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# On geometric and differentiation index of nonlinear differential-algebraic equations

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**Abstract:** We discuss two notions of index, i.e., the geometric index and the differentiation index for nonlinear differential-algebraic equations (DAEs). First, we analyze solutions of nonlinear DAEs by revising a geometric reduction method (see e.g. Rabier and Rheinboldt (2002), Rianza (2008)). Then we show that although both of the geometric index and the differentiation index serve as a measure of difficulties for solving DAEs, they are actually related to the existence and uniqueness of solutions in a different manner. It is claimed in (Campbell and Gear, 1995) that the two indices coincide when sufficient smoothness and assumptions are satisfied, we elaborate this claim and show that the two indices indeed coincide if and only if a condition of uniqueness of solutions is satisfied (under certain constant rank assumptions). Finally, an example of a pendulum system is used to illustrate our results on the two indices.

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*Keywords:* differential-algebraic equations, geometric method, differentiation index, geometric index, existence and uniqueness of solutions

## 1. INTRODUCTION

We consider a nonlinear differential-algebraic equation (DAE) of the following form

$$\Xi : E(x)\dot{x} = F(x), \quad (1)$$

where  $x \in X$  is called the generalized state and  $X$  is an open subset of  $\mathbb{R}^n$ , and  $E : TX \rightarrow \mathbb{R}^l$  and  $F : X \rightarrow \mathbb{R}^l$  are  $C^\infty$ -smooth maps, where  $TX$  is the tangent bundle of  $X$ . We denote a DAE of the form (1) by  $\Xi_{l,n} = (E, F)$  or, simply,  $\Xi$ . The DAE (1) is sometimes called a quasi-linear DAE (see e.g., Rabier and Rheinboldt (2002); Rianza (2008)), which is a special case of DAEs in the general form

$$\Xi^{gen} : G(t, x, x') = 0, \quad (2)$$

where  $G : I \times TX \rightarrow \mathbb{R}^l$  is  $C^\infty$ -smooth,  $I \subseteq \mathbb{R}$  is an open interval. Notice that  $\Xi^{gen}$  can be transformed into a DAE of the form (1) by extending the generalized state to  $(t, x, z) = (t, x, x')$ , i.e.,  $\dot{t} = 1$ ,  $\dot{x} = z$ ,  $0 = G(t, x, z)$ . which is a DAE of form (1) with  $E = \begin{bmatrix} I_{n+1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+l+1) \times (2n+1)}$  and  $F(t, x, z) = \begin{bmatrix} z \\ G(t, x, z) \end{bmatrix}$ .

Two main streams of researches on solutions of nonlinear DAEs are geometric methods (Rheinboldt, 1984; Reich, 1990, 1991; Rabier and Rheinboldt, 1994) and numerical methods (Gear, 1988; Brenan et al., 1996; Kunkel and Mehrmann, 2006). Moreover, some crossover results of the two methods can be consulted in Griepentrog (1991); Campbell and Griepentrog (1995); Rabier and Rheinboldt (2002) and the references therein. To characterize different properties of DAEs, various notions of index are proposed, see the survey or survey-like papers on index of DAEs as Griepentrog et al. (1992); Campbell (1995); Campbell and Gear (1995); Mehrmann (2015). The most commonly used

notions of index seem to be the geometric index (Reich, 1990; Rabier and Rheinboldt, 2002) and the differentiation index (see the two different definitions of differentiation index in Definition 4 (Campbell and Gear, 1995) and in Section 3 (Griepentrog, 1991)). Our formulations of the definitions of the two indices are given in Definition 6 and Definition 10 below, respectively.

The purpose of the present paper is to have a comprehensive understanding of the two notions of DAE index by analyzing their relations and differences. Moreover, Campbell and Gear (1995) have claimed that the geometric index and the differentiation index coincide when sufficient smoothness and assumptions are satisfied, we aim to elaborate this claim and present our results about the relations of the two indices. We give our analysis on the solvability of DAEs by revising the geometric reduction method and show how the method leads to the definition of geometric index in Section 2. We discuss differentiation index and show its relations and differences with geometric index in Section 3. The conclusions of this paper are given in Section 4. Throughout we use the following notations: We use  $x'$  or  $\dot{x}$  to denote the derivate of  $x$  with respect to  $t$ . The symbol  $C^k$  denotes the class of functions which are  $k$ -times differentiable. For a map  $A : X \rightarrow \mathbb{R}^{l \times n}$ ,  $\ker A(x)$ ,  $\text{Im } A(x)$  and  $\text{rank } A(x)$  are the kernel, the image and the rank of  $A$  at  $x$ , respectively. For a smooth map  $f : X \rightarrow \mathbb{R}$ , we denote its differential by  $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$  and for a vector-valued map  $f : X \rightarrow \mathbb{R}^m$ , where  $f = [f_1, \dots, f_m]^T$ , we denote its differential by  $Df$  and its Jacobian matrix by  $D_x f$ . For two column vectors  $v_1 \in \mathbb{R}^m$  and  $v_2 \in \mathbb{R}^n$ , we write  $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$ .

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## 2. GEOMETRIC REDUCTION METHOD AND GEOMETRIC INDEX OF DAES

A solution of a DAE  $\Xi$ , given by (1), is a  $\mathcal{C}^1$ -curve  $x : I \rightarrow X$  with an open interval  $I$  such that for all  $t \in I$ ,  $E(x(t))\dot{x}(t) = F(x(t))$ . A point  $x_a \in X$  is called *admissible* (or *consistent*) if there exists a solution  $x(\cdot)$  such that  $x(t_a) = x_a$  for a certain  $t_a \in I$ . Denote by  $S_a$  the set of all admissible (consistent) points of  $\Xi$ . It is clear that  $S_a$  is the set on which the solutions of DAEs exist.

The main idea of geometric analysis of DAE solutions is to view a DAE as a vector field defined on a submanifold<sup>1</sup>, which we will call the maximal invariant submanifold, its formal definition is given as follows.

