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Buring, R.; Kiselev, A. V.

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PHYSICS OF ELEMENTARY PARTICLES
AND ATOMIC NUCLEI. THEORY

Universal Cocycles and the Graph Complex Action on Homogeneous Poisson Brackets by Diffeomorphisms

R. Buring^{a, 1, *} and A. V. Kiselev^{b, 2, 3, **}

^aMathematical Institute, Johannes Gutenberg University of Mainz, Mainz, D-55128 Germany

^bBernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen,
AK Groningen, P.O. Box 407, 9700AK, The Netherlands

*e-mail: rburing@uni-mainz.de

**e-mail: A.V.Kiselev@rug.nl

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Abstract—The graph complex acts on the spaces of Poisson bi-vectors \mathcal{P} by infinitesimal symmetries. We prove that whenever a Poisson structure is homogeneous, i.e. $\mathcal{P} = L_{\vec{V}}(\mathcal{P})$ w.r.t. the Lie derivative along some vector field \vec{V} , but not quadratic (the coefficients of \mathcal{P} are not degree-two homogeneous polynomials), and whenever its velocity bi-vector $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$, also homogeneous w.r.t. \vec{V} by $L_{\vec{V}}(\mathcal{Q}) = n\mathcal{Q}$ whenever $\mathcal{Q}(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}^{\otimes n})$ is obtained using the orientation morphism Or from a graph cocycle γ on n vertices and $2n - 2$ edges, then the 1-vector $\vec{\mathcal{X}} = \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$ is a Poisson cocycle. Its construction is uniform for all Poisson bi-vectors \mathcal{P} satisfying the above assumptions, on all finite-dimensional affine manifolds M . Still, if the bi-vector $\mathcal{Q} \neq 0$ is exact in the respective Poisson cohomology, so there exists a vector field $\vec{\mathcal{Y}}$ such that $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$, then the universal cocycle $\vec{\mathcal{X}}$ does not belong to the coset of $\vec{\mathcal{Y}} \bmod \ker \llbracket \mathcal{P}, \cdot \rrbracket$. We illustrate the construction using two examples of cubic-coefficient Poisson brackets associated with the R -matrices for the Lie algebra $\mathfrak{gl}(2)$.

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1. INTRODUCTION

Bi-vector cocycles $\mathcal{Q}(\mathcal{P}) = \text{Or}(\gamma)(\mathcal{P}^{\otimes n}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ are obtained by Kontsevich's graph orientation morphism Or from graph cocycles γ on n vertices and $2n - 2$ edges in a way which is uniform for all finite-dimensional affine Poisson manifolds (M^r, \mathcal{P}) . The (non)triviality of cocycles $\mathcal{Q}(\mathcal{P})$ in the second Poisson cohomology w.r.t. the differential $\partial_{\mathcal{P}} = \llbracket \mathcal{P}, \cdot \rrbracket$ remains an open problem, twenty-five years after the discovery of the graph complex and orientation morphism (see [11]). In all the Poisson geometries probed so far, the known infinitesimal symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ of the Jacobi identity $\frac{1}{2} \llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$ are $\partial_{\mathcal{P}}$ -exact: there always

exists a vector field $\vec{\mathcal{Y}}$ such that $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$. The evolution $\mathcal{P}(\varepsilon = 0) \mapsto \mathcal{P}(\varepsilon > 0)$ of the tensor \mathcal{P} then amounts to its reparametrisations under the diffeomorphisms of Poisson manifold which are induced by the shifts along the integral trajectories of the vector field $\vec{\mathcal{Y}}$. This is why, instead of producing new Poisson brackets from a given one, the Kontsevich graph flows on the spaces of Poisson bi-vectors induce (non)linear diffeomorphisms of the base manifold M , although no more than its affine structure was the initial assumption and no possibility of smooth coordinate reparametrizations was presumed.

For a much used class of (scaling-)homogeneous Poisson bi-vectors $\mathcal{P} = L_{\vec{V}}(\mathcal{P})$, we obtain an explicit formula, $\vec{\mathcal{X}} = \text{Or}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$, of a 1-vector cocycle $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ which is built from the graph cocycles γ uniformly for all homogeneous Poisson bi-vectors \mathcal{P} on affine manifolds $M^{r < \infty}$. The cocycle $\vec{\mathcal{X}}$ is however not necessarily a 1-vector representative of the coset $\vec{\mathcal{Y}} \bmod \{\mathcal{L} \in \ker \llbracket \mathcal{P}, \cdot \rrbracket\}$ which would trivi-

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³ *Current address:* Institut des Hautes Études Scientifiques (IHÉS), Le Bois-Marie, 35 route de Chartres, Bures-sur-Yvette, 91440 France.

alise the value $\mathcal{Q}(\mathcal{P}) = \llbracket \bar{\mathcal{Y}}, \mathcal{P} \rrbracket$ of Kontsevich’s symmetries at homogeneous Poisson structures. Indeed, the Poisson cocycle $\mathcal{Q}(\mathcal{P})$ can be, we show, a nonzero bivector on M^r , whereas the bi-vector $\llbracket \bar{\mathcal{X}}, \mathcal{P} \rrbracket$ is identically zero on M^r by construction. We contrast the formulas of universal cocycles $\bar{\mathcal{X}}(\gamma, \bar{\mathcal{V}}, \mathcal{P})$ and trivialising vector fields $\bar{\mathcal{Y}}$ for nonzero symmetries $\dot{\mathcal{P}} = \text{O}\bar{\tau}(\gamma)(\mathcal{P})$ by two examples, namely, using cubic-coefficient Poisson brackets associated with the R -matrices for $\mathfrak{gl}(2)$.

This paper is organized as follows. In §1 we recall elements of Poisson cohomology theory in the context of Kontsevich’s universal deformations of bi-vectors by using the unoriented graph cocycles. In §2 we phrase the notion of structures which are homogeneous w.r.t. a 1-vector field, and we prove the main theorem. Finally, we illustrate the result (cf. [10]).

