On self-learning mechanism for the output regulation of second-order affine nonlinear systems

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Published in:
IEEE-Transactions on Automatic Control

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Haiwen Wu, Dabo Xu, and Bayu Jayawardhana

***Abstract*—This paper studies global robust output regulation of second-order nonlinear systems with input disturbances that encompass the fully-actuated Euler-Lagrange systems. We assume the availability of relative output (w.r.t. a family of reference signals) and output derivative measurements. Based on a specific separation principle and self-learning mechanism, we develop an internal model-based controller that does not require *apriori* knowledge of reference and disturbance signals and it only assumes that the kernels of these signals are a family of exosystems with unknown parameters (e.g., amplitudes, frequencies or time periods). The proposed control framework has a self-learning mechanism that extricates itself from requiring absolute position measurement nor precise knowledge of the feedforward kernel signals. By requiring the high-level task/trajectory planner to use the same class of kernels in constraining the trajectories, the proposed low-level controller is able to learn the desired trajectories, to suppress the disturbance signals, and to adapt itself to the uncertain plant parameters. The framework enables a plug-and-play control mechanism in both levels of control.**

***Index Terms*—Nonlinear systems, servo systems, output regulation, internal model principle, certainty equivalence principle, adaptive control.**

I. Introduction

Background In the design of modern control systems, the use of second-order systems as prototypical models has played an important role in the development of modern nonlinear control theories [2], [3]. They have been used in the study of geometric control theory, to obtain insights on dissipative and passive systems and to obtain constructive nonlinear control design such as, backstepping, feedback linearization and adaptive control. In this context, Euler-Lagrange (EL) systems form a large class of second-order systems which have been studied well for the past centuries and represent well electro-mechanical systems, see for instance, [4], [5], [6].

In recent decades, there have been progresses in the literature of EL nonlinear control systems that deal with the trajectory tracking control problem with ubiquitous applications in high-precision mechatronics systems and advanced robotic systems. We refer to the monographs [7], [5], [4], [6], [8] for a general overview of progresses in this field. In early studies, one may refer to [9], [10], [11] for a variety of adaptive inverse dynamics control methods, and refer to [12], [13], [14], [15], [16] for passivity-based adaptive control methods. Relevant to the present study, recent studies on control of EL systems can be found in [17], [18], [19], [20], [21], [22] with relevant references thereof, coming up with notable advanced nonlinear control developments. Other relevant nonlinear control systems besides the aforementioned EL systems are, to name a few, wing-rock motions in [23], chaotic Duffing systems in [24], and MEMS resonators in [25].

In all of the aforementioned results, the output regulator relies on the *apriori* knowledge of the reference signal and its derivatives, which becomes essentially the feedforward part of the tracking controller. Consequently the high-level controller, which pre-computes the reference signals to solve and optimize higher-level tasks, is not independent/separated from the low-level tracking controller [26], [27]. Such design is usually referred to as the so-called *feedforward approach* due to the requirement of the reference signal and its derivatives. In other words, these output regulators do not admit a self-learning mechanism of the references to enable an appealing separation principle between the high- (or management & supervision) and the low- (or control) levels in the hierarchy of information processing (see [26, pp. 18]).

Motivation For enabling the aforementioned self-learning capability, it motivates us to embed the classical internal model principle (see [28], [29]) in the design of tracking controller. Generally speaking, the internal model-part of our controller is responsible in predicting the common kernel signals that can subsequently be used in the output regulator. This allows us to realize plug-and-play mechanism between the high-level and low-level controllers, as long as, they agree on the common kernel or the exosystem which can be created provided that we know the number of frequencies relating to the reference and disturbance signals. In other words, a class of exosystems can firstly be defined as common kernels for both controllers, based on which, the high-level controller can use them for task and trajectory planning while the low-level controller employ them in the internal model-based controller.

Fig. 1 illustrates typical control architectures of mechatronic systems where high-level and low-level controllers interact with each other. The left figure shows the interconnection of both control levels when the standard feedforward approach is used for the low-level control. In this case, there are active information exchanges from the high-level controller to the low-level one: the exosystems state w, the reference trajectory q_{ref} and its derivatives $\dot{q}_{\text{ref}}, \ddot{q}_{\text{ref}}$. The right figure, on the other hand, shows the control architecture using the proposed *kernel-based output regulation approach* for the low-level control. In

This work was supported in part by EU SNN grant on Centre of Excellence on Smart Sustainable Manufacturing and in part by National Natural Science Foundation of China under Grant No. 61673216. A preliminary form of the paper is [1].

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this figure, the use of common kernels on both control levels enables the disconnection of information exchanges between them. The self-learning mechanism in the low-level control allows for the adaptation of the unknown parameters in the kernel as well as in the model-based feedback controller.

As an illustrative example, let us consider a simple single-link manipulator equipped with camera and (incremental) encoder sensors to provide displacement and velocity measurements as depicted in Fig. 2. In this figure, the relative displacement error between the end-effector and moving target effector can be measured by a camera. Based only on these measurements, our proposed controller will then be able to generate the desired trajectory and to track it robustly with respect to parameter uncertainties. In this perspective, the use of teaching pendant, which records all the motions of the target robotic behaviour, is no longer needed for training robotic systems as commonly used nowadays in industry.

Objective & Contribution A primary objective of the present study is to investigate the problem of globally asymptotically tracking of fully-actuated systems based only on the use of relative displacement and velocity feedback in order to track any reference signals generated by exosystems and be adaptive to system parameter uncertainties. Specifically, for a class of fully-actuated uncertain nonlinear systems, we pose the control problem as a global robust output regulation problem for strict-feedback nonlinear systems. The challenges in this problem include the restriction of error-velocity measurement and the uncertainties such as external disturbances and plant uncertain parameters. These factors complicate the internal model design and stabilizing control (w.r.t. an invariant manifold) as important steps for achieving the final control goal of global robust output regulation.

The main contribution of the present study is to develop a self-learning mechanism using an adaptive internal model approach for solving the asymptotic tracking and disturbance rejection problem for a general class of nonlinear electro-mechanical systems by displacement error and velocity feedback. In our proposed control design method, we present explicitly the construction of the adaptive internal model for a class of second-order nonlinear system (e.g., EL systems) to which existing approaches can not be applied, as far as we know of. Particularly, our proposed control design framework enables the separation of high-level and low-level control designs where the information exchange between them is only through the error signal. The use of exosystems as common kernels facilitates the self-learning mechanism in our proposed output feedback controller. Furthermore, our technical results surmount the substantial difficulties in cascade internal model design and transversal stability analysis for the output regulation of such multivariable nonlinear systems.

Comparative Literature Review For control problems of uncertain EL systems with input disturbances, relevant results can be found in [17], [18], [20], [21]. In these results, the tracking control relies on the availability of feedforward kernel signals and meanwhile the internal model is specific to realize the disturbance rejection. Such results can be referred to as the mixed internal model-based and feedforward approach, *cf.* [30], [31]. In comparison, the present study provides solely an internal model-based feedback approach for the same problem and a strictly larger class of nonlinear systems to achieve both asymptotic tracking and disturbance rejection using error and velocity measurement.

For solving the output regulation problem, the internal model design is a key step. In literature, there are extensive results on various internal model design techniques and we refer to [32] for a thorough recent survey in this subject. As will be shown later in this paper, the construction of suitable internal models, which is necessary for solving the output regulation problem, has been recognized as one of the main challenges. When the zero-error constraint input function is polynomial, the design of such internal models has been discussed in [33], [34], [35], [36], [37] to name but a few. In general, the plant nonlinearity may not be polynomial as shown later for a simple one-dimensional EL system in Example 2.1. Regarding nonlinear internal model construction for the non-polynomial case, it is in general never a trivial task, see [38], [39], [40], [41] for pioneering works. In this case, our work provides new techniques that generalize the aforementioned works for strict-feedback nonlinear systems using cascaded internal models.

Outline Section II gives the output regulation problem formulation and lists some standard assumptions. Section III presents the main results of this paper. Section IV illustrates the effectiveness of the proposed controllers by numerical simulations. The conclusions are given in Section V. All the technical proofs are put in the Appendix.

Notation $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ or the induced matrix norm in $\mathbb{R}^{n\times m}$. $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers. I is an identity matrix of appropriate dimension from the context. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. $\mathcal{K}_o$ and $\mathcal{K}_\infty$ are the subclasses of bounded and unbounded $\mathcal{K}$ functions, respectively. $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\gamma(s) \to 0$ as $s \to \infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, for each fixed s, $\beta(r, s)$ belongs to $\mathcal{K}$, and for each fixed r, function $\beta(r, s)$ belongs to $\mathcal{L}$. For two continuous and positive definite functions $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\alpha_1 \in \mathcal{O}(\alpha_2)$ implies $\limsup_{s\to 0^+} \frac{\alpha_1(s)}{\alpha_2(s)} < \infty$. $\mathcal{L}^n_\infty$ is defined as the set of all piecewise continuous $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\sup_{t\geq 0} |f(t)| < \infty$. $\mathcal{L}^n_2$ is defined as the set of all piecewise continuous $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\int_0^\infty |f(t)|^2 dt < \infty$. $f(\mathbb{A})$ stands for image of a set $\mathbb{A} \subseteq \mathbb{R}^n$ under the mapping $f : \mathbb{R}^n \to \mathbb{R}^m$. For column vectors $x_1, \ldots, x_n$, we write $(x_1, \ldots, x_n)^T := [x_1^T, \ldots, x_n^T]^T$ as the column stacking vector if no confusion arises from the context, and for matrix $A \in \mathbb{R}^{m\times n}$ with column vectors $A_1, \ldots, A_n$, we write $\mathrm{vec}(A) := (A_1, \ldots, A_n)^T$ and $\mathrm{mat}(\mathrm{vec}(A)) = A$ as the inverse.

