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van der Schaft, Arjan; Maschke, Bernhard

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Differential operator Dirac structures

Arjan van der Schaft, Bernhard Maschke**

*Bernoulli Institute for Mathematics, Computer Science and AI,
Jan C. Willems Center for Systems and Control,
University of Groningen, the Netherlands
(e-mail: a.j.van.der.schaft@rug.nl)*

*** Université Claude Bernard Lyon 1, CNRS, LAGEP, France
(e-mail: bernhard.maschke@univ-lyon1.fr)*

Abstract: As shown before, skew-adjoint linear differential operators, mapping efforts into flows, give rise to Dirac structures on a bounded spatial domain by a proper definition of boundary variables. In the present paper this is extended to *pairs* of linear differential operators defining a formally skew-adjoint *relation* between flows and efforts. Furthermore it is shown how the underlying repeated integration by parts operation is streamlined by the use of two-variable polynomial calculus. Dirac structures defined by formally skew adjoint operators together with differential operator effort constraints are treated within the same framework. Finally it is sketched how the approach can be also used for Lagrangian subspaces on bounded domains.

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Keywords: Dirac structures, boundary control systems, two-variable polynomial matrices, factorization, Lagrangian subspaces

1. INTRODUCTION

Dirac structures are an essential ingredient of port-Hamiltonian systems theory; see e.g. van der Schaft and Maschke (1995), van der Schaft and Jeltsema (2014), van der Schaft (2017). They capture the power-conserving interconnection structure in port-based modeling of complex physical systems. Dirac structures for the modeling of distributed parameter physical systems with bounded spatial domain were first introduced in van der Schaft and Maschke (2002). In the case of two coupled balance laws the relation between the distributed flows and efforts is given by a first-order differential operator. Using Stokes' theorem this leads to the definition of *boundary flows and efforts*. The resulting Dirac structure, called the Stokes-Dirac structure, involves the flows and efforts on the spatial domain *and* the boundary flows and efforts on the boundary of the spatial domain. In the case of *linear* distributed parameter systems and a one-dimensional spatial domain consisting of a finite interval the Stokes-Dirac structure was generalized in Le Gorrec et al. (2005). Starting from a general skew-adjoint linear differential operator, which is mapping the distributed efforts to the distributed flows, boundary flows and efforts are defined through repeated integration by parts. The present paper generalizes the set-up considered in Le Gorrec et al. (2005) to Dirac structures defined by *pairs* of linear differential operators relating the distributed flows and efforts. Furthermore, instead of relying on 'brute force' repeated integration by parts, we utilize two-variable polynomial calculus to define the boundary variables. This calculus was originally introduced in Willems and Trentelman (1998) for optimization and dissipativity analysis of finite-dimensional linear systems. It was also used for spectral factorization Trentelman and Rapisarda (1999), while in van der Schaft and Rapisarda (2011) it was shown how

this calculus admits to compute *state maps* from higher-order differential equations (in time) in input and output variables. In the present paper the *time* variable is replaced by the (scalar) *spatial* variable, and *state maps* become *boundary maps*, defining the boundary variables of the resulting Dirac structure. Furthermore it is shown how this construction can be extended to the definition of differential operator Lagrangian subspaces on spatial domains with boundary.

2. RECALL OF TWO-VARIABLE POLYNOMIAL MATRIX CALCULUS

In this section we will recall some relevant basics of two-variable polynomial calculus from Willems and Trentelman (1998); see also Trentelman and Rapisarda (1999); van der Schaft and Rapisarda (2011). A $p \times q$ two-variable polynomial matrix $\Phi(\zeta, \eta)$ is an expression in two indeterminates ζ and η of the form

$$\Phi(\zeta, \eta) := \sum_{k,l=0}^M \Phi_{k,l} \zeta^k \eta^l \quad (1)$$

for certain $p \times q$ matrices $\Phi_{k,l}$. The infinite matrix $\tilde{\Phi}$ whose (k, l) -th block is the matrix $\Phi_{k,l}$, $k, l = 0, \dots, M$, and is zero everywhere else, is called the *coefficient matrix* of $\Phi(\zeta, \eta)$. Associated to $\Phi(\zeta, \eta)$ and its coefficient matrix $\tilde{\Phi}$ is the *bilinear differential operator*¹ D_{Φ} defined as

$$D_{\Phi}(v, w)(z) = \sum_{k,l=0}^M \left[\frac{d^k}{dz^k} v(z) \right]^{\top} \Phi_{k,l} \frac{d^l}{dz^l} w(z), \quad (2)$$

acting on vector-valued functions $v : \mathbb{R} \rightarrow \mathbb{R}^p, w : \mathbb{R} \rightarrow \mathbb{R}^q$. A $p \times p$ two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\Phi(\zeta, \eta) = \Phi^{\top}(\eta, \zeta)$, or equivalently

¹ Called bilinear differential form in Willems and Trentelman (1998).

its coefficient matrix $\tilde{\Phi}$ is symmetric, and defines the quadratic differential operator $D_{\Phi}(v, v)(z)$, cf. Willems and Trentelman (1998).

An important fact in the calculus of bilinear differential operators is the following, cf. Willems and Trentelman (1998). The *derivative* of a bilinear differential operator D_{Φ} defines another bilinear differential operator

$$\frac{d}{dz}(D_{\Phi}(v, w))(z) =: D_{\Psi}(v, w)(z), \quad (3)$$

where $\Psi(\zeta, \eta)$ is the $p \times q$ two-variable polynomial matrix

$$\Psi(\zeta, \eta) = (\zeta + \eta)\Phi(\zeta, \eta) \quad (4)$$

This equality is the two-variable polynomial matrix version of the product rule of differentiation, or (in integral form) of 'integration by parts'.