*Definition 1.* (maximal invariant submanifolds). Consider a DAE  $\Xi = (E, F)$  defined on  $X$ , fix an admissible point  $x_a \in X$ . A smooth connected submanifold  $M$  of  $X$  is called *locally invariant* if there exists an open neighborhood  $U \subseteq X$  of  $x_a$  such that for any point  $x_0 \in M \cap U$ , there exists a solution  $x : I \rightarrow X$  of  $\Xi$  such that  $x(t_0) = x_0$  for a certain  $t_0 \in I$  and  $x(t) \in M \cap U$  for all  $t \in I$ . A locally invariant submanifold  $M^*$  is called *maximal* if there exists a neighborhood  $V$  of  $x_a$  such that for any other locally invariant submanifold  $M$ , we have  $M \cap V \subseteq M^* \cap V$ .

The following recursive method is called the geometric reduction method, which was frequently used (see e.g. Reich (1990); Rabier and Rheinboldt (2002); Riaza (2008); Berger (2016, 2017); Chen and Respondek (2021b)) to construct the locally maximal invariant submanifold  $M^*$  and to study the existence of solutions.

*Definition 2.* (geometric reduction method). For a DAE  $\Xi_{l,n} = (E, F)$ , fix a point  $x_p \in X$  and let  $U_0$  be an open connected subset of  $X$  containing  $x_p$ . Set  $M_0 = X$ ,  $M_0^c = U_0$ . Suppose that there exist an open neighborhood  $U_{k-1}$  of  $x_p$  and a sequence of smooth connected submanifolds  $M_{k-1}^c \subsetneq \dots \subsetneq M_0^c$  of  $U_{k-1}$  for a certain  $k \geq 1$ , has been constructed. Define recursively

$$M_k := \{x \in M_{k-1}^c : F(x) \in E(x)T_x M_{k-1}^c\}. \quad (3)$$

As long as  $x_p \in M_k$ , assume that there exists a neighborhood  $U_k$  of  $x_p$  such that  $M_k^c = M_k \cap U_k$  is a smooth connected submanifold.

*Remark 3.* If we apply the above method to a linear DAE  $E\dot{x} = Ax$ , where  $E, A \in \mathbb{R}^{l \times n}$ , and denote  $\mathcal{V} = M$ , we get the following sequence of subspaces:  $\mathcal{V}_0 := \mathbb{R}^n$  and for  $k \geq 1$ , set

$$\mathcal{V}_k = \{x \in \mathcal{V}_{k-1} \mid Ax \in E\mathcal{V}_{k-1}\} = A^{-1}E\mathcal{V}_{k-1}.$$

The above sequence  $\mathcal{V}_k$  is one of the Wong sequences (Wong, 1974), which plays an important role in the geometric theory of linear DAEs (see e.g., (Berger and Trenn, 2012) and (Chen and Respondek, 2021a)). Thus the sequence of submanifolds  $M_k$  is, clearly, a nonlinear generalization of the Wong sequence  $\mathcal{V}_k$ . Note that the limits of  $\mathcal{V}_k$ , i.e.,  $\mathcal{V}^* = \mathcal{V}_n$  is the largest subspace such that  $A\mathcal{V}^* \subseteq E\mathcal{V}^*$ , which coincides with the consistency space of the linear DAE.

The geometric descriptions of Definition 2 can be implemented through Algorithm 1 below.

<sup>1</sup> Throughout when talking about submanifolds, we will always mean embedded submanifolds, see (Lee, 2001)

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### Algorithm 1 Geometric reduction algorithm

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**Initialization:** Consider a DAE  $\Xi_{l,n} = (E, F)$ , fix  $x_p \in X$  and let  $U_0 \subseteq X$  be an open connected subset containing  $x_p$ . Set  $z_0 = x$ ,  $E_0(z_0) = E(x)$ ,  $F_0(z_0) = F(x)$ ,  $M_0^c = U_0$ ,  $r_0 = l$ ,  $n_0 = n$ , and  $\Xi_0 = (E_0, F_0)$ .

**Step  $k$  ( $k > 1$ ):** Suppose that we have defined at Step  $k-1$ : an open neighborhood  $U_{k-1} \subseteq X$  of  $x_p$ , a smooth connected submanifold  $M_{k-1}^c$  of  $U_{k-1}$  and a DAE  $\Xi_{k-1} = (E_{k-1}, F_{k-1})$  with

$$E_{k-1} : M_{k-1}^c \rightarrow \mathbb{R}^{r_{k-1} \times n_{k-1}}, \quad F_{k-1} : M_{k-1}^c \rightarrow \mathbb{R}^{r_{k-1}},$$

whose arguments are denoted  $z_{k-1} \in M_{k-1}^c$ . Rename  $\tilde{E}_k = E_{k-1}$ ,  $\tilde{F}_k = F_{k-1}$  and define  $\tilde{\Xi}_k := (\tilde{E}_k, \tilde{F}_k)$ .

**Assumption (A1):** There exists an open neighborhood  $U_k \subseteq U_{k-1} \subseteq X$  of  $x_p$  such that  $\dim E(x)T_x M_{k-1}^c = \text{rank } \tilde{E}_k(z_{k-1}) = \text{const.} = r_k$ ,  $\forall z_{k-1} \in W_k = U_k \cap M_{k-1}^c$ .

1: Find a smooth map  $Q_k : W_k \rightarrow GL(r_{k-1}, \mathbb{R})$ , such that  $\tilde{E}_k^1$  of

$$Q_k \tilde{E}_k = \begin{bmatrix} \tilde{E}_k^1 \\ 0 \end{bmatrix}$$

is of full row rank and denote  $Q_k \tilde{F}_k = \begin{bmatrix} \tilde{F}_k^1 \\ \tilde{F}_k^2 \end{bmatrix}$ , where  $\tilde{E}_k^1 : W_k \rightarrow \mathbb{R}^{r_k \times n_{k-1}}$ ,  $\tilde{F}_k^2 : W_k \rightarrow \mathbb{R}^{r_{k-1} - r_k}$  (so all the matrices depend on  $z_{k-1}$ ).

2: Following (3), define  $M_k = \{z_{k-1} \in W_k \mid \tilde{F}_k^2(z_{k-1}) = 0\}$ .

**Assumption (A2):**  $x_p \in M_k$  and  $\text{rank } D\tilde{F}_k^2(z_{k-1}) = \text{const.} = n_{k-1} - n_k$  for  $z_{k-1} \in M_k \cap U_k$ .