II. Formulation and Background

Consider uncertain nonlinear systems in the form (see [18])

$$H(q, w)\ddot{q} + G(q, \dot{q}, w) = u + d_E + d_0 \tag{1}$$

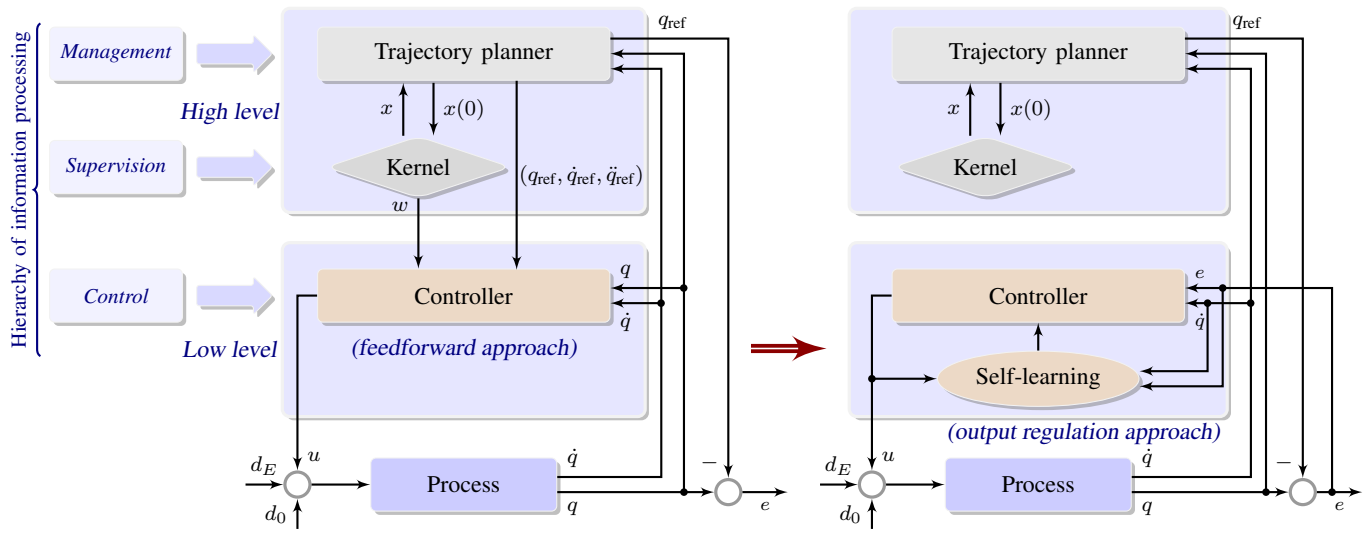

Fig. 1. Two different control architectures in the control of mechatronics systems. The left figure shows the standard architecture where high-level control actively exchanges information with the low-level controller for the computation of feedforward control signal. The right figure presents the control architecture that is enabled with the proposed kernel-based output regulation approach. Due to the use of common kernels in both control levels where the low-level control kernels are integrated in the self-learning part, active information exchange from the high-level controller to the low-level one is no longer needed.

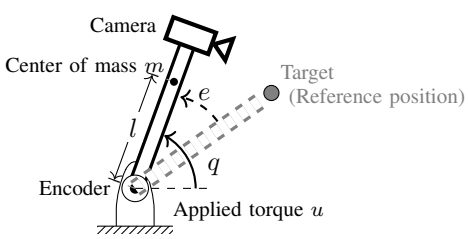

Fig. 2. Illustrative example of output regulation problem of a single-link manipulator where the motion of virtual reference needs to be tracked using only relative position and velocity in the feedback loop.

where $q(t) \in \mathbb{R}^n$ is the position vector, $\dot{q}(t) \in \mathbb{R}^n$ is the velocity vector and $u(t) \in \mathbb{R}^n$ is the control input. The signal $d_0(t)$ is an unmodeled disturbance that belongs to $\mathcal{L}_2^n$ and $d_E(t)$ is a periodic input disturbance that is generated by a *disturbance exosystem* with the exosystem state $w(t) \in \mathbb{R}^{n_w}$, which will be described shortly below. In (1), $H(q,w) \in \mathbb{R}^{n\times n}$ is a smooth and uncertain inertia matrix, $G(q,\dot{q},w)$ is locally Lipschitz continuous in all its arguments with $G(0,0,w)=0$, $\forall w \in \mathbb{R}^{n_w}$. System (1) is said to be *fully-actuated* if $H(q,w)$ is nonsingular for all (q,w).

The disturbance exosystem is given by

$$\dot{w} = A(\sigma)w, \quad \dot{\sigma} = 0, \quad d_E = D(w) \tag{2}$$

where A and D are assumed to be smooth. In addition to the disturbance exosystem (2), we assume that the reference trajectory $q_{\text{ref}} \in \mathbb{R}^n$ can be generated by a reference exosystem (or in the terminology of [42], the target system) as follows

$$\dot{v} = S(\sigma)v, \quad q_{\text{ref}} = Q(v) \tag{3}$$

where $v(t) \in \mathbb{R}^{n_v}$ is the reference exosystem state and the functions S and Q are assumed to be smooth. Consequently, the tracking error or regulated output is given by

$$e = q - q_{\text{ref}}. \tag{4}$$

In the exosystem description above, both the exosystem matrices $S(\sigma)$ and $A(\sigma)$ of (2) and (3) depend on parameter $\sigma \in \mathbb{R}^{l_\sigma}$.

In the present study, as in [43], [34] for nonlinear output regulation studies, we assume that the exosystem state variables $v(t)$, $w(t)$ and parameter σ are all unknown and evolve in fixed compact and (positively) invariant[1] sets $\mathbb{V} \subset \mathbb{R}^{n_v}$, $\mathbb{W} \subset \mathbb{R}^{n_w}$, $\mathbb{S} \subset \mathbb{R}^{l_\sigma}$, respectively. For presentation ease, we sometimes express both the exosystems as above in the following compact form

$$\dot{x} = f(x) \quad \text{with} \quad x = \begin{bmatrix} v \\ w \\ \sigma \end{bmatrix} \in \mathbb{X}, \quad f(x) = \begin{bmatrix} S(\sigma)v \\ A(\sigma)w \\ 0 \end{bmatrix} \tag{5}$$

where $\mathbb{X} = \mathbb{V} \times \mathbb{S} \times \mathbb{W}$ is compact and invariant for (5).

System (5) is regarded as the common kernel as mentioned in Section I, which should be known or at least estimable for any admissible trajectory tracking controller. More specifically, if system (5) is known, i.e., both the initial condition $x(0)$ and the vector field $f(x)$ are known, one can perform feedforward control to realized trajectory tracking. However, if both $x(0)$ and $f(x)$ are unknown, one should introduce a self-learning mechanism to realize the learning of system (5). The latter circumstance is the focus of the present study.

We note that the plant parametric uncertainties or uncertain parameters are all constants. Using the compact form (5), we introduce terms $q_0(v)$ and $a_0(w)$ satisfying

$$\frac{\partial q_0(v)}{\partial v} S(\sigma)v = 0, \quad \frac{\partial a_0(w)}{\partial w} A(\sigma)w = 0 \tag{6}$$

as the bias (*cf.* [33]) of the reference $q_{\text{ref}} = Q(v)$ and the plant uncertain parameters or parametric uncertainties arising in (1), respectively. By the above assumption, both the exosystems can generate any combination of sinusoids and steps signals

[1]Here, a set $\mathbb{V}$ is said to be (positively) invariant if, for every initial condition $v(0) \in \mathbb{V}$, the solution $v(t)$ of (2) satisfies $v(t) \in \mathbb{V}$ for all $t \geq 0$.

$\sum_i A_{\text{mi}} \sin(\sigma_i t + \phi_i)$ with *unknown parameters* of amplitudes A_{mi}, frequencies σ_i, and phases ϕ_i, relying on their initial conditions $v(0) \in \mathbb{V}$, $w(0) \in \mathbb{W}$ and parameters $\sigma \in \mathbb{S}$.

A. Standard Assumptions

For the rest of the paper, we list the following technical assumptions commonly used in literature.

H1 The matrix-valued function $H(q, w) \in \mathbb{R}^{n\times n}$ is positive definite, i.e., there exist constants $c_{\min}, c_{\max} > 0$ such that

$$c_{\min} I \leq H(q, w) \leq c_{\max} I, \quad \forall (q, w) \in \mathbb{R}^n \times \mathbb{W}.$$

H2 There are smooth functions $a(\cdot) \in \mathbb{R}^{l_a}, Y(\cdot) \in \mathbb{R}^{n\times l_a}$ such that for any reference $q_{\text{ref}}(t) \in \mathbb{R}^n$ and continuous derivatives $\dot{q}_{\text{ref}}(t)$ and $\ddot{q}_{\text{ref}}(t)$, and for any vector $v \in \mathbb{V}$,[2]

$$\begin{aligned} &H(q_{\text{ref}} + q_0(v), w)\ddot{q}_{\text{ref}} + G(q_{\text{ref}} + q_0(v), \dot{q}_{\text{ref}}, w) \\ &=: Y(q_{\text{ref}}, \dot{q}_{\text{ref}}, \ddot{q}_{\text{ref}})a(w, v) \end{aligned}$$

with $q_0(v)$ as noted in (6), where Y is the regressor matrix and $a(w, v)$ is such that, in line with (5), $[\partial a(w, v)/\partial x]f(x) = 0$. In other words, $a(w, v)$ contains only the plant parametric uncertainties and constant reference bias, relating to (6).