Any $p \times q$ two-variable polynomial matrix $\Phi(\zeta, \eta)$ can be *factorized* as a product $\Phi(\zeta, \eta) = X(\zeta)^T Y(\eta)$ of (single-variable) polynomial matrices $X(\zeta)$ (of dimension $k \times p$) and $Y(\eta)$ (of dimension $k \times q$). Such a factorization corresponds in a one-to-one manner to a factorization $\tilde{\Pi} = \tilde{X}^T \tilde{Y}$ of the coefficient matrix $\tilde{\Phi}$ of $\Phi(\zeta, \eta)$. Here \tilde{X}, \tilde{Y} are coefficient matrices of $X(\zeta)$ and $Y(\eta)$: let $X(\zeta) = \sum_{k=0}^M X_k \zeta^k$ then \tilde{X} is the infinite row matrix with k -th block given by $X_k, k = 0, 1, \dots, M$, and similarly for \tilde{Y} .

Proposition 1. Let $\Phi \in \mathbb{R}^{p \times q}(\zeta, \eta)$, with $\tilde{\Phi}$ its coefficient matrix. Any factorization $\Phi(\zeta, \eta) = X(\zeta)^T Y(\eta)$ corresponds to a factorization $\tilde{\Phi} = \tilde{X}^T \tilde{Y}$, where \tilde{X}, \tilde{Y} are the coefficient matrices of $X(\zeta)$, respectively $Y(\eta)$.

Factorizations which correspond to the minimal value $k = \text{rank}(\tilde{\Phi})$, are called *minimal*. They are unique up to premultiplication by a nonsingular matrix (Willems and Trentelman (1998), Trentelman and Rapisarda (1999)).

3. DIFFERENTIAL OPERATOR DIRAC STRUCTURES OVER THE INFINITE SPATIAL DOMAIN

Consider a finite-dimensional linear space $\mathcal{F} := \mathbb{R}^m$; called the *flow space*, with elements denoted by f . Consider the dual space $\mathcal{E} = \mathcal{F}^*$; called the *effort space*, with elements denoted by e . Denote the duality product between \mathcal{F} and \mathcal{E} by $\langle e|f \rangle$. Identifying $f \in \mathcal{F}$ and $e \in \mathcal{E}$ with vectors $f, e \in \mathbb{R}^m$ obviously $\langle e|f \rangle = e^T f$. Then the *bond space* $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is endowed with the symmetric bilinear form

$$\langle (f_1, e_1), (f_2, e_2) \rangle := \langle e_1|f_2 \rangle + \langle e_2|f_1 \rangle, \quad (5)$$

which has the matrix representation

$$Q_e = \begin{bmatrix} 0_m & I_m \\ I_m & 0_m \end{bmatrix}. \quad (6)$$

Note that Q_e has singular values $+1$ and -1 , both with multiplicity m . Hence the bilinear form $\langle \cdot, \cdot \rangle$ is *indefinite* and *symmetric*, as well as *non-degenerate*. Furthermore, $\langle \cdot, \cdot \rangle$ gives rise to the following symmetric bilinear form on the set $C^\infty(\mathbb{R}, \mathcal{F} \times \mathcal{E})$ of smooth functions $(f, e) : \mathbb{R} \rightarrow \mathcal{F} \times \mathcal{E}$ with compact support:

$$\begin{aligned} \langle \langle (f_1, e_1), (f_2, e_2) \rangle \rangle := \\ \int_{-\infty}^{\infty} \langle (f_1(z), e_1(z)), (f_2(z), e_2(z)) \rangle dz \end{aligned} \quad (7)$$

This is a non-degenerate form on $C^\infty(\mathbb{R}, \mathcal{F} \times \mathcal{E})$, in the sense that if $\langle \langle (f_1, e_1), (f_2, e_2) \rangle \rangle = 0$ for all compact support (f_1, e_1) , then $(f_2, e_2) = 0$. A subspace \mathcal{D} of the space of functions $(f, e) : \mathbb{R} \rightarrow \mathcal{F} \times \mathcal{E}$ of compact support is called a *Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$ where \perp denotes the orthogonal companion with respect to the form $\langle \langle \cdot, \cdot \rangle \rangle$. Equivalently, \mathcal{D} is a Dirac structure if $\langle \langle \cdot, \cdot \rangle \rangle$ is zero restricted to \mathcal{D} , and moreover \mathcal{D} is maximal with respect to this property.