3: By Assumption 2,  $M_k \cap U_k$  is a smooth submanifold and by taking a smaller  $U_k$ , we may assume that  $M_k^c = M_k \cap U_k$  is connected and choose new coordinates  $(z_k, \bar{z}_k) = \psi_k(z_{k-1})$  on  $W_k$ , where  $\bar{z}_k = (\bar{\varphi}_k^1(z_{k-1}), \dots, \bar{\varphi}_k^{n_{k-1} - n_k}(z_{k-1}))$ , with  $d\bar{\varphi}_k^1(z_{k-1}), \dots, d\bar{\varphi}_k^{n_{k-1} - n_k}(z_{k-1})$  being all independent rows of  $D\tilde{F}_k^2(z_{k-1})$ , and  $z_k$  are any complementary coordinates such that  $\psi_k$  is a local diffeomorphism.

4: Set  $\hat{E}_k = Q_k \tilde{E}_k \begin{pmatrix} \partial \bar{\varphi}_k \\ \partial z_{k-1} \end{pmatrix}^{-1}$ ,  $\hat{F}_k = Q_k \tilde{F}_k$ . Define

$$\hat{\Xi}_k : \begin{bmatrix} \hat{E}_k^1(z_k, \bar{z}_k) & \hat{E}_k^2(z_k, \bar{z}_k) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_k \\ \dot{\bar{z}}_k \end{bmatrix} = \begin{bmatrix} \hat{F}_k^1(z_k, \bar{z}_k) \\ \hat{F}_k^2(z_k, \bar{z}_k) \end{bmatrix} \quad (4)$$

with  $\hat{E}_k^1 : W_k \rightarrow \mathbb{R}^{r_k \times n_k}$ ,  $\hat{F}_k^1 \circ \psi_k = \tilde{F}_k^1$ ,  $\hat{F}_k^2 \circ \psi_k = \tilde{F}_k^2$  and  $[\hat{E}_k^1 \circ \psi_k \quad \hat{E}_k^2 \circ \psi_k] = \tilde{E}_k^1 \begin{pmatrix} \partial \psi_k \\ \partial z_{k-1} \end{pmatrix}^{-1}$ .

5: Set  $\bar{z}_k = 0$  to define the following restricted DAE on  $M_k^c = \{z_{k-1} \in W_k \mid \bar{z}_k = 0\}$  by

$$\hat{\Xi}_k|_{M_k^c} = \Xi_k : E_k(z_k)\dot{z}_k = F_k(z_k),$$

where  $E_k(z_k) = \hat{E}_k^1(z_k, 0)$ ,  $F_k(z_k) = \hat{F}_k^1(z_k, 0)$  are matrix-valued maps and  $E_k : M_k^c \rightarrow \mathbb{R}^{r_k \times n_k}$ ,  $F_k : M_k^c \rightarrow \mathbb{R}^{r_k}$ .

**Repeat:** Step  $k$  for  $k = 1, 2, 3, \dots$ , **until**  $n_{k+1} = n_k$ , set  $k^* = k$ .

**Result:** Set  $n^* = n_{k^*} = n_{k^*+1}$ ,  $r^* = r_{k^*+1}$ ,  $M^* = M_{k^*+1}^c$ ,  $z^* = z_{k^*+1} = z_{k^*}$  and  $\Xi^* = (E^*, F^*)$  with  $E^* = E_{k^*+1}$ ,  $F^* = F_{k^*+1}$ .

---

The following theorem shows that under Assumptions (A1) and (A2) of Algorithm 1, the sequence of submanifolds  $M_k$  converges to the locally maximal invariant submanifold  $M^*$ , which coincides locally with the admissible set  $S_a$ .

*Theorem 4.* Consider a DAE  $\Xi_{l,n} = (E, F)$  and fix a point  $x_p \in X$ . Suppose that Assumptions (A1) and (A2) in each Step  $k \geq 1$  of Algorithm 1 are satisfied. Then there always exists  $k^* \leq n$  such that  $k^*$  is the smallest integer such that  $x_p \in M_{k^*+1}^c$  and  $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$ . We have that  $x_p$  is an admissible point, i.e.,  $x_p = x_a$  and  $M^* = M_{k^*+1}^c$  is a locally maximal invariant submanifold around  $x_p$ . Moreover, there exists a neighborhood  $U^* \subseteq U_{k^*+1}$  of  $x_p$  such that

- (i)  $S_a \cap U^* = M^* \cap U^*$ ;
- (ii) there exist a local diffeomorphism defined on  $U^*$  mapping solutions of  $\Xi$  into solutions of

$$E^*(z^*)\dot{z}^* = F^*(z^*), \quad \bar{z}_1 = 0, \dots, \bar{z}_{k^*} = 0 \quad (5)$$

where  $z^* = z_{k^*+1} = z_{k^*}$  are local coordinates on  $M^*$ ,  $E^* = E_{k^*+1} : M^* \rightarrow \mathbb{R}^{r^* \times n^*}$ ,  $F^* = F_{k^*+1} : M^* \rightarrow \mathbb{R}^{r^*}$ , and  $\text{rank } E^*(z^*) = r^*$ ,  $\forall z^* \in M^*$ , i.e.,  $E^*(z^*)$  is of full row rank.

- (iii) for any  $x_0 \in M^* \cap U^*$ , there passes only one solution if and only if  $\dim M^* = \dim E(x)T_x M^*$  for all  $x \in M^* \cap U^*$ , i.e.,  $n^* = r^*$ .

The proof of Theorem 4 can be consulted in the proofs of Propositions 2.7 and 3.3, and Theorem 3.11(ii) of (Chen and Respondek, 2021b).

*Remark 5.* Item (i) of Theorem 4 illustrates that for all  $x$  in a neighborhood  $U^*$  of an admissible point  $x_a = x_p$ , the solutions of nonlinear DAEs exist on the maximal invariant submanifold  $M^*$  only. Therefore, for any point  $x_0 \in U^* \setminus M^*$ , there exist no solutions passing through  $x_0$ . From Theorem 4(ii), it is seen that solutions of  $\Xi$  are isomorphic to those of  $E^*(z^*)\dot{z}^* = F^*(z^*)$  with  $E^*$  being of full row rank, which is an “under-determined” DAE and can be expressed as an ordinary differential equation (ODE) with free variables. Item (iii) of Theorem 4 is a geometric characterization of the uniqueness of solutions for nonlinear DAEs.