1. POISSON COHOMOLOGY AND THE GRAPH COMPLEX

A Poisson bracket $\{, \}_{\mathcal{P}}$ on a real manifold M is a bi-linear skew-symmetric bi-derivation which takes $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ and satisfies the Jacobi identity $\frac{1}{2} \sum_{\sigma \in S_3} \{ \{ \sigma(f), \sigma(g) \}_{\mathcal{P}}, \sigma(h) \}_{\mathcal{P}} = 0$ for any $f, g, h \in C^\infty(M)$. The fact that both the arguments f, g and their bracket $\{f, g\}_{\mathcal{P}}$ are scalars dictates the tensor transformation law of the components \mathcal{P}^{ij} of a bi-vector $\mathcal{P} = \sum_{i,j} \mathcal{P}^{ij}(\mathbf{x}) \partial_i \otimes \partial_j = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij}(\mathbf{x}) \times (\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij} \partial_i \wedge \partial_j$ whenever the structure is referred to a system of coordinates $\mathbf{x} = (x^1, \dots, x^r)$ and $\partial_i = \partial/\partial x^i$ is a shorthand notation.

The calculus on the space of multivectors $\Gamma(\wedge^* TM) \cong C^\infty(\Pi T^*M)$ is simplified if one uses the parity-odd coordinates ξ_i along the directions dx^i in the fibres of the cotangent bundle T^*M over points $\mathbf{a} \in M$ (which are parametrized by x^i). The symbol ξ_i thus corresponds to $\partial/\partial x^i$ dual to dx^i , and bi-vectors are $\mathcal{P} = \frac{1}{2} \sum_{i,j} \mathcal{P}^{ij} \xi_i \xi_j$, so that $\{f, g\}_{\mathcal{P}}(\mathbf{a}) = (f) \bar{\partial}/\partial x^i \bar{\partial}/\partial \xi_\mu (\mathcal{P}) \bar{\partial}/\partial \xi_\nu \bar{\partial}/\partial x^\nu (g)$; here, both the coefficients \mathcal{P}^{ij} and derivatives $\partial/\partial x^k$ are evaluated at the point $\mathbf{a} \in M$ as in the left-hand side⁴.

⁴ The dot \cdot denotes the coupling of iterated variations of the objects f, \mathcal{P} , and g with respect to the canonically conjugate variables x^i and ξ_j , see [9] and references therein.

The space of multivectors is endowed with the parity-odd Poisson bracket $\llbracket \cdot, \cdot \rrbracket$ (the Schouten bracket, or antibracket) of own degree -1 . For arbitrary multivectors \mathcal{P}, \mathcal{Q} , the formula is $\llbracket \mathcal{P}, \mathcal{Q} \rrbracket = (\mathcal{P}) \bar{\partial}/\partial \xi_i \bar{\partial}/\partial x^i (\mathcal{Q}) - (\mathcal{Q}) \bar{\partial}/\partial x^i \bar{\partial}/\partial \xi_i (\mathcal{P})$; in particular, $\llbracket \bar{\mathcal{X}}, \bar{\mathcal{Y}} \rrbracket = [\bar{\mathcal{X}}, \bar{\mathcal{Y}}]$ is the usual commutator of vector fields $\bar{\mathcal{X}}, \bar{\mathcal{Y}}$ on M . The Schouten bracket $\llbracket \cdot, \cdot \rrbracket$ is shifted-graded skew-symmetric: $\llbracket \mathcal{Q}, \mathcal{P} \rrbracket = -(-1)^{(|\mathcal{P}|-1)(|\mathcal{Q}|-1)} \llbracket \mathcal{P}, \mathcal{Q} \rrbracket$ for \mathcal{P} and \mathcal{Q} grading-homogeneous. This is why, unlike the tautology $\llbracket \bar{\mathcal{X}}, \bar{\mathcal{X}} \rrbracket \equiv 0$, the equation $\llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$ is a nontrivial restriction for bi-vectors \mathcal{P} , containing the tri-vector in the l.-h.s. of the Jacobi identity $\frac{1}{2} \llbracket \llbracket \mathcal{P}, \mathcal{P} \rrbracket, \mathcal{P} \rrbracket(f, g, h) = 0$ for the bracket $\{f, g\}_{\mathcal{P}} = \llbracket \llbracket f, \mathcal{P} \rrbracket, g \rrbracket$. The Schouten bracket itself satisfies the graded Jacobi identity $\llbracket \llbracket \mathcal{P}, \llbracket \mathcal{Q}, \mathcal{R} \rrbracket \rrbracket - (-1)^{(|\mathcal{P}|-1)(|\mathcal{Q}|-1)} \llbracket \mathcal{Q}, \llbracket \mathcal{P}, \mathcal{R} \rrbracket \rrbracket = \llbracket \llbracket \mathcal{P}, \mathcal{Q} \rrbracket, \mathcal{R} \rrbracket$ with \mathcal{P} and \mathcal{Q} grading-homogeneous. This identity implies that for Poisson bi-vectors \mathcal{P} , their adjoint action by $\partial_{\mathcal{P}} = \llbracket \mathcal{P}, \cdot \rrbracket$ is a differential of degree $+1$ on the space of multivectors on M . The Poisson differential $\partial_{\mathcal{P}}$ gives rise to the Poisson cohomology $H^i_{\mathcal{P}}(M)$ of the manifold M (see [13])⁵.

If a bi-vector $\mathcal{Q} = \llbracket \bar{\mathcal{X}}, \mathcal{P} \rrbracket$ is a trivial Poisson cocycle, then it certainly is an infinitesimal symmetry of the Jacobi identity $\frac{1}{2} \llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$. But the infinitesimal change $\llbracket \bar{\mathcal{X}}, \mathcal{P} \rrbracket$ of the tensor \mathcal{P} then amounts to its reparametrisation under the infinitesimal change of coordinates $\mathbf{x}'(\mathbf{x}) \rightleftharpoons \mathbf{x}(\mathbf{x}')$ along the integral trajectories of the vector field $\bar{\mathcal{X}}$ on the manifold M . The following fact is true for all multivectors (regardless of the concept of Poisson cohomology).

