde{f}(\frac{x}{u}, \frac{y}{u}, \frac{z}{w})$ where x, y, u are $\mathbb{P}^2$ coordinates and z, w are $\mathbb{P}^1$ coordinates. Notice that every monomial which appears in f has degree a in the $\mathbb{P}^2$ coordinates and degree b in the $\mathbb{P}^1$ coordinates so f has bidegree (a, b).

1.3.5 Conic bundles

The character varieties for many of our examples have a component which is birational to a conic bundle. A conic is a curve defined by a polynomial over $\mathbb{P}^2$ of degree 2. Smooth conics have the genus zero ([24]) so are spheres. A degenerate conic consists of two spheres intersecting one one point. In this paper the term conic bundle will be used to mean a conic bundle over $\mathbb{P}^1$ i.e. over a sphere. Conic bundles are nice algebro-geometric objects. Whilst there is no classification of complex surfaces, there is a classification for the subclass of $\mathbb{P}^1$ bundles over $\mathbb{P}^1$ which are slightly different than conic bundles in the sense that conic bundles may can have fibers with singularities. Any $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ comes from a projectivized rank 2 vector bundle over $\mathbb{P}^1$. As the rank 2 vector bundles are parametrized by $\mathbb{Z}$, the $\mathbb{P}^1$ bundles over $\mathbb{P}^1$ are parameterized by $\mathbb{Z}$. Each vector bundle over $\mathbb{P}^1$ can be written as $E = \mathbb{O} \oplus \mathbb{O}(-e)$ ([1], [14]). Here $\mathbb{O}$ denotes the trivial rank 2 vector bundle over $\mathbb{P}^1$ and $\mathbb{O}(-e)$ denotes the vector bundle whose section has self-intersection number e.

Proposition 1.3.1. *A conic bundle is a rational surface.*

Proof. Any conic bundle T can be realized as a hypersurface defined by a polynomial f_T of bidegree $(2, m)$ over $\mathbb{P}^2 \times \mathbb{P}^1$. In particular a generic fiber of the coordinate projection of T to $\mathbb{P}^1$ is a nondegenerate conic. This means that T is locally, and hence birationally, equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ which is birationally equivalent to $\mathbb{P}^2$.

Another way to see that T is rational is by looking at the canonical divisor. The canonical divisor K_T of T is the canonical divisor $K_{\mathbb{P}^2 \times \mathbb{P}^1}$of $\mathbb{P}^2 \times \mathbb{P}^1$ twisted by the divisor class of T, all restricted to T. Namely $K_T = (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-3, -2) \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2, m))|_T = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, m-2)|_T$. In particular, the canonical divisor K_T corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, m-2)|_T$ the number of global sections of which are characterized by the number of polynomials of bidegree $(-1, m-2)$. Since there are no polynomials of bidegree $(-1, m-2)$ there are no global sections on T. The only surfaces in which the canonical bundle has no global sections are rational and ruled (i.e. birational to $\mathbb{P}^2$ and a fibration over a curve with $\mathbb{P}^2$ fibers). □

Corollary 1.3.2. *A conic bundle is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at n points for some integer n.*

It is a known fact ([14] Chapter V) that for two birational varieties the birational equivalence between them can be written as sequence of blow ups and blow downs. In particular, the varieties birational to $\mathbb{P}^2$ are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at n points. Hence any rational surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$ blown up at n points.

1.3.6 Blow ups

Blowing up varieties at points is a standard tool for resolving singularities and determining isomorphism classes of surfaces and we make repeated use of such in this paper.

Since blowing up is a local process, we can do all of our blow ups in affine neighborhoods. For our purposes, understanding what it means to blow up subvarieties of $\mathbb{A}^2$ and $\mathbb{A}^3$ at a point should be sufficient. For more details we refer to [14] or [24].

Intuitively blowing up $\mathbb{A}^2$ at a point can be described as replacing a point in $\mathbb{A}^2$ by an exceptional divisor (i.e. a copy of $\mathbb{P}^1$). To understand this more concretely, we will describe the blow up of $\mathbb{A}^2$ at the origin. Consider the product $\mathbb{A}^2 \times \mathbb{P}^1$. Take x, y as the affine coordinates of $\mathbb{A}^2$ and t, u as the homogeneous coordinates of $\mathbb{P}^1$. The blow up of $\mathbb{A}^2$ at $(0,0)$ is the closed subset $Y = \{[x, y : t, u] | xu = ty\}$ in $\mathbb{A}^2 \times \mathbb{P}^1$. The blow up comes with a natural map $\gamma : Y \to \mathbb{A}^2$ which is just projection onto the first factor. Notice that the fiber over any point $(x, y) \neq (0,0) \in \mathbb{A}^2$ is precisely one point in Y. However, the fiber over $(x, y) = (0,0)$, is a $\mathbb{P}^1$ worth of points in Y (i.e. $\{(0,0,t,u)\} \subset Y$). Since $\mathbb{A}^2 - \{(0,0)\} \simeq Y - \gamma^{-1}(0,0)$, γ is a birational map and $\mathbb{A}^2$ is birational to Y. Blowing up $\mathbb{A}^2$ at a point $p \neq 0$ simply amounts to a change in coordinates.

Suppose we want to blow up a subvariety $X \subset \mathbb{A}^2$ at a point, p. Take the blow up Y of $\mathbb{A}^2$ at p. Then the blow up $Bl|_p(X)$ of X at p is the closure

$\overline{\gamma^{-1}(X-p)}$ in Y where γ is as described above. We note that $Bl|_p(X)$ is birational to $X-p$ and if $Bl|_p(X)$ is smooth, $\gamma^{-1}(p)$ will intersect Y in a zero dimensional variety.

For our paper we need to understand how blowing up a surface at a smooth point affects the Euler characteristic.

Proposition 1.3.3. *The Euler characteristic of a surface X blown up at a smooth point p is*

$$\chi(Bl|_p(X)) = \chi(X) + 1 \ .$$

Proof. To blow up X at a smooth point p we work locally in an affine neighborhood about p. Near p, X is locally $\mathbb{A}^2$ at 0. Hence the result of blowing up X at p is the same as blowing-up $\mathbb{A}^2$ at 0. In terms of the Euler characteristic this amounts to replacing a point with an exceptional $\mathbb{P}^1$. In particular $\chi(Bl_p(X)) = \chi(X - \{p\}) + \chi(\mathbb{P}^1) = \chi(X) + 1$.

□

In order to resolve singularities we will need to blow up subvarieties of $\mathbb{A}^3$ at a point. Taking x_1, x_2, x_3 as affine coordinates for $\mathbb{A}^3$ and y_1, y_2, y_3 as projective coordinates for $\mathbb{P}^2$, the blow up of $\mathbb{A}^3$ at the origin is a closed subvariety, $Y' = \{[x_1, x_2, x_3 : y_1, y_2, y_3] | x_1y_2 = x_2y_1, x_1y_3 = x_3y_1, x_2y_3 = x_3y_2\}$ in $\mathbb{A}^3 \times \mathbb{P}^2$. Just as in the case of $\mathbb{A}^2$, this blow up comes with a natural map $\gamma : Y' \to \mathbb{A}^3$ which is simply projection onto the first factor. Just as before, the fiber over any point $(x_1, x_2, x_3) \neq (0, 0, 0) \in \mathbb{A}^3$ is precisely one point in

Y'. However, the fiber over $(x_1, x_2, x_3) = (0, 0, 0)$, is a $\mathbb{P}^2$ worth of points in Y' (i.e. $\{(0, 0, 0, y_1, y_2, y_3)\} \subset Y'$). Since $\mathbb{A}^3 - \{(0, 0, 0)\} \simeq Y' - \gamma^{-1}(0, 0, 0)$, γ is a birational map and $\mathbb{A}^3$ is birational to Y'. Blowing up $\mathbb{A}^3$ at a point $p \neq 0$ simply amounts to a change in coordinates. To blow up a subvariety $X \subset \mathbb{A}^3$ at a point, p. Take the blow up Y' of $\mathbb{A}^3$ at p. Then the blow up $Bl|_p(X)$ of X at p is the closure $\overline{\gamma^{-1}(X - p)}$ in Y'. We note that $Bl|_p(X)$ is birational to $X - p$ and if $Bl|_p(X)$ is smooth, $\gamma^{-1}(p)$ will intersect Y' in a smooth curve.

In this paper we obtain smooth surfaces by resolving singularities. As the Euler characteristic of these smooth surfaces helps us determine the isomorphism class we keep track of how blow up singular points affects the Euler characteristic.

Proposition 1.3.4. *If the blow up $Bl|_p(X)$ of a surface X at a singular point p is smooth, then the Euler characteristic of $Bl|_p(X)$ is $\chi(Bl|_p(X)) = \chi(X) + 2g + 1$ where g is the genus of the curve $\gamma^{-1}(p)$ in $Bl|_p(X)$.*

Proof. Away from the point p, X is isomorphic to $Bl_p(X) \backslash \gamma^{-1}(p)$. Hence, $\chi(Bl_p(X)) = \chi(X - p) + \chi(\gamma^{-1}(p))$. The preimage $\gamma^{-1}(p)$ in $Bl_p(X)$ is a smooth codimension-1 subvariety of the fiber over p in $Bl_p(\mathbb{A}^3)$. Since the fiber over p in $Bl_p(\mathbb{A}^3)$ is a $\mathbb{P}^2$, $\gamma^{-1}(p)$ in $Bl_p(X)$ is a smooth curve of genus g. Hence $\chi(\gamma^{-1}(p)) = 2g + 2$ and $\chi(Bl_p(X)) = \chi(X - \{p\}) + \chi(\gamma^{-1}(p)) = \chi(X) + 2g + 1$. □

1.3.7 Total transformations

For the proof of Proposition 2.2.4 we will use a total transform to extend a map ϕ between projective varieties. The description we provide here comes from [14] (pg 410). We begin by setting up some notation. Let X and Y be projective varieties.

Definition 1.3.1. A *birational transformation* T from X to Y is an open subset $U \subset X$ and a morphism $\phi : U \to Y$ which induces an isomorphism on the function fields of X and Y

Since different maps must agree on the overlap for different open sets, we take the largest open set U for which there is such a morphism ϕ. It is common to say that T is defined at the points of U and

Definition 1.3.2. The *fundamental points* of T are those in the set $X - U$.

For G the graph of ϕ in $U \times Y$, let $\overline{G}$ be the closure of G in $X \times Y$. Let $\rho_1 : \overline{G} \to X$ and $\rho_2 : \overline{G} \to Y$ be projections onto the first and second factors respectively.

Definition 1.3.3. For any subset $Z \subset X$ the *total transform* of Z is $T(Z) := \rho_2(\rho_1^{-1}(Z))$.

For a point $p \in U$, $T(p)$ is consistent with $\phi(p)$; while for a point $p \in X - U$, $T(p)$ is generally larger than a single point (in our examples it will be a copy of $\mathbb{P}^1$).

1.3.8 Intersection numbers of curves

A smooth curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ is cut out by a polynomial g which is homogenous in each of the $\mathbb{P}^1$ coordinates. We say g has bidgree (a, b) where a is the degree of g viewed as polynomial over the first factor and b is the degree of g viewed as a polynomial over the second factor. In the proof of Theorem 2.2.1 we will determine the number of intersections of two smooth curves in $\mathbb{P}^1 \times \mathbb{P}^1$ based solely on the bidegrees of their defining polynomials. Suppose C_1 and C_2 are two smooth curves cut out by irreducible polynomials g_1 and g_2 of bidegrees (a_1, b_1) and (a_2, b_2) respectively. Counting multiplicities, C_1 and C_2 intersect in $a_1b_2 + a_2b_1$ points ([14] 5.1).

Chapter 2

The Whitehead Link

This chapter focuses on the canonical component of the character variety for the Whitehead link complement. The Whitehead link is a two component 2-bridge link whose complement is a hyperbolic 3-manifold with 2 cusps. Hence, the canonical component of its character variety is a complex surface by Theorem 1.2.1 . Compactifying in $\mathbb{P}^2 \times \mathbb{P}^1$ we prove that canonical component for the Whitehead link complement is topologically $\mathbb{P}^2$ blown up at 10 points. We begin by constructing the affine model for the canonical component in $\mathbb{C}^3$.

2.1 The affine model

Let W denote the complement of the Whitehead link in S^3 and let $\Gamma_W = \pi_1(W)$. Then $\Gamma_W = \langle a, b | aw = wa \rangle$ where w is the word $w = bab^{-1}a^{-1}b^{-1}ab$.

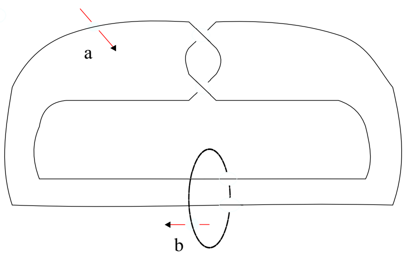

Figure 2.1: Whitehead link

Proposition 2.1.1. *$X(W)$ is a hypersurface in $\mathbb{C}^3$.*

Proof. To determine the defining polynomial for $X(W)$ in $\mathbb{C}^3$ we look at the image of $R(W)$ under the map $t = (t_1, \ldots, t_s) : R(W) \to \mathbb{C}^s$ as defined in Section 3.3.1. We begin by establishing the defining ideal for $R(W)$. Any representation of $\rho \in R(W)$ can be conjugated so that

$$\bar{a} = \rho(a) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \qquad \bar{b} = \rho(b) = \begin{pmatrix} s & 0 \\ r & s^{-1} \end{pmatrix}$$

The polynomials which define $R(W)$ then come from the relation $\bar{w}\bar{a} - \bar{a}\bar{w} = 0$. Writing $\rho(w) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$, we see that

$$\bar{w}\bar{a} - \bar{a}\bar{w} = \begin{pmatrix} -w_{21} & w_{11} + w_{12}(m^{-1} - m) - w_{22} \\ w_{21}(m - m^{-1}) & w_{21} \end{pmatrix}$$

Hence, the representation variety is cut out by the ideal $\langle p_1, p_2 \rangle \subset \mathbb{C}[m, m^{-1}, s, s^{-1}, r]$ where $p_1 = w_{21}$ and $p_2 = w_{11} + w_{12}(m^{-1} - m) - w_{22}$.

For the Whitehead link

$$\begin{aligned} p_1 &= m^{-2}s^{-2}r(r - m^2r + ms - m^3s + 2mr^2s - m^3r^2s - rs^2 \\ &\quad +4m^2rs^2 - m^4rs^2 + m^2r^3s^2 - ms^3 \\ &\quad +m^3s^3 - mr^2s^3 + 2m^3r^2s^3 - m^2rs^4 + m^4rs^4) \\ p_2 &= m^{-2}s^{-3}(-1+s)(1+s)(r - m^2r + ms - m^3s + 2mr^2s \\ &\quad -m^3r^2s - rs^2 + 4m^2rs^2 - m^4rs^2 + m^2r^3s^2 \\ &\quad -ms^3 + m^3s^3 - mr^2s^3 + 2m^3r^2s^3 - m^2rs^4 + m^4rs^4) \end{aligned}$$

Neither p_1 nor p_2 are irreducible. In fact their GCD is nontrivial. Let $p = GCD(p_1, p_2)$. That is

$$\begin{aligned} p \;=\; & m^2s^3(r - m^2r + ms - m^3s + 2mr^2s - m^3r^2s \\ & -rs^2 + 4m^2rs^2 - m^4rs^2 + m^2r^3s^2 - ms^3 \\ & +m^3s^3 - mr^2s^3 + 2m^3r^2s^3 - m^2rs^4 + m^4rs^4) \end{aligned}$$

Setting $g_1 = \frac{p_1}{p} = rs$ and $g_2 = \frac{p_2}{p} = s^2 - 1$, we can view the representation variety as $Z(\langle g_1p, g_2p\rangle) = Z(\langle g_1, g_2\rangle) \cup Z(\langle p\rangle)$. The ideal $\langle g_1, g_2\rangle$ defines the affine variety $R_a = \{(m, m^{-1}, s, s^{-1}, r) = (m, 1/m, \pm 1, \pm 1, 0)\} \subset \mathbb{C}^5$ which is just two copies of $\mathbb{A}^1$. Representations with r coordinate zero and s coordinate ± 1 are abelian as they send the generator b of Γ_W to $\pm I$. Hence the variety R_a consists of abelian representations of $R(W)$.