The definition of geometric index is given as follows (see also Rabier and Rheinboldt (2002); Riaza (2008)).

*Definition 6.* (geometric index). Consider the sequence of submanifolds  $M_k^c$  constructed via Definition 2 around a point  $x_p$ , then the (local) geometric index  $\nu_g$  of a DAE  $\Xi$  is defined by

$$\nu_g := \min \{k \geq 0 \mid (M_{k+1}^c = M_k^c) \wedge (M_k^c \neq \emptyset)\}.$$

Now combining the results of Theorem 4, we give some comments on the geometric index defined above.

*Remark 7.* (i) The definition of the geometric index needs only the assumption that for each  $k \geq 0$ ,  $M_k^c$  is locally a smooth connected submanifold. The constant rank assumptions (A1) and (A2) are not necessarily required for the existence of geometric index. Take for example the following DAE

$$\Xi : x\dot{x} = x^2, \tag{6}$$

where  $x \in X = \mathbb{R}$ . Clearly,  $M^* = M_1^c = M_0^c = X$  is a maximal invariant submanifold and any point  $x_a \in X$  is admissible. The geometric index of (6) is  $\nu_g = 0$ . However  $\dim E(x)T_x M^* \neq \text{const.}$ , for all  $x \in M^*$  since  $\dim E(x)T_x M^*$  equals 1 for  $x \neq 0$  and is 0 for  $x = 0$ .

(ii) The geometric index of DAEs allows for a conclusion about the existence of solutions. Suppose that  $\Xi$  has a well-defined geometric index  $\nu_g$  around  $x_p$ . For any  $x_0 \in U_k \setminus M_k^c$ ,  $0 \leq k \leq \nu_g$ , around  $x_p$ , we can conclude that there does not exist any solution passing through  $x_0$ .

(iii) The geometric index does not concern uniqueness of solutions. As an example, consider two DAEs

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_2, \end{cases} \quad \text{and} \quad \hat{\Xi}_{2,3} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_3, \end{cases}$$

where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth. Observe that both  $\Xi$  and  $\hat{\Xi}$  have the same geometric index  $\nu_g = 1$ . Nevertheless, the DAE  $\Xi$  has a unique solution for any admissible initial point  $(x_{10}, x_{20}) = (x_{10}, 0)$ , the DAE  $\hat{\Xi}$  has infinite numbers of solutions for any admissible initial point

$(x_{10}, x_{20}, x_{30}) = (x_{10}, x_{20}, 0)$  since  $x_2$  is a free variable of the ODE  $\dot{x}_1 = f(x_1, x_2)$ .

*Example 8.* Consider the following DAE  $\Xi_{5,5} = (E, F)$ , borrowed from Rabier and Rheinboldt (1994),

$$\Xi : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ x_4 \\ -x_5 x_3 - g \\ x_1^2 + x_3^2 - l^2 \end{bmatrix}. \tag{7}$$

This DAE describes the mathematical model of a pendulum with a mass attached to its end and where the length of the pendulum is given by the constant  $l$ . We consider a point  $x_p = (x_{1p}, x_{2p}, x_{3p}, x_{4p}, x_{5p})$ , where  $x_{1p} = 0$ ,  $x_{2p} = 0$ ,  $x_{3p} = -l$ ,  $x_{4p} = 0$ ,  $x_{5p} = g/l$  and apply Algorithm 1 to  $\Xi_1 = (E_1, F_1) = \Xi$ .

Step 1: since  $E_1$  is already in the desired form, we set  $Q_1 = I_5$ . It follows that

$$M_1 = \{x \in X \mid x_1^2 + x_3^2 - l^2 = 0\}.$$

We have  $x_p \in M_1$  and  $M_1^c = M_1 \cap U_1$  is a smooth connected submanifold, where  $U_1 = \{x \in X \mid x_3 < 0\}$ . Then choose a new coordinate  $\bar{x}_1 = \bar{x}_3 = x_1^2 + x_3^2 - l^2$  and keep the remaining coordinates  $z_1 = (x_1, x_2, x_4, x_5)$  unchanged, the DAE  $\Xi_1$  represented in the new coordinates (defined on  $U_1$ ) is

$$\hat{\Xi}_1 : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2x_1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ 2x_3 x_4 \\ -x_5 x_3 - g \\ \bar{x}_3 \end{bmatrix},$$

where  $x_3 = -(l^2 - \bar{x}_3^2 - x_1^2)^{1/2}$ . Set  $\bar{x}_3 = 0$  to get

$$\hat{\Xi}_1|_{M_1^c} : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2x_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ -2(l^2 - x_1^2)^{1/2} x_4 \\ x_5(l^2 - x_1^2)^{1/2} - g \end{bmatrix}.$$

Step 2: consider  $\Xi_2 = (E_2, F_2) = \hat{\Xi}_1|_{M_1^c}$ . Find  $Q_2$  such that

$$Q_2 E_2(z_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_2 F_2(z_1) = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ x_5(l^2 - x_1^2)^{1/2} - g \\ -(l^2 - x_1^2)^{1/2} x_4 + x_1 x_2 \end{bmatrix}.$$

It follows that

$$M_2 = \{x \in M_1^c \mid -(l^2 - x_1^2)^{1/2} x_4 + x_1 x_2 = 0\}.$$

We have  $x_p \in M_2$  and  $M_2^c = M_2 \cap U_2$  is a smooth connected submanifold, where  $U_2 = \{x \in U_1 \mid x_1 < l\}$ . Then we define new local coordinates on  $M_1^c$  via the local diffeomorphism  $\psi_2(z_1) = (z_2, \bar{z}_2) = (x_1, x_2, x_5, \bar{x}_4)$ , where  $\bar{z}_2 = \bar{x}_4 = -(l^2 - x_1^2)^{1/2} x_4 + x_1 x_2$ . The DAE  $\Xi_2$  in the new coordinates is

$$\hat{\Xi}_2 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a(z_1) & -x_1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ g(l^2 - x_1^2)^{1/2} - x_5(l^2 - x_1^2) \\ \bar{x}_4 \end{bmatrix},$$

where  $a(z_1) = x_1^2 x_2 (l^2 - x_1^2)^{-1} + x_2$ . Set  $\bar{x}_4 = 0$  to have

$$\hat{\Xi}_2|_{M_2^c} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a(x) & -x_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ -x_5(l^2 - x_1^2) + g(l^2 - x_1^2)^{1/2} \end{bmatrix}.$$