H3 For each parameter $\sigma \in \mathbb{S}$, all the eigenvalues of exosystem matrices $S(\sigma)$ and $A(\sigma)$ are distinct, on the imaginary axis, and each entry of $Q(v)$ and $D(w)$ is a nonlinear polynomial in its argument with unknown coefficients.

Both the assumptions **H1** and **H2** are easily verifiable for the conventional EL systems. Particularly, when $q_0(v) = 0$, **H2** is the standard parameter linearization property, fulfilled for general EL systems; see [6, Chapter 2] and [5, Chapter 9], but here system (1) pursued in the present study is independent of the so-called skew-symmetric property.

B. Problem and Motivation

Problem 2.1: For the composite system (1) and (5) with the tracking error (4), if possible, find a smooth controller of the form

$$\dot{x}_c = f_c(x_c, e, \dot{q}), \quad u = h_c(x_c, e, \dot{q}) \tag{7}$$

such that, for every initial condition $(v(0), w(0)) \in \mathbb{V} \times \mathbb{W}$, $q(0), \dot{q}(0) \in \mathbb{R}^n$ and for every $x_c(0)$, the closed-loop system (1) and (7) has the following properties:

- *(Stability Property)* the trajectory $(q(t), \dot{q}(t), x_c(t))$ exists for all $t \geq 0$ and is bounded over $[0, \infty)$;
- *(Regulation Property)* the tracking error $\lim_{t\to\infty} e(t) = 0$.

□

As noted in the preceding section, there are mainly two methodologies for tracking control of (1) when confined to model EL systems. One is the adaptive inverse dynamics control, see [9], [10], [11] and many others. The other is the passivity-based adaptive control, see [12], [13], [14], [16]. Besides, one may also refer to [19], [35, Example 4.3] for recent studies on position feedback design. All these studies are based on the combination of feedback and feedforward control method, see Fig. 1 again. More precisely, the availability of information on $q(t)$, $q_{\text{ref}}(t)$ and their time derivatives is prerequisite, and instead of (7), the control law has generically the form

$$\begin{aligned} \dot{x}_c &= f_c(x_c, q, \dot{q}, q_{\text{ref}}, \dot{q}_{\text{ref}}, \ddot{q}_{\text{ref}}) \\ u &= h_c(x_c, q, \dot{q}, q_{\text{ref}}, \dot{q}_{\text{ref}}, \ddot{q}_{\text{ref}}). \end{aligned} \tag{8}$$

On the one hand, the real-time information on $q, q_{\text{ref}}, \dot{q}_{\text{ref}}$ and $\ddot{q}_{\text{ref}}$ may not readily be available in order to implement (8). Firstly, we require a common frame of coordinates for defining q and q_{ref} which may not be accessible for industrial robots. Secondly, the computation of $q_{\text{ref}}, \dot{q}_{\text{ref}}$ and $\ddot{q}_{\text{ref}}$ by the high-level controller requires accurate knowledge of system parameters which are intrinsically uncertain. These limitations have restricted the wide adoption of (8) beyond bespoke robotic solutions as developed and deployed for space or advanced industrial sectors.

On the other hand, the well-known internal model principle plays a crucial role in the solvability of the robust output regulation problem by error feedback [28], [29]. The use of internal-model based controller has enabled the controller to recreate the reference trajectories internally within its dynamics [44]. It is able to self-learn the target's dynamical behavior based only on the output error feedback. In combination with adaptive control technique, the controller is able to learn both the target's behavior and the plant dynamics. Correspondingly, we will adopt these two approaches to solve the global robust output regulation problem for fully-actuated second-order nonlinear systems.

One may apply the internal model-based control framework proposed in [38] for the output regulation of lower triangular nonlinear systems. However it proves to be a non-trivial task as shown in the following example of a single-link manipulator. This design example will be further considered sequentially in the next two sections to demonstrate verifications of the proposed design conditions and results, respectively.

Example 2.1: Consider the single-link manipulator system as shown in Fig. 2 and modeled by

$$J\ddot{q} + mgl\cos(q) = u + d_E \tag{9}$$

where $m > 0$ is the mass, $J > 0$ is the moment of inertia about the joint axis and $l > 0$ is the distance from its axis of rotation to the center of mass. Notice that system (9) is non-polynomial, due to the trigonometric nonlinearity $\cos(q)$. Let us consider the asymptotic tracking control problem with a reference signal generated by (3) while rejecting a periodic disturbance signal generated by (2) using only the relative error measurement $e = q_{\text{ref}} - q$ and without having direct measurement of q.

To this end, following [43, pp. 83], the so-called zero-error constrained input can be written as

$$u^\star(x) = J\frac{\partial \xi^\star(v)}{\partial v} S(\sigma)v + mgl\cos(Q(v)) - D(w) \tag{10}$$

[2]Here, we use $q_0(v)$ to indicate the reference bias and $q_{\text{ref}} + q_0(v)$ is namely the biased reference.

where $\xi^\star(v) = [\partial Q(v)/\partial v]S(\sigma)v$, which is obtained from (3), (4) and (9). From (10), we observe that $u^\star(x)$ is not polynomial, and the internal model design conditions of [38], [45], [34] are not verifiable. Moreover, it is too difficult to apply standard Slotine-Li controller or its variants as we do not have the measurement of q. To the best of our knowledge, even when both w and σ are known, the construction of an internal model-based global robust output regulator is still an open issue.

Nonetheless, by exploiting the parameter linearization property in **H2**, the unknown system parameters (J, mgl) can be learned through adaptation. It provides the capability of generating the non-polynomial zero-error constrained input $u^\star(x)$. This will be further explained and resolved in Example 3.1 later. □

C. Internal Model Characterization and Stability Notions

For solving nonlinear output regulation problems, there are two key ingredients when error output feedback is concerned. The first one is the design of admissible internal models that provide self-learning of the desired feedforward control input. The second one is the design of a stabilizing control law for the augmented systems, comprising of the plant dynamics and the internal model. The latter one must ensure that the output-zeroing manifold is attractive for a given initial region.

Correspondingly, for solving Problem 2.1, we need the following technical notions. Consider (1) and (5) in the following compact form

$$\dot{x} = f(x) \tag{11a}$$
$$\dot{q} = \xi \tag{11b}$$
$$H(q,w)\dot{\xi} = u + D(w) - G(q,\xi,w) + d_0 \tag{11c}$$
$$e = q - Q(v) \tag{11d}$$

where e is the regulated output. Associated to (11) above, we need to solve the *regulator equations* (see [46] or [43, Chapter 3]). In fact, it has a globally defined solution

$$\left\{q^\star = q^\star(x), \xi^\star = \xi^\star(x), u^\star = u^\star(x)\right\}$$

with $x \in \mathbb{X}$ in line with (5), and

$$\begin{aligned} q^\star &= Q(v), \quad \xi^\star = \frac{\partial Q(v)}{\partial v} S(\sigma)v \\ u^\star &= H(q^\star(x), w)\psi^\star(x) + G(q^\star(x), \xi^\star(x), w) - D(w) \end{aligned} \tag{12}$$

where $\psi^\star(x) = [\partial \xi^\star(x)/\partial x]f(x)$. On the basis of (12), we define the *zero-error constrained state and input manifold* as $\left\{(q,\xi,u) = (q^\star(x), \xi^\star(x), u^\star(x)) : x \in \mathbb{X}\right\}$ to be invariant for system (11c), where the regulated output $e = 0$. In this regards, any admissible output regulator that solves Problem 2.1 must generate a control input signal $u(t)$ that converges asymptotically to $u^\star(x(t))$, which subsequently guarantees that $e(t) \to 0$ as $t \to \infty$. As mentioned before, since the information of $u^\star(x)$ is not directly available to the controller, we will provide a constructive design of an internal model that can provide $u^\star(x)$. Following [43, Definition 6.6], we provide a characterization of admissible internal models in this study.

Definition 2.1: (Internal Model Candidates) For system (11) with control input u, a dynamical compensator of the form

$$\dot{\eta} = \gamma(\eta, x, u) \tag{13}$$

is said to be *a pseudo internal model* with output u if, it satisfies the following internal model property:

- There exist smooth functions $\theta(x) \in \mathbb{R}^\ell$ and $\Gamma(\theta, x) \in \mathbb{R}^n$ such that, for all $x \in \mathbb{X}$,

  $$\frac{\partial \theta(x)}{\partial x} f(x) = \gamma(\theta(x), x, u^\star(x)), \quad u^\star(x) = \Gamma(\theta(x), x)$$

  referred to as a generator of $u^\star(x)$.

Moreover, if the vector field $\gamma(\eta, x, u)$ in (13) is independent of the exosystem state x, then it is said to be *an implementable internal model* with output u. □

Definition 2.1 has been given to characterize the *internal model property* only. It should be noted that this is not sufficient enough to succeed the output regulation synthesis because of the following reason. As elaborated in [38], after the internal model design, one needs to further solve the stabilizing control of the augmented system, composed of the plant dynamics and the designed internal model. In this study, although the fully-actuated plant in (11) itself is controllable or stabilizable (see [7, Chapter 12]), it may not be the case for the augmented system. Hence, an admissible internal model should be further specified to enjoy certain stability properties and in turn the solvability of the output regulation. Such a stability property is crucial to ensure certain stabilizability property relating to the associated augmented system. This indeed motivates us to carefully select or develop stability and stabilizing tools to carry out the current output regulation synthesis.