Since we are interested the components of the representation variety which contain discrete faithful representations, we can focus our attention on the component $R = Z(\langle p\rangle)$ of $R(W)$. In particular, the canonical component $X_0(W)$ of the character variety is in image of R under the regular map t.

As discussed in Section 3.3.1, we can express the map t in terms of generators of the coordinate ring T_w for $X(W)$. The coordinate ring T_w is generated by the trace maps $\{\tau_a, \tau_b, \tau_{ab}\}$. With these generators the map $t = (\tau_a, \tau_b, \tau_{ab}) : R \to \mathbb{C}^3$ is $t(\rho) = (m + m^{-1}, s + s^{-1}, ms + m^{-1}s^{-1} + r) = (x, y, z)$. Let X' denote the image of R under t. We determine the defining polynomial(s) for X' by appealing to the induced injective map $t^* : \mathbb{C}[X'] \to \mathbb{C}[R]$ on the coordinates rings of X' and R. The algebraic set R is defined by the polynomial ideal $\langle p \rangle$ and so its coordinate ring is $\mathbb{C}[R] = \mathbb{C}[m, m^{-1}, s, s^{-1}, r]/\langle p \rangle$. The coordinate ring $\mathbb{C}[X']$ is the image of $\mathbb{C}[R]$ under t^*, that is

$$\mathbb{C}[X'] = \mathbb{C}[m, m^{-1}, s, s^{-1}, r]/\langle p, x = m+m^{-1}, y = s+s^{-1}, z = ms+m^{-1}s^{-1}+r\rangle$$

which is isomorphic to $\mathbb{C}[x, y, z]/ < \tilde{f} >$ where $\tilde{f} = -xy - 2z + x^2z + y^2z - xyz^2 + z^3$. Since $\tilde{f}$ is smooth, it follows that X' is the affine variety $Z(\tilde{f})$. Now X' is a smooth affine surface in $\mathbb{C}^3$ containing the surface X_0. Hence $X' = X_0$ and so X_0 is the hypersurface $Z(\tilde{f})$. □

2.2 The smooth model

We use the compact model obtained by taking the projective closure of $X_0(W)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Throughout the rest of this section we will denote this projective closure of X_0 by S. With x, y, u the $\mathbb{P}^2$ coordinates and z, w the $\mathbb{P}^1$ coordinates, this compactification for the canonical component is defined by

$$f = -w^3xy - 2u^2w^2z + w^2x^2z + w^2y^2z - wxyz^2 + u^2z^3.$$

This surface S is not smooth. It has singularities at the four points:

$$\begin{aligned} s1 &= [1, 0, 0, 1, 0] \\ s2 &= [0, 1, 0, 1, 0] \\ s3 &= [1, -1, 0, 1, -1] \\ s4 &= [1, 1, 0, 1, 1] \end{aligned}$$

Our goal is to determine topologically the smooth surface $\tilde{S} = \tilde{X}(W)_0$ obtained by resolving the singularities of S . We do this in the following theorem.

Theorem 2.2.1. *The surface $\tilde{S}$ is a rational surface isomorphic to $\mathbb{P}^2$ blown up at* 10 *points.*

The Euler characteristic of $\tilde{S}$ together with the fact that $\tilde{S}$ is rational is enough to determine $\tilde{S}$ up to isomorphism.

Lemma 2.2.2. *$\tilde{S}$ is birational to a conic bundle.*

Proof. Consider the projection $\pi_{\mathbb{P}^1} : S \to \mathbb{P}^1$. The fiber over $[z_0, w_0] \in \mathbb{P}^1$ is the set of points $[x, y, u : z_0, w_0]$ which satisfy $-w_0^3 xy - 2u^2 w_0^2 z_0 + w_0^2 x^2 z_0 + w_0^2 y_0^2 z - w_0 xyz_0^2 + u^2 z_0^3 = 0$. This is the zero set of a degree 2 polynomial in $\mathbb{P}^2$ which is a conic. Away from the four singularities, S is isomorphic to a conic bundle. Hence, S is birational to a conic bundle. Since $\tilde{S}$ is obtained from S by a series of blow ups, $\tilde{S}$ is birational to S and so birational to a conic bundle. □

Applying Proposition 1.3.1 we now have that $\tilde{S}$ is rational surface. It follows from Corollary 1.3.2 that $\tilde{S}$ is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$ blown up at some number of n points. Since S has degenerate fibers, $\tilde{S}$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (see figure 2.2).

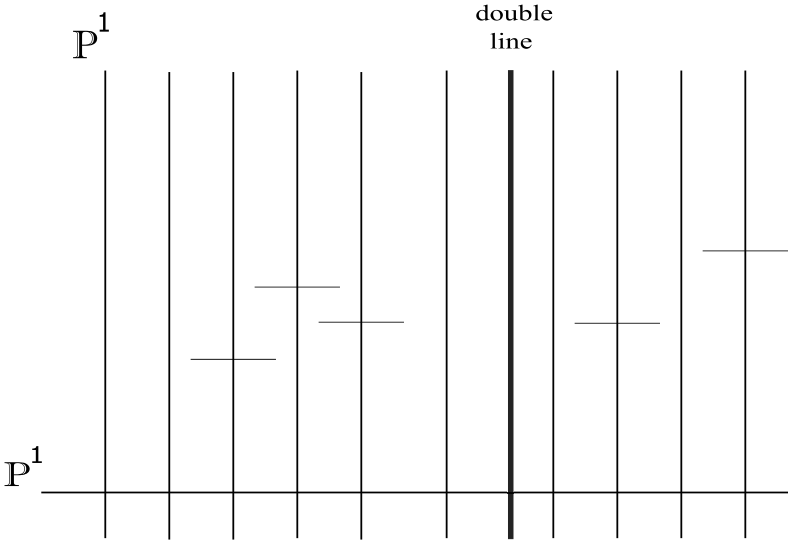

Figure 2.2: Canonical component of the Whitehead link.

So $\tilde{S}$ is topologically $\mathbb{P}^2$ blown up at n points and determining the isomorphism class reduces to determine n. It follows from Proposition 1.3.3 that $\chi(\tilde{S}) = \chi(\mathbb{P}^2) + n = 3 + n$. Thus we can determine n and from the Euler characteristic of $\tilde{S}$.

To calculate the Euler characteristic of $\tilde{S}$ we use the Euler characteristic of S. Since the smooth surface $\tilde{S}$ is obtained from S by a series of blow ups, we can use Proposition 1.3.4 to write $\chi(\tilde{S})$ in terms of $\chi(S)$.

Lemma 2.2.3. $\chi(\tilde{S}) = \chi(S) + 4$

Proof. The smooth surface, $\tilde{S}$, is obtained by resolving the four singularities, s_i, of S listed above. Above the singularities, a local model for $\tilde{S}$ can be obtained by blowing up S in an affine neighborhood of each of the singular points. Away from the singularities we can take the local model for S as a local model for $\tilde{S}$ since S and $\tilde{S}$ are locally isomorphic there. Each of the singularities is nice in the sense that it takes only one blow up to resolve them. Hence, in terms of the Euler characteristic, we have

$$\chi(\tilde{S}) = \Sigma_{i=1}^{4}\chi(S - \{s_i\}) + \Sigma_{i=1}^{4}\chi(\tilde{s_i}) \tag{2.1}$$

where for $i = 1 \ldots 4$, $\tilde{s_i}$ denotes the preimage of s_i in $\tilde{S}$. Determining the Euler characteristic of $\tilde{S}$ in terms of that for S reduces to determining $\tilde{s_i}$.

To blow up S at $s_1 = [1, 0, 0, 1, 0]$ we consider the affine open set A_1' where $x \neq 0$ and $z \neq 0$. Noticing that the singularities s_3 and s_4 are in A_1', we look at the blow up of S at s_1 in the affine open set $A_1 = A_1' \setminus \{s_3, s_4\}$. Local affine coordinates for $A_1 \cong \mathbb{A}^3$ are y, u, w. So to blow up S at s_1 we blow up $X_1 = Z(f|_{x=1,z=1})$ at $[y, u, w] = [0, 0, 0]$ in A_1. As described in Section 1.3.6, the blow up of X_1 at $[0, 0, 0]$ is the closure of the preimage of $X_1 - [0, 0, 0]$ in $Bl|_{[0,0,0]}(A_1)$. Using coordinates a, b, c for $\mathbb{P}^2$, the blow up Y_1 of X_1 at $[0, 0, 0]$

is the closed subset in $A_1 \times \mathbb{P}^2$ defined by the equations

$$f_1 = f|_{x=1,z=1} = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \quad (2.2)$$

$$e_1 = yb - ua \quad (2.3)$$

$$e_2 = yc - wa \quad (2.4)$$

$$e_3 = uc - wb \quad (2.5)$$

We determine the local model above s_1 and check for smoothness by looking at Y_1 in the affine open sets define by $a \neq 0$, $b \neq 0$, and $c \neq 0$.

First we look at Y_1 in the affine open set defined by $a \neq 0$ (i.e. we can set $a = 1$). In this open set the defining equations for Y_1 become

$$f_1 = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \quad (2.6)$$

$$e_1 = yb - u \quad (2.7)$$

$$e_2 = yc - w \quad (2.8)$$

$$e_3 = uc - wb \quad (2.9)$$

Using equations $e1$ and $e2$ and substituting for u and w in f_1 we obtain the local model, $y^2(-b^2 + c - c^2 - c^2y^2 + 2b^2c^2y^2 + c^3y^2)$. The first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Notice that E_1 and Y_1 meet in the smooth conic $-b^2+c-c^2$. So, in this affine open set, the local model above the singularity s_1 is a conic, and therefore isomorphic $\mathbb{P}^1$. Since the only places all the partial derivatives of the second factor vanish are over the singular points s_3 and s_4, this model is smooth in $A_1 \times \mathbb{P}^2$.

Next we look at Y_1 in the affine open set defined by $b \neq 0$. In this open set the defining equations for Y_1 become

$$f_1 = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \quad (2.10)$$

$$e_1 = y - ua \quad (2.11)$$

$$e_2 = yc - wa \quad (2.12)$$

$$e_3 = uc - w \quad (2.13)$$

Substituting into f_1, we obtain the local model, $u^2(1-ac+c^2-2c^2u^2+a^2c^2u^2-ac^3u^2)$. Again, the first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Notice that E_1 and Y_1 meet in the smooth conic $1 - ac + c^2$. So, in this affine open set, the local model above the singularity s_1 is a conic. Since all the partial derivatives of the second factor do not simultaneously vanish, this model is smooth in $A_1 \times \mathbb{P}^2$.

Finally we look at Y_1 in the affine open set defined by $c \neq 0$. In this open set the defining equations for Y_1 become

$$f_1 = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \quad (2.14)$$

$$e_1 = yb - ua \quad (2.15)$$

$$e_2 = y - wa \quad (2.16)$$

$$e_3 = u - wb \quad (2.17)$$

Substituting into f_1, we obtain the local model, $w^2(1 - a + b^2 - aw^2 + a^2w^2 - 2b^2w^2)$. The first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Notice that E_1 and Y_1 meet in the smooth conic $1 - a + b^2$. So, in this affine open set, the local model above the singularity s_1 is a conic. Since the only places all the partial derivatives of the second factor vanish simultaneously are s_3 and s_4, this model is smooth in $A_1 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing up yields a smooth local model which intersects the exceptional plane above s_1 in the conic defined by $c^2 - a + b^2$. Hence $\chi(\tilde{s_1}) = 2$.

Blowing up S at s_2, s_3 and s_4 is similar to blowing up S at s_1. For the sake of completion, we provide detailed calculations here.

To blow-up S at $s_2 = [0, 1, 0, 1, 0]$ we consider the affine open set A_2' where $y \neq 0$ and $z \neq 0$. Noticing that the singularities s_3 and s_4 are in A_2', we look at the blow-up of S at s_2 in the affine open set $A_2 = A_2' - s_3, s_4$. Local affine coordinates for $A_2 \cong A^3$ are x, u, w. So to blow-up S at s_2 we blow-up $X_2 = Z(f|_{y=1,z=1})$ at $[x, u, w] = [0, 0, 0]$ in A_2. Using coordinates a, b, c for $\mathbb{P}^2$, the blow-up Y_2 of X_2 at [0,0,0] is the closed subset in $A_2 \times \mathbb{P}^2$ defined by the equations

$$f_2 = f|_{y=1,z=1} = u^2 + w^2 - 2u^2w^2 - wx - w^3x + w^2x^2 \tag{2.18}$$

$$q_1 = xb - au \tag{2.19}$$

$$q_2 = xc - aw \tag{2.20}$$

$$q_3 = uc - wb \tag{2.21}$$

In the affine open set defined by $a \neq 0$, the local model is defined by $x^2(-b^2 + c - c^2 - c^2x^2 + 2b^2c^2x^2 + c^3x^2)$. The first factor is the exceptional plane E_2 and the second factor is the local model for Y_2. The exceptional plane E_2 and Y_2 meet in the smooth conic $-b^2 + c - c^2$. Since all the partial derivatives of the second factor do not vanish simultaneously anywhere, this model is smooth in $A_2 \times \mathbb{P}^2$.

In the affine open set defined by $b \neq 0$, the local model is defined by $u^2(1 - ac + c^2 - 2c^2u^2 + a^2c^2u^2 - ac^3u^2)$. The first factor is the exceptional plane E_2 and the second factor is the local model for Y_2. Here, the exceptional plane E_2 and Y_2 meet in the smooth conic $1 - ac + c^2$. No where do all the partial derivatives of the second factor vanish simultaneously and so this model is smooth in $A_2 \times \mathbb{P}^2$.

In the affine open set defined by $c \neq 0$, the local model is defined by $w^2(1 - a + b^2 - aw^2 + a^2w^2 - 2b^2w^2)$. The first factor is the exceptional plane E_2 and the second factor is the local model for Y_2. Here, the exceptional plane E_2 and Y_2 meet in the smooth conic $1 - a + b^2$. Since all the partial derivatives of the second factor vanish simultaneously only above the singularities s_3 and s_4, this model is smooth in $A_2 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing-up yields a smooth local model which intersects the exceptional plane above s_2 in the conic defined by $c^2 - ac + b^2$. Hence $\chi(\tilde{s_2}) = 2$.