Step 3: Consider  $\Xi_3 = (E_3, F_3) = \hat{\Xi}_2|_{M_2^c}$ . Then via a similar procedure as in Step 1 and 2, we get

$$M_3 = \{x \in M_2^c \mid x_1^2 x_2^2 (l^2 - x_1^2)^{-1} + x_2^2 - x_5 l^2 - g x_3 = 0\}$$

$$= \{x \in M_2^c \mid x_4^2 + x_2^2 - x_5 l^2 - g x_3 = 0\}.$$

It follows that  $x_p \in M_3$  and  $M_3^c = M_3 \cap U_3$ , where  $U_3 = U_2$ , is a smooth connected submanifold. Set  $\bar{x}_5 = x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x_2^2)^{1/2} - l^2 x_5$  and define the new local coordinates  $(z_3, \bar{z}_3) = (x_1, x_2, \bar{x}_5)$  on  $M_2^c$ , then we denote  $\Xi_3$  in the new coordinates by  $\tilde{\Xi}_3$ . It follows that

$$\tilde{\Xi}_3|_{M_3^c} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \end{bmatrix}, \quad (8)$$

where  $x_5 = \frac{1}{l^2}(x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x_1^2)^{1/2})$ .

Step 4: Since  $\Xi_4 = \tilde{\Xi}_3|_{M_3^c}$  is an ODE, it is seen that  $k^* = 3$  and  $M^* = M_4^c = M_3^c$ .

Notice that  $x_p \in M^*$  and assumptions (A1) and (A2) are satisfied around  $x_p$ , we can conclude by Theorem 4 that the solution of  $\Xi$  passing through any point  $x_0 \in M^* \cap U^*$ , where  $U^* = U_3$ , exists and is unique (since  $\dim M^* = \dim E(x)T_x M^* = 2$  for all  $x \in M^* \cap U^*$ ), and this unique solution is mapped via a diffeomorphism defined on  $U^*$  into the solution of  $\Xi^* = \tilde{\Xi}_3|_{M_3^c}$ , given by (8), with the constraints  $\bar{x}_3 = \bar{x}_4 = \bar{x}_5 = 0$ . Moreover, the geometric index  $\nu_g = k^* = 3$ . Note that our sequence of submanifolds  $M_k$  coincides with that of the projections  $W_k$  of the tangent bundles  $TW_k$  in (Rabier and Rheinboldt, 1994). But the two methods of constructing sequences of submanifolds are different, although it is interesting to compare them, it is not our aim to discuss their differences in details in the present paper.

### 3. GEOMETRIC INTERPRETATION OF THE DIFFERENTIATION INDEX

A classical definition of differentiation index (Gear, 1988) is given for DAEs of the general form (2): define the differential array of (2) by

$$H_k(t, x, x', w) = \begin{bmatrix} H \\ D_t H + D_x H x' + D_{x'} H x'' \\ \vdots \\ \frac{d^k}{dt^k} H \end{bmatrix} (t, x, x', w) = 0, \quad (9)$$

where  $w = [x^{(2)}, \dots, x^{(k+1)}]$ , the differentiation index  $\nu_d$  is the least integer  $k$  such that equation (9) uniquely determines  $x'$  as a function of  $(x, t)$ , i.e.,  $x' = v(x, t)$ . Note that some other definition of differentiation index in, e.g., (Campbell and Gear, 1995), requires, a priori, the solvability of DAEs. We do not state such a requirement since an index is a measure of difficulties for solving a DAE, whose definition should be independent of the solvability of the DAE (Rabier and Rheinboldt, 2002). We now illustrate two deficiencies or ambiguities of the above classical definition of differentiation index.

*Remark 9.* (i) In contrast to the geometric index of Definition 6, the above classical definition of differentiation index does not allow for a conclusion about the existence of solutions. Take the following DAEs for example,

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \end{cases} \quad \text{and} \quad \tilde{\Xi}_{3,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \\ 0 = x_1 \end{cases},$$

where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth and we assume  $f(0, 0) \neq 0$ . The two DAEs  $\Xi$  and  $\tilde{\Xi}$  have the same differentiation index  $\nu_d = 0$  by the definition above. However,  $\Xi$  has a unique

solution and  $\tilde{\Xi}$  has no solutions for the admissible initial point  $(x_{10}, x_{20}) = (0, 0)$ . The reason is that the above definition of differentiation index only implies the existence and uniqueness of some vector field determined by the differential array but does not indicate where the solutions of the DAE should exist (solutions of DAE exists on the maximal invariant submanifold  $M^*$  which we discussed in Section 2). In particular, if the defined vector field  $v(x, t)$  is not tangent to  $M^*$  at  $x_0 \in M^*$  (as in our example, the vector field  $(x_2, f(x_1, x_2))$  determined by  $\tilde{\Xi}$  is not tangent to  $M^* = \{x \in \mathbb{R}^2 \mid x_1 = x_2 = 0\}$  at  $(x_{10}, x_{20}) = (0, 0)$ ), then the trajectory of  $\dot{x} = v(x, t)$  will leave  $M^*$ , indicating that the DAE does not have a solution at  $x_0$ .

(ii) The classical definition of differentiation index does not indicate the difference between ODEs and “over-determined” DAEs. Consider the DAEs  $\Xi_{2,2}$  and  $\tilde{\Xi}_{3,2}$  of item (i) above, we suppose now  $f(0, 0) = 0$ . Then both  $\Xi$  and  $\tilde{\Xi}$  have the same differentiation index  $\nu_d = 0$ . The DAE  $\Xi$  is clearly an ODE which does not need any differentiation to be solved, while  $\tilde{\Xi}$  needs two times of differentiations (of the algebraic constraint  $x_1 = 0$ ) in order to deduce a solution. Note that the “over-determined” (and also “under-determined”) properties of nonlinear DAEs can be characterized by the notion of strangeness index, see e.g., (Kunkel and Mehrmann, 2006).