In the rest of this section, we revisit some useful stability notions for nonlinear systems transformable in the form

$$\dot{z} = F(z, x, u), \quad \dot{x} = f(x) \tag{14}$$

with state $(z, x) \in \mathbb{R}^n \times \mathbb{X}$ and input $u \in \mathbb{R}^m$, where $\mathbb{X}$ is a restricted invariant set for subsystem $\dot{x} = f(x)$, and the vector field $F : \mathbb{R}^n \times \mathbb{X} \times \mathbb{R}^m \to \mathbb{R}^n$, $F(0, x, 0) = 0$ for all x, is locally Lipschitz in its arguments. Let $Z(t) = Z(z(0), x(0), u, t)$, $t \geq 0$, be the (unique) solution with initial condition $(z(0), x(0))$ and input $u \in \mathbb{R}^m$.

The following transversal stability definitions are technically inspired and given in the spirit of [47] and [48] on zero-input transversal (local) uniform exponential stability (0-TUES), transversal ISS (TISS), and TiISS.

Definition 2.2: (0-TUES) System (14) is 0-TUES if, the zero-input system is forward complete and there exist numbers $r, k, \lambda > 0$ such that, for every $(z(0), x(0)) \in \mathbb{R}^n \times \mathbb{X}$ with $z(0)$ restricted in a neighborhood of its origin,

$$|Z(z(0), x(0), 0, t)| \leq k|z(0)| \exp(-\lambda t), \quad t \geq 0.$$

□

Definition 2.3: (TISS & TiISS) System (14) is said to be

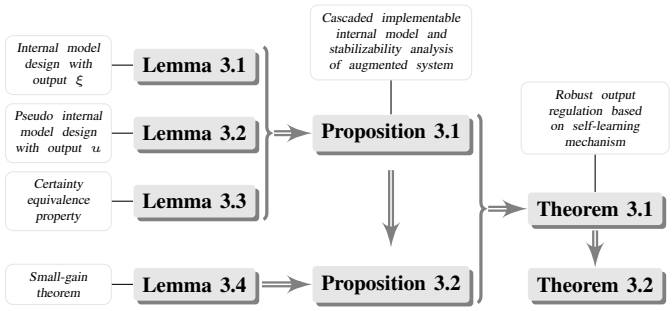

Fig. 3. Roadmap and descriptions for technical results.

- TISS if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for any initial state $(z(0), x(0)) \in \mathbb{R}^n \times \mathbb{X}$ and for any input $u(t) \in L_\infty^m$, it is forward complete and satisfies

$$|Z(t)| \leq \beta(|z(0), x(0)|, t) + \gamma\Big(\sup_{0\leq s\leq t} |u(s)|\Big),\ t \geq 0.$$

- TiISS if there exist functions $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any initial state $(z(0), x(0)) \in \mathbb{R}^n \times \mathbb{X}$ and for any input $u(t) \in \mathcal{L}_\infty^m$, it is forward complete and satisfies

$$\alpha(|Z(t)|) \leq \beta(|z(0), x(0)|, t) + \int_0^t \gamma(|u(s)|)ds,\ t \geq 0.$$

□

The proceeding stability properties can be validated based on Lyapunov-like functions, analogous to the iISS notion and its equivalent or sufficient Lyapunov function characterization.

III. Main Results

A. Cascading Internal Models via Certainty Equivalence

Let us begin with introducing a specific nonlinear internal model candidate serving the output regulation of system (11) in accordance with the characterization in Definition 2.1.

Specifically, we first present Lemma 3.1 for the design of a reference internal model with output ξ, and Lemma 3.2 for the design of a pseudo disturbance internal model with output u. It is then followed by Lemma 3.3 that gives a crucial certainty equivalence property. We refer to Fig. 3 for the roadmap on the technical results presented in this section.

Lemma 3.1: Consider the subsystem (11a) *and* (11b) *with ξ as virtual input and* (11d) *as regulated output. Then there is a smooth nonlinear internal model of the affine form*

$$\dot{\eta} = \varphi_{\rm a}(\eta) + N_{\rm a}\xi =: \gamma_{\rm a}(\eta, \xi), \quad \eta \in \mathbb{R}^\ell \tag{15}$$

with output ξ per Definition 2.1, satisfying the internal model property: there exist smooth functions $\theta(x) \in \mathbb{R}^\ell$, $\Gamma(\theta) \in \mathbb{R}^n$ such that, for all $x \in \mathbb{X}$,

$$\frac{\partial\theta(x)}{\partial x} f(x) = \gamma_{\rm a}(\theta(x), \xi^\star(x)), \quad \xi^\star(x) = \Gamma \circ \theta(x). \tag{16}$$

Moreover, the output function $\Gamma(\cdot)$ can be chosen to be smooth, globally defined, and compactly supported. [3]

[3] A continuous function is said to be compactly supported if it is zero outside a compact set.

Lemma 3.2: Consider system (11) *satisfying* ***H2****. Let*

$$c(x) = \big(q^\star(x) - q_0(v), \xi^\star(x), \psi^\star(x)\big)^T \in \mathbb{R}^{3n} \tag{17}$$

where $q_0(v)$ is a reference bias. Then one can construct a smooth pseudo internal model of an affine form

$$\dot{\zeta} = \varphi_{\rm b}(\zeta, c) + N_{\rm b}u =: \gamma_{\rm b}(\zeta, c, u), \quad c = c(x), \quad \zeta \in \mathbb{R}^l \tag{18}$$

with output u per Definition 2.1, satisfying the internal model property: there exist smooth functions $\vartheta : \mathbb{X} \to \mathbb{R}^l$, $\rho : \vartheta(\mathbb{X}) \times c(\mathbb{X}) \times \mathbb{X} \to \mathbb{R}^n$ such that, for all $x \in \mathbb{X}$,

$$\frac{\partial\vartheta(x)}{\partial x} f(x) = \gamma_{\rm b}(\vartheta(x), c(x), u^\star(x))$$
$$u^\star(x) = \rho(\vartheta(x), c(x), x)$$

where $\rho(\cdot)$ takes the form

$$\rho(\vartheta(x), c(x), x) = \rho_1(\vartheta(x)) + \rho_2(\vartheta(x), c(x))\Omega \tag{19}$$

with $\Omega := \Omega(a(w, v), \sigma)$, for smooth functions $\rho_1(\cdot)$, $\rho_2(\cdot)$, and $\Omega(\cdot)$ of appropriate dimensions. Moreover, the function $\rho(\cdot)$ can be chosen to be smooth, globally defined, and compactly supported.

At this place, for the internal model provided in Lemma 3.1, we note an appealing transversal stability property. For this purpose, consider the systems (5) and (15), and let $\xi = \xi^\star(x) + \tilde{\xi}$ where $\tilde{\xi}$ defines the incremental state with respect to $\xi^\star$. We have then the following augmented system

$$\dot{x} = f(x), \quad \dot{\eta} = \gamma_{\rm a}(\eta, \xi^\star(x) + \tilde{\xi}). \tag{20}$$

By taking the error coordinate $\bar{\eta} = \eta - \theta(x)$, we obtain

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\bar{\eta}} &= \gamma_{\rm a}(\bar{\eta} + \theta(x), \xi^\star(x) + \tilde{\xi}) - \gamma_{\rm a}(\theta(x), \xi^\star(x)) \end{aligned} \tag{21}$$

which takes the same form as (14) in Definition 2.2. Consequently, the transversal uniform exponential stability notion can become a useful concept to address the output regulation design. In this regards, system (20) has a (positively) invariant manifold

$$\big\{(\eta, x) : \eta = \theta(x), x \in \mathbb{X}\big\} \tag{22}$$

which should be made attractive for fulfilling the regulation property. In light of [47], we require the manifold (22) to be 0-TUES in the sense of Definition 2.2. As far as the stabilizing control is concerned, we can show that the system (21) admits an iISS property with input $\tilde{\xi}$ and output $\bar{\eta}$. Specifically, inspired by [49], the so-called PE condition (see [50, pp. 265]) of $\theta(x)$ can ensure the above properties and (21) will become TiISS. To be shown shortly, such internal model stability properties is of importance for us to reach the TiISS stability property of Proposition 3.1.

Next, to validate the pseudo internal model (18), we explore the following interesting certainty equivalence property from (15) to (18) based on Lemmas 3.1 and 3.2.

Lemma 3.3: (Certainty Equivalence) Consider the generator (16) *as in Lemma 3.1 and the function $c(x)$ as in Lemma 3.2. Then there are smooth mappings $L_1(\cdot)$ and $L_2(\cdot)$ such that, for all $x \in \mathbb{X}$,*

$$q^\star(x) = L_1 \circ \theta(x) + q_0(v), \quad \psi^\star(x) = L_2 \circ \theta(x) \tag{23}$$

where $q_0(v)$ is a reference bias of $q_{ref} = Q(v)$ as specified in (6)*. Moreover, there exists a smooth, globally defined, and compactly supported function $L(\cdot)$ such that the following equivalence condition*

$$L \circ \theta(x) = c(x) = \begin{bmatrix} L_1(\theta(x)) \\ \Gamma(\theta(x)) \\ L_2(\theta(x)) \end{bmatrix}, \quad \forall x \in \mathbb{X} \tag{24}$$

is ensured.

Based on the above technical results, we are ready to design an implementable internal model that can be used to resolve output regulation problem of systems (11).