To blow-up S at $s_3 = [1, -1, 0, 1, -1]$ we consider the affine open set

A_3' where $x \neq 0$ and $z \neq 0$. Noticing that the singularities s_1 and s_4 are in A_3', we look at the blow-up of S at s_3 in the affine open set $A_3 = A_3' - s_1, s_4$. Local affine coordinates for $A_3 \cong A^3$ are y, u, w. So to blow-up S at s_3 we blow-up $X_3 = Z(f|_{x=1,z=1})$ at $[y, u, w] = [-1, 0, -1]$ in A_3. Using coordinates a, b, c for $\mathbb{P}^2$, the blow-up Y_3 of X_3 at [-1,0,-1] is the closed subset in $A_3 \times \mathbb{P}^2$ defined by the equations

$$f_3 = f|_{x=1,z=1} = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \qquad (2.22)$$

$$e_1 = ((y+1)b) - ua \qquad (2.23)$$

$$e_2 = ((y+1)c) - (w+1)a \qquad (2.24)$$

$$e_3 = uc - (w+1)b \qquad (2.25)$$

In the affine open set defined by $a \neq 0$, the local model is defined by $(1+y)^2(-1+b^2+2c-4b^2c-c^2+2b^2c^2+2cy-4b^2cy-3c^2y+4b^2c^2y+c^3y-c^2y^2+2b^2c^2y^2+c^3y^2)$. The first factor is the exceptional plane E_3 and the second factor is the local model for Y_3. The exceptional plane E_3 and Y_3 meet in the smooth conic $-1+b^2+c^2$. Since they only place all the partial derivatives of the second factor vanish simultaneously is above the singularity s_1, this model is smooth in $A_3 \times \mathbb{P}^2$.

In the affine open set defined by $b \neq 0$, the local model is defined by $u^2(-1+a^2-c^2+4cu-2a^2cu+ac^2u+c^3u-2c^2u^2+a^2c^2u^2-ac^3u^2)$. The first factor is the exceptional plane E_3 and the second factor is the local model for Y_3. Here, the exceptional plane E_3 and Y_3 meet in the smooth conic

$-1+a^2-c^2$. No where do all the partial derivatives of the second factor vanish simultaneously and so this model is smooth in $A_3 \times \mathbb{P}^2$.

In the affine open set defined by $c \neq 0$, the local model is defined by $(1+w)^2(-b^2 - w + aw + aw^2 - a^2w^2 + 2b^2w^2)$. The first factor is the exceptional plane E_3 and the second factor is the local model for Y_3. Here, the exceptional plane E_3 and Y_3 meet in the smooth conic $1 - a^2 + b^2$. Since all the partial derivatives of the second factor vanish simultaneously only above the singularities s_1 and s_4, this model is smooth in $A_3 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing-up yields a smooth local model which intersects the exceptional plane above s_3 in the conic defined by $c^2 - a^2 + b^2$. Hence $\chi(\tilde{s_3}) = 2$.

Finally, to blow-up S at $s_4 = [1,1,0,1,1]$ we consider the affine open set A_4' where $x \neq 0$ and $z \neq 0$. Noticing that the singularities s_1 and s_3 are in A_4', we look at the blow-up of S at s_4 in the affine open set $A_4 = A_4' - s_1, s_3$. Local affine coordinates for $A_4 \cong A^3$ are y, u, w. So to blow-up S at s_4 we blow-up $X_4 = Z(f|_{x=1,z=1})$ at $[y,u,w] = [1,0,1]$ in A_4. Using coordinates a, b, c for $\mathbb{P}^2$, the blow-up Y_4 of X_4 at [1,0,1] is the closed subset in $A_4 \times \mathbb{P}^2$ defined by the equations

$$f_4 = f|_{x=1,z=1} = u^2 + w^2 - 2u^2w^2 - wy - w^3y + w^2y^2 \qquad (2.26)$$

$$e_1 = ((y-1)b) - ua \qquad (2.27)$$

$$e_2 = ((y-1)c) - (w-1)a \qquad (2.28)$$

$$e_3 = uc - (w-1)b \qquad (2.29)$$

In the affine open set defined by $a \neq 0$, the local model is defined by $(-1+y)^2(-1+b^2+2c-4b^2c-c^2+2b^2c^2-2cy+4b^2cy+3c^2y-4b^2c^2y-c^3y-c^2y^2+2b^2c^2y^2+c^3y^2)$. The first factor is the exceptional plane E_4 and the second factor is the local model for Y_4. The exceptional plane E_4 and Y_4 meet in the smooth conic $-1+b^2+c^2$. Since they only place all the partial derivatives of the second factor vanish simultaneously are above the singularities s_1 and s_3, this model is smooth in $A_4 \times \mathbb{P}^2$.

In the affine open set defined by $b \neq 0$, the local model is defined by $u^2(-1+a^2-c^2-4cu+2a^2cu-ac^2u-c^3u-2c^2u^2+a^2c^2u^2-ac^3u^2)$. The first factor is the exceptional plane E_4 and the second factor is the local model for Y_4. Here, the exceptional plane E_4 and Y_4 meet in the smooth conic $-1+a^2-c^2$. No where do all the partial derivatives of the second factor vanish simultaneously and so this model is smooth in $A_4 \times \mathbb{P}^2$.

In the affine open set defined by $c \neq 0$, the local model is defined by $(-1+w)^2(-b^2+w-aw+aw^2-a^2w^2+2b^2w^2)$. The first factor is the exceptional plane E_4 and the second factor is the local model for Y_4. Here, the exceptional plane E_4 and Y_4 meet in the smooth conic $1-a^2+b^2$. Since all the partial derivatives of the second factor vanish simultaneously only above the singularities s_1 and s_3, this model is smooth in $A_4 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing-up yields a smooth local model which intersects the exceptional plane above s_4 in the conic defined by $c^2-a^2+b^2$. Hence $\chi(\tilde{s_4})=2$.

In each case, we have shown that the local model for $Bl|_{s_i}(S)$ intersects the exceptional plane above s_i in a smooth conic. Hence $\chi(\tilde{s_i}) = 2$ for $i = 1, \ldots, 4$ and

$$\begin{aligned}
\chi(\tilde{S}) &= \chi(S - \{s_i\}) + \Sigma_{i=1}^{4}\chi(\tilde{s_i}) & (2.30)\\
&= \chi(S) - \Sigma_{i=1}^{4}\chi(s_i) + \Sigma_{i=1}^{4}\chi(\tilde{s_i}) & (2.31)\\
&= \chi(S) - 4 + 4(2) & (2.32)\\
&= \chi(S) + 4 & (2.33)
\end{aligned}$$

□

Proposition 2.2.4. *The Euler characteristic of the surface S is $\chi(S) = 9$.*

To calculate the Euler characteristic we will appeal to the map $\phi : S \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $[x, y, u : z, w] \to [x, y : z, w]$ on a dense open subset of S. That the map ϕ is generically 2-to-1 makes it an attractive tool in determining the Euler characteristic of S. However, in order to calculate the Euler characteristic of S we must understand the map ϕ everywhere not just generically. To this affect, there are four aspects we need to consider. The map ϕ is neither surjective nor defined at the three points $P = \{(0, 0, 1, 0, 1), (0, 0, 1, 1, \pm\frac{1}{\sqrt{2}})\}$. Over six points in the $\mathbb{P}^1 \times \mathbb{P}^1$ the fiber is a copy of $\mathbb{P}^1$. Finally, the map is branched over three copies of $\mathbb{P}^1$. We explain how to alter the Euler characteristic calculation to account for each of these situations.

Lemma 2.2.5. *The image of ϕ on $U = S - P$ is $\mathbb{P}^1 \times \mathbb{P}^1 - Q$ where*

$$\begin{aligned} Q &= \mathbb{P}^1 \times \{[0,1]\} \diagdown \{[1,0,0,1],[0,1,0,1]\} \\ &\cup \ \mathbb{P}^1 \times \{[1, \tfrac{1}{\sqrt{2}}]\} \diagdown \{[\tfrac{1}{\sqrt{2}},1,1,\tfrac{1}{\sqrt{2}}],[\sqrt{2},1,1,\tfrac{1}{\sqrt{2}}]\} \\ &\cup \ \mathbb{P}^1 \times \{[1, -\tfrac{1}{\sqrt{2}}]\} \diagdown \{[-\tfrac{1}{\sqrt{2}},1,1,-\tfrac{1}{\sqrt{2}}],[-\sqrt{2},1,1,-\tfrac{1}{\sqrt{2}}]\} \end{aligned}$$

Proof. We can see that this is in fact the image by viewing f as a polynomial in u with coefficients in $\mathbb{C}[x,y,z,w]$. Namely $f = g + u^2 h$ where $g = -w^3xy + w^2x^2z + w^2y^2z - wxyz^2$ and $h = z(z^2 - 2w^2)$. The image of ϕ is the collection of all points $[x,y,z,w] \in \mathbb{P}^1 \times \mathbb{P}^1$ except those for which $f(x,y,z,w) \in \mathbb{C}[u]$ is a nonzero constant. The polynomial $f(x,y,z,w)$ is a nonzero constant whenever $h = 0$ and $g \neq 0$. It is easy to see that $h = 0$ whenever $[z,w] = \{[0,1],[1,\pm\frac{1}{\sqrt{2}}]\}$. For each of the z,w coordinates which satisfy h, there are two x,y coordinates which satisfy $g(z,w)$. Hence the image of ϕ on U is all of $\mathbb{P}^1 \times \mathbb{P}^1$ less the three twice punctured spheres as listed above.

□

Lemma 2.2.6. *The map ϕ smoothly extends to all of S.*

Proof. We can extend the map ϕ to all of S by using a total transformation. Let $U = S - P$. Then U is the largest open set in S on which ϕ is defined. Let $\overline{G(\phi,U)}$ be the closure of the graph of ϕ on U. We can then smoothly extend the map ϕ to all of S by definingϕ at each $p_i \in P$ to be $\phi(p_i) := \rho_2\rho_1^{-1}(p_i)$

where $\rho_1 : \overline{G} \to S$ and $\rho_1 : \overline{G} \to \mathbb{P}^1 \times \mathbb{P}^1$ are the natural projections. Note that, for $s \in U$, $\rho_2\rho_1^{-1}(s)$ coincides with the original map so that this extension makes sense on all of S. Now, the closure of the graph is $\overline{G} = \{[x, y, u, z, w : a, b, c, d] | f = 0, ay = bx, cw = dz\}$. So, ϕ extends to S as follows:

$$\begin{aligned} \phi((0,0,1,0,1)) &= \{[a,b,0,1]\} \\ \phi((0,0,1,1,\tfrac{1}{\sqrt{2}})) &= \{(a,b,1,\tfrac{1}{\sqrt{2}})\} \\ \phi((0,0,1,1,-\tfrac{1}{\sqrt{2}})) &= \{(a,b,1,-\tfrac{1}{\sqrt{2}})\} \end{aligned}$$

□

Notice that the set $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is not contained in the image of ϕ on U, is contained in the image of ϕ on P. That the extension ϕ maps three points in S to not just three disjoint $\mathbb{P}^1$'s in $\mathbb{P}^1 \times \mathbb{P}^1$ but to the three disjoint $\mathbb{P}^1$'s which are are missing from the image of ϕ on U will be important for the Euler characteristic calculation.

Lemma 2.2.7. *There are six points in $\mathbb{P}^1 \times \mathbb{P}^1$, the collection of which we will call L, whose fiber in S is infinite.*

Proof. Thinking of f as a polynomial in the variable u with coefficients in $\mathbb{C}[x, y, z, w]$, we see that the points in $\mathbb{P}^1 \times \mathbb{P}^1$ which are simultaneously zeros of these coefficient polynomials are precisely the points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite. We note here that the points of L are precisely the punctures of the three punctured spheres which are not in the image of $\phi|_U$. The preimage of L in S is the union of six $\mathbb{P}^1$'s each intersecting exactly one other $\mathbb{P}^1$ in one point. These three points of intersection are the points on the $\mathbb{P}^1$'s where the coordinate u goes to infinity which is equivalent to the points where the x and y coordinates go to zero. Thus these intersection points are precisely the points in P. The points in L along with their infinite fibers in $\mathbb{P}^2 \times \mathbb{P}^1$ are listed below.

$[1,0,0,1]$ has fiber $\{[1,0,u,0,1]\} \supset [0,0,1,0,1]$

$[0,1,0,1]$ has fiber $\{[0,1,u,0,1]\} \supset [0,0,1,0,1]$

$[1,\sqrt{2},1,\frac{1}{\sqrt{2}}]$ has fiber $\{[1,\sqrt{2},u,1,\frac{1}{\sqrt{2}}]\} \supset [0,0,1,1,\frac{1}{\sqrt{2}}]$

$[1,\frac{1}{\sqrt{2}},1,\frac{1}{\sqrt{2}}]$ has fiber $\{[1,\frac{1}{\sqrt{2}},u,1,\frac{1}{\sqrt{2}}]\} \subset [0,0,1,1,\frac{1}{\sqrt{2}}]$

$[1,-\sqrt{2},1,-\frac{1}{\sqrt{2}}]$ has fiber $\{[1,-\sqrt{2},u,1,-\frac{1}{\sqrt{2}}]\} \supset [0,0,1,1,-\frac{1}{\sqrt{2}}]$

$[1,-\frac{1}{\sqrt{2}},1,-\frac{1}{\sqrt{2}}]$ has fiber $\{[1,-\frac{1}{\sqrt{2}},u,1,-\frac{1}{\sqrt{2}}]\} \supset [0,0,1,1,-\frac{1}{\sqrt{2}}]$

$\square$

In calculating the Euler characteristic we will use the fact that the preimage of L in S are six $\mathbb{P}^1$'s which intersect in pairs at ideal points in the set $P \subset S$. In fact, each point in P appears as the intersection of two of these fibers and the image of P under ϕ is precisely L.

Let B denote the branch set of ϕ in $\mathbb{P}^1 \times \mathbb{P}^1$. We have the following lemma.

Lemma 2.2.8. $\chi(B) = 2$.

Proof. The branch set, or at least the places where ϕ is not one-to-one, consists of the points in S which also satisfy the coordinate equation $u = 0$. The image, $B \subset \mathbb{P}^1 \times \mathbb{P}^1$, of this branch set, is the union of the three varieties, B_1, B_2, and B_3 defined by the respective three polynomials $f_1 = wy - xz$, $f_2 = wx - yz$, and $f_3 = w$ which are all $\mathbb{P}^1$'s. From the bidegrees of the f_i we know that B_3 intersects each of B_1 and B_2 in one point ($[0, 1, 1, 0]$ and $[1, 0, 1, 0]$ respectively) while B_1 and B_2 intersect in two points ($[1, -1, -1, 1]$ and$[1, 1, 1, 1]$). Again thinking of f as a polynomial in u we can write f as $f = A + u^2 B$ where A and B are polynomials in $\mathbb{C}[x, y, z, w]$. Since L cut out by the ideal $< A, B >$ and B is cut out by the ideal $< A >$, L is a subvariety of B. That each of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite is also a branch point is necessary for the Euler characteristic calculation.

□

Now that we understand the map ϕ everywhere we can calculate the Euler characteristic of S and prove Proposition 2.2.4.

Proof. (Proposition 2.2.4) Since the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fibers are infinite coincide with the image L of the fundamental set P, and L is the intersection of Q and the branch set B ,

$$\begin{aligned} \chi(s) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 - B - Q) + \chi(Q + B - L) + \chi(\phi^{-1}(L)) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(Q) - \chi(B) - \chi(L) + \chi(\phi^{-1}(L)) \end{aligned}$$

The Euler characteristic of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4$. As Q is the disjoint union of three twice-punctured spheres, $\chi(Q) = 3(\chi(\mathbb{P}^1) - 2) = 0$. Since B is three $\mathbb{P}^1$'s which intersect at four points, $\chi(B) = 3\chi(\mathbb{P}^1) - 4\chi(point) = 2$. Now L is just six points so $\chi(L) = 6$. That $\phi^{-1}(L)$ is the union of six $\mathbb{P}^1$'s which intersect in pairs at a point implies that $\chi(\phi^{-1}(L)) = 6\chi(\mathbb{P}^1) - 3\chi(points) = 9$. All together this gives $\chi(S) = 9$.

