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Existence of a Smooth Hamiltonian Circle Action near Parabolic Orbits and Cuspidal Tori

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Abstract—We show that every parabolic orbit of a two-degree-of-freedom integrable system admits a C^∞-smooth Hamiltonian circle action, which is persistent under small integrable C^∞ perturbations. We deduce from this result the structural stability of parabolic orbits and show that they are all smoothly equivalent (in the non-symplectic sense) to a standard model. As a corollary, we obtain similar results for cuspidal tori. Our proof is based on showing that every symplectomorphism of a neighbourhood of a parabolic point preserving the first integrals of motion is a Hamiltonian whose generating function is smooth and constant on the connected components of the common level sets.

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1. INTRODUCTION

Parabolic orbits and associated cuspidal tori are one of the simplest degenerate singularities of integrable systems, which often appear in concrete integrable models, including rigid body dynamics [2, 3] and systems admitting rotational symmetry [6, 9] (see also [7, 8, 10]). Such singularities have been extensively studied in the literature from different points of view, ranging from topological and stability properties [10, 13, 16] to symplectic geometry and semi-classical analysis [1, 4]. Nevertheless, some of the fundamental questions about these singularities, such as the ones addressed in this and our related work [14], have remained (or remain) open until now.

A typical example of a parabolic orbit is given by the (singular) fibration induced by the energy-momentum map

$$F = (H, J)\colon \mathbb{R}^3 \times S^1 \to \mathbb{R}^2, \tag{1.1}$$

where $H = x^2 - y^3 + \lambda y$ and $J = \lambda$; here x, y, λ are Euclidean coordinates on $\mathbb{R}^3$. If φ denotes the standard angle coordinate on S^1, then the symplectic structure can be of the form

$$\omega_0 = dx \wedge dy + d\lambda \wedge d\varphi$$

or, more generally,

$$\omega = g(x, y, \lambda)dx \wedge dy + d\lambda \wedge \big(d\varphi + A(x, y, \lambda)dx + B(x, y, \lambda)dy\big), \tag{1.2}$$

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where g, A and B are smooth functions[1)]. Note that in many mechanical systems parabolic orbits lie on compact critical fibers, called *cuspidal tori*. A simple fibration with compact fibers can be given by considering

$$\hat{F} = (\hat{H} = x^2 - y^3 + \lambda y + y^4, J) \tag{1.3}$$

instead of F (with the symplectic structure as above). The cuspidal torus is then $\hat{F}^{-1}(0,0)$; it contains the parabolic orbit $x = y = \lambda = 0$.

The singular fibration $F\colon \mathbb{R}^3 \times S^1 \to \mathbb{R}^2$ is schematically shown in Fig. 1, together with the corresponding *bifurcation diagram*, which is the set of the critical values of F, and the *bifurcation complex* — the space of the connected components of the pre-images $F^{-1}(f)$, $f \in \mathbb{R}^2$ (the local bifurcation diagram and the bifurcation complex of $\hat{F}$ are essentially the same as those of F).