Proposition 3.1: Consider (11) *satisfying assumption* ***H2****. Then one can construct an implementable internal model*

$$\begin{aligned} \dot{\eta} &= \gamma_{\rm a}(\eta, \xi) \\ \dot{\zeta} &= \gamma_{\rm b}(\zeta, L(\eta), u) \\ u &= \rho_1(\zeta) + \rho_2(\zeta, L(\eta))\Omega \end{aligned} \tag{25}$$

with output u and Ω as in (19) *relying on the plant/exosystem parameters, satisfying both the following conditions:*

- *(Internal Model Property) For the smooth functions $\theta(\cdot)$, $\vartheta(\cdot)$ and $\rho(\cdot)$ specified in Lemmas 3.1 and 3.2, we have, for all $x \in \mathbb{X}$,*

$$\begin{aligned} \frac{\partial \theta(x)}{\partial x} f(x) &= \gamma_{\rm a}(\theta(x), \xi^\star(x)) \\ \frac{\partial \vartheta(x)}{\partial x} f(x) &= \gamma_{\rm b}(\vartheta(x), L \circ \theta(x), u^\star(x)) \\ u^\star(x) &= \rho(\vartheta(x), L \circ \theta(x), x) \\ &= \rho_1(\vartheta(x)) + \rho_2(\vartheta(x), c(x))\Omega. \end{aligned} \tag{26}$$

- *(Stability Property) For system* (18)*, let $x(t)$ be any trajectory of exosystem* (5) *and*

$$\begin{aligned} z(t) &= \big(z_1(t), z_2(t)\big)^T, \tilde{u}(t) = \big(\tilde{u}_1(t), \tilde{u}_2(t)\big)^T \\ z_1(t) &= \eta(t) - \theta(x(t)), \;\; z_2(t) = \zeta(t) - \vartheta(x(t)) \\ \tilde{u}_1(t) &= \xi(t) - \xi^\star(x(t)), \;\; \tilde{u}_2(t) = u(t) - u^\star(x(t)) \\ \bar{N} &= \text{block diag}(N_{\rm a}, N_{\rm b}) \end{aligned} \tag{27}$$

be the new coordinates and inputs. Further let

$$\begin{aligned} F(z, x, \tilde{u}) &= \big(F_1(z_1, x, \tilde{u}), F_2(z, x, \tilde{u})\big)^T \\ F_1(z_1, x, \tilde{u}) &= \gamma_{\rm a}(z_1 + \theta, \tilde{u}_1 + \xi^\star) - \gamma_{\rm a}(\theta, \xi^\star) \\ F_2(z, x, \tilde{u}) &= \gamma_{\rm b}(z_2 + \vartheta, L(z_1 + \theta), \tilde{u}_2 + u^\star) \\ &\quad - \gamma_{\rm b}(\vartheta, L(\theta), u^\star). \end{aligned}$$

If the trajectory $x(t)$ of (5) *is such that $\theta(x(t))$ is of PE in the sense of [50, pp. 265] then for an external input $\nu \in \mathbb{R}^{2n}$, the following dynamics*

$$\dot{z} = F(z + \bar{N}\nu, x, \tilde{u}), \quad \dot{x} = f(x) \tag{28}$$

where matrix $\bar{N}$ is given in (27)*, is TiISS w.r.t. input $(\tilde{u}, \nu)$ per Definition 2.3 and moreover is 0-TUES.*

We remark here that the PE condition in Proposition 3.1 is mild and frequently used in the adaptive output regulation design. This can be fulfilled if the degree of minimal zeroing polynomial is known, or equivalent to the following condition: the number of excited modes of the reference exosystem or the order of the reference exosystem is known.

Proposition 3.1 has two key ingredients for resolving the output regulation problem. One is to provide a specific internal model taking a cascaded interconnection structure. The other is that its stability ensures the important stabilizability of the augmented system, composed of the plant dynamics and the implementable internal model. The latter will be shown shortly in the next subsection.

B. Transveral Stability Analysis for the Augmented System

Combining the internal model (25) to the composite system (11) gives the following augmented system

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\eta} &= \gamma_{\rm a}(\eta, \xi) \\ \dot{q} &= \xi \\ \dot{\zeta} &= \gamma_{\rm b}(\zeta, L(\eta), u) \\ H(q, w)\dot{\xi} &= u + D(w) - G(q, \dot{q}, w) + d_0 \\ e &= q - Q(v). \end{aligned} \tag{29}$$

Correspondingly, it is sufficient enough to design an output regulator for u that only uses the available measurement (e, ξ, η, ζ) such that the zeroing output manifold

$$\{(\eta, q, \zeta, \xi) = (\theta(x), q^\star(x), \vartheta(x), \xi^\star(x)) : x \in \mathbb{X}\} \tag{30}$$

is globally attractive or stable in some sense.

At this moment, it remains to stabilize system (29) w.r.t. its invariant manifold (30) to imply the solvability of Problem 2.1. Let us first consider the basic situation of the system with known parameter vector Ω in (19) or (26).

Toward that end, based on (11d) and (27) with $\tilde{u}_1, \tilde{u}_2$ thereof replaced by

$$\tilde{\xi} = \xi - \Gamma(\eta), \quad \bar{u} = u - \rho_1(\zeta) - \rho_2(\zeta, L(\eta))\Omega$$

respectively, we write the translated system as

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{z}_1 &= F_1(z_1, x, \tilde{\xi} + \tilde{\Gamma}(z_1, x)) \\ \dot{e} &= \tilde{\xi} + \tilde{\Gamma}(z_1, x) \\ \dot{z}_2 &= F_2(z, x, \bar{u} + \tilde{\rho}(z, x)) \\ H(q, w)\dot{\tilde{\xi}} &= \bar{u} + \tilde{\rho}(z, x) + \Delta_0(z_1, e, \tilde{\xi}, x) + d_0 \end{aligned} \tag{31}$$

where

$$\begin{aligned} \tilde{\Gamma}(z_1, x) &= \Gamma(z_1 + \theta) - \Gamma(\theta) \\ \tilde{\rho}(z, x) &= \rho(z_2 + \vartheta, L(z_1 + \theta), x) - \rho(\vartheta, L(\theta), x) \\ \Delta_0(z_1, e, \tilde{\xi}, x) &= G(L_1(\theta), \Gamma(\theta), w) + H(L_1(\theta), w)L_2(\theta) \\ &\quad - G(e + L_1(\theta), \tilde{\xi} + \Gamma(z_1 + \theta), w) \\ &\quad - H(e + L_1(\theta), w)[d\Gamma(z_1 + \theta)/dt]. \end{aligned} \tag{32}$$

In (32), it is easy to show that $\tilde{\Gamma}(0, x) = 0$, $\tilde{\rho}(0, x) = 0$, $\Delta_0(0, 0, 0, x) = 0$, $\forall x \in \mathbb{X}$. Thus, the system (31) has an equilibrium at $(z, e, \tilde{\xi}) = (0, 0, 0)$. Clearly, the *global uniform asymptotic stability* of this equilibrium, if it can be done, assures the attractivity of manifold (30) and leads to

the solvability of the output regulation problem (see [43, Corollary 7.4] for similar arguments). Hence, we turn to the stabilization problem.

More specifically, we look for a stabilizing control law

$$\bar{u} = k_c(e, \tilde{\xi}) \tag{33}$$

that renders the system (31) 0-TUES per Definition 2.2. In this way, it leads to a control law of the form

$$u = k_c(e, \tilde{\xi}) + \rho_1(\zeta) + \rho_2(\zeta, L(\eta))\Omega \tag{34}$$

for system (29). Once (34) is derived and it is possible to achieve a modified adaptive control law of the form

$$u = k_c(e, \tilde{\xi}) + \rho_1(\zeta) + \rho_2(\zeta, L(\eta))\widehat{\Omega}$$

for system (29) with $\widehat{\Omega}(t)$ to be an estimate of Ω.

To make the above idea clear, for system (31), further let

$$\mathbf{z}_1 = \begin{bmatrix} z_1 - N_{\mathrm{a}} e \\ z_2 - N_{\mathrm{b}} H(q, w)\delta \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} e \\ \delta \end{bmatrix}, \quad \delta = \tilde{\xi} + k_0(e)$$

as the new coordinates, where $k_0(\cdot) \in \mathbb{R}^n$ together with subsequent $k(\cdot) \in \mathbb{R}^n$ are the designing functions to be specified. Then we write the closed-loop system of system (31) under the control law (33) or $\bar{u} = -k(\delta)$ as

$$\dot{x} = f(x) \tag{35a}$$

$$\dot{\mathbf{z}}_1 = \mathbf{f}_1(\mathbf{z}_1, \mathbf{z}_2, x, d_0) \tag{35b}$$

$$\dot{\mathbf{z}}_2 = \mathbf{f}_2(\mathbf{z}_1, \mathbf{z}_2, x, d_0) \tag{35c}$$

whose vector fields are given in (69). Compared with system (31), system (35) is of greater interest due to its interconnection structure for performing the small-gain theorem based stability analysis.

Specifically, we can show an iISS property for subsystem (35b) and an ISS one for subsystem (35c), respectively, validated by the design parameters $k_0(\cdot), k(\cdot) \in \mathbb{R}^n$. Based on that, the key idea of managing the stability analysis of (35) is to adapt the general nonlinear small-gain theorem proposed in [51, Theorem 2]. Particularly, the iISS network scenario encountered in system (35) is distinguished from those addressed in [51], [52], [53] and the special ISS network in [54].

In what follows, inspired by [51], [54], we modify a specific small-gain theorem serving the current stability analysis and moreover, succeeding the Lyapunov function construction. We use the following verifiable conditions for system (35) in terms of Lyapunov-like functions.