□

Corollary 2.2.9. *The Euler characteristic of $\tilde{S}$ is $\chi(\tilde{S}) = 13$*

Proof. We have $\chi(\tilde{S}) = \chi(S) + 4 = 9 + 4 = 13$. □

We are now ready to prove Theorem 2.2.1.

Proof. (Theorem 2.2.1) It follows from Lemma 2.2.2 and Corollary 1.3.3 that $\chi(\tilde{S}) = \chi(\mathbb{P}^2) + n$. By Corollary 2.2.9, n must be 10 and $\tilde{S}$ must be $\mathbb{P}^2$ blown up at 10 points. □

Chapter 3

Character Varieties for Other Two component 2-Bridge Links

We examine the character varieties associated to the two families of link complements obtained by $1/n$ Dehn filling on the Borromean rings and on the Magic manifold (3-chain link complement). Studying these families of varieties lends insight into the problem of identifying which subvarieties of a character variety correspond to character varieties of manifolds obtained by Dehn filling.

The Whitehead link complement can be realized as $1/1$ Dehn filling on one cusp of both the Borromean rings and the Magic manifold. It is natural to ask whether the character varieties of link complements obtained by $1/n$ Dehn surgery on these two manifolds exhibit any similarities. For $n = 1, \ldots, 4$ we are able to determine explicit models for certain components of these character varieties. One striking similarity among these varieties is that they exhibit components birational to conic bundles. For the character varieties associated to manifolds obtained by $1/n$ Dehn filling on the Magic manifold the rational component coincides with the canonical component.

The process for determining the isomorphism class of these rational components is similar to that for determining the isomorphism class for the canonical component associated to the Whitehead link. Detailed calculations can be obtained from the author upon request. Here we summarize the process. Using the standard techniques (cf. [23], [6]) we obtain explicit coordinates in $\mathbb{C}^3$ for the canonical component of the character varieties of these 2-bridge links. Following the work of [17] we take the projective completion in $\mathbb{P}^2 \times \mathbb{P}^1$ and realize the canonical component as a hypersurface, S, cut out by a single polynomial of bidegree $(2, n)$. That S is birational to a conic bundle indicates S is a rational surface [14]. One of the primary complications is that S is not smooth. After resolving the singularities to obtain a smooth rational surface, it is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown-up at n points [14]. Knowing how the Euler characteristic behaves under blow-ups allows us to determine the isomorphism class from the Euler characteristic of S. The surface S exhibits a double line fiber. Otherwise we could calculate $\chi(S)$ using its conic bundle structure as $\chi(f_g)\chi(B) + d$ where f_g is a generic fiber, B is the base and d is the number of degenerate fibers. Rather than using normal stabilization to resolve this double line, we determine $\chi(S)$ by appealing to the total transform of a map that projects S onto $\mathbb{P}^1 \times \mathbb{P}^1$.

Section 3.1 concerns the character varieties for the links obtained by $1/n$ Dehn filling on the Magic manifold and Section 3.2 focuses on those for links obtained by $1/n$ filling on the Borromean rings. We explain how these results relate to the canonical components for the Magic manifold and twist knot character varieties in section 3.3.

3.1 Links obtained by $1/n$ Dehn Surgery on the Magic Manifold

The Magic manifold, M_m in this thesis, is the complement of the 3-chain link in S^3. Its volume, 5.33, is the smallest currently known among hyperbolic 3-cusped manifolds and most hyperbolic manifolds as well as non-hyperbolic fillings on cusped manifolds can be realized by Dehn filling on M_m. The Whitehead link is realized as 1/1 Dehn filling on one cusp of M_m. The manifolds obtained by $1/n$ Dehn filling on one cusp of M_m are all hyperbolic two component 2-bridge links of Schubert normal form $(3, 6n + 2)$.

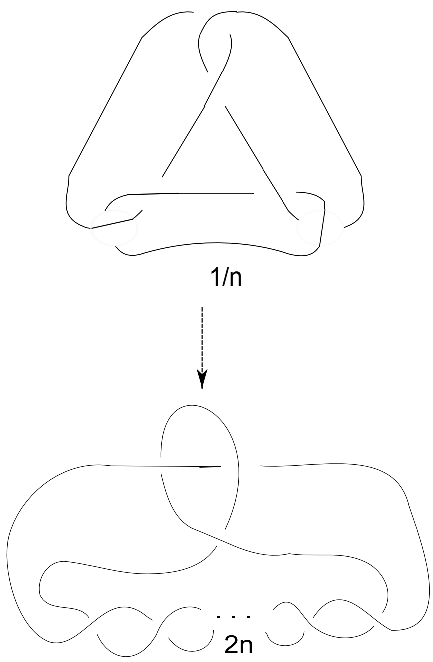

Figure 3.1: $1/n$ Dehn surgery on the Magic manifold.

That these links are hyperbolic follows from [19]. To see that the links obtained by $1/n$ Dehn filling on one component of M_m have Schubert normal form $(3, 6n+2)$ we notice that they are isotopic to links pictured in Figure 3.2.

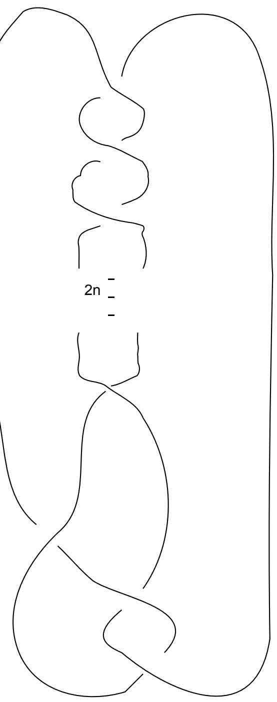

Figure 3.2: Link diagram with Conway notation $[2n\ 1\ 2]$.

The link in Figure 3.2 is the diagram corresponding to the Conway notation $[2n\ 1\ 2]$. This tuple $[2n\ 1\ 2]$ gives rise to the partial fraction $\frac{1}{2n+\frac{1}{2+\frac{1}{2}}} = \frac{3}{6n+2}$ and hence determines the Schubert normal form $(3, 6n+2)$.

The fundamental groups of these two component 2-bridge links have a presentation of the form $\Gamma = \langle a, b\ | aw = wa \rangle$ with $w = b^{\epsilon_1} a^{\epsilon_2} \ldots b^{\epsilon_{6n+1}}$ where $\epsilon_i = (-1)^{\lfloor \frac{i(4n-1)}{8n} \rfloor}$. For these links this word is $w = (ba)^n (b^{-1}a^{-1})^n b^{-1} (ab)^n$.

For $n = 1, \ldots, 4$ we were able to use Mathematica to determine the polynomials which define the character varieties of $M_m(1/n)$. All of these character varieties have exactly one component; the canonical component. Compactifying in $\mathbb{P}^2 \times \mathbb{P}^1$, the resulting singular projective models S_{m_n} for canonical components associated to the links obtained by $1/n$ Dehn filling on M_m are hypersurfaces defined by a polynomial of bidegree $(2, 3n)$. As reflected in the bidegree, all of these components are birational to conic bundles and so are rational surfaces. Since the surfaces S_{m_n} exhibit fibers with singularities, they all have minimal model $\mathbb{P}^2$. Hence the smooth projective models $\tilde{S}_{m_n}$ obtained by resolving the singularities of S_{m_n} are rational surfaces with minimal model $\mathbb{P}^2$. Determining the isomorphism class for $\tilde{S}_{m_n}$ requires understanding how all the singularities resolve. That these surfaces S_{m_n} exhibit so many singularities makes it difficult to actually specify the isomorphism class. At this point, we can only offer a conjecture regarding the isomorphism class for $\tilde{S}_{m_n}$. In Section 3.1.1, we describe the models for S_{m_n} and discuss this conjecture. At the end of this chapter we provide specific details for the case $n = 2$ in Section 3.1.2.

These link complement examples exhibit similar yet more complicated recursive relations than the knots studied in [17]. Hence we conjecture that $\tilde{S}_{m_n}$ are rational surfaces with minimal model $\mathbb{P}^2$ for all n, and verifying this would yield a 2-dimensional generalization of our understanding of the twist knot character varieties. We discuss this conjecture and its consequences in next section . In Section 3.3.1, we consider the canonical component of the character variety for the Magic manifold. This variety yields a 3-dimensional analogue. It is not only rational but also birational to a fiber bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with conic fibers.

3.1.1 Models S_{m_n} for $n = 2, 3, 4$

For $n = 2, 3, 4$, we give specific coordinates and discuss the structure for the singular projective models S_{m_n}. Each of these varieties is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ cut out by a single polynomial $f_{m_n} \in \mathbb{C}[x, y, u : z, w]$ of bidegree $(2, 3n)$. The polynomials f_{m_n} can all be expressed as $g_{m_n} + u^2 h_{m_n}$ where $h_{m_n} \in \mathbb{C}[z, w]$ and $g_{m_n} \in \mathbb{C}[x, y, z, w]$. For $n = 2, 3, 4$ the defining polynomials are

$$\begin{aligned}
f_{m_2} &= wz(w^2x + wyz - xz^2)(w^2y + wxz - yz^2) \\
&\quad +u^2[w^6 + 6w^4z^2 - 5w^2z^4 + z^6] \\
f_{m_3} &= w(w - z)(w + z)(w^3y - 2w^2xz - wyz^2 + xz^3) \\
&\qquad\qquad \times (w^3x - 2w^2yz - wxz^2 + yz^3) \\
&\quad +u^2[z(w^2 - wz - z^2)(w^2 + wz - z^2)(5w^4 - 5w^2z^2 + z^4)] \\
f_{m_4} &= wz(2w^2 - z^2)(w^4x + 2w^3yz - 3w^2xz^2 - wyz^3 + xz^4) \\
&\qquad\qquad \times (w^4y + 2w^3xz - 3w^2yz^2 - wxz^3 + yz^4) \\
&\quad +u^2[(w^6 - 3w^5z - 6w^4z^2 + 4w^3z^3 + 5w^2z^4 - wz^5 - z^6) \\
&\qquad\qquad \times (w^6 + 3w^5z - 6w^4z^2 - 4w^3z^3 + 5w^2z^4 + wz^5 - z^6)]
\end{aligned}$$

That all of these varieties are birational to conic bundles follows easily from the bidgree of f_{m_n}. Consider the projection map $S_{m_n} \to \mathbb{P}^1$. The fiber over a generic point $[z_0, w_0] \in \mathbb{P}^1$ is the curve cut out by $f_{m_n}(x, y, u, z_0, w_0)$ which is a conic since the degree of f_{m_n} over $\mathbb{P}^2$ is 2. Locally S_{m_n} is isomorphic to a conic bundle and therefore birational to such. The surfaces S_{m_n}

are all singular with 4, 8, 12, and 16 singular points for $n = 1\ 2,\ 3,\ 4$ respectively. Hence none of the surfaces S_{m_n} are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. From Proposition 1.3.1 and Corollary 1.3.2 the resolved varieties $\tilde{S_{m_n}}$ are all rational surfaces isomorphic to $\mathbb{P}^2$ blown up at n_{m_n} points. By Proposition 1.3.3, $n_{m_n} = \chi(\tilde{S}_{m_n}) - 3$. Using Proposition 1.3.4 we can calculate $\chi(\tilde{S}_{m_n})$ from $\chi(S_{m_n})$ depending on how each of the singularities resolve. While we have yet to resolve the singularities explicitly, it is reasonable to guess that each of the singularities for S_{m_n} resolves into a conic after a single blow up so that

Conjecture 3.1.1. *For $n = 1, \ldots, 4$, $\chi(\tilde{S}_{m_n}) = 4n + \chi(S_{m_n})$.*

The surfaces S_{m_n} have a similar structure to that of the Whitehead link. The Euler characteristic can be computed by appealing to the same map $\phi : S_{m_n} \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $[x, y, u : z, w] \mapsto [x, y : z, w]$. As in the case of the Whitehead link, ϕ is a 2-to-1 map almost everywhere. By extending ϕ so that it is defined on all of S_{m_n}, examining the branch set B_{m_n}, and determining the set of points L_{m_n} in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite, we can use ϕ to calculate the Euler characteristic of S_{m_n}.

Proposition 3.1.2. *For $n = 1, \ldots, 4$, the Euler characteristic of S_{m_n} is $\chi(S_{m_n}) = 4 + 5n$.*

Proof.

$$\begin{aligned} \chi(S_{m_n}) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 - B_{m_n}) + \chi(B_{m_n} - L_{m_n}) + \chi(\phi^{-1}(L_{m_n})) \\ &= 8 - \chi(B_{m_n}) - \chi(L_{m_n}) + \chi(\phi^{-1}(L_{m_n})) \end{aligned}$$

The points $[x, y : z, w] \in L_{m_n}$ are those which simultaneously satisfy g_{m_n} and h_{m_n}. Since $h_{m_n} \in \mathbb{C}[z, w]$ has degree $3n$ and $g_{m_n} \in \mathbb{C}[x, y : zw]$ has degree 2, L_{m_n} consists of $6n$ points and $\chi(L_{m_n}) = 6n$. The preimage of L_{m_n} under ϕ is the union of $6n$ $\mathbb{P}^1$'s which intersect in pairs at points of the form $[0, 0, 1, z, w]$ where $[z, w]$ satisfy h_{m_n}. Hence $\chi(\phi^{-1}(L_{m_n})) = 6n\chi(\mathbb{P}^1) - 3n = 9n$. The branch set $B_{m_n} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the zero set of g_{m_n} and thus contains L_{m_n} as a subset. For $n = 1, \ldots, 4$, the polynomials g_{m_n} have $2 + n$ factors. From the bidegrees of these factors we see that B_{m_n} is the union of $2 + n$ $\mathbb{P}^1$'s which intersect in a total of $4n$ points. Hence, $\chi(B_{m_n}) = (2 + n)\chi(\mathbb{P}^1) - 4n = 4 - 2n$. All together we have

$$\begin{aligned}
\chi(S_{m_n}) &= 8 - \chi(B_{m_n}) - \chi(L_{m_n}) + \chi(\phi^{-1}(L_{m_n})) \\
&= 8 - (4 - 2n) - 6n + 9n \\
&= 4 + 5n
\end{aligned}$$

□

Conjecture 3.1.1 together with Propositions 3.1.2 and 1.3.3 would imply that, for $n = 1, \ldots, 4$, $\tilde{S}_{m_n}$ is isomorphic to $\mathbb{P}^2$ blown up at $9n + 1$ points. That these link complements examples exhibit similar yet more complicated recursive relations than the knots studied in [17] leads us to make the following conjecture:

Conjecture 3.1.3. *The canonical components $\tilde{S}_{m_n}$ for the link complements obtained by $1/n$ Dehn filling on one cusp of the Magic manifold are isomorphic to $\mathbb{P}^2$ blown up at $9n + 1$ points.*

In addition to understanding the isomorphism class of the canonical components for these link complements, we want to understand how the algebro-geometric structure reflects the topological structure of the associated manifolds. For $n = 1, \ldots, 4$, the surfaces S_{m_n} are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$. One striking common feature among these examples is the existence of a double line fiber, a fiber in which every point is a singularity. Double lines are a fairly rare feature to conic bundles. More precisely, all conics can be parameterized by $\mathbb{P}^5$ and double lines correspond to a codimension 3 subvariety ([12]). Hence a conic bundle with a double line fiber corresponds to a line which passes through a particular codimension 3 subvariety in $\mathbb{P}^5$ which is a rare occurrence. We are still working to determine what topological properties these double lines may reflect.