To give a geometric background to the notion of differentiation index, a new definition was proposed in (Griepentrog, 1991). In the present paper, in order to clear out the deficiencies mentioned in Remark 9, we reform Griepentrog’s definition of differentiation index as follows. Consider a nonlinear DAE  $\Xi_{l,n} = (E, F)$  and fix a point  $x_p \in X$ , let  $U_0 \subseteq X$  be a neighborhood of  $x_p$ , set  $H(x, \zeta_1) = E(x)\zeta_1 - F(x)$ , denote  $(\frac{d^k}{dt^k} H) = H^{(k)}$  and for  $k \geq 0$ , define

$$H_k(x, \bar{\zeta}_{k+1}) = \begin{bmatrix} H^{(0)}(x, \zeta_1) \\ H^{(1)}(x, \zeta_1, \zeta_2) \\ \vdots \\ H^{(k)}(x, \bar{\zeta}_{k+1}) \end{bmatrix} = 0, \quad (10)$$

where  $\bar{\zeta}_{k+1} = (\zeta_1, \dots, \zeta_{k+1})$ . Set  $\mathcal{M}_0 = X$ ,  $M_0^c = U_0$ ,  $\mathcal{Z}_1^0 = \mathbb{R}^n$  and for  $k > 0$ , define

$$\mathcal{M}_k := \{x \in X \mid H_{k-1}(x, \bar{\zeta}_k) = 0\}, \quad (11)$$

As long as  $x_p \in \mathcal{M}_k$ , assume that there exists a neighborhood  $U_k \subseteq U_{k-1}$  such that  $M_k^c = \mathcal{M}_k \cap U_k$  is a smooth connected submanifold. Set

$$\mathcal{Z}_1^k := \{\zeta_1 \in \mathbb{R}^n \mid H_{k-1}(x, \bar{\zeta}_k) = 0, x \in \mathcal{M}_k^c\}.$$

*Definition 10.* (Differentiation index). The (local) differentiation index  $\nu_d$  of a DAE  $\Xi$  around a point  $x_p$  is defined by  $\nu_d :=$

$$\min \left\{ k \geq 0 \mid (\mathcal{M}_k^c \neq \emptyset) \wedge (\mathcal{Z}_1^k = \mathcal{Z}_1^k(x) \text{ is a singleton}) \wedge (\mathcal{Z}_1^k(x) \in T_x \mathcal{M}_k^c, \forall x \in \mathcal{M}_k^c) \right\}.$$

Now we state our main theorem of this subsection.

*Theorem 11.* For a DAE  $\Xi_{l,n} = (E, F)$ , fix a point  $x_p$ , assume that for each  $k \geq 0$ ,

- (A1)’  $\dim E(x)T_x \mathcal{M}_k^c = \text{const.}$  for all  $x \in \mathcal{M}_k^c$  around  $x_p$ .
- (A2)’  $\mathcal{M}_k^c$  is a smooth connected submanifold and  $x_p \in \mathcal{M}_k^c$ .

Then we have that for each  $k \geq 0$ , the set  $M_k^c$  of the geometric reduction method of Definition 2 is a smooth connected submanifold and

$$\mathcal{M}_k^c = M_k^c,$$

and there exists a smallest integer  $k^*$  such that  $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$ . It follows that  $x_p \in M^*$  is admissible and  $M^* = M_{k^*+1}$  is a locally maximal invariant submanifold, and

- (i) the geometric index  $\nu_g = k^*$ ;
- (ii) the differentiation index  $\nu_d$  exists and  $\nu_d = \nu_g$  if and only if  $\dim M^* = \dim E(x)T_x M^*$  for all  $x \in M^*$  around  $x_p$ .

**Proof.** First, we construct  $M_k$  explicitly via the following procedure: Fix  $x_p \in X$  and let  $U_0$  be an open connected subset of  $X$  containing  $x_p$ . Set  $\mathcal{M}_0^c = U_0, E_0 = E, F_0 = F$ . Assume in Step  $k - 1$ , we have constructed a smooth submanifold  $M_{k-1}^c$ , and maps  $E_{k-1} : M_{k-1}^c \rightarrow \mathbb{R}^{l \times n}$  and  $F_{k-1} : M_{k-1}^c \rightarrow \mathbb{R}^{l \times n}$ . Step  $k$  ( $k > 0$ ): Rename  $\hat{E}_k = E_{k-1}, \hat{F}_k = F_{k-1}$ . Assume that  $\text{rank } \hat{E}_k(x) = \text{const.} = r_k$  for all  $x \in M_{k-1}^c$  around  $x_p$ . After a suitable permutation, it is possible to assume that the first  $r_k$  rows of  $\hat{E}_k$  are linearly independent. Then we can rewrite  $\hat{E}_k v + \hat{F}_k = 0$  as

$$\begin{bmatrix} \hat{E}_k^1 \\ \hat{E}_k^2 \end{bmatrix} v + \begin{bmatrix} \hat{F}_k^1 \\ \hat{F}_k^2 \end{bmatrix} = 0,$$

where  $\hat{E}_k^1 : M_{k-1}^c \rightarrow \mathbb{R}^{r_k \times n}$  is of full row rank and  $\hat{F}_k^1 : M_{k-1}^c \rightarrow \mathbb{R}^{r_k}$ . Find  $a_k : M_{k-1}^c \rightarrow \mathbb{R}^{(l-r_k) \times r_k}$  such that  $a_k \hat{E}_k^1 = \hat{E}_k^2$ . Denote

$$v_k = \hat{E}_k^1 v + \hat{F}_k^1,$$

we have  $\hat{E}_k^2 v + \hat{F}_k^2 = a_k(\hat{E}_k^1 v + \hat{F}_k^1) - a_k \hat{F}_k^1 + \hat{F}_k^2 = a_k v_k + \tilde{F}_k^2$ , where  $\tilde{F}_k^2 = \hat{F}_k^2 - a_k \hat{F}_k^1$ . Define

$$\begin{aligned} M_k &= \left\{ x \in M_{k-1}^c \mid \hat{E}_k(x)v + \hat{F}_k(x) = 0 \right\} \\ &= \left\{ x \in M_{k-1}^c \mid \tilde{F}_k^2(x) = 0 \right\}. \end{aligned} \tag{12}$$

Assume that there exists a neighborhood  $U_k \subseteq U_{k-1}$  of  $x_p$  such that  $M_k^c = M_k \cap U_k$  is a smooth connected submanifold. Set

$$E_k = \begin{bmatrix} \hat{E}_k^1 \\ D_x \tilde{F}_k^2 \end{bmatrix} \Big|_{M_k^c} \quad \text{and} \quad F_k = \begin{bmatrix} \hat{F}_k^1 \\ 0 \end{bmatrix} \Big|_{M_k^c},$$

where  $E_k : M_k^c \rightarrow \mathbb{R}^{l \times n}$  and  $F_k : M_k^c \rightarrow \mathbb{R}^l$ .