Proposition 1. *Let $\mathbf{a} \in M$ be a point of an r -dimensional manifold and $\bar{\mathcal{X}} \in \Gamma(TM)$ be a vector field on it. For every $\varepsilon \in \mathcal{F} \subseteq \mathbb{R}$ such that there is the integral trajectory bringing $\mathbf{b}(-\varepsilon) := \exp(-\varepsilon \bar{\mathcal{X}})(\mathbf{a})$ to \mathbf{a} by the $(+\varepsilon)$ -*

⁵ The group $H^0_{\mathcal{P}}(M)$ spans the Casimirs, i.e. the functions which Poisson-commute with any $f \in C^\infty(M)$; the group $H^1_{\mathcal{P}}(M)$ consists of vector fields which preserve the Poisson structure but do not amount to the Hamiltonian vector fields $\bar{\mathcal{X}}_h = \llbracket \mathcal{P}, h \rrbracket$; the second group $H^2_{\mathcal{P}}(M) \ni \mathcal{Q}$ contains infinitesimal symmetries $\mathcal{P} \mapsto \mathcal{P} + \varepsilon \mathcal{Q} + \bar{o}(\varepsilon)$ of Poisson bi-vectors, whereas the next group $H^3_{\mathcal{P}}(M)$ stores the obstructions to formal integration $\mathcal{P} \mapsto \mathcal{P}(\varepsilon) = \mathcal{P} + \sum_{k \geq 1} \varepsilon^k \mathcal{Q}_{(k)}$ of infinitesimal symmetries $\mathcal{Q} = \mathcal{Q}_{(1)}$ to Poisson bi-vector formal power series satisfying $\llbracket \mathcal{P}(\varepsilon), \mathcal{P}(\varepsilon) \rrbracket = 0$.

shift, and for any choice of the r -tuple $\mathbf{x} = (x^1, \dots, x^r)$ of local coordinates in a chart U_α around $\mathbf{a} \in M$ (and for $|\varepsilon|$ small enough for the points $\mathbf{b}(-\varepsilon)$ to not yet run out of the chart U_α), introduce a new parametrization⁶ for the point \mathbf{a} by using the new r -tuple \mathbf{x}' . By definition, put $\mathbf{x}'(\mathbf{a}) := \mathbf{x}(\mathbf{b}(-\varepsilon))$. Let Ω be any multi-vector field near \mathbf{a} on M . Under the reparametrization $\mathbf{x}'(\mathbf{x})$, the speed at which the components of Ω at the point \mathbf{a} change in ε , as $\varepsilon \rightarrow 0$, equals $\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Omega(\mathbf{a}) = \llbracket \tilde{\mathcal{X}}, \Omega \rrbracket(\mathbf{a})$. In particular, a 1-vector field \bar{Y} near \mathbf{a} would change at \mathbf{a} as fast as its commutator with the vector field $\tilde{\mathcal{X}}$: $\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \bar{Y}(\mathbf{a}) = \llbracket \tilde{\mathcal{X}}, \bar{Y} \rrbracket(\mathbf{a})$.

The geography of the set of Poisson structures near a given bracket $\{\cdot, \cdot\}_\mathcal{P}$ on a given manifold M^r is, generally speaking, unknown. All the more it was a priori unclear whether Poisson bi-vectors \mathcal{P} , irrespective of the dimension $r \geq 3$, topology of M^r , etc., can be infinitesimally shifted by Poisson 2-cocycles $\mathcal{Q}(\mathcal{P})$, the construction of which would be universal for all \mathcal{P} . The discovery of the graph complex in 1993–1994 allowed Kontsevich to state (in [11]) the affirmative answer to the above question. Namely, the graph orientation morphism $\text{O}\bar{r}(\cdot)(\mathcal{P}): \ker d \ni \gamma \mapsto \mathcal{Q}(\mathcal{P}) \in \ker \partial_\mathcal{P}$ takes graph cocycles on n vertices and $2n - 2$ edges in each term (e.g., the tetrahedron, cf. [1, 3, 5, 6]) to Poisson cocycles whenever the bi-vector \mathcal{P} itself is Poisson. Willwacher [15] revealed that the generators of Drinfeld’s Grothendieck–Teichmüller Lie algebra grt are source of at least countably many such cocycles in the vertex-edge bi-grading $(n, 2n - 2)$; these cocycles are marked by the $(2\ell + 1)$ -wheel graphs (e.g., see [6, 7]). Brown proved in [2] that, under the Willwacher isomorphism $\text{grt} \cong H^0(G_{\text{RA}})$ these graph cocycles with wheels generate a free Lie subalgebra in grt , which means effectively that the iterated commutators of already known cocycles—under the bracket in the differential graded Lie algebra G_{RA} of graphs—would never vanish. The commutator of two cocycles is a cocycle by the Jacobi identity. All of them again being of the bi-grading $(n, 2n - 2)$, these graph cocycles determine countably many infinitesimal symmetries of a given Poisson bi-vector \mathcal{P} ; the construction is uniform for all the geometries (M^r, \mathcal{P}) .

Lemma 2. *For a given Poisson bi-vector \mathcal{P} , the graph orientation mapping $\text{O}\bar{r}(\cdot)(\mathcal{P}) \ker d \ni \gamma \mapsto$*

⁶ Actually, this is a way to construct new coordinates for all points of M near \mathbf{a} in U_α , i.e. not only those which lie on a piece of the integral trajectory of $\tilde{\mathcal{X}}$ passing through \mathbf{a} .

$\mathcal{Q}(\mathcal{P}) \in \ker \partial_\mathcal{P}$ is a Lie algebra morphism that takes the bracket of two cocycles in bi-grading $(n, 2n - 2)$ to the commutator $\left[\frac{d}{d\varepsilon_1}, \frac{d}{d\varepsilon_2} \right](\mathcal{P})$ of two symmetries $\frac{d}{d\varepsilon_i}(\mathcal{P}) = \mathcal{Q}_i(\mathcal{P})$ ⁷.