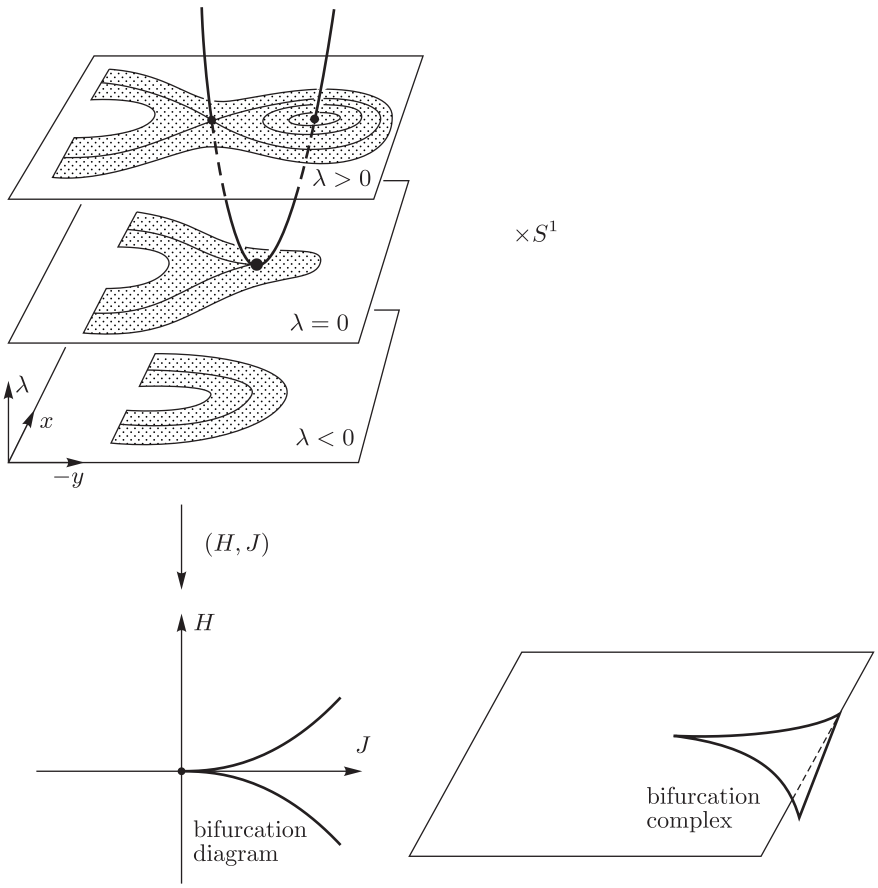

Fig. 1. The singular Lagrangian fibration (top), the bifurcation diagram (bottom left) and the bifurcation complex (bottom right) of the energy-momentum map (H, J).

As we will show in this paper, the fibration $F\colon \mathbb{R}^3 \times S^1 \to \mathbb{R}^2$ is, in fact, a “standard” model of a parabolic orbit in the sense that a neighbourhood of such an orbit can always be put into the form (1.1)–(1.2) (similarly, (1.2)–(1.3) is a standard model of a cuspidal torus).

Remark 1. This smooth normal form result can be compared with [5], where a similar approach of bringing the singular fibration to a “standard form” is used. If we fix the symplectic structure to be canonical instead, that is, if $\omega = \omega_0 = dx \wedge dy + d\lambda \wedge d\varphi$, then we get that every parabolic orbit arises (up to a fiberwise C^∞ symplectomorphism) from an energy-momentum map

$$\tilde{F} = (\tilde{H}, J = \lambda)\colon \mathbb{R}^3 \times S^1 \to \mathbb{R}^2,$$

[1)] In this paper, we consider integrable systems that are of the C^∞ differentiability class; in particular, the Hamiltonian and the first integrals as well as the symplectic form are always assumed to be smooth.

where $\tilde{H}$ is a smooth function in (x, y, λ) such that $\beta = (0,0,0) \times S^1$ is a parabolic orbit in the sense of [1, Definition 2.1]). These two ("preliminary") normal forms are, in fact, equivalent (see the proof of Theorem 3 below), but the first one has the advantage that it immediately implies that all parabolic orbits are fiberwise C^∞ diffeomorphic (in general, $\tilde{H}$ is no longer of the form $x^2 - y^3 + \lambda y$ by [1, 14]).

We note that our normal form result mentioned above is well-known when we make the additional assumption that the integrable system admits a smooth and free Hamiltonian circle action near a parabolic orbit. (In the above models (1.1)–(1.2) and (1.2)–(1.3), the Hamiltonian circle action is given by the periodic integral J.) It is also known that parabolic orbits admitting a Hamiltonian circle action are *smoothly structurally stable* in the space of integrable systems with such an action; see [13, 16]. The main motivation for the present work is to remove the extra assumption on the existence of a smooth circle action in the above results.

We note that it is not difficult to show that in a neighborhood of a parabolic orbit (resp., cuspidal torus), excluding the orbit itself, a smooth circle action always exists. Indeed, it is given by the flow of the 2π-periodic first integral

$$J = \frac{1}{2\pi} \int_c \alpha, \tag{1.4}$$

where α is a primitive one-form for the symplectic structure and $c \subset F^{-1}(f)$ is a cycle homologous to the parabolic orbit. The smoothness of J (equivalently, of the circle action) follows [5, 17] from the non-degeneracy of co-rank 1 singularities on the complement of the parabolic orbit. The main problem is therefore to prove the smoothness of the periodic integral J near the parabolic orbit itself. We remark that in the analytic category, the corresponding result is known: J and the circle action are analytic in the case when the integrals and symplectic form are analytic [23]. It follows that the analytic equivalence of parabolic orbits and their analytic structural stability hold without the additional assumption on the existence of a circle action [1, 15, 23]. The same can be said about the topological equivalence and topological structural stability [13, 16] (one can show this independently, even without proving the existence of a C^0 circle action). What has remained open until now is whether or not the corresponding results are also true in the smooth C^∞ situation.