We now show that under assumptions (A1)' and (A2)', for each  $k > 0$ , the set  $M_k$ , constructed by (12), is indeed the one defined by (11). Observe that for  $k \geq 0$ ,

$$\ker E_k(x) = \ker E(x) \cap T_x M_k^c, \quad \forall x \in M_k^c.$$

Thus assumption (A1)' implies that  $\text{rank } E_k(x) = \text{const.}$  for all  $x \in M_k^c$ . Then consider the differentiation array  $H_k$ , given by (10), denote  $\zeta_1 = v$  and use the same notations  $v_k$  as in the above procedure to have

$$\begin{aligned} H^{(0)}(x, v) &= E(x)v + F(x) = \begin{bmatrix} \hat{E}_k^1(x) \\ \hat{E}_k^2(x) \end{bmatrix} v + \begin{bmatrix} \hat{F}_k^1(x) \\ \hat{F}_k^2(x) \end{bmatrix} \\ &= \begin{bmatrix} v_1 \\ \tilde{a}_1(x, v_1) + \tilde{F}_k^2(x) \end{bmatrix}, \end{aligned}$$

where  $\tilde{a}_1(x, v_1) = a_1(x)v_1$ . Now by assumption (A2)' that  $M_k^c = M_k \cap U_k$  is a smooth connected submanifold, we have (note that for any smooth map  $L$  defined on  $X$ , its restriction  $L|_M$  to a smooth submanifold  $M \subseteq X$  and the differentiation of  $L$  with respect to  $t$  are two commutative operations, i.e.,  $\frac{d}{dt}(L|_M) = (\frac{d}{dt}L)|_M$ )

$$H^{(1)}|_{M_1^c} = (H|_{M_1^c})^{(1)} = \begin{bmatrix} v_1^{(1)} \\ \tilde{a}_1^{(1)} + \begin{bmatrix} \bar{v}_1 \\ \tilde{F}_2^2 \end{bmatrix} \end{bmatrix},$$

where  $\bar{a}_2(x, v_2) = a_2(x)v_2$  and  $v_2 = (v_1, \bar{v}_1)$ ; in general,

$$H^k|_{M_k^c} = \begin{bmatrix} v_1^{(k)} \\ \tilde{a}_1^{(k)} + (\tilde{F}_1^2)^{(k)} \end{bmatrix},$$

where  $(\tilde{F}_1^2)^{(k)}$  is given by the following iterative formula:

$$(\tilde{F}_i^2)^{(k)} = \begin{bmatrix} \bar{v}_i^{(k-1)} \\ \tilde{a}_{i+1}^{(k-1)} + (\tilde{F}_{i+1}^2)^{(k-1)} \end{bmatrix},$$

where  $\bar{a}_i = \bar{a}_i(x, v_i) = a_i(x)v_i$  and  $v_i = (v_{i-1}, \bar{v}_{i-1})$  for  $i > 1$ . Notice that for each  $k$ , since  $\hat{E}_k^1$  is of full row rank, the differentiation  $v_1^{(k)}$  of  $v_1$  and  $\bar{v}_i^{(k-1)}$  of  $\bar{v}_i$  for  $i \geq 1$  do not create any constraint for  $x$ . So it is clear to see that for  $k > 0$ ,

$$\begin{aligned} M_k^c &= \{x \in U_k \mid H_{k-1}(x, \bar{\zeta}_k) = 0\} \\ &= \{x \in M_{k-1}^c \cap U_k \mid H^{(k-1)}(x, \bar{\zeta}_k) = 0, \} \\ &= \{x \in M_{k-1}^c \cap U_k \mid \tilde{F}_k^2(x) = 0\}, \end{aligned}$$

$$\begin{aligned} Z_1^k &= \{\zeta_1 \mid H_{k-1}(x, \bar{\zeta}_k) = 0\} \\ &= \{\zeta_1 \mid E_{k-1}(x)\zeta_1 + F_{k-1}(x) = 0, x \in M_{k-1}^c\} \\ &= \{\zeta_1 \mid v_k(\zeta_1, x) = 0, x \in M_{k-1}^c\}. \end{aligned} \tag{13}$$

Next we show that for each  $k > 0$ ,  $M_k^c$  coincides with the submanifold  $M_k^c$  of the geometric reduction method of Definition 2. It is clear that  $M_1^c = \{x \in U_0 \mid \hat{F}_1(x) \in \text{Im } \hat{E}_1(x)\} = \{x \in X \mid \tilde{F}_1^2(x) = 0\} = M_1^c$ . For  $k > 1$ , suppose that  $M_{k-1}^c = M_{k-1}^c$ , then we have

$$\begin{aligned} M_k &= \left\{ x \in M_{k-1}^c \mid \hat{E}_k(x)v + \hat{F}_k(x) = 0 \right\} \\ &= \left\{ x \in M_{k-1}^c \mid \begin{bmatrix} \hat{E}_k^1(x) \\ D_x \tilde{F}_k^2(x) \end{bmatrix} v + \begin{bmatrix} \hat{F}_k^1(x) \\ 0 \end{bmatrix} = 0 \right\} \\ &= \left\{ x \in M_{k-1}^c \mid E_{k-1}^1(x)v + F_{k-1}^1(x) = 0, v \in T_x M_{k-1}^c \right\} \\ &= \left\{ x \in M_{k-1}^c \mid F_{k-1}^1(x) \in E_{k-1}^1(x)T_x M_{k-1}^c \right\} \\ &= \left\{ x \in M_{k-1}^c \mid F(x) \in E(x)T_x M_{k-1}^c \right\} = M_k. \end{aligned}$$