3.1.2 Calculation: 1/2 Dehn surgery on the Magic manifold

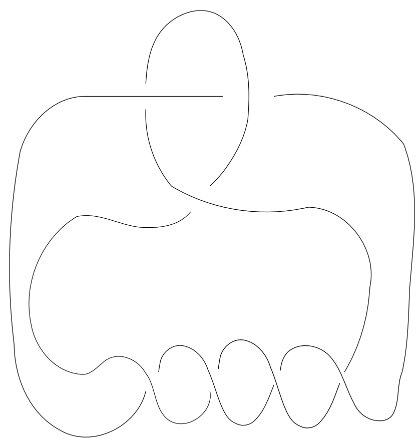

Figure 3.3: $M_m(1/2)$

The manifold $M_m(1/2)$ (shown in Figure 3.3) has partial fraction coefficients [4 1 2] and Schubert normal form $(3,14)$. Its fundamental group is $\pi_1(M) = [a, b\,|aw = wa]$ where $w = (ba)^2(b^{-1}a^{-1})^2b^{-1}(ab)^2$. The character variety consists of a single component, the canonical component. The affine model for the canonical component is a hypersurface in $\mathbb{C}^3$ defined by the polynomial

$$-1 - xyz + 6z^2 - x^2z^2 - y^2z^2 + xyz^3 - 5z^4 + x^2z^4 + y^2z^4 - xyz^5 + z^6$$

We obtain a projective model S_{m_2} by compactifying in $\mathbb{P}^2 \times \mathbb{P}^1$. The defining polynomial for S_{m_2} is

$$\begin{aligned} f_{m_2} \;\; = \;\; & wz(w^2x + wyz - xz^2)(w^2y + wxz - yz^2) \\ & + u^2(w^6 + 6w^4z^2 - 5w^2z^4 + z^6) \end{aligned}$$

where x, y, u are $\mathbb{P}^2$ coordinates and z, w are $\mathbb{P}^1$ coordinates. Examining the partial derivatives of f_{m_2} , we find that S_{m_2} has 8 singular points $[x, y, u : z, w]$

$$\begin{array}{l} [0,1,0:1,0] \\ [0,1,0:0,1] \\ [1,0,0:1,0] \\ [1,0,0:0,1] \\ [1,-1,0:\frac{-1-\sqrt{5}}{2},1] \\ [1,-1,0:\frac{-1+\sqrt{5}}{2},1] \\ [1,-1,0:\frac{1-\sqrt{5}}{2},1] \\ [1,-1,0:\frac{1+\sqrt{5}}{2},1] \end{array}$$

Theorem 3.1.4. *The singular projective model S_{m_2} for the canonical component of the manifold $M_m(1/2)$ is a rational surface with Euler characteristic 16.*

The proof of Theorem 3.1.4 is similar to the proof of Theorem 2.2.1. We show S_{m_2} is a rational surface and then determine its Euler characteristic.

Lemma 3.1.5. *S_{m_2} is a rational surface.*

Proof. To show that S_{m_2} is rational, we show that S_{m_2} is birational to a conic bundle. Consider the projection $\pi_{\mathbb{P}^1} : S_{m_2} \to \mathbb{P}^1$. The fiber over $[z_0, w_0] \in \mathbb{P}^1$ is the set of points $[x, y, u : z_0, w_0]$ which satisfy $-u^2w_0^6 - w_0^5xyz_0 + 6u^2w_0^4z_0^2 - w_0^4x^2z_0^2 - w_0^4y^2z_0^2 + w_0^3xyz_0^3 - 5u^2w_0^2z_0^4 + w_0^2x^2z_0^4 + w_0^2y^2z_0^4 - w_0xyz_0^5 + u^2z_0^6 = 0$. This is the zero set of a degree 2 polynomial in $\mathbb{P}^2$ which is a conic. Hence S_{m_2} is birational to a conic bundle. By Proposition 1.3.1, S_{m_2} is a rational surface. □

Since the surface S_{m_2} exhibits singular fibers, it is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It follows that $\mathbb{P}^2$ is the minimal model for both the singular projective surface S_{m_2} and the smooth projective surface $\tilde{S}_{m_2}$. The isomorphism class of $\tilde{S}_{m_2}$ is determined by its Euler characteristic. We can determine the Euler characteristic of $\tilde{S}_{m_2}$ from that for S_{m_2} by keeping track of how the Euler characteristic changes with each resolution of a singularity. Resolving each of the 8 singularities is tedious. In this thesis, we merely calculate the Euler characteristic for S_{m_2}.

Proposition 3.1.6. *The Euler characteristic of the surface S_{m_2} is $\chi(S_{m_2}) = 14$.*

We prove Proposition 3.1.6 through a series of lemmas. The proof of Lemma 3.1.5 shows that S_{m_2} is a fiber bundle over $\mathbb{P}^1$ with conic fibers. If none of the fibers were double lines we could calculate $\chi(S_{m_2})$ as $\chi(B)\chi(f)+d$ where B is the base of the fiber bundle, f is a generic fiber and d is the number of fibers exhibiting a single singularity. However, like the canonical component for the Whitehead link, the projective model S_{m_2} exhibits a double line fiber over the point $[z,w]=[1,0]$. To calculate the $\chi(S_{m_2})$ we appeal to the map $\phi : S_{m_2} \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $[x,y,u : z,w] \mapsto [x,y : z,w]$. The map ϕ is almost everywhere 2-to-1. By understanding this map everywhere we can use it to calculate $\chi(S_{m_2})$ in terms of $\chi(\mathbb{P}^1 \times \mathbb{P}^1)$. There are four aspects we need to consider. The map ϕ is neither surjective nor defined at 6 points, the set of which we denote by P_{m_2}. Over 12 points, in $\mathbb{P}^1 \times \mathbb{P}^1$ the fiber is a copy of $\mathbb{P}^1$. Finally, the map is branched over three $\mathbb{P}^1$'s.

The map ϕ is not defined at the points $P_{m_2} = \{[0,0,1 : z,w] \,|f_{m_2}=0\}$. Namely, P_{m_2} consists of the 6 points $[0,0,1 : z,1]$ where z is

$$
\begin{aligned}
z_1 &= \tfrac{1}{3F} + \tfrac{7^{2/3}}{3(\frac{1}{2}(-1+3i\sqrt{3}))^{1/3}} + \tfrac{1}{3}(\tfrac{7}{2}(-1+3i\sqrt{3}))^{1/3} \\
z_2 &= \tfrac{1}{3} - \tfrac{(7/2)^{2/3}(1+i\sqrt{3})}{3(-1+3i\sqrt{3})^{1/3}} - \tfrac{1}{6}(1-i\sqrt{3})(\tfrac{7}{2}(-1+3i\sqrt{3}))^{1/3} \\
z_3 &= \tfrac{1}{3} - \tfrac{(7/2)^{2/3}(1-i\sqrt{3})}{3(-1+3i\sqrt{3})^{1/3}} - \tfrac{1}{6}(1+i\sqrt{3})(\tfrac{7}{2}(-1+3i\sqrt{3}))^{1/3} \\
z_4 &= -\tfrac{1}{3} + \tfrac{7^{2/3}}{3(\frac{1}{2}(1+3i\sqrt{3}))^{1/3}} + \tfrac{1}{3}(\tfrac{7}{2}(1+3i\sqrt{3}))^{1/3} \\
z_5 &= -\tfrac{1}{3} - \tfrac{(7/2)^{2/3}(1+i\sqrt{3})}{3(1+3i\sqrt{3})^{1/3}} - \tfrac{1}{6}(1-i\sqrt{3})(\tfrac{7}{2}(1+3i\sqrt{3}))^{1/3} \\
z_6 &= -\tfrac{1}{3} - \tfrac{(7/2)^{2/3}(1-i\sqrt{3})}{3(1+3i\sqrt{3})^{1/3}} - \tfrac{1}{6}(1+i\sqrt{3})(\tfrac{7}{2}(1+3i\sqrt{3}))^{1/3}
\end{aligned}
$$

Lemma 3.1.7. *The image of ϕ on $U_{m_2} = S_{m_2} - P_{m_2}$ is $\mathbb{P}^1 \times \mathbb{P}^1 - Q_{m_2}$ where Q_{m_2} is the union the 6 twice punctured spheres*

$$Q_i = \mathbb{P}^1 \times \{[z_i, 1]\} \diagdown \{[1, \tfrac{z_i}{z_i^2-1}, z_i, 1], [1, \tfrac{z_i^2-1}{z_i}, z_i, 1]\} \ .$$

Proof. To see that $\mathbb{P}^1 \times \mathbb{P}^1 - Q_{m_2}$ is the image of ϕ we view f_{m_2} as a polynomial in u with coefficients in $\mathbb{C}[x, y, z, w]$. Namely $f_{m_2} = g + u^2 h$ where $g = zw(w^2x + wyz - xz^2)(w^2y + wxz - yz^2)$ and $h = -w^6 + 6w^4z^2 - 5w^2z^4 + z^6$. The image of ϕ is the collection of all points $[x, y, z, w] \in \mathbb{P}^1 \times \mathbb{P}^1$ except those for which $f_{m_2}(x, y, u, z, w) \in \mathbb{C}[u]$ is a nonzero constant. The polynomial f_{m_2} is a nonzero constant whenever $h = 0$ and $g \neq 0$. Using Mathematica we see $h = 0$ for $[z, w] = \{[z_i, 1]\}$. For each of the z, w coordinates that satisfy h, there are two solutions $[x, y] = \{[1, \frac{z_i}{z_i^2-1}], [1, \frac{z_i^2-1}{z_i}]\}$ that satisfy $g(z, w)$. Hence the image of ϕ on U_{m_2} is all of $\mathbb{P}^1 \times \mathbb{P}^1$ less the six twice punctured spheres Q_i listed above.

□

Lemma 3.1.8. *The map ϕ smoothly extends to all of S_{m_2}.*

Proof. We can extend the map ϕ to all of S_{m_2} by using a total transformation. Let $U_{m_2} = S_{m_2} - P_{m_2}$. Then U_{m_2} is the largest open set in S_{m_2} on which ϕ is defined. Let $\overline{G(\phi, U_{m_2})}$ be the closure of the graph of ϕ on U_{m_2}. We can then smoothly extend the map ϕ to all of S_{m_2} by definingϕ at each $p_i \in P_{m_2}$ to be

$\phi(p_i) := \rho_2\rho_1^{-1}(p_i)$ where $\rho_1 : \overline{G} \to S_{m_2}$ and $\rho_1 : \overline{G} \to \mathbb{P}^1 \times \mathbb{P}^1$ are the natural projections. Note that, for $s \in U_{m_2}$, $\rho_2\rho_1^{-1}(s)$ coincides with the original map so that this extension makes sense on all of S_{m_2}. Now, the closure of the graph is $\overline{G} = \{[x, y, u, z, w : a, b, c, d] | f_{m_2} = 0, ay = bx, cw = dz\}$. So, ϕ extends to S_{m_2} as follows:

$$\begin{aligned}
\phi((0,0,1,z_1,1)) &= \{[a,b,z_1,1]\} \\
\phi((0,0,1,z_2,1)) &= \{[a,b,,z_2,1]\} \\
\phi((0,0,1,z_3,1)) &= \{[a,b,z_3,1]\} \\
\phi((0,0,1,z_4,1)) &= \{[a,b,0,z_4,1]\} \\
\phi((0,0,1,z_5,1)) &= \{[a,b,0,z_5,1]\} \\
\phi((0,0,1,z_6,1)) &= \{[a,b,0,z_6,1]\}
\end{aligned}$$

□

Notice that the set $Q_{m_2} \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is not contained in the image of ϕ on U_{m_2}, is contained in the image of ϕ on P_{m_2}. That the extension ϕ maps six points in S_{m_2} to not just six disjoint $\mathbb{P}^1$'s in $\mathbb{P}^1 \times \mathbb{P}^1$ but to the six disjoint $\mathbb{P}^1$'s which are are missing from the image of ϕ on U_{m_2} will be important for the Euler characteristic calculation.

Lemma 3.1.9. *There are* 12 *points in* $\mathbb{P}^1 \times \mathbb{P}^1$*, the collection of which we will call* L_{m_2}*, whose fiber in* S_{m_2} *is infinite.*

Proof. Thinking of f_{m_2} as a polynomial in the variable u with coefficients in $\mathbb{C}[x, y, z, w]$, we see that the points in $\mathbb{P}^1 \times \mathbb{P}^1$ which are simultaneously zeros of these coefficient polynomials are precisely the points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite. We note here that the points of L_{m_2} are precisely the punctures of the six punctured spheres which are not in the image of $\phi|_{U_{m_2}}$. The preimage of L_{m_2} in S_{m_2} is the union of 12 $\mathbb{P}^1$'s each intersecting exactly one other $\mathbb{P}^1$ in one point. These six points of intersection are the points on the $\mathbb{P}^1$'s where the coordinate u goes to infinity which is equivalent to the points where the x and y coordinates go to zero. Thus these intersection points are precisely the points in P_{m_2}.

The points $[x, y, z, w]$ in L_{m_2} are $\{[1, \frac{z_i}{z_i^2-1}, z_i, 1], [1, \frac{z_i^2-1}{z_i}, z_i, 1]\}$ where the z_i are those listed in 3.1.2. The infinite fiber over $[1, \frac{z_i}{z_i^2-1}, z_i, 1]$ is $\{[1, \frac{z_i}{z_i^2-1}, u, z_i, 1]\{$ and the infinite fiber over $[1, \frac{z_i^2-1}{z_i}, z_i, 1]$ is $\{[1, \frac{z_i^2-1}{z_i}, u, z_i, 1]\}$. Both of these fibers contain the point $[0, 0, 1, z_i, 1] \in P_{m_2}$.

□

In calculating the Euler characteristic we will use the fact that the preimage of L_{m_2} in S_{m_2} are 12 $\mathbb{P}^1$'s which intersect in pairs at ideal points in the set $P_{m_2} \subset S_{m_2}$. In fact, each point in P_{m_2} appears as the intersection of two of these fibers and the image of P_{m_2} under ϕ is precisely L_{m_2}.

Let B_{m_2} denote the branch set of ϕ in $\mathbb{P}^1 \times \mathbb{P}^1$. We have the following lemma.

Lemma 3.1.10. $\chi(B_{m_2}) = 0$.

Proof. The branch set, or at least the places where ϕ is not one-to-one, consists of the points in S_{m_2} which also satisfy the coordinate equation $u = 0$. The image, $B_{m_2} \subset \mathbb{P}^1 \times \mathbb{P}^1$, of this branch set, is the union of the four varieties, B_0, B_1, B_2, and B_3 defined by the respective three polynomials $g_0 = z$, $g_1 = w$, $g_2 = (w^2x + wyz - xz^2)$, $g_3 = (w^2y + wxz - yz^2)$ which are all $\mathbb{P}^1$'s. From the bidegrees of the g_i we know that B_0 intersects each of B_2 and B_3 in one point ($[0, 1, 0, 1]$ and $[1, 0, 0, 1]$ respectively), B_1 intersects each of B_2 and B_3 in one point ($[0, 1, 1, 0]$ and $[1, 0, 1, 0]$ respectively), and B_2 and B_3 intersect in four points ($[1, 1, 1, -\frac{1}{2}(1 + \sqrt{5})]$, $[1, -1, 1, \frac{1}{2}(1 + \sqrt{5})]$, $[1, 1, 1, \frac{1}{2}(-1 + \sqrt{5})]$, $[1, -1, 1, \frac{1}{2}(1 - \sqrt{5})]$). Since B_{m_2} is the union of four $\mathbb{P}^1$'s which intersect in a total of 8 points, $\chi(B_{m_2}) = 4\chi(\mathbb{P}^1) - 8 = 0$.