By construction, the components of universal symmetry bi-vectors $\mathcal{Q}(\mathcal{P})$ are differential polynomials w.r.t. the components \mathcal{P}^{ij} of the Poisson bi-vector \mathcal{P} that evolves. It can of course be that a graph flow $\dot{\mathcal{P}} = \text{O}\bar{r}(\gamma)(\mathcal{P})$ vanishes identically over the manifold M^r whenever \mathcal{Q} is evaluated at a particular class of Poisson structures \mathcal{P} ⁸. Nevertheless, there is no mechanism which would force a given Kontsevich’s graph flow to vanish at all Poisson structures on all manifolds of all dimensions⁹. Independently, it remains an open problem (cf. [10]) whether there is a Poisson manifold (M^r, \mathcal{P}) and a graph cocycle γ such that the Poisson cohomology class of $\mathcal{Q}(\mathcal{P}) := \text{O}\bar{r}(\gamma)(\mathcal{P})$ would be *nontrivial* in $H_\mathcal{P}^2(M)$. In other words, for all the shifts $\mathcal{Q} = \text{O}\bar{r}(\gamma)$ and all Poisson bi-vectors tried so far, the Poisson coboundary equation $\mathcal{Q}(\mathcal{P}) = \llbracket \tilde{\mathcal{X}}, \mathcal{P} \rrbracket$ did have vector field solutions $\tilde{\mathcal{X}}$ on the manifolds M .

Remark 1. Obtained from the graphs $\gamma \in \ker d$, the symmetries $\mathcal{Q}(\mathcal{P}) = \text{O}\bar{r}(\gamma)(\mathcal{P}) \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ are independent of a choice of local coordinates x^i (hence ξ_i) on a chart if, the Kontsevich construction requires, the manifold M^r is endowed with an *affine* structure: all

⁷ By Brown [2], the commutator does in general not vanish for Willwacher’s odd-sided wheel cocycles.

⁸ **Example.** So it is for the Kontsevich tetrahedral flow ([11] and [1]) evaluated at the Kirillov–Kostant linear Poisson brackets on the duals \mathfrak{g}^* of Lie algebras because in every term within the cocycle $\mathcal{Q}(\mathcal{P})$ under study, at least one copy is \mathcal{P} is differentiated at least twice with respect to the global coordinates on \mathfrak{g}^* .

⁹ **Example.** The Poisson bi-vectors $\mathcal{P} = da_1 \wedge \dots \wedge da_m / \text{dvol}(\mathbb{R}^{m+2})$ of Nambu type with arbitrary Casimirs $a_1, \dots, a_m \in C^\infty(\mathbb{R}^{m+2})$ and an arbitrary density in the volume element can have polynomial components $\mathcal{P}^{ij} \in \mathbb{R}[x^1, \dots, x^{m+2}]$ of degrees as high as need be w.r.t. the global Cartesian coordinates x^α on the vector space \mathbb{R}^{m+2} . The universal symmetries $\dot{\mathcal{P}} = \text{O}\bar{r}(\gamma)(\mathcal{P})$ obtained from Kontsevich’s graph cocycles deform the symplectic foliation (which is given in \mathbb{R}^{m+2} by the intersections of the level sets for the Casimirs a_1, \dots, a_m) in a regular way on an open dense subset of \mathbb{R}^{m+2} , so that the symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ preserve this Nambu class of Poisson brackets: the flows force the evolution of the Casimirs and the volume density. Its integrability is an open problem; by Lemma 2 and [2], the evolutions induced by different graph cocycles do not commute.

the coordinate transformations amount to $\mathbf{x}' = A\mathbf{x} + \bar{\mathbf{b}}$ with a constant (over the intersection of charts) Jacobian matrix A . The parity-odd fibre variables are transformed using the inverse Jacobian matrix,

$\xi_i = A_i^j \xi'_j$, making sense of the couplings $\bar{\partial}/\partial\xi_i, \bar{\partial}/\partial x^i$ which decorate the oriented edges of Kontsevich's graphs after the morphism Of works (see [3, 11]). The problem of Poisson cohomology class (non)triviality for the Kontsevich infinitesimal symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ thus acquires two diametrically opposite interpretations:

1 (as in [11]). The Poisson manifold $M^{r < \infty}$ is equipped with *both* the smooth and affine structures¹⁰. By definition, two Poisson bi-vectors are equivalent, $\mathcal{P}_1 \sim \mathcal{P}_2$, if they are related by a diffeomorphism of the manifold M : using its smooth structure, the diffeomorphism identifies points in two copies of M , then relating the Poisson tensors by local coordinate reparametrizations near the respective points. The affine structure on M is now used to run the Kontsevich flows in two initial value problems $\dot{\mathcal{P}}_i(\varepsilon) = \mathcal{Q}(\mathcal{P}_i(\varepsilon)), \mathcal{P}_i(\varepsilon = 0) = \mathcal{P}_i$. The Poisson triviality $\mathcal{P}(\mathcal{P}(\varepsilon)) = [\![\bar{\mathcal{X}}(\varepsilon), \mathcal{P}(\varepsilon)]\!]$ would relate either of bi-vectors $\mathcal{P}_i(\varepsilon)$ back to the Cauchy datum \mathcal{P}_i by diffeomorphisms (as long as $|\varepsilon|$ is small enough). Consequently, the Poisson bi-vectors $\mathcal{P}_1(\varepsilon) \sim \mathcal{P}_2(\varepsilon)$ do not run out of the old equivalence class. In conclusion, the goal is to produce essentially new Poisson brackets by using a nontrivial cocycle \mathcal{Q} , two given structures on the manifold M^r , and its diffeomorphism. No examples of nontrivial action, so that $\mathcal{P}_2(\varepsilon) \neq \mathcal{P}_i + \mathcal{P}_1(\varepsilon)$ at $\varepsilon > 0$, have ever been produced since 1996 (see [7, 11]).