In the present paper, we prove that this is indeed the case. More specifically, we show that every parabolic orbit of an integrable two-degree-of-freedom system admits a smooth system-preserving free Hamiltonian circle action. We deduce from this result that

i) from the smooth point of view, all parabolic orbits are equivalent, i. e., any two such orbits admit fiberwise diffeomorphic neighbourhoods (which is the direct product of a "standard" 3-dimensional Poincaré cross-section and a circle; see Fig. 1). Note that this implies directly the existence of a C^∞ circle action by the explicit formula (1.4), so the existence of such a circle action is, in fact, necessary and sufficient for this statement;

ii) parabolic orbits are smoothly structurally stable in the space of all smooth 2-degree-of-freedom integrable systems (this means that a small integrable C^∞ perturbation of a parabolic singularity is again a parabolic singularity, which is moreover fiberwise diffeomorphic to the unperturbed one).

As a corollary, we obtain similar results for cuspidal tori. Specifically, there always exists a smooth circle action in a neighbourhood of a cuspidal torus, all cuspidal tori are C^∞ equivalent and they are also C^∞ structurally stable.

The main ingredient in our proof is to show that any F-preserving symplectomorphism of a neighbourhood of a parabolic *point*[2)] (and therefore also of a parabolic orbit) is, in fact, a Hamiltonian symplectomorphism whose generating function is constant on the connected components of the common level sets $\{F = f\}$, $f \in \mathbb{R}^2$. This implies that any such symplectomorphism is smoothly isotopic to the identity in the class of F-preserving symplectomorphisms.

[2)] This is a rank-one singular point that locally admits the (non-canonical) coordinates (x, y, λ, φ) as above.

We note that a similar result is known for elliptic, non-degenerate co-rank 1, and focus-focus singularities [5, 11, 17, 21]. However, it is false in general: the symplectomorphism $(x, y) \mapsto (-y, x)$ of $(\mathbb{R}^2, dx \wedge dy)$ preserves the function $H = x^4 + y^4$, but the corresponding generating function is not smooth (not even C^2 differentiable) at the origin. This means that this symplectomorphism cannot be included into a smooth H-preserving Hamiltonian flow. In fact, it cannot be connected to the identity by a smooth (or even C^3) H-preserving homotopy. This shows that, in the context of integrable systems, the problem of the inclusion of a smooth or analytic (symplectic) map into a smooth/analytic flow (cf. [18, 19] and references therein) does not admit a universal solution, even in the case of polynomial first integrals.

2. MAIN RESULTS

In this section, we prove that a neighbourhood of a parabolic orbit of a two-degree-of-freedom system admits a free Hamiltonian circle action (and, in particular, a periodic integral) in the smooth C^∞ case. Such a result will be used in a subsequent work on the symplectic classification of parabolic orbits and cuspidal tori in the smooth category [14]; cf. work [1] for the analytic case.

Let $(\tilde{H}, G)\colon U \to \mathbb{R}^2$ be an integrable system with a parabolic orbit β (for a formal definition of a parabolic orbit, see [1, Definition 2.1]). Assume dG is non-zero along β. Then (due to [1, 16]) near each point $P \in \beta$, one can introduce (non-canonical) coordinates $(x, y, \lambda, \varphi) \in D^4$ centred at this point (note that here φ is only a local coordinate) and a smooth function $H(\tilde{H}, G)$ with $\partial_{\tilde{H}} H(0, 0) \neq 0$ such that

$$H(\tilde{H}, G) = x^2 - y^3 + \lambda y \text{ and } G = \pm\lambda + \text{const}$$

and the symplectic structure has the form

$$g(x, y, \lambda)dx \wedge dy + d\lambda \wedge (d\varphi + A(x, y, \lambda)dx + B(x, y, \lambda)dy).$$

The Hamiltonian flow of G gives rise to the first return map $\mu\colon D^3 \to D^3$, where D^3 is a cross-section given by $\varphi = 0$. The map μ is smooth. Our goal is to first prove the following.