H0 There are smooth iISS-Lyapunov functions $V_i = V_i(t, \mathbf{z}_i)$, $i = 1, 2$ for subsystems (35b) and (35c), respectively. More specifically, there are smooth comparison functions $\bar{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i, \gamma_i, r_i \in \mathcal{K}$, $i = 1, 2$ with $\lim_{s\to\infty} \alpha_2(s) = \infty$ and $\lim_{s\to\infty} \gamma_2(s) < \infty$ such that, for all the arguments,

$$\begin{aligned} &\underline{\alpha}_i(|\mathbf{z}_i|) \le V_i(t, \mathbf{z}_i) \le \bar{\alpha}_i(|\mathbf{z}_i|) \\ &\dot{V}_i|_{(35)} \le -\alpha_i(V_i) + \gamma_i(V_{3-i}) + r_i(|d_0|), \ i = 1, 2. \end{aligned} \tag{36}$$

Furthermore, there are constants $\kappa_1, \kappa_2 > 0$ rendering (36) with gain functions

$$\gamma_1(s) = \kappa_1 \alpha_2(s), \quad \gamma_2(s) = \kappa_2 \alpha_1(s), \quad s \ge 0. \tag{37}$$

Lemma 3.4: (Small-Gain Theorem) Consider system (35) *satisfying all the conditions of* **H0**. *If the gain condition*

$$\kappa_1 \kappa_2 < 1 \tag{38}$$

holds, then there are constants $\tilde{\kappa}_1, \tilde{\kappa}_2$ such that the sum-type function

$$V(t, \mathbf{z}_1, \mathbf{z}_2) = \sum_{i=1}^{2} \tilde{\kappa}_i V_i(t, \mathbf{z}_i), \quad \tilde{\kappa}_1, \tilde{\kappa}_2 > 0 \tag{39}$$

is a well-defined smooth iISS-Lyapunov function for the composite $(\mathbf{z}_1, \mathbf{z}_2)$ system.

We point out that Lemma 3.4 can be regarded as a modest extension from the analogous ISS one proposed in [54, Theorem 3.1 & Remark 3.2]. In comparison with the most general small-gain theorem of [51, Theorem 2] for the iISS networks, Lemma 3.4 is special but it lends itself to a direct construction of the sum-type iISS-Lyapunov functions [52].

Now we state a result on the transversal stability analysis for system (35). It finally offers a set of design parameters by verifying all the conditions posed in Lemma 3.4.

Proposition 3.2: Consider system (35) *satisfying assumptions **H1** to **H3** and the PE condition posed in Proposition 3.1. Then there exist smooth design functions $k_0(\cdot), k(\cdot) \in \mathbb{R}^n$ such that the closed-loop system* (35) *is TiISS w.r.t. d_0. Moreover, it is 0-TUES.*

The transversal stability notions have been adopted here for managing the by-product stability analysis arising in the output regulation design. Particularly, a Lyapunov function approach has been developed to prove the TiISS and 0-TUES properties. It is worth noting that, as elaborated in [47], such transveral stability is somehow essential for nonlinear observer design and synchronization, which can provide necessary and sufficient conditions for the design of observers and synchronizers. From this viewpoint, our study can be a concrete case study for nonlinear output regulation design analysis in the terminology of transversal or incremental stability theory. We shall refer to [55] for an interesting result in this direction.

C. Main Theorems and Discussions

As a summary of the preceding subsections, we state the following two main theorems of this study for two extreme circumstances on the knowledge of parameter vector Ω in (19). Based on them, the design with partially known parameters can be easily modified.

Theorem 3.1: Consider the composite system (1) *and* (5) *or its equivalent representation* (11) *satisfying assumptions **H1** to **H3**. Further suppose that the PE condition in Proposition 3.1 holds. If Ω in* (19) *is given, then Problem 2.1 can be solved by a smooth control law of the form*

$$\begin{aligned} &\dot{\eta} = \gamma_{\mathrm{a}}(\eta, \xi) \\ &\dot{\zeta} = \gamma_{\mathrm{b}}(\zeta, L(\eta), u), \quad \delta = k_0(e) + \xi - \Gamma(\eta) \\ &u = -k(\delta) + \rho_1(\zeta) + \rho_2(\zeta, L(\eta))\Omega \end{aligned}$$

with design parameters as given in Propositions 3.1 and 3.2.

Theorem 3.1 is a direct consequence of Proposition 3.2. This can be shown in the same spirit of the problem conversion from the output regulation problem to a tractable stabilization one (w.r.t. an equilibrium), *cf.* [38, Proposition 5.1], whose proof is omitted. Based on that, when Ω in (19) is unknown, we can further approach the direct adaptive control redesign to deal with the controller gain/parameter estimation for system (11) as the main conclusion of the present study.

Theorem 3.2: Under the same conditions as those in Theorem 3.1 but with unknown parameter Ω, Problem 2.1 is still solvable by a smooth control law of the form

$$\begin{aligned}
&\dot{\eta} = \gamma_{\rm a}(\eta, \xi)\\
&\dot{\zeta} = \gamma_{\rm b}(\zeta, L(\eta), u)\\
&\dot{\widehat{\Omega}} = -\lambda \rho_2^T(\zeta, L(\eta))\delta, \quad \delta = k_0(e) + \xi - \Gamma(\eta)\\
&u = -k(\delta) + \rho_1(\zeta) + \rho_2(\zeta, L(\eta))\widehat{\Omega}
\end{aligned} \tag{40}$$

with design parameter λ of a positive definite matrix and all the others are as given in Theorem 3.1.

To illustrate the results in this section, let us accomplish the design example of Example 2.1.

Example 3.1: This example illustrates the proposed approach by solving the output regulation problem of the single-link manipulator example of Example 2.1

As required in Theorem 3.2, we first verify conditions **H1** and **H2** for the single-link manipulator in (9). Note that the condition **H1** naturally holds because the moment of inertia J is always positive. To show condition **H2**, let us consider the zero-error constrained input (10) in Example 2.1. Let $q_0 = q_0(v)$ be the constant bias of reference $q_{\rm ref} = Q(v)$ as in (6). Using (12) and (23), it can be shown that

$$u^\star(x) = \underbrace{\begin{bmatrix} L_2(\theta) & \cos(L_1(\theta)) & -\sin(L_1(\theta)) \end{bmatrix}}_{=:Y(L_1(\theta),\Gamma(\theta),L_2(\theta))} \underbrace{\begin{bmatrix} J \\ mgl\cos(q_0) \\ mgl\sin(q_0) \end{bmatrix}}_{=:a(w,v)} - D(w) \tag{41}$$

where $a(w, v)$ collects all the constant uncertainties due to system parameters and reference bias. It verifies the condition **H2**. Moreover, the certainty equivalence property in Lemma 3.3 is also verified. Explicit expressions of functions L_1 and L_2 are given in (56) and (57), respectively.

Once conditions **H1** and **H2** are satisfied, a controller of the form (40) can be constructed for solving the global output regulation problem of single-link manipulator. Explicit expressions of subsystems η and ζ are given in (46) and (50), respectively. □

We stress that the control law is internal model-based, and particularly contains a pair of internal models in a cascade interconnection structure. As noted before, such characteristic is different from that of the nonlinear internal models introduced in [38] for the general lower triangular systems. In fact, if we apply [38] for our problem, the internal model would take the following isolated structure

$$\dot{\eta} = \gamma_{\rm a}(\eta, \xi), \quad \dot{\zeta} = \gamma_{\rm b}(\zeta, u). \tag{42}$$

However, in this way the required condition (see the conditions of [38, Lemma 3.1] together with the relevant stability condition iii) in [38, Proposition 5.1]) may fail here even for the uncertain single-link manipulator as shown in Example 2.1.

At this point, if we insist on the internal model design within the structure like (42), it can be shown that the internal model design conditions proposed in [41, *Assumptions A1) & A2)*] are verifiable, not expanded here in details, with the semi-global output regulation control goal. However, it leads to a relaxed practical regulation property and does not assure the asymptotic one, i.e., zero steady-state tracking error. This is beyond the scope of this research, and it turns out to be an interesting future direction of combining the internal model technique proposed in [41] and the learning mechanism in the present study toward semi-global output regulation design for electro-mechanical systems.

Besides the above, in comparison with the global control studies based on the polynomial conditions, see, for example, [56, Equation (15)], Theorem 3.1 or 3.2 of this paper is still of interest and potential to offer a distinguished solution by virtue of the cascade internal models. Similarly, for the relevant semi-global output regulation studies, we further point out that the proposed method is also promising to be applicable for more nonlinear systems than before. For example, the usual internal model design condition for semi-global output regulation by error feedback, such as those in [34] and [35, Chapter 5], fails and thus it does not lead to asymptotic tracking due to the same reason as that encountered in constructing (42). In summary, the proposed internal model is more constructive thanks to the use of additional velocity measurement and the embedding of the self-learning mechanism in the proposed regulator as well.

Finally, we shall note that the proposed approach possesses certain scalability, *cf.* [57]. For example, the required conditions **H1** and **H2** are verifiable when the number of DOF grows to the popular 6 or 7-DOF robotic manipulators, and the proposed controller performs well. Nevertheless, the computational complexity of course increases as expected in the computation of the dynamic regressor matrix in the condition **H2**.

IV. Simulation Setup and Results

In this section, the effectiveness of the proposed approach is demonstrated by applying the Theorem 3.2 to the tracking and disturbance rejection control of single- and two-link manipulators, and van der Pol oscillator. [4]

Example 4.1: In this example, we present simulation results for single- and two-link manipulators. For a computational setup, all the reference signals are piecewise continuous as shown in Fig. 4 and 5, respectively, and the disturbance in each channel is a pure unity harmonic.

For the single-link manipulator presented in Example 2.1 and solved in Example 3.1, we apply Theorem 3.2 with parameters $\lambda = 1$, $m_{1i} = (1, 1.4142)^T$, $i = 1$, $\Lambda = 1$, $k_0(e) = 10(1 + e^2)e$, $k(\delta) = 10(1 + \delta^2)\delta$. Although the reference bias changes at 30 and 60 second, respectively, and

[4]Matlab codes of all the simulations are available at https://github.com/haiwenwu/ELOR2021.

the frequency changes at 60 second, Fig. 4 shows that the tracking error tends to zeros in each time interval.