2 (as in [10]). The Poisson manifold $M^{r < \infty}$ is equipped only with an affine structure. The countably many grt -related graph cocycles on n vertices and $2n - 2$ edges in every term (the tetrahedron, the pentagon-wheel cocycle, etc., see [6, 15]) generate a non-commutative Lie algebra of infinitesimal symmetries $\mathcal{Q}(\mathcal{P}) = \text{Of}(\gamma)(\mathcal{P})$ for a given Poisson structure \mathcal{P} . Consider the extreme case when *all* the cocycles $\mathcal{Q}(\mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ are exact in the cohomology group $H_{\mathcal{P}}^2(M)$ w.r.t. the Poisson differential $\partial_{\mathcal{P}}$. This assumption gives rise to the countable set of vector fields $\bar{\mathcal{Y}}(\gamma, \mathcal{P})$ on M such that $\mathcal{Q}(\mathcal{P}) = [\![\bar{\mathcal{Y}}, \mathcal{P}]\!]$. (Some

of these vector fields can be identically zero over M .) But if at least one such vector field is not constant w.r.t. the affine structure on M , then the shifts along its integral trajectories are nonlinear diffeomorphisms of M . The evolution of bi-vector \mathcal{P} is $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = [\![\bar{\mathcal{Y}}, \mathcal{P}]\!]$ or similarly, $\dot{\Omega} = [\![\bar{\mathcal{Y}}, \Omega]\!]$ for any multi-vector Ω on M (see Proposition 1); this evolution is now seen as multivectors' response to the diffeomorphism whose construction refers only to the simple, affine local portrait of M . Summarizing, the store of flows $\text{Of}(\gamma)(\mathcal{P})$ from the grt -related graph cocycles γ could be enough to approximate arbitrary smooth vector fields on M^r , that is, imitate its smooth structure. Whether this theoretical possibility is actually realised in relevant Poisson models is an open problem.

The Kontsevich symmetry construction is, therefore, either a generator of new Poisson brackets or the mechanism that provides diffeomorphisms of the underlying manifold.

2. HOMOGENEOUS POISSON STRUCTURES

By definition, a bi-vector \mathcal{P} on a manifold M is called *homogeneous* (of scale λ) with respect to a vector field $\bar{\mathcal{V}}$ on M if $[\![\bar{\mathcal{V}}, \mathcal{P}]\!] = \lambda \cdot \mathcal{P}$.

Example 1. Let $M = \mathbb{R}^r$ be a vector space (only linear reparametrizations $\mathbf{x}' = A\mathbf{x}$ are allowed, so that the polynomial degrees of monomials in the ring $\mathbb{R}[x^1, \dots, x^r]$ is well defined). Introduce the Euler vector field $\bar{\mathcal{E}} = \sum_{i=1}^r x^i \partial/\partial x^i$, and let all the components \mathcal{P}^{ij} of a bi-vector \mathcal{P} be homogeneous polynomials of degree d in the variables x^i . Then we have that $[\![\bar{\mathcal{E}}, \mathcal{P}]\!] = (d - 2) \cdot \mathcal{P}$, which means that \mathcal{P} is homogeneous of scale $d - 2$ w.r.t. the Euler vector field $\bar{\mathcal{E}}$. In particular, if $d \neq 2$ (i.e. if the coefficients of bi-vector \mathcal{P} are not quadratic), then we set $\bar{\mathcal{V}} = (d - 2)^{-1} \cdot \bar{\mathcal{E}}$ and from the equality $\mathcal{P} = [\![\bar{\mathcal{V}}, \mathcal{P}]\!]$ we obtain that the same bi-vector \mathcal{P} has homogeneity scale $\lambda = 1$ w.r.t. the multiple $\bar{\mathcal{V}}$ of the Euler vector field $\bar{\mathcal{E}}$ on \mathbb{R}^r .

Example 2. Under the same assumptions, suppose further that $\gamma = \sum_a c_a \gamma_a$ is a graph cocycle with n vertices and $2n - 2$ edges in every term γ_a (e.g., take the tetrahedron). Orient the ordered (by First $< \dots <$ Last) edges in every γ_a using the edge decoration operators $\bar{\Delta}_{ij} = \sum_{\mu=1}^r (\bar{\partial}/\partial \xi_{\mu}^{(i)} \otimes \bar{\partial}/\partial x_{(j)}^{\mu} + \bar{\partial}/\partial \xi_{\mu}^{(j)} \otimes \bar{\partial}/\partial x_{(i)}^{\mu})$. By placing a copy of bi-vector $\mathcal{P} = \frac{1}{2} \mathcal{P}^{kl}(\mathbf{x}) \xi_k \xi_l$ in each vertex $v^{(i)}$ of γ_a and taking the sum (over the graph

¹⁰On the circle \mathbb{S}^1 , the affine coordinate 'angle' is obvious whereas the smooth structure is used in the realm of Poincaré topology. A smooth atlas is always available for the spheres \mathbb{S}^r , but not for all $r \in \mathbb{N}$ would the r -dimensional sphere admit an affine structure.

index a) of products of the content of vertices in γ_a after all the edge operators $\bar{\Delta}_{ij}$ work, we obtain¹¹ the bi-vector $\mathcal{Q}(\mathcal{P}) := \text{O}\bar{\Gamma}(\gamma)(\mathcal{P})$. Then the coefficients of the bi-vector $\mathcal{Q}(\mathcal{P})$ are homogeneous polynomials of degree $n \cdot d - (2n - 2)$ with respect to x^1, \dots, x^r , so that $\llbracket \bar{\mathcal{E}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n(d - 2)\mathcal{Q}(\mathcal{P})$. In particular, if $d \neq 2$, then $\llbracket \bar{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n \cdot \mathcal{Q}(\mathcal{P})$, whereas quadratic-coefficient bi-vectors \mathcal{P} (with $d = 2$) are deformed within their subspace by the quadratic bi-vectors $\mathcal{Q}(\mathcal{P})$ which are obtained from the Kontsevich graph cocycles.

Lemma 3. *If a Poisson bi-vector $\mathcal{P} = \llbracket \bar{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket$ is homogeneous and $\mathcal{Q}(\mathcal{P}) = \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^{\otimes n})$ is built from a graph cocycle γ on n vertices, now containing a copy of \mathcal{P} in each vertex, then the bi-vector $\mathcal{Q}(\mathcal{P})$ is also homogeneous: $\llbracket \bar{\mathcal{V}}, \mathcal{Q}(\mathcal{P}) \rrbracket = n \cdot \mathcal{Q}(\mathcal{P})$, so that its scale is n^{12} .*

Remark 2 ([14, Rem. 4.9]). Consider a Nambu-type Poisson bi-vector $\mathcal{P} = da/dxdydz$ on \mathbb{R}^3 with Cartesian coordinates x, y, z ; here $a \in \mathbb{R}[x, y, z]$ is a weight-homogeneous polynomial with an isolated singularity at the origin¹³, so that $(w_{(x)} \cdot x\partial/\partial x + w_{(y)} \cdot y\partial/\partial y + w_{(z)} \cdot z\partial/\partial z)(a) = w_{(a)} \cdot a$. Then a vector field $\bar{\mathcal{V}}$ with polynomial components satisfying the first-order PDE $\mathcal{P} = \llbracket \bar{\mathcal{V}}, \mathcal{P} \rrbracket$ exists if and only if¹⁴ the weight degree $w_{(a)}$ of the polynomial a is not equal to the sum $w_{(x)} + w_{(y)} + w_{(z)}$ of weight degrees for the variables x, y, z ¹⁵.