We will show that such infinite dimensional Dirac structures can be generated by pairs of linear *differential operators*². Indeed, consider systems of differential equations over the spatial domain \mathbb{R} given by

$$F\left(\frac{d}{dz}\right)f(z) + E\left(\frac{d}{dz}\right)e(z) = 0, \quad z \in \mathbb{R}, \quad (8)$$

where $F\left(\frac{d}{dz}\right)$ and $E\left(\frac{d}{dz}\right)$ are square linear differential operators. Denote the corresponding $m \times m$ polynomial matrices by $F(s)$ and $E(s)$ in the indeterminate s . Now let $F(s)$ and $E(s)$ satisfy

$$[E(s) \ F(s)] \begin{bmatrix} 0_m & I_m \\ I_m & 0_m \end{bmatrix} \begin{bmatrix} E^T(-s) \\ F^T(-s) \end{bmatrix} = 0, \quad \forall s \in \mathbb{C}, \quad (9)$$

together with the maximal rank assumption

$$\text{rank} [F(s) \ E(s)] = m, \quad \forall s \in \mathbb{C} \quad (10)$$

Hence

$$\ker [F(s) \ E(s)] = \text{im} \begin{bmatrix} E^T(-s) \\ F^T(-s) \end{bmatrix}, \quad \forall s \in \mathbb{C} \quad (11)$$

As a result the *kernel representation* (8) of $\mathcal{S}_{\mathcal{D}}$ can be equivalently replaced by the *image representation*

$$\begin{bmatrix} f(z) \\ e(z) \end{bmatrix} = \begin{bmatrix} E^T(-\frac{d}{dz}) \\ F^T(-\frac{d}{dz}) \end{bmatrix} \ell(z), \quad (12)$$

with vector of latent variables $\ell(z) \in \mathbb{R}^m$ ranging through the set of C^∞ functions. Furthermore, it follows from Willems and Trentelman (1998); van der Schaft and Rapisarda (2011)³ that the set $\mathcal{S}_{\mathcal{D}}$ of solutions with compact support of (8) satisfies

$$\int_{-\infty}^{\infty} \langle (f_1(z), e_1(z)), (f_2(z), e_2(z)) \rangle dz = 0 \quad (13)$$

for all $(f_1, e_1), (f_2, e_2) \in \mathcal{S}_{\mathcal{D}}$. Hence $\mathcal{S}_{\mathcal{D}} \subset \mathcal{S}_{\mathcal{D}}^\perp$. Furthermore, assuming for the moment that the rational matrix $G(s) := -E^{-1}(s)F(s)$ is *proper*, it follows from standard linear system realization theory that the behavior $\mathcal{S}_{\mathcal{D}}$ is generated by a minimal 'state' space system of the form

$$\begin{aligned} \frac{d}{dz}b(z) &= Ab(z) + Bf(z) \\ e(z) &= Cb(z) + Df(z) \end{aligned} \quad (14)$$

with 'state' vector $b(z) \in \mathbb{R}^n$, where n is the McMillan degree of $G(s)$, and A, B, C, D are appropriately dimensioned matrices. Here 'state' has been denoted with quotation marks, since actually b corresponds to the *boundary* vector of the behavior \mathcal{S} as we will see in the next section. (In ordinary realization theory the *spatial* variable z is replaced by the *time* variable t .) Then, following the techniques

² An abstract algebraic theory about Dirac structures defined by pairs of differential operators can be found in Dorfman (1993).

³ Here the spatial variable $z \in \mathbb{R}$ is replaced by the time variable t .

exploited in van der Schaft and Polyuga (2009) (again in a time-domain setting), it follows that actually $\mathcal{S}_{\mathcal{D}}^{\perp} \subset \mathcal{S}_{\mathcal{D}}$, thus yielding equality $\mathcal{S}_{\mathcal{D}}^{\perp} = \mathcal{S}_{\mathcal{D}}$, proving that the solution set $\mathcal{S}_{\mathcal{D}}$ of (8) is actually a Dirac structure.

Proposition 2. Consider $F(s), E(s)$ satisfying (9) and (10). Let $\mathcal{S}_{\mathcal{D}}$ be the set of smooth solutions of (8) with compact support within the spatial domain \mathbb{R} . Then $\mathcal{S}_{\mathcal{D}}$ is a Dirac structure with respect to the bilinear form (7).

Note furthermore that by (9) the 'transfer matrix' $G(s)$ satisfies $G(s) = -G^{\top}(-s)$. Therefore the system (47) is cyclo-lossless in the sense that there exists an invertible matrix $\Sigma = \Sigma^{\top}$ satisfying

$$A^{\top}\Sigma + \Sigma A = 0, \quad B^{\top}\Sigma = C, \quad D = -D^{\top} \quad (15)$$

Hence by defining the skew-symmetric matrix $J := A\Sigma^{-1}$ the system (47) can be rewritten into port-Hamiltonian form

$$\begin{aligned} \frac{d}{dz}b(z) &= J\Sigma b(z) + Bf(z) \\ e(z) &= B^{\top}\Sigma b(z) + Df(z) \end{aligned} \quad (16)$$

with Hamiltonian $\frac{1}{2}b^{\top}\Sigma b$. In case $G(s)$ is *not* proper, the same results continue to hold. This follows from the fact, see Bloch & Crouch (1999); van der Schaft and Jeltsema (2014) for the constant case, that there always exists a partitioning $\{1, \dots, n\} = I \cup I^c$ such that by defining the polynomial matrix $\widehat{F}(s)$ as the matrix with k -th column given by the k -th column of $F(s)$ whenever $k \in I$, and equal to the k -th column of $E(s)$ whenever $k \in I^c$, and similarly $\widehat{E}(s)$ as the matrix with k -th column given by the k -th column of $E(s)$ whenever $k \in I$, and equal to the k -th column of $F(s)$ whenever $k \in I^c$, then $\widehat{G}(s) = -\widehat{E}^{-1}(s)\widehat{F}(s)$ is proper. Hence by realization theory we obtain a system as in (47) with inputs $f_k, k \in I, e_k, k \in I^c$, and outputs $e_k, k \in I, f_k, k \in I^c$.