For the two-link robot manipulator, we refer to [5, Example 6.2] for system model and physical parameters. By applying Theorem 3.2 with parameters $\lambda = 30I$, $m_{1i} = (1, 1.4142)^T$, $i = 1, 2$, $\Lambda = 10I$, $k_0(e) = (10(1+e_1^2)e_1, 10(1+e_2^2)e_2)^T$, $k(\delta) = (10(8+\delta_1^2)\delta_1, 10(8+\delta_2^2)\delta_2)^T$, the position tracking error response is shown in Fig. 5. □

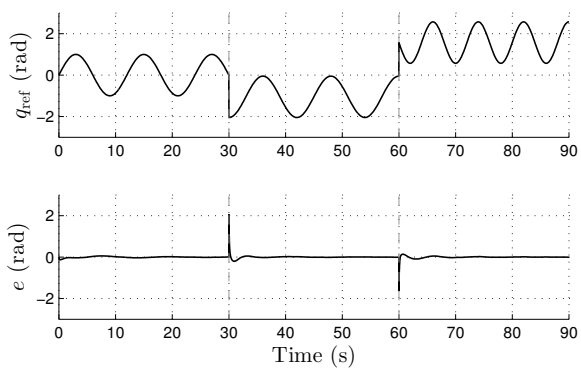

Fig. 4. Reference signal $q_{\text{ref}}(t)$ and tracking error $e(t)$ for the single-link manipulator in Example 4.1.

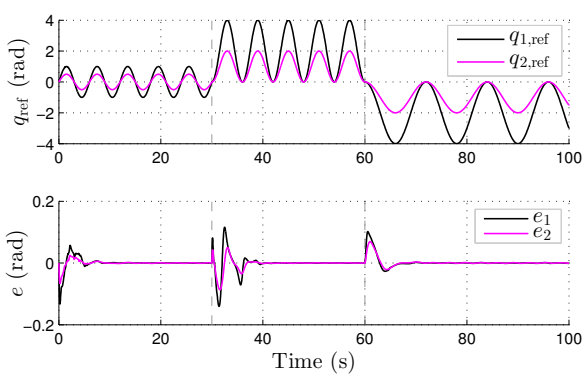

Fig. 5. Reference signal $q_{\text{ref}}(t)$ and tracking error $e(t)$ for the two-link manipulator in Example 4.1.

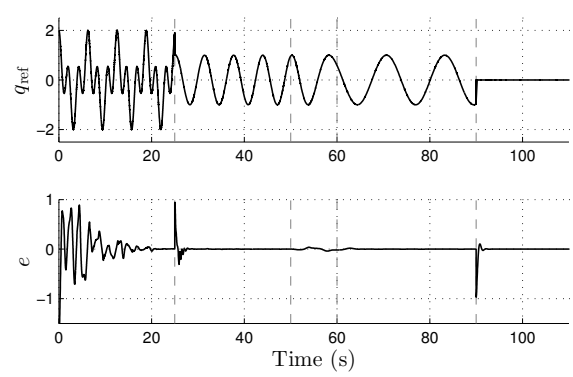

Fig. 6. Reference signal $q_{\text{ref}}(t)$ and tracking error $e(t)$ in Example 4.2.

Example 4.2: Borrowed from [34] and for a global regulator design, consider the controlled van der Pol oscillator

$$\ddot{q} + q - w_1\dot{q} + \dot{q}^3 + w_2 q\dot{q}^2 = u$$

perturbed by $w_2 q\dot{q}^2$, where $q \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, w_1, w_2 are constant uncertain parameters whose values range in a compact set. As in [34], the objective is to make the state q asymptotically track a reference signal q_{ref} that is generated by an exosystem (3). Assume that $w = (w_1, w_2)^T$ is such that $0.5 \le w_1 \le 1.5$ and $1.5 \le w_2 \le 2.5$ and the desired steady-state input $u^\star(x)$ is given by

$$u^\star = q^\star(x) + \xi^{\star 3}(x) + \psi^\star(x) - w_1\xi^\star(x) + w_2 q^\star(x)\xi^{\star 2}(x).$$

Applying Theorem 3.2, the controller can be modified as

$$\begin{aligned}
&\dot{\eta} = \gamma_{\text{a}}(\eta, \xi)\\
&\dot{\widehat{\Omega}} = -\lambda\rho_2^T(\eta)\delta, \quad \delta = k_0(e) + \xi - \Gamma(\eta)\\
&u = -k(\delta) + \rho_1(\eta) + \rho_2(\eta)\widehat{\Omega}
\end{aligned}$$

since the external input disturbance is absent, where ρ_1, ρ_2 are specified as $\rho_1 = L_1(\eta) + \Gamma^3(\eta) + L_2(\eta)$, $\rho_2 = \begin{bmatrix}-\Gamma(\eta) & L_1(\eta)\Gamma^2(\eta)\end{bmatrix}$. The simulating reference is set the same as that in [34], shown in Fig. 6. The design parameters in controller are chosen as $\lambda = 100I$, $\Lambda = 500I$, $k_0(e) = 5e$, $k(\delta) = 5\delta$. The parameter in M_1 is chosen as $m_1 = (1, 2.7, 3.4, 2.1)^T$ during $[0, 25)$ (s) and $m_1 = (1, 1.4142)^T$ during $[25, 90)$ (s). In the last time interval $[90, 110]$ (s), the reference signal is set zero, and in this case the internal model is no need. The tracking error response is shown in Fig. 6.

To close this example, we note that the design method of [37] is applicable here using internal models of the structure (42). In sharp contrast to that, the proposed learning-based design is distinguished and it provides an alternative but reduced-order controller, e.g., the controller order is reduced from 11 to 6 in the case of the period $[25, 90)$ (s). □

V. Conclusion

We have studied a self-learning mechanism-based global robust output regulation design for second-order nonlinear systems subject to external input disturbances by error and velocity feedback. Specifically, based on a certainty equivalence principle method, we proposed a novel class of nonlinear internal models taking a cascade interconnection structure with strictly relaxed conditions than before. It can get through the hurdles for constructive internal model design in nonlinear output regulation of electro-mechanical systems.

Appendix A
Proof of Lemma 3.1

The proof is sketched as a practical modification of [49, Theorem 3.1], leading to an interesting reduced-order redesign.

For the sake of presentation ease, we denote

$$\langle x\rangle = (x_1, 0, x_2, 0, \ldots, x_n, 0)^T \in \mathbb{R}^{2n}$$
$$\text{odd}[x] = (x_1, x_3, x_5, \cdots)^T, \quad \text{even}[x] = (x_2, x_4, x_6, \cdots)^T$$

as induced vectors of a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$. Recalling (12), we write $\xi^\star(x) = (\xi_1^\star(x), \ldots, \xi_n^\star(x))^T$ whose

entries are all polynomials in v. For $i=1,\dots,n$, suppose that $\xi_i^\star(x)$ is unbiased, then each $\xi_i^\star(x)$ has a *minimal zeroing polynomial* (see [43, Remark 6.15]) $P_i(s)=s^{\ell_i}-\sum_{j=1}^{\ell_i/2} b_{ij}s^{2j-2}$ for an even integer ℓ_i and a set of real numbers $\{b_{ij}=b_{ij}(\sigma)\}_{j=1}^{\ell_i/2}$. Let

$$\Xi_i(x)=\begin{bmatrix}\xi_i^\star(x) & \dfrac{d\xi_i^\star(x)}{dt} & \cdots & \dfrac{d^{(\ell_i-1)}\xi_i^\star(x)}{dt^{(\ell_i-1)}}\end{bmatrix}^T \tag{43}$$

$$\Phi_i(b_i)=\left[\begin{array}{c|c}0_{\ell_i-1} & I_{\ell_i-1}\\ \hline -b_{i1} & 0,\dots,-b_{i\ell_i/2},0\end{array}\right],\ \Psi_i=\begin{bmatrix}1\\0_{\ell_i-1}\end{bmatrix}^T$$

where $b_i:=b_i(\sigma)=(b_{i1},\dots,b_{i\ell_i/2})^T$. Then we have to be a generator of $\xi_i^\star(x)$ in the sense of Definition 2.1.

Select any controllable pair (M_{1i},N_{1i}) with

$$M_{1i}=\left[\begin{array}{cc}0_{\ell_i-1} & I_{\ell_i-1}\\ \hline \multicolumn{2}{c}{-m_{1i}^T}\end{array}\right],\quad N_{1i}=\begin{bmatrix}0_{\ell_i-1}\\1\end{bmatrix}$$

where $m_{1i}\in\mathbb{R}^{\ell_i}$ is given so that M_{1i} is Hurwitz. Solving the Sylvester equation $T_i(b_i)\Phi_i(b_i)=M_iT_i(b_i)+N_i\Psi_i$ determines the unique and nonsingular solution $T_i(b_i)$ (see [58, Theorem 2]). In a similar manner as the proof of [49, Theorem 3.1] but for a reduced-order design, we claim that, for each $i=1,\dots,n$, the following system

$$\begin{aligned}[\partial\theta_{ia}(x)/\partial x]f(x)&=M_{1i}\theta_{ia}(x)+N_{1i}\theta_{ic}(x)\\ 0&=-\mathrm{odd}[\theta_{ia}(x)]\big\{\mathrm{odd}[\theta_{ia}(x)]^T\theta_{ib}(\sigma)\\ &\quad+\mathrm{even}[\theta_{ia}(x)]^T\mathrm{even}[m_{1i}]-\theta_{ic}(x)\big\}\\ [\partial\theta_{ic}(x)/\partial x]f(x)&=-\theta_{ic}(x)+\xi_i^\star(x)\\ \xi_i^\star(x)&=\Gamma_{1i}\circ\theta_i(x),\quad x\in\mathbb{X}\end{aligned} \tag{44}$$

can be made a well-defined generator of $\xi_i^\star(x)$, where $\theta_i(x)=(\theta_{ia}(x),\theta_{ib}(\sigma),\theta_{ic}(x))^T$, and the output mapping $\Gamma_{1i}(\theta_i)$ is smooth, globally defined, and compactly supported such that, for all $x\in\mathbb{X}$,

$$\begin{aligned}&\Gamma_{1i}(\theta_i(x))\\ &=[m_{1i}-\langle\varrho_{1i}(\theta_{ib}(\sigma))\rangle]^T[\Phi_i\circ\varrho_{1i}(\theta_{ib}(\sigma))+I]\theta_{ia}(x)\end{aligned} \tag{45}$$

where $\varrho_{1i}(\theta_{ib}):=\mathrm{odd}[m_{1i}]-\theta_{ib}$. This can be shown by means of the closed-form solution $\theta_i(x)$ of each (44).