Theorem 1. *The first return map μ can be written as the time-1 map of a smooth family of Hamiltonian vector fields (with respect to the symplectic structure $g(x, y, \lambda)dx \wedge dy$) that are tangent to the level curves $H(x, y, \lambda) = h$. Here λ is regarded as a parameter.*

Proof. Step 1. Consider the family of Lagrangian sections:

$$L_\lambda = \{(x, y, \lambda)\colon x = 0\}$$

and its image under μ. Since μ is a diffeomorphism preserving the functions H and G, the fixed point set of μ contains the parabola $\{x = 0, 3y^2 - \lambda = 0\}$; see Fig. 1.

Let μ^x and μ^y denote the x- and the y-components of μ, respectively. It can be shown (using that μ preserves the functions H and G, and that the y-axis and, hence, its μ-image are "squeezed" between the two branches of the invariant level set $\{\lambda = H = 0\}$, see Fig. 2) that $\mu^y(0, y, \lambda)$ is monotone with respect to y for all small (y, λ). The monotonicity implies that the following formula

$$u(y, \lambda) = \eta \int_y^{\mu^y(0,y,\lambda)} \frac{g(\eta\sqrt{t^3 - \lambda t + \lambda y - y^3}, t, \lambda)dt}{2\sqrt{t^3 - \lambda t + \lambda y - y^3}},$$

where $\eta = \operatorname{sign}(\mu^x(0, y, \lambda))$, is well defined for $\lambda \neq 3y^2$; see Fig. 3. We claim that $u = u(y, \lambda)$ extends to a smooth function in a neighbourhood of the origin; this is the content of Lemma 1 below.

Observe that $u = u(y, \lambda)$ admits a natural extension to a function $\hat{u} = \hat{u}(x, y, \lambda)$ that is constant on the connected components of $H = h$ for fixed λ; the function $\hat{u}$ is defined by the condition $\hat{u}(0, y, \lambda) = u(y, \lambda)$. We claim that $\hat{u} = \hat{u}(x, y, \lambda)$ is also smooth. The required family of Hamiltonian vector fields is then defined by

$$\hat{u}(x, y, \lambda)X_H,$$

where X_H denotes the Hamiltonian vector field of the function H with respect to the symplectic structure $g(x, y, \lambda)dx \wedge dy$; recall that here λ appears as a parameter. Indeed, outside the parabola

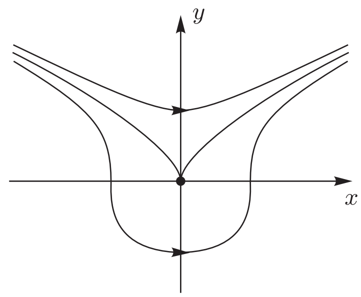

Fig. 2. The slice of the singular Lagrangian fibration for $\lambda = 0$.

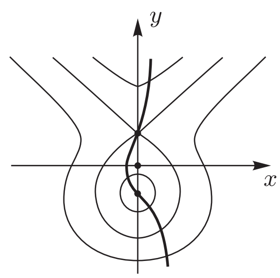

Fig. 3. The slice of the singular Lagrangian fibration for fixed $\lambda > 0$ and the μ-image of the y-axis.

$\{x = 0, 3y^2 = \lambda\}$, $\hat{u}(x, y, \lambda)$ is the time needed to reach $\mu(x, y, \lambda)$ from a point (x, y, λ) along the flow of X_H.

Step 2. To show that $\hat{u} = \hat{u}(x, y, \lambda)$ is smooth, observe that it can be written as $\tilde{u}(H, \lambda)$ and $\tilde{u}_\circ(H, \lambda)$ on the closures of each of the two open strata of the bifurcation complex; see Fig. 1. We will first show[3)] in steps 2 and 3 that the functions $\tilde{u}(H, \lambda)$ and $\tilde{u}_\circ(H, \lambda)$ are smooth on these closures (in the sense that each of these functions admits a smooth extension to an open neighbourhood of the closure of the corresponding stratum, or equivalently, to $\mathbb{R}^2$). Moreover, we shall show that the corresponding partial derivatives of $\tilde{u}$ and $\tilde{u}_\circ$ coincide on the "common boundary" $(\lambda \geqslant 0, H = 2(\lambda/3)^{3/2})$ of the two strata.