4. DIFFERENTIAL OPERATOR DIRAC STRUCTURES OVER A BOUNDED INTERVAL

In this section we will show how the Dirac structure $\mathcal{S}_{\mathcal{D}}$ over the *infinite* spatial domain \mathbb{R} turns into a new Dirac structure when restricting to any *finite* spatial domain $[\alpha, \beta] \subset \mathbb{R}$, involving *boundary variables* at the end points α, β . This idea was first introduced in the context of Stokes-Dirac structures on a bounded domain (for arbitrary dimensions) in van der Schaft and Maschke (2002). Subsequently this was extended in Le Gorrec et al. (2005) to Dirac structures on a finite interval induced by skew-adjoint linear differential operators. The present paper extends this to Dirac structures induced by *pairs* of linear differential operators $F(\frac{d}{dz}), E(\frac{d}{dz})$ on a finite interval, satisfying (9), (10). Furthermore, the treatment is extended and simplified by the use of two-variable polynomial matrix calculus. In fact, the mathematics employed in this section is similar to the one in van der Schaft and Rapisarda (2011).

Consider the differential operators $F(\frac{d}{dz})$ and $E(\frac{d}{dz})$ satisfying (9) and (10), defining a differential operator Dirac structure over \mathbb{R} . Obviously (9) is the same as

$$F(s)E^{\top}(-s) + E(s)F^{\top}(-s) = 0 \quad (17)$$

It follows that the two variable expression $F(\zeta)E^{\top}(\eta) + E(\zeta)F^{\top}(\eta)$ is zero for $\zeta + \eta = 0$, and thus, cf. Willems and Trentelman (1998); van der Schaft and Rapisarda (2011),

$$F(\zeta)E^{\top}(\eta) + E(\zeta)F^{\top}(\eta) = (\zeta + \eta)\Pi(\zeta, \eta) \quad (18)$$

for some two-variable symmetric polynomial matrix $\Pi(\zeta, \eta)$. By using the theory of factorization of two-variable polynomial matrices, cf. Section 2, it follows that we can write

$$\Pi(\zeta, \eta) = Z^{\top}(\zeta)\Sigma Z(\eta) \quad (19)$$

for some polynomial matrix $Z(s)$ and invertible symmetric matrix Σ . Hence, following the developments in van der Schaft and Rapisarda (2011), the differential operator $Z(\frac{d}{dz})$ defines a minimal *boundary map* such that $b(z)$ in (16) is determined as

$$b(z) = Z(\frac{d}{dz})\ell(z), \quad (20)$$

where $\ell(z)$ is the vector of latent variables of the image representation (12).

Proposition 3. Consider $F(\frac{d}{dz}), E(\frac{d}{dz})$ satisfying (9), (10). Consider the factorization (18), (19), (20). Then for any $\alpha, \beta \in \mathbb{R}$ the space of functions $f(z), e(z), z \in [\alpha, \beta]$, and vectors $b(\alpha), b(\beta)$, which are solutions of

$$\begin{bmatrix} f(z) \\ e(z) \end{bmatrix} = \begin{bmatrix} E^{\top}(-\frac{d}{dz}) \\ F^{\top}(-\frac{d}{dz}) \end{bmatrix} \ell(z), \quad (21)$$

$$b(\alpha) = Z(\frac{d}{dz})\ell(\alpha), \quad b(\beta) = Z(\frac{d}{dz})\ell(\beta),$$

defines a Dirac structure with respect to the bilinear form

$$\begin{aligned} \int_{\alpha}^{\beta} \langle (f_1(z), e_1(z)), (f_2(z), e_2(z)) \rangle dz \\ - b_1^{\top}(\beta)\Sigma b_2(\beta) + b_1^{\top}(\alpha)\Sigma b_2(\alpha) \end{aligned} \quad (22)$$

Note that by using two-variable polynomial calculus we do not have to rely on the 'state' space realization of the 'transfer' matrix $G(s)$, but we may directly construct the 'state' $b(z)$ and the matrix Σ in (16) from the factorization (18); mirroring the treatment (in the time-domain!) in van der Schaft and Rapisarda (2011). In particular, we do not have to rely on $G(s)$ (or $\widehat{G}(s)$) being proper.

A special case is provided by formally *skew-adjoint* differential operators, corresponding to $E(s) = I_m$ (hence $e(z) = \ell(z)$), and $F^{\top}(s) = -F(-s)$.

Example 4. (Stokes-Dirac structure). The simplest example of a differential operator Dirac structure is the one defined by the formally skew-adjoint differential operator

$$F(s) = \begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix}, \quad E(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (23)$$

In this case (18) amounts to

$$F(\zeta) + F^{\top}(\eta) = (\zeta + \eta) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (24)$$

with $\Sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and boundary map $Z(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

This is the scalar spatial domain version of the Stokes-Dirac structure introduced in van der Schaft and Maschke (2002).

Remark 5. The Stokes-Dirac structure as introduced in van der Schaft and Maschke (2002) is defined on spatial

domains of *arbitrary* dimension, by replacing the scalar spatial differentiation $\frac{d}{dz}$ by the exterior derivative d . Similarly, much of the theory developed in the present paper can be extended to higher-dimensional spatial domains by replacing $\frac{d}{dz}$ by d , and considering polynomials in d .