¹¹We refer to the original paper [11] and to [3] for illustrations and discussion how the graph orientation morphism works in practice.

¹²The proof amounts to the Leibniz rule: let us inspect how fast the bi-vector $\mathcal{Q}(\mathcal{P})$, which by construction is a homogeneous differential polynomial of degree n in \mathcal{P} , evolves along the vector field $\bar{\mathcal{V}}$.

¹³The Milnor number is the dimension $\dim_{\mathbb{R}} \mathbb{R}[x, y, z]/(\partial a/\partial x, \partial a/\partial y, \partial a/\partial z)$ – here, $< \infty$ by assumption.

¹⁴This means that not all Nambu-type Poisson bi-vectors $\mathcal{P} = da/dxdydz$ are homogeneous w.r.t. a vector field $\bar{\mathcal{V}}$ with polynomial components; the PDE $\mathcal{P} = \llbracket \bar{\mathcal{V}}, \mathcal{P} \rrbracket$ with polynomial coefficients and unknown $\bar{\mathcal{V}}$ can in principle admit non-polynomial solutions.

¹⁵**Example.** If the weights of (x, y, z) are $(1, 1, 1)$ and $a = \frac{1}{3}(x^3 + y^3 + z^3)$ is cubic-homogeneous, then the components of Poisson bi-vector \mathcal{P} are quadratic and (by the above and also by [12]) not of the form $\mathcal{P} = \llbracket \bar{\mathcal{V}}, \mathcal{P} \rrbracket$ for any polynomial-coefficient vector field $\bar{\mathcal{V}}$. The non-existence of a solution $\bar{\mathcal{V}}$ with smooth non-polynomial coefficients is a separate problem.

Summarizing, the homogeneity assumption about bi-vectors \mathcal{P} is restrictive; it is not always satisfied in Poisson models.

Theorem 4. *Let (M, \mathcal{P}) be an affine finite-dimensional real Poisson manifold with $\mathcal{P} = \llbracket \bar{\mathcal{V}}, \mathcal{P} \rrbracket$ homogeneous. Let $\gamma = \sum_a c_a \cdot \gamma_a$ be a graph cocycle consisting of unoriented graphs γ_a over n vertices and $2n - 2$ edges (with a fixed ordering of edges in each γ_a). Then the 1-vector $\bar{\mathcal{X}}(\gamma, \bar{\mathcal{V}}, \mathcal{P}) = \text{O}\bar{\Gamma}(\gamma)(\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n-1})$, which is obtained by representing each edge $i - j$ with the operator $\bar{\Delta}_{ij}$ and by (graded-)symmetrizing over all the ways $\sigma \in \mathbb{S}_n$ to send the n -tuple $\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n-1}$ into the n vertices in each γ_a , is a Poisson cocycle: $\bar{\mathcal{X}} \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$ ¹⁶.*

The vector field $\bar{\mathcal{X}}$ is defined up to adding arbitrary Poisson 1-cocycles $\bar{\mathcal{L}} \in \ker \llbracket \mathcal{P}, \cdot \rrbracket$.

Proof. The expansion $0 = \text{O}\bar{\Gamma}(d\gamma)(\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n})$ for $\gamma \in \ker d$ goes along the lines of [11] and [3, 7, 8], but the $(n + 1)$ -tuple of multivectors now contains one 1-vector and only n copies of the Poisson bi-vector \mathcal{P} . By assumption, $d\gamma = 0 \in \mathbb{G}_{\text{RA}}$; recall that $\text{O}\bar{\Gamma}(0)$ (any multivectors) $= 0 \in \Gamma(\wedge^* TM)$. This zero l.-h.s. equates $0 = (\pi_S \circ \text{O}\bar{\Gamma}(\gamma) - (-)^{(-1)(-N)} \text{O}\bar{\Gamma}(\gamma) \circ \pi_S) \times (\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n})$ ¹⁷.

The appointment of graded (multi)vectors into the vertices of $d\gamma$ (hence, into the argument slots of the endomorphism $\text{O}\bar{\Gamma}(d\gamma)$) is achieved by the graded symmetrization using $((n + 1)!)^{-1} \text{O}\bar{\Gamma}(d\gamma)(\pm \sigma(\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n}))$. Fortunately, the field $\bar{\mathcal{V}}$ is the only parity-odd object, so its transpositions with the parity-even bi-vectors \mathcal{P} produce no sign factor: these \pm are all $+$. Likewise, the $n!$ permutations of n indistinguishable copies of \mathcal{P} leave only $n + 1$ from $(n + 1)!$ in the denominator; to get rid of it, let us multiply by $n + 1$ both sides of the equality $0 = \text{O}\bar{\Gamma}(d\gamma)(\bar{\mathcal{V}} \otimes \mathcal{P}^{\otimes n})$. The symmetrization thus amounts, by the linearity of $\text{O}\bar{\Gamma}(\gamma)$, to its evaluation at the

¹⁶**Open problem** (for \mathcal{P} homogeneous and Poisson). Is the universal 1-vector field $\bar{\mathcal{X}}(\gamma, \bar{\mathcal{V}}, \mathcal{P}) \in \ker \partial_{\mathcal{P}}$ Hamiltonian, i.e. $\bar{\mathcal{X}} = \llbracket \mathcal{P}, h \rrbracket$ for $h \in C^\infty(M)$ or at least, $\bar{\mathcal{X}} = \mathcal{P} \lrcorner \eta$ for a maybe not exact 1-form η on M ?