Consider the stratum that is not the swallow-tail domain, and let $\tilde{u}(H, \lambda)$ be the corresponding function defined on it. The smoothness of $\tilde{u}(H, \lambda)$ follows readily from the formula

$$\tilde{u}(H(\varepsilon, y, \lambda), \lambda) = \int_y^{\mu^y(\varepsilon, y, \lambda)} \frac{g(\sqrt{t^3 - \lambda t - y^3 + \lambda y}, t, \lambda)dt}{2\sqrt{t^3 - \lambda t - y^3 + \lambda y}};$$

recall that μ^y is the y-component of μ. Indeed, the right-hand side is smooth as a function of (y, λ) since $x = \varepsilon > 0$. Furthermore, at the point $(x = \varepsilon, y = \varepsilon^{2/3}, \lambda = 0)$, we have $H = 0$, but $\partial_y H = \lambda - 3y^2 \neq 0$. So we can take H as a local coordinate instead of y.

Now consider the swallow-tail stratum, on which $\tilde{u}_\circ(H, \lambda)$ is defined. Then we have smoothness at least in the open half-plane $\lambda > 0$ (in the above sense), since the singularities are 2D non-degenerate; cf. [5] and [17, Corollary 3.5]. Indeed, near the elliptic family, this can be shown separately, and near the hyperbolic family, this can be shown using the Lagrangian section $y = 0$ transversal to the fibers. We note that, using the section $y = 0$, we also have that the partial derivatives of $\tilde{u}$ and $\tilde{u}_\circ$ coincide on the set $(\lambda > 0, H = 2(\lambda/3)^{3/2})$.

[3)]In fact, to prove that $\hat{u}$ is smooth, we will need less information from $\tilde{u}_\circ$: it suffices to show that $\tilde{u}_\circ$ is smooth for $\lambda > 0$ and that its partial derivatives have a continuous limit at $(H = 0, \lambda = 0)$. We will still need the well-known property of non-degenerate singularities that all partial derivatives of $\tilde{u}$ and $\tilde{u}_\circ$ (more precisely, their limits) exist and coincide on the hyperbolic branch $(\lambda > 0, H = 2(\lambda/3)^{3/2})$, while all partial derivatives of $\tilde{u}_\circ$ continuously extend to the elliptic branch $(\lambda > 0,\ H = -2(\lambda/3)^{3/2})$ of the bifurcation complex.

Step 3. Let us now prove that the partial derivatives of $\tilde{u}_\circ(H,\lambda)$ extend continuously to the origin. We will then use this to prove that $\hat{u}$ is smooth (and also that the function $\tilde{u}_\circ(H,\lambda)$ admits a smooth extension, which, as we have noticed earlier, is not really needed for our purposes).

To this end, consider again the case $\lambda > 0$ and observe that

$$\partial_y u = \begin{cases} \partial_H \tilde{u}_\circ|_{(-y^3+\lambda y,\lambda)}(-3y^2+\lambda), & -2\sqrt{\lambda/3} < y < \sqrt{\lambda/3}, \\ \partial_H \tilde{u}|_{(-y^3+\lambda y,\lambda)}(-3y^2+\lambda), & \text{otherwise}, \end{cases}$$

where the left-hand side is a smooth function for all (y,λ) by Lemma 1. It follows that

$$\partial_y u = 0 \text{ for } \lambda = 3y^2.$$

Hence, $\partial_y u = A(y,\lambda)(\lambda - 3y^2)$ for some smooth function $A = A(y,\lambda)$ (this follows from a parametric version of Hadamard's lemma, which is the integral form of the first-order remainder term in Taylor's formula; see also Malgrange's preparation theorem [12]). The function A must then satisfy

$$A = \begin{cases} \partial_H \tilde{u}_\circ|_{(-y^3+\lambda y,\lambda)}, & -2\sqrt{\lambda/3} < y < \sqrt{\lambda/3}, \\ \partial_H \tilde{u}|_{(-y^3+\lambda y,\lambda)}, & \text{otherwise}. \end{cases}$$

We thus get that $\partial_H \tilde{u}_\circ$ extends continuously to $(H=0, \lambda=0)$, with the same limit as that of $\partial_H \tilde{u}$. Similarly one can prove the continuity of all partial derivatives. We note that Whitney's extension theorem [22] now implies an even stronger form of differentiability, namely, that $\tilde{u}_\circ(H,\lambda)$ admits a smooth extension to an open set, but we do not need this to prove that $\hat{u}(x,y,\lambda)$ is a smooth function.