A very important case in the definition of the boundary variables $b(z)$ occurs if Σ has *as many positive singular values as negative singular values*. In this case we can always take Σ to be in the canonical form

$$\Sigma = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad 2p = n \quad (25)$$

Then by denoting

$$\begin{bmatrix} f_\delta \\ e_\delta \end{bmatrix} := b(z) = Z\left(\frac{d}{dz}\right)\ell(z), \quad f_\delta, e_\delta \in \mathbb{R}^p, \quad (26)$$

we have

$$e_1^\top f_2 + e_2^\top f_1 = \frac{d}{dz} (e_{\delta 1}^\top f_{\delta 2} + e_{\delta 2}^\top f_{\delta 1}) \quad (27)$$

Equation (27) has an immediate interpretation in terms of *power balance*. Indeed, in integral form it amounts

$$\int_\alpha^\beta e_1^\top(z) f_2(z) + e_2^\top(z) f_1(z) dz = e_{\delta 1}^\top(\beta) f_{\delta 2}(\beta) + e_{\delta 2}^\top(\beta) f_{\delta 1}(\beta) - e_{\delta 1}^\top(\alpha) f_{\delta 2}(\alpha) - e_{\delta 2}^\top(\alpha) f_{\delta 1}(\alpha) \quad (28)$$

for any interval $[\alpha, \beta] \subset \mathbb{R}$. In particular, by taking $f := f_1 = f_2, e := e_1 = e_2, f_\delta := f_{\delta 1} = f_{\delta 2}, e_\delta := e_{\delta 1} = e_{\delta 2}$, this implies the power balance

$$\int_\alpha^\beta e^\top(z) f(z) dz = e_\delta^\top(\beta) f_\delta(\beta) - e_\delta^\top(\alpha) f_\delta(\alpha), \quad (29)$$

where the left-hand side is the incoming power on the interval (α, β) , and the right-hand side is the difference of the outgoing power at the right-end β and the outgoing power at the left-end α . The variables f_δ, e_δ are called *boundary power variables*, since $e_\delta^\top f_\delta$ equals power.

The condition that Σ has as many positive as negative singular values can be verified as follows. Consider the two-variable polynomial matrix $\Pi(\zeta, \eta)$ in (18) obtained from dividing $F(\zeta)E^\top(\eta) + E(\zeta)F^\top(\eta)$ by $\zeta + \eta$. Then

Proposition 6. (Trentelman and Rapisarda (1999)). (Prop. 2.1) Σ has as many positive as negative singular values if and only if the coefficient matrix $\tilde{\Pi}$ of $\Pi(\zeta, \eta)$ has.

Remark 7. This should be compared with the approach taken in Le Gorrec et al. (2005), where boundary variables similar to f_δ, e_δ are defined *in general*. However, this is enforced at the expense of ‘mixing’ the values at the left boundary α and the right boundary β ; in this way extending the matrix Σ to a matrix of double dimension $\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$, which always has as many positive as negative singular values.

The boundary map $b(z) = Z\left(\frac{d}{dz}\right)\ell(z)$ can be given the following interpretation. Consider $f_1(z), e_1(z)$ satisfying (8) on an interval $[\alpha, \gamma]$ and $f_2(z), e_2(z)$ satisfying (8) on an interval $[\gamma, \beta]$. When does the *concatenation* of $f_1(z), e_1(z)$ and $f_2(z), e_2(z)$, i.e., f, z on $[\alpha, \beta]$ defined as

$$\begin{aligned} f(z) &= f_1(z), & z \in [\alpha, \gamma], & & f(z) &= f_2(z), & z \in (\gamma, \beta] \\ e(z) &= e_1(z), & z \in [\alpha, \gamma], & & e(z) &= e_2(z), & z \in (\gamma, \beta] \end{aligned}$$

satisfy (8) on $[\alpha, \beta]$ in a *weak sense*? This holds if and only if

$$b_1(\gamma) = b_2(\gamma), \quad b_i(z) = Z\left(\frac{d}{dz}\right) \begin{bmatrix} f_i(z) \\ e_i(z) \end{bmatrix}, \quad i = 1, 2 \quad (30)$$

Thus in general $b(\gamma)$ provides exactly the information needed to extend a solution $f(z), e(z)$ on an interval $[\alpha, \gamma]$ to a larger interval $[\alpha, \beta]$. This has been discussed in more detail in the related context of linear partial differential equations involving both spatial and time variables in van der Schaft and Rapisarda (2013).

4.1 Differential operator effort constraints

Another interesting case to be considered concerns differential operator Dirac structures arising from skew-adjoint differential operators and differential operator constraints on the effort variables e . This case is well-motivated from an applications point of view. Let $J\left(\frac{d}{dz}\right)$ be a linear differential operator which is formally skew adjoint, i.e., $J(s) = -J^\top(-s)$. Consider the linear space of functions $f(z), e(z)$ satisfying the implicit set of differential equations

$$\{(f(z), e(z)) \mid \exists \lambda(z) \text{ such that}$$

$$f(z) = J\left(\frac{d}{dz}\right)e(z) + G^\top\left(-\frac{d}{dz}\right)\lambda(z), \quad G\left(\frac{d}{dz}\right)e(z) = 0\} \quad (31)$$

where $G\left(\frac{d}{dz}\right)$ is a linear differential operator representing the *effort constraints* $G\left(\frac{d}{dz}\right)e(z) = 0$. The vector $\lambda(z)$ represents a vector of Lagrange multiplier functions. For any such $f_i(z), e_i(z), i = 1, 2$, belonging to the set (31) for some $\lambda_i(z), i = 1, 2$, we compute