□

Again thinking of f_{m_2} as a polynomial in u we can write f_{m_2} as $f = g + u^2h$ where g and h are polynomials in $\mathbb{C}[x, y, z, w]$. Since L_{m_2} is cut out by the ideal $< g, h >$ and B_{m_2} is cut out by the ideal $< g >$, L_{m_2} is a subvariety of B_{m_2}. That each of 12 points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite is also a branch point is necessary for the Euler characteristic calculation.

Now that we understand the map ϕ everywhere we can calculate the Euler characteristic of S_{m_2} and prove Proposition 3.1.6.

Proof. (Proposition 3.1.6) Since the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fibers are infinite coincide with the image L_{m_2} of the fundamental set P_{m_2}, and L_{m_2} is the intersection of Q_{m_2} and the branch set B_{m_2} ,

$$\begin{aligned}\chi(S_{m_2}) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 - B_{m_2} - Q_{m_2}) + \chi(Q_{m_2} + B_{m_2} - L_{m_2}) + \chi(\phi^{-1}(L_{m_2})) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(Q_{m_2}) - \chi(B_{m_2}) - \chi(L_{m_2}) + \chi(\phi^{-1}(L_{m_2}))\end{aligned}$$

The Euler characteristic of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4$. As Q_{m_2} is the disjoint union of six twice-punctured spheres, $\chi(Q_{m_2}) = 6(\chi(\mathbb{P}^1) - 2) = 0$. Since B_{m_2} is four $\mathbb{P}^1$'s which intersect at 8 points, $\chi(B_{m_2}) = 4\chi(\mathbb{P}^1) - 8\chi(point) = 0$. Now L_{m_2} is just 12 points so $\chi(L_{m_2}) = 12$. That $\phi^{-1}(L_{m_2})$ is the union of 12 $\mathbb{P}^1$'s which intersect in pairs at a point implies that $\chi(\phi^{-1}(L)) = 12\chi(\mathbb{P}^1) - 6\chi(points) = 18$. All together this gives $\chi(S_{m_2}) = 14$.

□

3.2 Links obtained by $1/n$ Dehn Surgery on the Borromean Rings

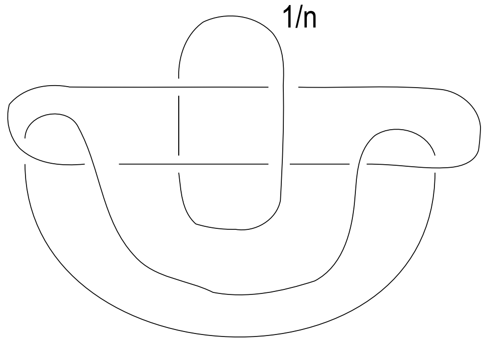

Figure 3.4: $1/n$ Dehn surgery on the Borromean rings.

Let M_{br} denote the complement of the Borromean rings. The manifolds which results from $1/n$ Dehn filling on one of the cusps of M_{br} are ([15]) two component 2-bridge link with Schubert normal form $S(8n, 4n+1)$. The fundamental group of these two component 2-bridge links has a presentation of the form $\Gamma = \langle a, b | aw = wa \rangle$ with $w = b^{\epsilon_1} a^{\epsilon_2} \dots b^{\epsilon_{8n-1}}$ where $\epsilon_i = (-1)^{\lfloor \frac{i(4n-1)}{8n} \rfloor}$. For $n = 1, \dots, 4$ we were able to use Mathematica to determine the polynomials which define the character varieties of $M_{br}(1/n)$. For $n \geq 4$ the polynomials are a bit too large for mathematica to handle.

Although we have few examples, there are some trends among these character varieties which make them worth noting. We can look at the number of components and the number of canonical components which comprise these character varieties. Although the defining polynomials for certain components may not be smooth, their bidegree and how they change with the surgery coefficient n is still of interest. Below we summarize this information.

Character Varieties for $M_{br}(1/n)$

Manifold	component	canonical component	bidegree
$M_{br}(1/1)$	1	$\surd$	(2,3)
$M_{br}(1/2)$	1		(2,2)
	2	$\surd$	(4,5)
$M_{br}(1/3)$	1		(2,2)
	2		(2,2)
	3	$\surd$	(6,7)
$M_{br}(1/4)$	1		(2,2)
	2		(4,4)
	3	$\surd$	(8,9)

Possibly the most interesting characteristic these character varieties share is the existence of a component which is defined by a polynomial of bidegree $(2, k)$ where $k = \{2, 3\}$. All of these components are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$. Although they are not conic bundles due to the existence of singularities, they all share a common feature. Over the same $\mathbb{P}^1$ coordinate $([z, w] = [1, 0])$, they all have a double line fiber. See figure 3.5.

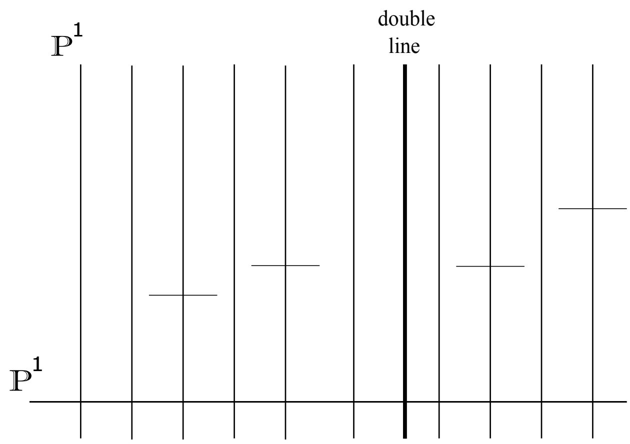

Figure 3.5: component birational to a conic bundle for $M_{br}(1/n)$

Away from a few points all of these components look like conic bundles. All conics are parameterized by $\mathbb{P}^5$ and so we can think of conic bundles as curves in $\mathbb{P}^5$. All the degeneracies live in a hypersurface in $\mathbb{P}^5$ and all the double lines live in a codimension two subvariety inside this hypersurface. Hence, it is fairly uncommon for a curve in $\mathbb{P}^5$ to intersect the subvariety which corresponds to double line fibers.

All of these components are defined by polynomials that have singularities (four for the Whitehead link and two for each of the other Dehn surgery components). While these $\mathbb{P}^1$ bundles are not isomorphic to conic bundles, they are birational to such. Since surfaces birational to conic bundles are rational, all of these components are rational surfaces and thus isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at some number of points. As we did for the Whitehead link complement, we can use the Euler characteristic to determine these character varieties components topologically. Aside from the Whitehead link all of these components of these character varieties are hypersurfaces defined by a singular polynomial of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Each of these defining polynomials has two singularities which each resolve into a conic after a single blow up. Hence the Euler characteristic of the smooth models are equal to that of the singular models plus two. The way we calculate the Euler characteristic of these singular models is very similar to way to we calculated such for the Whitehead link. It turns out that all of these singular (and so smooth) models have Euler characteristic 10 and hence are all isomorphic to $\mathbb{P}^2$ blown up at 7 points. From an algebro-geometric perspective $\mathbb{P}^2$ blown up at 7 points is interesting in the sense that is has only finitely many (precisely 49) (-1) curves.

What we have just described is a brief outline of the proof of Theorem 1.1.3.

Theorem 3.2.1. *For $n = 2, \ldots, 4$, the character variety of $M_{br}(1/n)$ has a component which is a rational surface isomorphic to $\mathbb{P}^2$ blown up at 7 points.*

The proof is very similar to that for 1.1.1. We include the the details in Section 3.2.1. Below we list the singular defining polynomials for these conic bundle components.

Conic bundle components for $M_{br}(1/n)$

Manifold	Singular defining polynomial for conic bundle component	χ	Smooth Surface $\mathbb{P}^2$ blown up at
$M_{br}(1/1)$	$-w^3xy + w^2x^2z + w^2y^2z - wxyz^2 + u^2(z^3 - 2w^2z)$	13	10 points
$M_{br}(1/2)$	$w^2x^2 + w^2y^2 - wxyz + u^2(z^2 - 2w^2)$	10	7 points
$M_{br}(1/3)$	$w^2x^2 + w^2y^2 - wxyz + u^2(z^2 - 3w^2)$	10	7 points
	$w^2x^2 + w^2y^2 - wxyz + u^2(z^2 - w^2)$	10	7 points
$M_{br}(1/4)$	$w^2x^2 + w^2y^2 - wxyz + u^2(z^2 - 2w^2)$	10	7 points

3.2.1 Proof of Theorem 3.2.1

.

Throughout this section we will refer to the projective completion in $\mathbb{P}^2 \times \mathbb{P}^1$ of the rational component of $M_{br}(1/2)$ as S_2. The defining polynomial for S_2 is $f_2 = w^2x^2+w^2y^2-wxyz+u^2(z^2-2w^2)$ where x, y, u are $\mathbb{P}^2$ coordinates are z, w are $\mathbb{P}^1$ coordinates. Now, f_2 is not smooth. It is singular at the two points

$$\begin{aligned} s_1 &= [1, 0, 0, 1, 0] \\ s_2 &= [0, 1, 0, 1, 0] \end{aligned}$$

Our goal is to determine topologically the smooth surfaces $\tilde{S}_2$ obtained by resolving the singularities of S_2.

Theorem 3.2.2. *The surface $\tilde{S_2}$ is a rational surface isomorphic to $\mathbb{P}^2$ blown up at 7 points.*

Like the canonical component for the Whitehead link, the Euler characteristic of $\tilde{S_2}$ together with the fact that $\tilde{S_2}$ is rational is enough to determine $\tilde{S_2}$ topologically. Since $\tilde{S_2}$ is obtained from S_2 by a series of blow ups,

Lemma 3.2.3. *$\tilde{S_2}$ is birational to a conic bundle.*

Proof. Consider the projection $\pi_{\mathbb{P}^1} : S_2 \to \mathbb{P}^1$. The fiber over $[z_0, w_0] \in \mathbb{P}^1$ is the set of points $[x, y, u : z_0, w_0]$ which satisfy

$$w_0^2x^2 + w_0^2y^2 - w_0xyz_0 + u^2(z_0^2 - 2w_0^2) = 0$$

This is the zero set of a degree 2 polynomial in $\mathbb{P}^2$ which is a conic. Away from the two singularities, S_2 is isomorphic to a conic bundle. Hence, S_2 is birational to a conic bundle. Since $\tilde{S_2}$ is obtained from S_2 by a series of blow ups, $\tilde{S_2}$ is birational to S and so birational to a conic bundle. □

Proposition 1.3.1 implies that $\tilde{S_2}$ is rational. Since S_2 has degenerate fibers, it is isomorphic to $\mathbb{P}^2$ blown-up at n points by Corollary 1.3.2. By proposition 1.3.3 $\chi(\tilde{S_2}) = 3 + n$. Hence we can determine $\tilde{S_2}$ topologically by calculating $\chi(\tilde{S_2})$.

To calculate the Euler characteristic of $\tilde{S_2}$ we use the Euler characteristic of S_2. Since the smooth surface $\tilde{S_2}$ is obtained from S_2 by a series of blow ups, we can use proposition 1.3.4 to write $\chi(\tilde{S_2})$ in terms of $\chi(S_2)$.

Lemma 3.2.4. $\chi(\tilde{S_2}) = \chi(S_2) + 2$

Proof. The smooth surface, $\tilde{S_2}$, is obtained by resolving the two singularities, s_i, of S_2 listed above. This process of blowing up singularities is a local one. Above the singularities, a local model for $\tilde{S_2}$ can be obtained by blowing up S_2 in an affine neighborhood of each of the singular points. Away from the singularities we can take the local model for S_2 as a local model for $\tilde{S_2}$ since S_2 and $\tilde{S_2}$ are locally isomorphic there. Each of the singularities is nice in the sense that it takes only one blow up to resolve them. Hence, in terms of the Euler characteristic, we have

$$\chi(\tilde{S_2}) = \chi(S_2 - \{s_1, s_2\}) + \chi(\tilde{s_1}) + \chi(\tilde{s_2}) \tag{3.1}$$

where, $\tilde{s_i}$ denotes the preimage of s_i in $\tilde{S_2}$. Determining the Euler characteristic of $\tilde{S_2}$ in terms of that for S_2 amounts to determining $\tilde{s_i}$.

To blow up S_2 at $s_1 = [1, 0, 0, 1, 0]$ we consider the affine open set A_1 where $x \neq 0$ and $z \neq 0$. Local affine coordinates for $A_1 \cong A^3$ are y, u, w. So to blow up S_2 at s_1 we blow up $X_1 = Z(f_2|_{x=1,z=1})$ at $[y, u, w] = [0, 0, 0]$ in A_1. Using coordinates a, b, c for $\mathbb{P}^2$, the blow up Y_1 of X_1 at [0,0,0] is the closed subset in $A_1 \times \mathbb{P}^2$ defined by the equations

$$g_1 = f_2|_{x=1,z=1} \;=\; u^2 + w^2 - 2u^2w^2 - wy + w^2y^2 \tag{3.2}$$

$$e_1 \;=\; yb - ua \tag{3.3}$$

$$e_2 \;=\; yc - wa \tag{3.4}$$

$$e_3 \;=\; uc - wb \tag{3.5}$$

We determine the local model above s_1 and check for smoothness by looking at Y_1 in the affine open sets define by $a \neq 0$, $b \neq 0$, and $c \neq 0$.

First we look at Y_1 in the affine open set defined by $a \neq 0$. In this open set the defining equations for Y_1 become

$$g_1 = u^2 + w^2 - 2u^2w^2 - wy + w^2y^2 \tag{3.6}$$

$$e_1 = yb - u \tag{3.7}$$

$$e_2 = yc - w \tag{3.8}$$

$$e_3 = uc - wb \tag{3.9}$$

Substituting, we obtain the local model, $y^2(-b^2+c-c^2-c^2y^2+2b^2c^2y^2)$. The first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Notice that E_1 and Y_1 meet in the smooth conic $-b^2+c-c^2$. In this affine open set, the local model above the singularity s_1 is a smooth conic; a $\mathbb{P}^1$. Since all the partial derivatives of the second factor do not simultaneously vanish, this model is smooth in $A_1 \times \mathbb{P}^2$.

Next we look at Y_1 in the affine open set defined by $b \neq 0$. In this the local model for Y_1 above $[0, 0, 0]$ is $u^2(1 - ac + c^2 - 2c^2u^2 + a^2c^2u^2)$. Again, the first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Here, E_1 and Y_1 meet in the smooth conic $1 - ac + c^2$. In this affine open set, the local model above the singularity s_1 is a conic. Since all the partial derivatives of the second factor do not simultaneously vanish, this model is smooth in $A_1 \times \mathbb{P}^2$.

Finally we look at Y_1 in the affine open set defined by $c \neq 0$. In this open set the local model for Y_1 above $[0,0,0]$ is $(w^2(1-a+b^2+a^2w^2-2b^2w^2)$. The first factor is the exceptional plane, E_1 and the other factor is the local model for Y_1. Here, E_1 and Y_1 meet in the smooth conic $1-a+b^2$. In this affine open set, the local model above the singularity s_1 is a conic. Since all the partial derivatives of the second factor do not simultaneously vanish, this model is smooth in $A_1 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing up yields a smooth local model which intersects the exceptional plane above s_1 in the conic defined by $c^2 - ac + b^2$. Hence $\chi(\tilde{s_1}) = 2$.