¹⁷Here, π_S is the graded-symmetric Schouten bracket (so $\pi_S(F, G) = (-)^{|F|-1} \llbracket F, G \rrbracket$), the graph insertion \circ into vertices is now the endomorphism insertion into argument slots, $|\pi_S| = -1$, and $N = 2n - 2$ is the even number of edges in γ , hence minus the even number of $\partial/\partial \xi_{\mu}$ in the edge operators $\bar{\Delta}_{ij}$ making $\text{O}\bar{\Gamma}(\gamma)$.

sum of arguments, $\vec{V} \cdot \mathcal{P}^n + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-1} + \dots + \mathcal{P}^n \cdot \vec{V}$, in which the ordering of (multi)vectors now matches an arbitrary fixed enumeration of the vertices.

The rest of the proof is standard¹⁸. There remains $0 = \text{O}\bar{\Gamma}(\gamma)(\pi_S(\vec{V}, \mathcal{P}) \cdot \mathcal{P}^{n-1}) + \mathcal{P} \cdot \pi_S(\vec{V}, \mathcal{P}) \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \pi_S(\vec{V}, \mathcal{P}) - (-)^N [\pi_S(\text{O}\bar{\Gamma}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P}) + \pi_S(\text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^n), \vec{V})]$. By the homogeneity assumption, $\pi_S(\vec{V}, \mathcal{P}) = (-)^{1-1} \llbracket \vec{V}, \mathcal{P} \rrbracket = \mathcal{P}$, and by construction, $\text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^n) = \mathcal{Q}(\mathcal{P})$, whence the minuend equals $n \cdot \mathcal{Q}(\mathcal{P})$. By Lemma 3, the graph flow is also homogeneous: $\llbracket \vec{V}, \mathcal{Q}(\mathcal{P}) \rrbracket = \lambda \cdot \mathcal{Q}(\mathcal{P})$ with the vertex count $\lambda = n$. We obtain the equality

$$\begin{aligned} & (-)^{2n-2} \cdot \llbracket \text{O}\bar{\Gamma}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \\ & \quad \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P} \rrbracket = n \cdot \mathcal{Q}(\mathcal{P}) - (-)^{2n-2} \lambda \cdot \mathcal{Q}(\mathcal{P}) \\ & \quad = (n - (-)^{\text{even}} n) \cdot \mathcal{Q}(\mathcal{P}) \equiv 0. \end{aligned}$$

We conclude that the 1-vector $\vec{\mathcal{X}} := \text{O}\bar{\Gamma}(\gamma)(\vec{V} \otimes \mathcal{Q}^{\otimes n-1})$ lies in $\ker \llbracket \mathcal{P}, \cdot \rrbracket^{\text{19}}$.

Example 3. Take the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ with its four-dimensional vector space structure; denote by x, y, z, v the Cartesian coordinates. Consider the R -matrix $\begin{pmatrix} x & y \\ z & v \end{pmatrix} \mapsto \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix}$ known from [12]; the standard construction then yields the Poisson bi-vector in the algebra of coordinate functions, $\mathcal{P} = (x^2y + y^2z)\partial_x \wedge \partial_y + (x^2z + yz^2)\partial_x \wedge \partial_z + (2xyz + 2yzv)\partial_x \wedge \partial_v + (y^2z + yv^2)\partial_y \wedge \partial_v + (yz^2 + zv^2)\partial_z \wedge \partial_v$. This bracket has cubic-nonlinear homogeneous polynomial coeffi-

¹⁸We have $0 = \text{O}\bar{\Gamma}(\gamma)(\pi_S(\vec{V}, \mathcal{P}), \mathcal{P}^{n-1}) + \text{O}\bar{\Gamma}(\gamma)(\pi_S(\mathcal{P}, \vec{V}), \mathcal{P}^{n-1}) + \text{O}\bar{\Gamma}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \vec{V}, \mathcal{P}^{n-2}) + \dots + \text{O}\bar{\Gamma}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}, \vec{V}) + \text{O}\bar{\Gamma}(\gamma)(\vec{V}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}) + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}, \pi_S(\vec{V}, \mathcal{P}), \mathcal{P}^{n-2}) + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \vec{V}), \mathcal{P}^{n-2}) + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \vec{V}, \mathcal{P}^{n-3}) + \dots + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-3}, \vec{V}) + \dots$ (the Schouten bracket π_S passes along the slots towards the end) $+ \text{O}\bar{\Gamma}(\gamma)(\vec{V}, \mathcal{P}^{n-2}, \pi_S(\mathcal{P}, \mathcal{P})) + \text{O}\bar{\Gamma}(\dots + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^{n-2}, \vec{V}, \pi_S(\mathcal{P}, \mathcal{P})) + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\vec{V}, \mathcal{P})) + \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\mathcal{P}, \vec{V})) - (-)^N \cdot [\pi_S(\text{O}\bar{\Gamma}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}), \mathcal{P}) + \pi_S(\text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^n), \vec{V}) + \pi_S(\vec{V}, \text{O}\bar{\Gamma}(\gamma)(\mathcal{P}^n)) + \pi_S(\mathcal{P}, \text{O}\bar{\Gamma}(\gamma)(\vec{V} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{V} \cdot \mathcal{P}^{n-2} + \dots + \mathcal{P}^{n-1} \cdot \vec{V}))]$. For \mathcal{P} Poisson, $\pi_S(\mathcal{P}, \mathcal{P}) = 0$, so we exclude all such terms [4]. The remaining graded-symmetric Schouten brackets π_S contain a bi-vector as one of the arguments, hence those can be swapped at no sign factor; all doubles, so let us divide by 2.