$$\begin{aligned} &< (f_1(z), e_1(z)), (f_2(z), e_2(z)) > = \\ &f_1(z)^\top e_2(z) + e_1(z)^\top f_2(z) = \\ &e_2^\top [J\left(\frac{d}{dz}\right)e_1 + G^\top\left(-\frac{d}{dz}\right)\lambda_1] + e_1^\top [J\left(\frac{d}{dz}\right)e_2 + G^\top\left(-\frac{d}{dz}\right)\lambda_2] \\ &= e_2^\top J\left(\frac{d}{dz}\right)e_1 + e_1^\top J\left(\frac{d}{dz}\right)e_2 + \\ &e_2^\top G^\top\left(-\frac{d}{dz}\right)\lambda_1 + e_1^\top G^\top\left(-\frac{d}{dz}\right)\lambda_2 \end{aligned} \quad (32)$$

The integral of the term $e_2^\top J\left(\frac{d}{dz}\right)e_1 + e_1^\top J\left(\frac{d}{dz}\right)e_2$ over any finite interval $[\alpha, \beta]$ can be computed as follows. Integration by parts and use of $J(s) = -J^\top(-s)$ yields

$$\begin{aligned} &\int_\alpha^\beta [e_2(z)^\top J\left(\frac{d}{dz}\right)e_1(z) + e_1(z)^\top J\left(\frac{d}{dz}\right)e_2(z)] dz = \\ &[Z_J\left(\frac{d}{dz}\right)e_1(z)]^\top \Pi_J Z_J\left(\frac{d}{dz}\right)e_2(z) \Big|_\alpha^\beta, \end{aligned} \quad (33)$$

where the differential operator $Z_J\left(\frac{d}{dz}\right)$ and the matrix Σ_J are obtained by the two-variable polynomial factorization

$$J(\zeta) + J^\top(\eta) = (\zeta + \eta) Z_J^\top(\zeta) \Sigma_J Z_J(\eta) \quad (34)$$

Analogously, the integral of $e_2^\top G^\top\left(-\frac{d}{dz}\right)\lambda_1 + e_1^\top G^\top\left(-\frac{d}{dz}\right)\lambda_2$ over $[\alpha, \beta]$ yields by integration by parts

$$\begin{aligned} &\int_\alpha^\beta [e_2^\top G^\top\left(-\frac{d}{dz}\right)\lambda_1 + e_1^\top G^\top\left(-\frac{d}{dz}\right)\lambda_2] dz = \\ &[Z_G\left(\frac{d}{dz}\right)e_2(z)]^\top \Pi_G V_G\left(\frac{d}{dz}\right)\lambda_1(z) + \\ &[Z_G\left(\frac{d}{dz}\right)e_1(z)]^\top \Pi_G V_G\left(\frac{d}{dz}\right)\lambda_2(z) \Big|_\alpha^\beta, \end{aligned} \quad (35)$$

where $Z_G(\frac{d}{dz})$, $V_G(\frac{d}{dz})$ and the matrix Π_G are obtained by the two-variable polynomial factorization

$$G^\top(-\eta) - G^\top(\zeta) = (\zeta + \eta)Z_G^\top(\zeta)\Pi_G V_G(\eta) \quad (36)$$

Thus two sources for boundary terms are arising. The first one defined by $Z_J(\frac{d}{dz})$ and Σ_J , resulting from the formally skew-adjoint linear differential operator $J(\frac{d}{dz})$, and the second defined by $Z_G(\frac{d}{dz})$, $V_G(\frac{d}{dz})$ and Π_G , resulting from the effort constraints $G(\frac{d}{dz})e(z) = 0$ in (31). By including these boundary terms one then obtains a differential operator Dirac structure on any spatial domain $[\alpha, \beta]$.

Proposition 8. Consider a formally skew-adjoint differential operator $J(\frac{d}{dz})$ and a differential operator $G(\frac{d}{dz})$ defining the effort constraints $G(\frac{d}{dz})e(z) = 0$. Consider the two variable polynomial factorizations (34), (36).

Then for any $\alpha, \beta \in \mathbb{R}$ the space defined by (31) and the boundary variables

$$\begin{aligned} b_J(z) &= Z_J(\frac{d}{dz})e(z) \\ b_G(z) &= Z_G(\frac{d}{dz})e(z), \quad c_G(z) = V_G(\frac{d}{dz})\lambda(z) \end{aligned} \quad (37)$$

defines a Dirac structure with respect to the bilinear form

$$\begin{aligned} \int_\alpha^\beta &< (f_1(z), e_1(z)), (f_2(z), e_2(z)) > dz \\ &- b_{J_1}^\top(\beta)\Sigma_J b_{J_2}(\beta) + b_{J_1}^\top(\alpha)\Sigma_J b_{J_2}(\alpha) \\ &- b_{G_2}^\top(\beta)\Pi_G c_{G_1}(\beta) + b_{G_2}^\top(\alpha)\Pi_G c_{G_1}(\alpha) \\ &- b_{G_1}^\top(\beta)\Pi_G c_{G_2}(\beta) + b_{G_1}^\top(\alpha)\Pi_G c_{G_2}(\alpha) \end{aligned} \quad (38)$$

Furthermore, we conjecture that any differential operator Dirac structure defined by a pair $F(\frac{d}{dz}), E(\frac{d}{dz})$ as above can be also represented as a differential operator Dirac structure defined by a skew-adjoint operator $J(\frac{d}{dz})$ and some $G(\frac{d}{dz})$; analogously to the finite-dimensional case exposed in van der Schaft and Maschke (2018).