So  $M_k^c = M_k \cap U_k$  coincides with  $M_k^c = M_k \cap U_k$ . Then  $k^*$  is the smallest integer such that  $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$  and the geometric index  $\nu_g = k^*$  by Definition 6. It can be deduced from  $M^* = M_{k^*+1}^c = M_{k^*}^c$  on  $U_{k^*+1}$  that  $Z_1^{k^*}(x) = Z_1^{k^*+1}(x) \in T_x M^*$ . Thus by (13),

$$\begin{aligned} Z_1^{k^*}(x) &= Z_1^{k^*+1}(x) \\ &= \{\zeta_1 \in \mathbb{R}^n \mid E_{k^*}(x)\zeta_1 + F_{k^*}(x) = 0, x \in M^*\} \end{aligned}$$

It follows that  $Z_1^{k^*} = Z_1^{k^*}(x)$  is a singleton if and only if  $E_{k^*}$  is invertible, i.e.,  $\text{rank } E_{k^*} = n$ , or, equivalently,  $\dim \ker E_{k^*}(x) = 0$ . We can now conclude by Definition 10 that the differentiation index  $\nu_d = k^* = \nu_g$  if and

only if  $\dim \ker E_{k^*}(x) = \dim(\ker E(x) \cap T_x \mathcal{M}^*) = 0$ , i.e.,  $\dim \mathcal{M}^* = \dim E(x)T_x \mathcal{M}^*$  for all  $x \in \mathcal{M}^*$  around  $x_p$ .  $\square$

*Remark 12.* (i) Assumptions (A1)' and (A2)' of Theorem 11 correspond to assumptions (A1) and (A2) of Theorem 4, respectively. If, a priori, we assume that for each  $k \geq 0$ ,  $M_k^c = \mathcal{M}_k^c$ , then it is clear that (A1) coincides with (A1)', and (A2) is equivalent to (A2)' since  $\mathcal{M}_k^c = M_k^c$  is a smooth (embedded) submanifold if and only if  $\text{rank } D\tilde{F}_k^2(z_{k-1}) = \text{const.}$  for  $z_{k-1} \in M_k \cap U_k$  (see e.g., (Lee, 2001)).

(ii) The differentiation index  $\nu_d$  associated with a geometric background, given by Definition 10, allows to have a conclusion on the existence of solutions as the sequence of submanifolds  $\mathcal{M}_k$  coincides with  $M_k$  and  $\mathcal{M}^* = M^*$  is a locally maximal invariant submanifold, which coincides locally with the admissible set  $S_a$ .

(iii) The differentiation index and the geometric index differ from each other by their relations with uniqueness of solutions. Namely, the geometric index may exist even a DAE does not have a unique solution, while the differentiation index exists only if the DAE is uniquely solvable. As seen from Theorem 11, the geometric index  $\nu_g$  always exists if the submanifolds  $\mathcal{M}_k$  are not empty and the sequence of submanifolds converges. While in order to have a well-defined differentiation index, the condition  $\dim \mathcal{M}^* = \dim E(x)T_x \mathcal{M}^*$  for uniqueness of solutions is necessary.

*Example 13.* (continuation of Example 8) Consider the DAE  $\Xi = (E, F)$ , given by (7), since the derivatives  $(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)$  are already determined by the first four equations of  $\Xi$ , we only differentiate the last equation  $x_1^2 + x_3^2 - l^2 = 0$  to determine  $\dot{x}_5$ . It is not hard to verify that assumptions (A1)' and (A2)' are satisfied around  $x_0$ , and  $\mathcal{M}_1^c = \{x \in U_0 \mid x_1^2 + x_3^2 - l^2 = 0\} = M_1^c$ ,

$$\begin{aligned} \mathcal{M}_2^c &= \{x \in \mathcal{M}_1^c \cap U_2 \mid 2x_1\dot{x}_1 + 2x_3\dot{x}_3 = 0\} \\ &= \{x \in \mathcal{M}_1^c \cap U_2 \mid x_1x_2 + x_3x_4 = 0\} = M_2^c, \end{aligned}$$

$$\begin{aligned} \mathcal{M}_3^c &= \{x \in \mathcal{M}_2^c \mid x_1\dot{x}_2 + x_2\dot{x}_3 + x_3\dot{x}_4 + x_4\dot{x}_3 = 0\} \\ &= \{x \in \mathcal{M}_2^c \mid -x_5x_1^2 + x_2^2 - x_5x_3^2 - x_3g + x_4^2 = 0\} \\ &= \{x \in \mathcal{M}_2^c \mid x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0\} = M_3^c. \end{aligned}$$

Moreover, by differentiating  $x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0$ , we get that

$$\dot{x}_5 = \frac{1}{l^2}(-2x_4(x_5x_3 + g)2x_2(x_5x_1) - gx_4).$$

Combining the above equation with the first 4-equations of (7), we have

$$\mathcal{Z}_1^3 = \{\zeta_1 \mid \zeta_1 = (x_2, -x_5x_1, x_4, -x_5x_3 - g, \dot{x}_5), x \in \mathcal{M}_3^c\}.$$

We can conclude that  $\mathcal{Z}_1^3 = \mathcal{Z}_1^3(x)$  is a singleton and  $\mathcal{Z}_1^3(x) \in T_x \mathcal{M}_3^c, \forall x \in M_3^c$ . Therefore the differentiation index  $\nu_d = 3$ , which coincides with the geometric index  $\nu_g$ .

#### 4. CONCLUSIONS

In this paper, we revised the geometric reduction method and give our conditions for the existence and uniqueness of the solutions of nonlinear DAEs. Then we discuss the notion of geometric index via its relations with solutions of DAEs. We modify the classical definition of the differentiation index of DAEs and show that under some constant rankness and smoothness assumptions, the sequence of submanifolds defined by the differentiation array coincides

with those defined by the geometric reduction method and the differentiation index coincides with the geometric index if and only if a condition for uniqueness of solutions is satisfied.

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