¹⁹**Exercise.** Extend the proof to the case $n=1$, $\gamma = \bullet$, $d\gamma = -\bullet - \bullet$ (so that the l.-h.s. was nonzero).

icients, hence $d = 3$. The vector field $\vec{V} = (d-2)^{-1} \cdot \vec{\mathcal{E}}$ is the (multiple of the) Euler vector field on \mathbb{R}^4 . As the graph cocycle γ , we take the tetrahedron (see [1, 11]); then the symmetry flow is $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = (-48x^5y - 288x^3y^2z - 240xy^3z^2 + 192y^3z^2v - 384xy^2zv^2 - 192y^2zv^3)\partial_x \wedge \partial_y + (-48x^5z - 288x^3yz^2 - 240xy^2z^3 + 192y^2z^3v - 384xyz^2v^2 - 192yz^2v^3)\partial_x \wedge \partial_z + (-336x^4yz - 480x^2y^2z^2 - 576x^3yzv + 480y^2z^2v^2 + 576xyzv^3 + 336yzv^4)\partial_x \wedge \partial_v + (192x^3y^2z - 192xy^3z^2 + 288y^2zv^3 + 48yv^5 + 48(8x^2y^2z + 5y^3z^2)v)\partial_y \wedge \partial_v + (192x^3yz^2 - 192xy^2z^3 + 288yz^2v^3 + 48zv^5 + 48(8x^2yz^2 + 5y^2z^3)v)\partial_z \wedge \partial_v$. We detect that this bi-vector is a coboundary, $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$ with the vector $\vec{\mathcal{Y}} = (-24x^4 + 120y^2z^2 - 96yzv^2)\partial_x + (96x^3y - 96yv^3)\partial_y + (96x^3z - 96zv^3)\partial_z + (96x^2yz - 120y^2z^2 + 24v^4)\partial_v \pmod{\ker \llbracket \mathcal{P}, \cdot \rrbracket}$. The vector field $\vec{\mathcal{Y}} \notin \ker \partial_{\mathcal{P}}$ cannot be Poisson-exact (clearly, $\mathcal{Q}(\mathcal{P}) \neq 0$), hence $\vec{\mathcal{Y}}$ does not mark the Poisson cocycle of zero 1-vector.²⁰

But the universal vector field $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P}) \in \ker \partial_{\mathcal{P}}$ is identically zero on \mathbb{R}^4 . Indeed, the Euler field $\vec{\mathcal{E}} = \vec{V}$ is linear, yet it is readily seen from the figures in [1]

²⁰Likewise, by using another R -matrix for $\mathfrak{gl}_2(\mathbb{R})$, namely $\begin{pmatrix} x & y \\ z & v \end{pmatrix} \mapsto \begin{pmatrix} x & y \\ -z & v \end{pmatrix}$ also from [12], we obtain the Poisson bi-vector $\mathcal{P} = 2x^2y\partial_x \wedge \partial_y + 2yz^2\partial_x \wedge \partial_z + (2xyz + 2yzv)\partial_x \wedge \partial_v + (-2xyz + 2yzv)\partial_y \wedge \partial_z + 2yv^2\partial_y \wedge \partial_v + 2yz^2\partial_z \wedge \partial_v$ on \mathbb{R}^4 with Cartesian coordinates x, y, z, v . The tetrahedral flow then equals $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = (-384x^5y - 384x^3y^2z - 1536xy^2zv^2 + 384(x^2y^2z - 4y^3z^2)v)\partial_x \wedge \partial_y + (-384x^3yz^2 - 2688xy^2z^3 + 1152xyz^2v^2 + 384yz^2v^3 - 384(3x^2yz^2 - 7y^2z^3)v)\partial_x \wedge \partial_z + (-384x^4yz - 2688x^2y^2z^2 - 1536x^3yzv + 2688y^2z^2v^2 + 1536xyz \times v^3 + 384yzv^4)\partial_x \wedge \partial_v + (384x^4yz + 384x^2y^2z^2 + 1536y^3z^3 - 384xyzv^3 + 384yzv^4 + 384(x^2yz + y^2z^2)v^2 - 384(x^3yz - 2xy^2z^2)v)\partial_y \wedge \partial_z + (1536xy^3z^2 + 1536x^2y^2zv - 384xy^2zv^2 + 384y^2zv^3 + 384yv^5)\partial_y \wedge \partial_v + (-384x^3yz^2 - 2688xy^2z^3 + 1152xyz^2v^2 + 384yz^2v^3 - 384(3x^2yz^2 - 7y^2z^3)v)\partial_z \wedge \partial_v$. It is Poisson-trivial: $\mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$ with a representative $\vec{\mathcal{Y}} = (-96x^4 + 576y^2z^2 - 384yzv^2)\partial_x + (-192xy^2z + 192y^2zv - 384yv^3)\partial_y + (-96x^3z - 96xzv^2 + 96zv^3 + 96(x^2z - 4yz^2)v)\partial_z + (-576y^2z^2 - 384xyzv + 96v^4)\partial_v$. These explicit examples of Poisson-exact bi-vector flows $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = \llbracket \vec{\mathcal{Y}}, \mathcal{P} \rrbracket$ will be useful in the future study of the mechanism $\vec{\mathcal{Y}} = \vec{\mathcal{Y}}(\gamma, \vec{V}, \mathcal{P})$ of their observed $\partial_{\mathcal{P}}$ -triviality.

that in every orgraph from the 1-vector $\text{O}\bar{\Gamma}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes n-1})$, the vertex with \vec{V} is differentiated at least twice (and at most thrice), so $\vec{\mathcal{L}} \equiv 0$.

Proposition 5. *The flow $\dot{\mathcal{P}} = \text{O}\bar{\Gamma}(\text{tetrahedron } \gamma_3)(\mathcal{P})$ preserves the Nambu class of Poisson brackets, $\{f, g\}_{\mathcal{P}} = \varrho(x, y, z) \cdot \det(\partial(a, f, g)/\partial(x, y, z))$ with arbitrary ϱ and global Casimir a on \mathbb{R}^3 : the flow forces the nonlinear evolution \dot{a} , $\dot{\varrho}$ with differential-polynomial r.-h.s.*

• *This flow $\dot{P} = Q(P)$ is not Poisson-exact in terms of any vector field \vec{Y} with differential-polynomial coefficients (cubic in both a and ϱ , of total differential order eight).*

The cocycle equation at hand, $\mathcal{E}(\gamma_3, a, \varrho) = \{\dot{\mathcal{P}} = \llbracket \vec{Y}, \mathcal{P} \rrbracket\}$, is a first-order PDE with differential-polynomial coefficients (their skew-symmetry under permutations of x, y, z is inherited from the property of the Jacobian determinant and from the transformation law for the density ϱ in \mathcal{P}). Whether this equation \mathcal{E} does not admit any non-polynomial solutions $\vec{Y}(a_{|\sigma| \leq 3}, \varrho_{|\tau| \leq 2})$ is an open problem.

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