5. DIFFERENTIAL OPERATOR LAGRANGIAN SUBSPACES

Consider again $\mathcal{F} \times \mathcal{E}$, where $\mathcal{E} = \mathcal{F}^*$. In view of applications we will replace the notation $\mathcal{F} = \mathbb{R}^m$ by $\mathcal{X} = \mathbb{R}^m$, with \mathcal{X} standing for a linear state space. In fact, in port-Hamiltonian systems theory the flow space \mathcal{F} is actually the *tangent space* to the *space of energy variables* \mathcal{X} . In the present linear case \mathcal{F} obviously can be identified with \mathcal{X} . Apart from the bilinear form $\langle \cdot, \cdot \rangle$ in (5) there is *another* canonically defined bilinear form on $\mathcal{X} \times \mathcal{E}$, defined as

$$[(x_1, e_1), (x_2, e_2)] := \langle e_1 | x_2 \rangle - \langle e_2 | x_1 \rangle, \quad (39)$$

which has the matrix representation

$$J_e = \begin{bmatrix} 0_m & I_m \\ -I_m & 0_m \end{bmatrix} \quad (40)$$

As in the case of the bilinear form $\langle \cdot, \cdot \rangle$, the skew-symmetric bilinear form $[\cdot, \cdot]$ with matrix representation J_e on $\mathcal{X} \times \mathcal{E} = \mathbb{R}^{2m}$ gives rise to the following bilinear form on the set $\mathcal{C}(\mathbb{R}, \mathcal{X} \times \mathcal{E})$ of smooth functions $(x, e) : \mathbb{R} \rightarrow \mathcal{X} \times \mathcal{E}$ with compact support:

$$[[(x_1, e_1), (x_2, e_2)]] := \int_{-\infty}^{\infty} [(x_1(z), e_1(z)), (x_2(z), e_2(z))] dz$$

This is again a skew-symmetric non-degenerate form, in the sense that if $[[(x_1, e_1), (x_2, e_2)]] = 0$ for all compact

support (x_1, e_1) , then $(x_2, e_2) = 0$. Thus it defines a *symplectic form* on $\mathcal{C}(\mathbb{R}, \mathcal{X} \times \mathcal{E})$. Recall that a subspace \mathcal{L} of a linear space \mathcal{V} with symplectic form ω is called *Lagrangian* if $\mathcal{L} = \mathcal{L}^\perp$ where $^\perp$ denotes the orthogonal complement with respect to the symplectic form ω . Equivalently, \mathcal{L} is Lagrangian if ω is zero when restricted to \mathcal{L} , and moreover \mathcal{L} is maximal with respect to this property. This leads to the following definition of a *differential operator Lagrangian subspace* over the infinite spatial domain \mathbb{R} . Consider systems of differential equations over \mathbb{R} given by

$$S^\top(-\frac{d}{dz})x(z) = P^\top(-\frac{d}{dz})e(z), \quad z \in \mathbb{R}, \quad (41)$$

where $P(\frac{d}{dz})$ and $S(\frac{d}{dz})$ are square linear differential operators. Denote the corresponding $m \times m$ polynomial matrices by $P(s)$ and $S(s)$. Now let $P(s)$ and $S(s)$ satisfy

$$\begin{bmatrix} P^\top(-s) & S^\top(-s) \end{bmatrix} \begin{bmatrix} 0_m & I_m \\ -I_m & 0_m \end{bmatrix} \begin{bmatrix} P(s) \\ S(s) \end{bmatrix} = 0, \quad \forall s \in \mathbb{C}, \quad (42)$$

together with the maximal rank assumption

$$\text{rank} \begin{bmatrix} P(s) \\ S(s) \end{bmatrix} = m, \quad \forall s \in \mathbb{C} \quad (43)$$

Similar as before, it follows that

$$\ker [-S^\top(-s) \quad P^\top(-s)] = \text{im} \begin{bmatrix} P(s) \\ S(s) \end{bmatrix}, \quad (44)$$

and thus the kernel representation (41) can be replaced by the image representation

$$\begin{bmatrix} x(z) \\ e(z) \end{bmatrix} = \begin{bmatrix} P(\frac{d}{dz}) \\ S(\frac{d}{dz}) \end{bmatrix} \ell(z) \quad (45)$$

Furthermore, cf. van der Schaft and Rapisarda (2011), the set $\mathcal{S}_{\mathcal{L}}$ of solutions with compact support of (41) satisfies

$$\int_{-\infty}^{\infty} [(x_1(z), e_1(z)), (x_2(z), e_2(z))] dz = 0 \quad (46)$$

for all $(x_1, e_1), (x_2, e_2) \in \mathcal{S}_{\mathcal{L}}$. Hence $\mathcal{S}_{\mathcal{L}} \subset \mathcal{S}_{\mathcal{L}}^\perp$. By the same reasoning as in the differential operator Dirac structure case it follows that $\mathcal{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}}^\perp$, and thus $\mathcal{S}_{\mathcal{L}}$ is an infinite-dimensional Lagrangian subspace.