To blow up S_2 at $s_2 = [0,1,0,1,0]$ we consider the affine open set A_2 where $y \neq 0$ and $z \neq 0$. Local affine coordinates for $A_2 \cong A^3$ are x, u, w. So to blow up S_2 at s_2 we blow up $X_2 = Z(f_2|_{y=1,z=1})$ at $[x,u,w] = [0,0,0]$ in A_2. Using coordinates a, b, c for $\mathbb{P}^2$, the blow up Y_2 of X_2 at [0,0,0] is the closed subset in $A_2 \times \mathbb{P}^2$ defined by the equations

$$g_2 = f_2|_{y=1,z=1} = u^2 + w^2 - 2u^2w^2 - wx + w^2x^2 \quad (3.10)$$

$$q_1 = xb - au \quad (3.11)$$

$$q_2 = xc - aw \quad (3.12)$$

$$q_3 = uc - wb \quad (3.13)$$

In the affine open set defined by $a \neq 0$, the local model is defined by $x^2(-b^2 + c - c^2 - c^2x^2 + 2b^2c^2x^2)$. The first factor is the exceptional plane E_2

and the second factor is the local model for Y_2. The exceptional plane E_2 and Y_2 meet in the smooth conic $-b^2+c-c^2$. Since all the partial derivatives of the second factor do not vanish simultaneously, this model is smooth in $A_2 \times \mathbb{P}^2$.

In the affine open set defined by $b \neq 0$, the local model is defined by $u^2(1 - ac + c^2 - 2c^2u^2 + a^2c^2u^2)$. The first factor is the exceptional plane E_2 and the second factor is the local model for Y_2. Here, the exceptional plane E_2 and Y_2 meet in the smooth conic $1 - ac + c^2$. No where do all the partial derivatives of the second factor vanish simultaneously and so this model is smooth in $A_2 \times \mathbb{P}^2$.

In the affine open set defined by $c \neq 0$, the local model is defined by $w^2(1-a+b^2+a^2w^2-2b^2w^2)$. The first factor is the exceptional plane E_2 and the second factor is the local model for Y_2. Here, the exceptional plane E_2 and Y_2 meet in the smooth conic $1 - a + b^2$. Since all the partial derivatives of the second factor do not vanish simultaneously, this model is smooth in $A_2 \times \mathbb{P}^2$.

Rehomogenizing we see that blowing up yields a smooth local model which intersects the exceptional plane above s_2 in the conic defined by $c^2 - ac + b^2$. Hence $\chi(\tilde{s_2}) = 2$.

We have shown that for $i = 1, 2$, $\tilde{s_i}$ is a smooth conic. Hence $\chi(\tilde{s_i}) = 2$ for $i = 1, 2$ and

$$
\begin{aligned}
\chi(\tilde{S_2}) &= \chi(S_2 - \{s_1, s_2\}) + \chi(\tilde{s_2} + \chi(\tilde{s_2} && (3.14)\\
&= \chi(S_2) - \chi(\{s_1, s_2\}) + 2\chi(conic) && (3.15)\\
&= \chi(S) - 2 + 2(2) && (3.16)\\
&= \chi(S) + 2 && (3.17)
\end{aligned}
$$

□

Proposition 3.2.5. *The Euler characteristic of the surface S_2 is $\chi(S_2) = 8$.*

To do this we again appeal to the map $\phi : S_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $[x, y, u : z, w] \to [x, y : z, w]$ on a dense open set U of S_2. In order to use ϕ to calculate the Euler characteristic we must understand ϕ everywhere. Let G denote the graph of ϕ. Rather than first extending ϕ to all of S_2 and then using the extension to calculate $\chi(S_2)$ we will use the projection map $\rho_2 : \overline{G} \to S_2$ to calculate $\chi(S_2)$ in terms of $\chi(\overline{G})$ which we will then determine using the projection $\rho_2 : \overline{G} \to \mathbb{P}^1 \times \mathbb{P}^1$.

Here, ϕ is generically 2-to-1. In calculating the Euler characteristic of S_2 there are four aspects we need to consider. The map ϕ is neither surjective nor defined at the two points $P = \{(0, 0, 1, \pm\sqrt{2}, 1)\}$. Over four points in the $\mathbb{P}^1 \times \mathbb{P}^1$ the fiber is a copy of $\mathbb{P}^1$. Finally, the map is branched over two copies of $\mathbb{P}^1$.

Lemma 3.2.6. *The image of* ϕ *on* $U = S_2 - P$ *is* $\mathbb{P}^1 \times \mathbb{P}^1 - Q$ *where*

$$\begin{aligned} Q \quad = \quad & \mathbb{P}^1 \times \{[\sqrt{2}, 1]\} \setminus \{[\tfrac{\sqrt{2}}{2}(1 \pm i), 1, \sqrt{2}, 1], \} \\ \cup \quad & \mathbb{P}^1 \times \{[\sqrt{2}, 1]\} \setminus \{[-\tfrac{\sqrt{2}}{2}(1 \pm i), 1, -\sqrt{2}, 1]\} \end{aligned}$$

Proof. We can see that this is in fact the image by viewing f as a polynomial in u with coefficients in $\mathbb{C}[x, y, z, w]$. Namely $f = g + u^2 h$ where $g = w^2 x^2 + w^2 y^2 - wxyz$ and $h = u^2(z^2 - 2w^2)$. The image of ϕ is the collection of all points $[x, y, z, w] \in \mathbb{P}^1 \times \mathbb{P}^1$ except those for which $f(x, y, z, w) \in \mathbb{C}[u]$ is a nonzero constant. The polynomial $f(x, y, z, w)$ is a nonzero constant whenever $h = 0$ and $g \neq 0$. It is easy to see that $h = 0$ whenever $[z, w] = \{[\pm\sqrt{2}, 1]\}$. For each of the z, w coordinates which satisfy h, there are two x, y coordinates which satisfy $g(z, w)$. Hence the image of ϕ on U is all of $\mathbb{P}^1 \times \mathbb{P}^1$ less the two twice punctured spheres as listed above. □

Lemma 3.2.7. *The map* ϕ *smoothly extends to all of* S_2.

Proof. Let $U = S_2 - P$. Then U is the largest open set in S_2 on which ϕ is defined. Let $\overline{G(\phi, U)}$ be the closure of the graph of ϕ on U. We can then smoothly extend the map ϕ to all of S_2 by taking a total transform. By abuse of notation we will again call this total transform ϕ. For $s \in U$, $\phi(s) = \rho_2 \rho_1^{-1}(s)$ coincides with the original map. Since the closure of the graph is $\overline{G} = \{[x, y, u, z, w : a, b, c, d] | f = 0, ay = bx, cw = dz\}$, ϕ extends to S_2 as follows:

$$\begin{aligned} \phi((0,0,1,\sqrt{2},1)) &= \{[a,b,\sqrt{2},1]\} \\ \phi((0,0,1,-\sqrt{2},1) &= \{(a,b,-\sqrt{2},1)\} \end{aligned}$$

□

Notice that the set $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is not contained in the image of ϕ on U, is contained in the image of ϕ on P. That the extension ϕ maps two points in S_2 to not just two disjoint $\mathbb{P}^1$'s in $\mathbb{P}^1 \times \mathbb{P}^1$ but to the two disjoint $\mathbb{P}^1$'s which are are missing from the image of ϕ on U will be important for the Euler characteristic calculation.

Lemma 3.2.8. *There are four points, the collection of which we will call L, in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber in S_2 is infinite.*

Proof. Thinking of f as a polynomial in the variable u with coefficients in $\mathbb{C}[x, y, z, w]$, we see that the points in $\mathbb{P}^1 \times \mathbb{P}^1$ which are simultaneously zeros of these coefficient polynomials are precisely the points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber is infinite. The points of L are precisely the punctures of the two punctured spheres which are not in the image of $\phi|_U$. The preimage of L in S_2 is the union of four $\mathbb{P}^1$'s each intersecting exactly one other $\mathbb{P}^1$ in one point. These two points of intersection are the points on the $\mathbb{P}^1$'s where the coordinate u goes to infinity which is equivalent to the points where the x and y coordinates go to zero. Thus these intersection points are precisely the points in P. The points in L along with their infinite fibers in $\mathbb{P}^1 \times \mathbb{P}^1$ are listed below.

$$\begin{array}{lll} [1, \frac{\sqrt{2}}{2}(1+i), \sqrt{2}, 1] & \text{has fiber} & \{[1, \frac{\sqrt{2}}{2}(1+i), 1, \sqrt{2}, 1]\} \supset [0,0,1,\sqrt{2},1] \\ [1, \frac{\sqrt{2}}{2}(1-i), \sqrt{2}, 1] & \text{has fiber} & \{[1, \frac{\sqrt{2}}{2}(1-i), 1, \sqrt{2}, 1]\} \supset [0,0,1,\sqrt{2},1] \\ [1, \frac{\sqrt{2}}{2}(-1+i), -\sqrt{2}, 1] & \text{has fiber} & \{[1, \frac{\sqrt{2}}{2}(-1+i), -\sqrt{2}, 1]\} \supset [0,0,1,-\sqrt{2},1] \\ [1, \frac{\sqrt{2}}{2}(-1-i), -\sqrt{2}, 1] & \text{has fiber} & \{[1, \frac{\sqrt{2}}{2}(-1-i), -\sqrt{2}, 1]\} \supset [0,0,1,-\sqrt{2},1] \end{array}$$

□

In calculating the Euler characteristic we will use the fact that the preimage of L in S_2 are four $\mathbb{P}^1$'s which intersect in pairs at ideal points in the set $P \subset S_2$. In fact, each point in P appears as the intersection of two of these fibers and the image of P under ϕ is precisely L.

Lemma 3.2.9. *The branched set B in $\mathbb{P}^1 \times \mathbb{P}^1$ is the union of two $\mathbb{P}^1$'s which intersect in two points.*

Proof. The branched set $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the zero set of $w^2x^2 + w^2y^2 - wxyz$. From the factorization $w(wx^2 + wy^2 - xyz)$, we see that B is the union of two $\mathbb{P}^1$'s of bidegree $(0,1)$ and $(2,1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Hence these two $\mathbb{P}^1$'s intersect each other in two points. □

Proposition 3.2.10. $\chi(S_2) = \chi(\overline{G}) - 2$

Proof. In $\mathbb{P}^2 \times \mathbb{P}^1$, taking the closure, $\overline{G}$, of the graph of ϕ is equivalent to blowing up S_2 at the points $p \in P$. Each blow up increases the Euler characteristic by one. Since $|P| = 2$, we have $\chi(\overline{G}) = \chi(S_2) + 2$. We can see this by looking at the coordinates of $\overline{G}$ as well. Let f denote the polynomial which cuts out S_2 in $\mathbb{P}^2 \times \mathbb{P}^1$. The graph $G = \{[x, y, u, z, w : a, b, c, d] | f = 0, a = x, b = y, z = c, w = d\} \subset (\mathbb{P}^2 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1)$. The closure of G is cut out by the smallest prime ideal whose solutions contains G and thus $\overline{G} = \{[x, y, u, z, w : a, b, c, d] | f = 0, ay = bx, cw = dz\}$. For every point $s \in U = S_2 - P$, there is exactly one point $\overline{s} \in \overline{G}$. The Euler characteristic $\chi(\overline{G}) = \chi(S_2 - P) + \chi(\rho_1^{-1}(P)) = \chi(S_2) - \chi(P) + \chi(\rho^{-1}(P))$ where $\rho_1 : \overline{G} \to S_2$ is the projection map. Now, for each of the two points $p_i \in P$, there is a copy of $\mathbb{P}^1$ in $\overline{G}$. Assuming these two copies of $\mathbb{P}^1$ are disjoint, $\chi(\overline{G}) = \chi(S_2) - 2\chi(point) + 2\chi(\mathbb{P}^1) = \chi(S_2) + 2$. Looking at the coordinates (listed below) for $p_i \in P$ and their corresponding images, $\overline{p_i}$ in $\overline{G}$ we see that these exceptional divisors are indeed disjoint .

$$
\begin{aligned}
for \quad p_1 &= (0, 0, 1, \sqrt{2}, 1) \qquad & \overline{p_1} &= \{[a, b : \sqrt{2}, 1]\} \\
for \quad p_2 &= (0, 0, 1, -\sqrt{2}, 1) \qquad & \overline{P_2} &= \{[a, b, -\sqrt{2}, 1]\}
\end{aligned}
$$

□

Proposition 3.2.11. $\chi(\overline{G}) = 10$

Proof. In order to calculate the Euler characteristic of $\overline{G}$ we appeal to the projection map $\rho_2 : \overline{G} \to \mathbb{P}^1 \times \mathbb{P}^1$. On $\Gamma = \overline{G} - \{\rho_1^{-1}(P)\}$, ρ_1 is injective and

ϕ is generically 2:1 so $\rho_2|_\Gamma = \phi \circ \rho_1|_\Gamma$ is generically 2:1. The image of Γ under ρ_2 is $\mathbb{P}^1 \times \mathbb{P}^1 - Q$. In the branched set B, there are four points, L, whose preimage under ϕ is the union of two pairs of two $\mathbb{P}^1$'s which intersect in a point. In $U = S_2 - P$, the preimage of L under ϕ is four disjoint punctured $\mathbb{P}^1$'s, which we will denote L_U. Now, the closure of pullback of L_U in the graph G is precisely the preimage of L in $\overline{G}$ under ρ_2. Since this closure of these four punctured $\mathbb{P}^1$'s in G is the collection of four disjoint $\mathbb{P}^1$'s, the preimage of L in $\overline{G}$ is four disjoint $\mathbb{P}^1$'s.

The Euler characteristic of $\overline{G}$ is then

$$\begin{aligned} \chi(\overline{G}) &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 - B - Q) + \chi(Q + B - L) + \chi(\rho_2^{-1}(L)) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(Q) - \chi(B) - \chi(L) + \chi(\rho_2^{-1}(L)) \end{aligned}$$

We know that $2\chi(\imath^1 \times \mathbb{P}^1) = 8$. Since B is the union of two $\mathbb{P}^1$'s which intersect in two points, $\chi(B) = 2\chi(\mathbb{P}^1) - 2 = 2$. With L a set of four points, $\chi(L) = 2$. Now, Q is the union of two twice punctured spheres so $\chi(Q) = 2(\chi(\mathbb{P}^1) - 2) = 0$. Since $\rho_2^{-1}(L)$ is the union of four disjoint $\mathbb{P}^1$'s, $\chi(\rho_2^{-1}(L)) = 4\chi(\mathbb{P}^1) = 8$. All together this gives $\chi(\overline{G}) = 10$.

$\square$

We can now complete the proof of Proposition 3.2.5.

Proof. (Proposition 3.2.5) We have that $\chi(\overline{G}) = 10$ from Proposition 3.2.11 and $\chi(S_2) = \chi(\overline{G}) - 2$ from Proposition 3.2.10. Hence $\chi(S_2) = 8$. $\square$

Corollary 3.2.12. *The Euler characteristic of* $\tilde{S}_2$ *is* $\chi(\tilde{S}_2) = 10$

Proof. From Lemma and Proposition 3.2.5 we have $\chi(\tilde{S}_2) = \chi(S_2) + 2 = 8 + 2 = 10$. □

We are now ready to prove Theorem 3.2.2.