Step 4. To show that $\hat{u} = \hat{u}(x,y,\lambda)$ is smooth, it is left to observe that, for each (x,y,λ), $\hat{u}(x,y,\lambda) = \tilde{u}(x^2-y^3+\lambda y,\lambda)$ or $\tilde{u}_\circ(x^2-y^3+\lambda y,\lambda)$. Indeed, outside the origin $(0,0,0)$, the smoothness of $\hat{u}$ follows since $\tilde{u}$ and $\tilde{u}_\circ$ are smooth and the restrictions of (the extensions of) the partial derivatives to $(\lambda \geqslant 0, H = 2(\lambda/3)^{3/2})$ coincide. Moreover, all of the partial derivatives of $\hat{u}$ will extend continuously to $(0,0,0)$ since we have proved that the partial derivatives of $\tilde{u}$ and $\tilde{u}_\circ$ extend continuously to $(H=0,\lambda=0)$. This implies (see, for example, [22, Section 3]) that $\hat{u} \in C^\infty$. □

In steps 1 and 3 of the proof, we used the following lemma.

Lemma 1. *The function*

$$u(y,\lambda) = \eta \int_y^{\mu^y(0,y,\lambda)} \frac{g(\eta\sqrt{t^3-\lambda t - y^3 + \lambda y}, t, \lambda)dt}{2\sqrt{t^3 - \lambda t - y^3 + \lambda y}},$$

where $\eta = \operatorname{sign}(\mu^x(0,y,\lambda))$ *and* $\lambda \neq 3y^2$*, admits a smooth extension to a neighbourhood of the origin.*

Proof. Let $t = y + z^2(\mu^y(0,y,\lambda) - y)$. Denote the difference $\mu^y(0,y,\lambda) - y$ by ν. Then, for $\nu \neq 0$,

$$u = \eta\nu \int_0^1 \frac{g(\eta z\sqrt{z^4\nu^3 + 3z^2\nu^2 y + 3\nu y^2 - \lambda\nu}, y + z^2\nu, \lambda)dz}{\sqrt{z^4\nu^3 + 3z^2\nu^2 y + 3\nu y^2 - \lambda\nu}}.$$

Observe that $\nu(3y^2 - \lambda) \geqslant 0$. Clearly,

$$z^4\nu^3 + 3z^2\nu^2 y + 3\nu y^2 - \lambda\nu = \nu(3y^2-\lambda)(1 + \frac{\nu}{3y^2-\lambda}(z^4\nu + 3z^2 y))$$

and

$$\frac{\eta\nu}{\sqrt{\nu(3y^2-\lambda)}} = \frac{\eta\nu\sqrt{\nu(3y^2-\lambda)}}{\nu(3y^2-\lambda)} = \frac{\eta\sqrt{\nu(3y^2-\lambda)}}{3y^2-\lambda}.$$

Hence, for $\lambda \neq 3y^2$ (including the case $\nu = 0, \lambda \neq 3y^2$),

$$u = \frac{\eta\sqrt{\nu(3y^2-\lambda)}}{3y^2-\lambda}\int_0^1 \frac{g(\eta z\sqrt{z^4\nu^3+3z^2\nu^2y+3\nu y^2-\lambda\nu}, y+z^2\nu, \lambda)dz}{\sqrt{1+\dfrac{\nu}{3y^2-\lambda}(z^4\nu+3z^2y)}}.$$

Now, $\nu = \nu(y,\lambda)$ is a smooth function that is zero on $3y^2 = \lambda$. By Hadamard's lemma,

$$\frac{\nu}{3y^2-\lambda} \text{ and } \left(1+\frac{\nu}{3y^2-\lambda}(z^4\nu+3z^2y)\right)^{\pm 1/2},$$

which are well defined for $\lambda \neq 3y^2$, admit smooth extensions to a small neighbourhood of the origin (when (y,λ) are small enough).