Proposition 9. Consider $P(s), S(s)$ satisfying (42) and (43). Let $\mathcal{S}_{\mathcal{L}}$ be the set of smooth solutions of (41) with compact support within \mathbb{R} . Then $\mathcal{S}_{\mathcal{L}}$ is a Lagrangian subspace with respect to the bilinear form (7).

Also, assuming for the moment that the rational matrix $K(s) := -S^{-\top}(s)P^\top(s)$ is *proper*, it follows from standard linear system realization theory that the behavior $\mathcal{S}_{\mathcal{L}}$ is generated by a minimal 'state' space system of the form

$$\begin{aligned} \frac{d}{dz}c(z) &= Ac(z) + Bx(z) \\ e(z) &= Cc(z) + Dx(z) \end{aligned} \quad (47)$$

with 'state' vector $c(z) \in \mathbb{R}^n$, where n is the McMillan degree of $K(s)$. Furthermore, by (42) the 'transfer matrix' $K(s)$ satisfies $K(s) = K^\top(-s)$. Therefore the system (47) is an input-output Hamiltonian system (van der Schaft and Jeltsema (2014)) in the sense that there exists an invertible matrix $J_i = -J_i^\top$ satisfying

$$A^\top J_i + J_i A = 0, \quad B^\top J_i = C, \quad D = D^\top \quad (48)$$

In case $K(s)$ is not proper, the results continue to hold as before. By two-variable polynomial calculus the skew-symmetric matrix J_i can be directly inferred from the fact that by (42) $P^\top(-s)S(s) - S^\top(-s)P(s) = 0$, and thus

$$P^\top(\zeta)S(\eta) - S^\top(\zeta)P(\eta) = (\zeta + \eta)W^\top(\zeta)\Pi W(\eta) \quad (49)$$

for some polynomial matrix $W(s)$ and full-rank skew-symmetric matrix Π . Hence, without loss of generality (note that this is fundamentally different from the Dirac structure case!) we can take

$$\Pi = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} =: J_i \quad (50)$$

Then by defining the image representation

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} P(\frac{d}{dz}) \\ S(\frac{d}{dz}) \end{bmatrix} \ell, \quad \begin{bmatrix} x_\delta \\ e_\delta \end{bmatrix} = W(\frac{d}{dz})\ell, \quad (51)$$

we have

$$x_1^\top e_2 - x_2^\top e_1 = \frac{d}{dz} (x_{\delta 1}^\top e_{\delta 2} - x_{\delta 2}^\top e_{\delta 1}) \quad (52)$$

Proposition 10. Consider $P(\frac{d}{dz}), S(\frac{d}{dz})$ satisfying (42), (43). Consider the factorization (49), (50), (51). Then for any $\alpha, \beta \in \mathbb{R}$ the space of $x(z), e(z), b(\alpha), b(\beta)$ satisfying

$$\begin{bmatrix} x(z) \\ e(z) \end{bmatrix} = \begin{bmatrix} P(\frac{d}{dz}) \\ S(\frac{d}{dz}) \end{bmatrix} \ell(z), \quad z \in [\alpha, \beta] \quad (53)$$

$$\begin{bmatrix} x_\delta(\alpha) \\ e_\delta(\alpha) \end{bmatrix} = W(\frac{d}{dz})\ell(\alpha), \quad \begin{bmatrix} x_\delta(\beta) \\ e_\delta(\beta) \end{bmatrix} = W(\frac{d}{dz})\ell(\beta)$$

defines a Lagrangian subspace with respect to

$$\int_\alpha^\beta [(x_1(z), e_1(z)), (x_2(z), e_2(z))] dz + x_{\delta 1}^\top(\beta)e_{\delta 2}(\beta) - e_{\delta 1}^\top(\beta)x_{\delta 2}(\beta) - x_{\delta 1}^\top(\alpha)e_{\delta 2}(\alpha) + e_{\delta 1}^\top(\alpha)x_{\delta 2}(\alpha) \quad (54)$$

A special case of the above is provided by $P(s) = I_m$ and $S(\frac{d}{dz})$ a formally self-adjoint differential operator, i.e., $S^\top(-s) = S(s)$. All of this will be further investigated in Maschke and van der Schaft (2021). In particular, in Maschke and van der Schaft (2021) the exposed theory of differential operator Dirac structures will be combined with the theory of differential operator Lagrangian subspaces in order to give a general definition of port-Hamiltonian systems defined by differential operators This combination is based on coupling a differential operator Dirac structure with elements $(f(z, t), e(z, t))$ with a differential operator Lagrangian subspace with elements $(x(z, t), e(z, t))$ by setting $f(z, t) = -\dot{x}(z, t)$; in the spirit of the theory of linear DAE port-Hamiltonian systems exposed in van der Schaft and Maschke (2018), Beattie et al. (2017); see van der Schaft and Maschke (2020) for the nonlinear extension.

6. CONCLUSIONS

It has been shown how, by using two-variable polynomial matrix calculus, differential operator Dirac structures can be defined on finite intervals, starting from general pairs of linear differential operators defining a Dirac structure on the whole real line. Of particular interest for applications is

the closely related class of Dirac structures with boundary variables derived from formally skew-adjoint operators together with differential operator effort constraints. In order that the boundary variables can be split into boundary flow and effort variables an extra condition is identified on the coefficient matrix of a two-variable polynomial matrix. Using a different canonical bilinear form it is shown how the same construction can be also used for the definition of Lagrangian subspaces on bounded intervals.

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