Proof. (Theorem 3.2.2) Since $\tilde{S}_2$ is both rational and ruled we know that $\tilde{S}_2$ is topologically equivalent to $\mathbb{P}^2$ blown-up at some n points. This together with the fact that each blow up of $\mathbb{P}^2$ increases the Euler Characteristic by one gives $\chi(\tilde{S}_2) = \chi(\mathbb{P}^2) + n$. With $\chi(\tilde{S}_2) = 10$, n must be 7 and $\tilde{S}_2$ must be $\mathbb{P}^2$ blown-up at 7 points. □

3.3 Painting the Bigger Picture

The canonical components of character varieties associated to the two families of link complements obtained by $1/n$ Dehn filling on the Borromean rings and on the Magic manifold all exhibit rational components. Rational surfaces are well understood algebro-geometric objects. The isomorphism classes are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at n points. For $n \leq 8$, the surfaces $\mathbb{P}^2$ blown up at n points are nice in the sense they exhibit only finitely many (-1) curves that is curves with self-intersection number -1. The rational components associated to the two families of link complements obtained by $1/n$ Dehn filling on the Borromean rings and on the Magic manifold are all $\mathbb{P}^1$ bundles over $\mathbb{P}^1$ which exhibit the uncommon feature of a double line fiber (refer to the end of Section 3.1.1 for a discussion of this). In the case of the Borromean rings, the rational components do not coincide with the canonical components. However, being isomorphic to $\mathbb{P}^2$ blown up at 7 points, they do have only finitely many (i.e. 56) (-1) curves.

In the case of the Magic manifold the rational components coincide with the canonical component. As these surfaces are birational to conic bundles, they can be realized as smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$, cut out by a polynomial of bidegree $(2, n)$. The twist knot character varieties are all hyperelliptic, meaning they can be realized as smooth hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by a polynomial of bidegree $(2, n)$ ([17]). These surfaces birational to conic bundles are subvarieties of the canonical component of the character variety associated to the Magic manifold, which yields a 3-dimensional analogue.

Namely, the canonical component for the Magic manifold is not only rational but also birational to a fiber bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with conic fibers (see Section 3.3.1)

3.3.1 The character variety for the Magic manifold

In this section we briefly describe the canonical component of the character variety for the Magic manifold M_m, which by Theorem 1.2.1 is a 3-dimensional complex variety. The fundamental group for M_m has presentation $\langle a, b, c | ac^{-1} = c^{-1}a, ab^2cb = bcb^2a \rangle$ ([5]). In these coordinates, the meridians and longitudes of the three cusps are

$$\begin{aligned} m_0 &= a & l_0 &= a^2c^{-1} \\ m_1 &= c^{-1}ab^{-1} & l_1 &= ab^2ab^{-1} \\ m_2 &= ab & l_2 &= abab^{-1}cb^{-1} \end{aligned}$$

First we establish the defining ideal for the representation variety $R(M_m)$. Any representation $\rho \in R(M_m$ can be conjugated so that

$$\bar{a} = \rho(a) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \qquad \bar{b} = \rho(b) = \begin{pmatrix} p & 0 \\ q & p^{-1} \end{pmatrix}$$

Since a and c^{-1} commute, they have the same fixed points. Hence ρ maps c^{-1} to

$$\bar{c^{-1}} = \rho(c^{-1}) = \begin{pmatrix} s & \frac{s-s^{-1}}{m-m^{-1}} \\ 0 & s^{-1} \end{pmatrix}$$

In these coordinates, the polynomials which define $R(M_m)$ come from the relation $\bar{a}\bar{b}^2\bar{c}\bar{b} - \bar{b}\bar{c}\bar{b}^2\bar{a} = 0$. For $i, j \in \{1, 2\}$, let p_{ij} denote the coordinate polynomials of $\rho(ab^2cb - bcb^2a)$. Then $R(M_m)$ is cut out by the ideal $\langle p_{ij}\rangle \in \mathbb{C}[m, m^{-1}, s, s^{-1}, p, p^{-1}, q]$ where

$$\begin{aligned}
p_{11} &= \frac{q(-p^2-p^4+m^2p^4+mpq+mp^3q-s^2+m^2s^2+m^2p^2s^2-mpqs^2-mp^3qs^2)}{(-1+m)(1+m)p^2s} \\
p_{12} &= \frac{(-1+p)(1+p)(-p^2-p^4+m^2p^4+mpq+mp^3q-s^2+m^2s^2+m^2p^2s^2-mpqs^2-mp^3qs^2)}{(-1+m)(1+m)p^3s} \\
p_{21} &= \frac{q(-p^2-p^4+m^2p^4+mpq+mp^3q-s^2+m^2s^2+m^2p^2s^2-mpqs^2-mp^3qs^2)}{mp^2s} \\
p_{22} &= \frac{q(-p^2-p^4+m^2p^4+mpq+mp^3q-s^2+m^2s^2+m^2p^2s^2-mpqs^2-mp^3qs^2)}{(-1+m)(1+m)p^2s}
\end{aligned}$$

None of the p_{ij} are irreducible. In fact their GCD is nontrivial. Let $p_m = GDC(p_{ij})$ that is

$$p_m = \frac{-p^2-p^4+m^2p^4+mpq+mp^3q-s^2+m^2s^2+m^2p^2s^2-mpqs^2-mp^3qs^2}{(-1+m)m(1+m)p^3s}$$

Setting $g_{ij} = \frac{p_{ij}}{p_m}$, we can view the representation variety as $Z(\langle g_{ij}p_m\rangle) = Z(\langle g_{ij}\rangle) \cup Z(\langle p_m\rangle)$. Since $g_{1,1} = -g_{2,2} = mpq$, $g_{1,2} = -m(-1+p)(1+p)$, and $g_{2,1} = -(-1+m)(1+m)pq$, the ideal $\langle g_{ij}\rangle$ defines an affine variety in $\mathbb{C}^4$ which is the union of four $\mathbb{P}^1$'s and two $\mathbb{P}^2$'s. This affine subvariety of $R(M_m)$ consists of reducible representations. We are interested in the components of $R(M_m)$ which contain discrete faithful representations all of which lie in the subvariety $R_m = Z(\{p_m\})$. Hence the canonical component $X_0(M_m)$ is in the image of R_m under the map t.

As discussed in Section , we can express the map t in terms of generators of the coordinate ring $T_m = T_{M_m}$ for $X(M_m)$. We know from [6] that trace maps $\{\tau_a, \tau_b, \tau_c, \tau_{ab}, \tau_{ac}, \tau_{bc}, \tau_{abc}\}$ generate the coordinate ring T_m ([6]). In this case, however, the coordinate ring is generated by a smaller subset of trace maps. Namely, $\{\tau_a, \tau_b, \tau_{ac}, \tau_{bc}\}$ is a generating set. With these generators the map $t = (\tau_a, \tau_b, \tau_c, \tau_b c) : R_m \to \mathbb{C}^4$ is

$$t(\rho) = (m + m^{-1}, p + p^{-1}, ms^{-1} + sm^{-1}, sp^{-1} + ps^{-1} + \frac{q(s - s^{-1})}{m - m^{-1}}).$$

Let X_m denote the image of R_m under t.

We determine the defining polynomial(s) for X_m by appealing to the induced injective map $t^* : \mathbb{C}[X_m] \to \mathbb{C}[R_m]$ on the coordinates rings of X_m and R_m. The algebraic set R_m is defined by the polynomial ideal $\langle p_m \rangle$ and so its coordinate ring is $\mathbb{C}[R_m] = \mathbb{C}[m, m^{-1}, s, s^{-1}, p, p^{-1}, q]/\langle p_m \rangle$. The coordinate ring $\mathbb{C}[X_m]$ is the image of $\mathbb{C}[R_m]$ under t^*, that is $\mathbb{C}[X_m]$ is
$\mathbb{C}[m, m^{-1}, s, s^{-1}, p, p^{-1}, q]/$
$\langle p_m, x = m + m^{-1}, y = p + p^{-1}, v = ms^{-1} + sm^{-1}, w = sp^{-1} + ps^{-1} + \frac{q(s-s^{-1})}{m-m^{-1}} \rangle$
which is isomorphic to $\mathbb{C}[x, y, z, w]/\langle \tilde{f_m} \rangle$ where

$$\tilde{f_m} = \tilde{f}_{m_o} \tilde{f}_{m_1} = (4 - v^2 - 4w^2y^2 + w^2x^2y^2)(v^2 - x^2).$$

Hence the image X_m has two components, $Z(\tilde{f}_{m_0})$ and $Z(\tilde{f}_{m_1})$. By specializing to the case where the meridian a is parabolic (and $x = 2$), we determine $Z(\tilde{f}_{m_0})$ as the affine canonical component for the character variety of the Magic manifold. Since $\tilde{f}_{m_o}$ is nonsingular, X_{0_m} is a smooth affine model.

To determine the birational equivalence class of X_{0_m} we consider the projective model obtained by compactifying in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. With v, w, r the $\mathbb{P}^2$ coordinates, x, h one set of $\mathbb{P}^1$ coordinates, and y, k the second set of

$\mathbb{P}^1$ coordinates, this compactification S_m for the canonical component X_{0_m} is defined by $f_{m_0} = 4r^2h^2k^2 - v^2h^2k^2 - 4w^2y^2h^2 + w^2x^2y^2$. That S_m is birational to a fiber bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with conic fibers follows easily form the tridegree, $(2, 2, 2)$ of f_{m_0}. Consider the projection map $S_m \to \mathbb{P}^1 \times \mathbb{P}^1$. The fiber over a generic point $[x_0, h_0 : y_0, k_0] \in \mathbb{P}^1 \times \mathbb{P}^1$ is the curve cut out by $f_{m_0}(v, w, r, x_0, h_0 : y_0, k_0)$ which is a conic f_{m_0} has degree 2 over $\mathbb{P}^2$. Locally S_m is isomorphic to a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore birational to such. Since S_m is birational to conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, it is birational to $\mathbb{P}^3$ and hence is a rational variety.

Chapter 4

Future Research

We plan to continue to extend our understanding of character varieties by developing examples and tools to study character varieties of link complements since most of the previous work is particular to one dimension. Complex surfaces, while still tractable algebro-geometric objects, are more complicated than complex curves. Within a birational equivalence class, different compactifications yield different smooth models with different points at infinity.

Question 4.1. *Which compactification do we take as the canonical component of a character variety associated to a hyperbolic 3-manifold with* $n > 1$ *cusps?*

In their work, Macasieb, Petersen and van Luijk noticed a potential relationship between the number of components of the character variety and the symmetry group of the corresponding link complement. For knots $J(k, l)$ with $l \neq k$ the symmetry group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the character variety exhibits one component. When $l = k$, the knot complements exhibit an extra rotational symmetry and the character varieties exhibit two components. We studied the character varieties for two component 2-bridge links obtained by $1/n$ Dehn surgery on one cusp of both the Magic manifold and the Borromean rings. Aside from the Whitehead link, those with character varieties containing exactly one component all have a dihedral symmetry group of order 4 while those with character varieties containing multiple components all have a dihedral symmetry group of order 8.

Question 4.2. *For a hyperbolic link complement, is a symmetry group of order greater than* 4 *a necessary condition for the associated character variety to have more than one component?*

In [17] the authors showed that for hyperbolic 2-bridge knots $J(2,l)$ with $l \neq -1, 0, 1, 2$, the genus of the canonical component is equal to the $[F(J(2,l)) : \mathbb{Q}] - 1$ where $F(J(2,l))$ is the invariant trace field. Upon realizing the conical components as smooth hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1$, both the genus and the degree of the invariant trace field can be determined from the defining polynomial ([24], [27]). Similarly, the canonical components of the link complements obtained by $1/n$ Dehn filling on one cusp of the Borromean rings can all be realized as singular hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ and the bidegree of the defining polynomial appears to coincide with the degree of the invariant trace field ([16], [5]). Drawing connections between the geometric genus is much more difficult in this case since the defining polynomials are not smooth. Hence we ask the following.

Question 4.3. *For hyperbolic link complements, how is the degree of the invariant trace field related to the canonical component of the character variety?*

Complex curves of every geometric genus p_g are realized as character varieties of hyperbolic knot complements ([17]). Unlike the 1-dimensional case where the geometric genus is the topological genus, the geometric genus for complex surfaces does not identify the isomorphism class.

Question 4.4. *What isomorphism classes of which geometric genera arise as canonical components of character varieties associated to hyperbolic 3-manifolds.*

Question 4.5. *What are the obstructions for a complex surface to be a component of a character variety of a hyperbolic 3-manifold?*

Many of our examples of character varieties of hyperbolic link complements have components which are rational surfaces. These varieties have $p_g = 0$ and a defining algebro-geometric feature is (-1) curves. Rational surfaces isomorphic to $\mathbb{P}^2$ blown-up at $n < 8$ points exhibit finitely many (-1) curves. The rational components associated to $1/n$ Dehn filling on the Borromean rings are $\mathbb{P}^2$ blown-up at 7 points which has exactly 56 (-1) curves. When these surfaces can be viewed as fiber bundles over $\mathbb{P}^1$, a few of the (-1) curves appear as degenerate fibers. We hope to identify all the (-1) curves and determine what topological information they reflect.

The results of Theorem 1.1.1 and the conjecture extending Theorem 1.1.2 provide a partial answer to the question of which rational surfaces arise as canonical components of character varieties of hyperbolic link complements. For $n \geq 1$ rational surfaces isomorphic to $\mathbb{P}^2$ blown-up at $1 + 9n$ points are realized as canonical components of character varieties of hyperbolic 2-bridge links. For examples of canonical components with $p_g > 0$ we look to the character varieties of manifolds which result from $1/n$ Dehn filling on one cusp of the Borromean rings, M_{br}. These are hyperbolic two component 2-bridge links with Schubert normal form $S(8n, 4n+1)$ ([15]). Our explicit Mathematica calculations for $M_{br}(1/n)$ for $n = 1, \ldots, 4$ support the following conjecture.

Conjecture 4.0.1. *The canonical component for the hyperbolic link complements obtained by* $1/n$ *Dehn filling on one cusp of the Borromean rings is a hypersurface in* $\mathbb{P}^2 \times \mathbb{P}^1$ *defined by a singular polynomial of bidegree* $(2n, 2n+1)$

Doing Dehn filling amounts to adding a relation to the fundamental group of the original manifold. It seems reasonable to expect to be able to realize the Dehn filled character variety as a subvariety; however even determining how the twist knot character varieties sit inside the Whitehead link character variety is still an open problem because finding the common zero set is nontrivial. In our work, the rational surfaces we see as components of character varieties associated to the hyperbolic links obtained by $1/n$ Dehn filling are all birational to conic bundles which exhibit a double line i.e. a fiber in which every point is singular. All conics can be parameterized by $\mathbb{P}^5$ and those which are double lines correspond to a codimension 3 subvariety ([12]). Hence a conic bundle with a double line fiber corresponds to a line which passes through a particular codimension 3 subvariety in $\mathbb{P}^5$ which is a rare occurrence. That all of these conic bundle components exhibit a double line lends insight into the affect Dehn surgery has on the character variety.

We have studied character varieties only for hyperbolic 3-manifolds with torus boundary. We would like to extend our work to hyperbolic manifolds M exhibiting boundary components of higher genus. The deformation space of hyperbolic structures on M can be viewed as a set of discrete and faithful representation. Although, geometers have studied the subset of these varieties consisting of representations that are discrete and faithful, little is currently known about how this subset relates to the algebraic structure of the variety. We hope to use the dynamics of the action of $Out(\pi_1(M))$ to gain insight into the algebro-geometric structure of the character variety and as well as deformation space of hyperbolic structures on M. In particular we hope to address the open problem of determining when the all the connected components of the deformation space of M lie in the same component of the character variety.

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Vita

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