Next, observe that upon substitution of $z = 1$ in the expression

$$\eta z\sqrt{z^4\nu^3+3z^2\nu^2y+3\nu y^2-\lambda\nu} = \eta z\sqrt{\nu(3y^2-\lambda)}\sqrt{1+\frac{\nu}{3y^2-\lambda}(z^4\nu+3z^2y)}$$

we get $\mu^x(0,y,\lambda)$, which is smooth. It follows that $\eta\sqrt{\nu(3y^2-\lambda)}$ (and hence also the expression itself) is smooth. Moreover, $\eta\sqrt{\nu(3y^2-\lambda)}$ vanishes when $3y^2-\lambda = 0$ since $\mu^x(0,y,\lambda)$ does. Applying Hadamard's lemma again, we get that

$$\frac{\eta\sqrt{\nu(3y^2-\lambda)}}{3y^2-\lambda}$$

admits a smooth extension to $\lambda = 3y^2$. We conclude that $u = u(y,\lambda)$ extends to a smooth function (as a product of functions admitting a smooth extension). □

After we have shown that μ is the time-1 map of $\hat{u}X_H$, we can consider a smooth fiberwise isotopy on $D^3 \times [0,\varepsilon] \subset D^4$ connecting Id with μ (it is given by the smooth family of vector fields $\alpha(\varphi)\hat{u}X_H$ with α a bump function). This shows the existence of a smooth fibration by circles lying on the common level sets of the first integrals of a neighborhood of a parabolic orbit and hence a smooth periodic integral J. We have thus proven the following result.

Theorem 2. *A parabolic orbit of an integrable two-degree-of-freedom Hamiltonian system $F\colon U \to \mathbb{R}^2$ admits a smooth 2π-periodic first integral. More specifically, there exists a free F-preserving C^∞ Hamiltonian circle action in a neighbourhood of such an orbit.*

Remark 2. Theorem 2 implies that one of the action variables of $F\colon U \to \mathbb{R}^2$ is non-singular in a neighbourhood of the parabolic orbit, i. e., it is C^∞ smooth in the whole neighborhood, including all singular fibers therein, and defines a free circle action on this neighbourhood. We note that this result implies that the same is true in a neighbourhood of a cuspidal torus: if $F\colon U \to \mathbb{R}^2$ is proper and admits a parabolic orbit β on a critical fiber $F^{-1}(f_0)$ (a cuspidal torus) such that dF has rank 2 on the complement $F^{-1}(f_0) \setminus \beta$, then the smooth 2π-periodic integral existing by Theorem 2 generates a free C^∞ Hamiltonian circle action in a neighbourhood of the whole cuspidal torus $F^{-1}(f_0)$.

3. SMOOTH STRUCTURAL STABILITY AND NORMAL FORM

An important consequence of Theorem 2 is the existence of a smooth ("preliminary") normal form of a parabolic singularity. Specifically, we get the following

Theorem 3. *Let $F = (\tilde{H}, G)\colon U \to \mathbb{R}^2$ be an integrable two-degree-of-freedom Hamiltonian system with a parabolic orbit β. Then there exist:*

(i) a small neighbourhood $V \subset U$ of β diffeomorphic to a solid torus $D^3 \times S^1$,

(ii) smooth coordinates (x,y,λ,φ) on V, with φ being an angle coordinate and $\beta = (0,0,0)\times S^1$,

(iii) smooth functions H and J on V that are constant on the connected components of $F^{-1}(f)$,

such that $H = x^2 - y^3 + \lambda y$ *and* $J = \lambda$ *is a* 2π*-periodic first integral. Moreover, the symplectic structure can be written as*

$$\omega = g(x, y, \lambda)dx \wedge dy + d\lambda \wedge (d\varphi + A(x, y, \lambda)dx + B(x, y, \lambda)dy).$$

Proof. First note that the existence of a 2π-periodic first integral J in a neighbourhood of a parabolic orbit allows us to bring the symplectic form to the canonical form; this is essentially the Darboux–Carathéodory theorem, see also [15, Theorem 3.4(a)]. Indeed, by the Darboux–Carathéodory theorem, we can include the function J into a set $(\tilde{x}, \tilde{y}, J = \tilde{\lambda}, \tilde{\varphi})$ of canonical coordinates in a neighbourhood of a parabolic point $P \in \beta$. Since the Hamiltonian flow of J is 2π-periodic, we can extend these coordinates to a neighborhood of the parabolic orbit β, using this flow. Thus, we get extended coordinates

$$(\tilde{x}, \tilde{y}, J = \tilde{\lambda}, \tilde{\varphi})\colon \tilde{V} \to D^3 \times S^1, \ \tilde{\varphi} \in S^1 = \mathbb{R}/2\pi\mathbb{Z},$$

on a small neighbourhood $\tilde{V}$ of β such that, in these coordinates, the symplectic structure has the canonical form

$$\omega = d\tilde{x} \wedge d\tilde{y} + d\tilde{\lambda} \wedge d\tilde{\varphi}.$$

In particular, $\tilde{H}$ is a function of $(\tilde{x}, \tilde{y}, \tilde{\lambda})$ only.

Next, we can assume that $G = J$. Applying a parametric Morse lemma and the versality theorem (see [1] and references therein), one shows that there exists a suitable change of coordinates

$$x = x(\tilde{x}, \tilde{y}, \tilde{\lambda}), \ y = y(\tilde{x}, \tilde{y}, \tilde{\lambda}), \ \lambda = \pm\tilde{\lambda} + \text{const}, \ \varphi = \pm\tilde{\varphi}$$

(on a possibly smaller neighbourhood $V \subset \tilde{V}$) such that

$$\tilde{H} = \pm\big(x^2 - y^3 + \lambda c(\lambda)y + a(\lambda)\big)$$

for some smooth germs $a = a(\lambda)$ and $c = c(\lambda)$ with $c(0) > 0$. It is left to apply a quasi-homogeneous rescaling (cf. [15, §4], where this rescaling was also used) $x \mapsto x/c^{3/4}(\lambda)$, $y \mapsto y/c^{1/2}(\lambda)$, $\tilde{H} \mapsto (\pm\tilde{H} - a(\lambda))/c^{3/2}(\lambda)$ and rename the variables accordingly. Note that

$$\omega = g(x, y, \lambda)dx \wedge dy + d\lambda \wedge (d\varphi + A(x, y, \lambda)dx + B(x, y, \lambda)dy)$$

is then automatically satisfied. □

A direct consequence of Theorem 3 is that all parabolic singularities are locally, i. e., near a parabolic orbit, fiberwise C^∞ diffeomorphic to each other. In view of Remark 2, we get that the same is true semi-locally, i.e., near a cuspidal torus (one can use a similar proof as in, e. g., [10], since we have proven the existence of a C^∞ circle action). As a corollary, using that parabolic points are structurally stable under small integrable perturbations [16], we obtain the following stability result.

Corollary 1. *Let* $F\colon U \to \mathbb{R}^2$ *define an integrable two-degree-of-freedom system with a parabolic orbit* $\beta \subset U$. *Then every integrable system* $\tilde{F}\colon U \to \mathbb{R}^2$ *sufficiently close to* F *in the* C^∞ *topology also admits a parabolic orbit* $\tilde{\beta} \subset U$. *The fibration induced by* $\tilde{F}$ *is locally fiberwise* C^∞ *diffeomorphic to the fibration induced by* F *in a small neighbourhood of the orbit* $\tilde{\beta}$.

In the semi-local case, we similarly have the following. Assume that F *is proper and that the parabolic orbit* β *is the only singularity of* F *on the critical fiber* $F^{-1}(F(\beta))$ *(so that* $F^{-1}(F(\beta))$ *is a cuspidal torus). Then every integrable perturbation* $\tilde{F}$ *sufficiently close to* F *in the* C^∞ *topology also admits a cuspidal torus* $\tilde{F}^{-1}(\tilde{F}(\tilde{\beta}))$. *The fibration induced by* $\tilde{F}$ *is semi-locally fiberwise* C^∞ *diffeomorphic to that of* F *in a small neighbourhood of the cuspidal torus* $\tilde{F}^{-1}(\tilde{F}(\tilde{\beta}))$. □

4. DISCUSSION

In this paper, we have shown that, in a neighbourhood of a parabolic point of a two-degree-of-freedom integrable system $F\colon U \to \mathbb{R}^2$, every F-preserving symplectomorphism is Hamiltonian with a smooth generating function that is constant on the connected components of $\{F = f\}, f \in \mathbb{R}^2$. We deduced from this result the existence of a C^∞ Hamiltonian circle action near parabolic orbits and cuspidal tori as well as a smooth ("preliminary") normal form and structural stability results; see Theorem 3 and Corollary 1.

We conjecture that more is true in fact, and that "uniform" versions of Theorem 3 and Corollary 1 hold as well. In particular, this would imply that the fiberwise diffeomorphism in Corollary 1 can be chosen to be close to the identity. These results would follow from a "uniform" version of the versality theorem (similar to [20, §8.1]) and the continuous dependence of the smooth periodic first integral J on the system in the C^∞ topology.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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