Denote $\theta_a=(\theta_{1a},\dots,\theta_{na})^T$, $\theta_b=(\theta_{1b},\dots,\theta_{nb})^T$, $\theta_c=(\theta_{1c},\dots,\theta_{nc})^T$, $m_1=(m_{11},\dots,m_{1n})^T$, and let $M_1,N_1,\Psi,\Phi,\chi_1(\theta_a),\chi_2(\theta_a)$ be block matrices with diagonal blocks $M_{1i},N_{1i},\Psi_i,\Phi_i,\mathrm{odd}[\theta_{ia}],\mathrm{even}[\theta_{ia}]$ for $i=1,\dots,n$, respectively. Then we rewrite (44) in a compact form

$$\begin{aligned}[\partial\theta_a(x)/\partial x]f(x)&=M_1\theta_a(x)+N_1\theta_c(x)\\ 0&=-\Lambda\chi_1(\theta_a(x))\big\{\chi_1^T(\theta_a(x))\theta_b(\sigma)\\ &\quad+\chi_2(\theta_a(x))^T\mathrm{even}[m_1]-\theta_c(x)\big\}\\ [\partial\theta_c(x)/\partial x]f(x)&=-\theta_c(x)+\xi^\star(x).\end{aligned}$$

It finally leads to the following internal model

$$\begin{aligned}\dot\eta_a&=M_1\eta_a+N_1\eta_c\\ \dot\eta_b&=-\Lambda\chi_1(\eta_a)\big\{\chi_1^T(\eta_a)\eta_b+\chi_2^T(\eta_a)\mathrm{even}[m_1]-\eta_c\big\}\\ \dot\eta_c&=-\eta_c+\xi,\quad \eta=(\eta_a,\eta_b,\eta_c)^T\in\mathbb{R}^\ell\end{aligned} \tag{46}$$

taking the form (15) for any positive definite matrix Λ, with output mapping $\Gamma(\cdot)=(\Gamma_{11}(\cdot),\dots,\Gamma_{1n}(\cdot))^T$ to be smooth, globally defined, and compactly supported, satisfying (45). The proof is complete.

Appendix B
Proof of Lemma 3.2

Consider function $u^\star(x)$ as a solution to the regulator equations (12) and $c(x)$ as in (17). Our main objective in this proof is to construct a pseudo internal model with output u by means of establishing a suitable generator of $u^\star(x)$. Toward this end, we first note that

$$\begin{aligned}u^\star(x)&=H(q^\star,w)\psi^\star+G(q^\star,\xi^\star,w)-D(w)\\ &=Y(q^\star-q_0(v),\xi^\star,\psi^\star)a-D(w)=:\bar Y(c)a-D(w).\end{aligned}$$

Since each entry of $D(w)$ is polynomial in w, as shown in Appendix A, it has a minimal zeroing polynomial of degree l_i for $i=1,\cdots,n$. One can select any controllable pair (M_{2i},N_{2i}) with $M_{2i}\in\mathbb{R}^{l_i\times l_i}$ being Hurwitz, and denote $M_2=\text{block diag}(M_{21},\dots,M_{2n})\in\mathbb{R}^{l_b\times l_b}$, $N_2=\text{block diag}(N_{21},\dots,N_{2n})$, $M_{2a}=I_{l_a}\otimes M_2$, $N_{2a}=I_{l_a}\otimes N_2$ with l_a as the dimension of $a(w,v)$ in **H2** and $l_b=\sum_{i=1}^n l_i$.

Now we claim that the following equations

$$\begin{aligned}[\partial\vartheta_a(x)/\partial x]f(x)&=M_{2a}\vartheta_a(x)+N_{2a}\mathrm{vec}(\bar Y(c(x)))\\ [\partial\vartheta_b(x)/\partial x]f(x)&=M_2\vartheta_b(x)+N_2u^\star(x)\end{aligned} \tag{47}$$

where $M_{2a}=I_{l_a}\otimes M_2$, $N_{2a}=I_{l_a}\otimes N_2$ with l_a as the dimension of $a(w,v)$ in **H2**, are solvable with solution $\vartheta(x)=(\vartheta_a(x),\vartheta_b(x))^T$. In fact, different from deriving the closed-form solution for (44) in the proof of Lemma 3.1, this can be shown in the same manner as that of [41, Proposition 1] to assure the solvability of (47) and thus is omitted. At this moment, we remark that the proof of Lemma 3.2 here is independent of the closed-form solution $\vartheta(x)$ of (47) which is as a matter of fact never realistic for the current study because of the strong nonlinearities involed in $(u^\star(x),\bar Y(c(x)))$.

Next, to make (47) a valid generator of $u^\star(x)$, we need to assign an output mapping in accordance with Definition 2.1. For this purpose, we observe that (47) is a linear system w.r.t. the external input $(u^\star,\bar Y(c))$. Moreover, by using $u^\star(x)=\bar Y(c)a-D(w)$, generator (47) satisfies $[\partial\bar\vartheta(x)/\partial x]f(x)=M_2\bar\vartheta(x)+N_2[u^\star(x)-\bar Y(c(x))a]$ where $\bar\vartheta(x)=\vartheta_b(x)-\mathrm{mat}\big(\vartheta_b(x)\big)a$ with output $u^\star(x)=\bar Y(c(x))a-D(w)=\bar Y(c(x))a+\Gamma_2(\sigma)\vartheta(x)$. Employing the classical additivity and homogeneity properties relating to linear dynamical systems, we can set an output mapping of system (47) as

$$u^\star(x)=\Gamma_2(\sigma)[\vartheta_b(x)-\mathrm{mat}(\vartheta_a(x))a]+\bar Y(c(x))a \tag{48}$$

where $\Gamma_2(\sigma)\in\mathbb{R}^{n\times l_b}$ can be parameterized as $\Gamma_2(\sigma)=\Gamma_0+\sum_{i=1}^{n_\mu}\breve\Gamma_{2i}\mu_i(\sigma)$ for an integer $n_\mu>0$. Note that the right hand side of (48) is a function of $(\vartheta(x),c(x))$ of linearly parameterized. For another representation of this function, using the facts that $\Gamma_0\in\mathbb{R}^{n\times l_b}$ and $\breve\Gamma_{2i}\in\mathbb{R}^{n\times l_b}$

for $i = 1, \ldots, n_\mu$ are all matrices independent of σ, and $\mu_i(\sigma) \in \mathbb{R}$ are functions of σ, we have

$$\begin{aligned} u^\star(x) &= \Gamma_0 \vartheta_b(x) + [\breve{\Gamma}_{21}\vartheta_b(x), \cdots, \breve{\Gamma}_{2n_\mu}\vartheta_b(x)]\mu(\sigma) \\ &\quad - \big[\breve{\Gamma}_{21}\mathrm{mat}(\vartheta_a(x)), \cdots, \breve{\Gamma}_{2n_\mu}\mathrm{mat}(\vartheta_a(x))\big] \\ &\quad \cdot (\mu(\sigma) \otimes a) - \Gamma_0 \mathrm{mat}(\vartheta_a(x))a + \bar{Y}(c(x))a \\ &=: \rho_1(\vartheta(x)) + \rho_2(\vartheta(x), c(x))\Omega(a, \sigma) \end{aligned} \tag{49}$$

where $\Omega(a,\sigma) = \big(\mu(\sigma), (\mu(\sigma) \otimes a), a\big)^T$, $\mu(\sigma) = (\mu_1(\sigma), \ldots, \mu_{n_\mu}(\sigma))^T$, $\rho_1(\vartheta) = \Gamma_0 \vartheta_b$ and $\rho_2(\vartheta, c) = \big[\rho_{21}, \rho_{22}, \rho_{23}\big]$ with $\rho_{21} = \big[\breve{\Gamma}_{21}\vartheta_b, \cdots, \breve{\Gamma}_{2n_\mu}\vartheta_b\big]$, $\rho_{21} = \big[\breve{\Gamma}_{21}\mathrm{mat}(\vartheta_a), \cdots, \breve{\Gamma}_{2n_\mu}\mathrm{mat}(\vartheta_a)\big]$ and $\rho_{23} = \big[\bar{Y}(c) - \Gamma_0 \mathrm{mat}(\vartheta_a)\big]$. As a summary, the system (47) with output (49) is a generator of $u^\star(x)$, and gives the following internal model

$$\begin{aligned} \dot{\zeta}_a &= M_{2a}\zeta_a + N_{2a}\mathrm{vec}(\bar{Y}(L(\eta))) \\ \dot{\zeta}_b &= M_2\zeta_b + N_2 u, \quad \zeta = (\zeta_a, \zeta_b)^T \in \mathbb{R}^l \end{aligned} \tag{50}$$

with output u and the output mapping is specified in (49). It exactly takes the form (18). The proof is complete.

Appendix C
Proof of Lemma 3.3

We need to solve $q^\star(x)$ and $\psi^\star(x)$ in (12) from $\theta(x)$ satisfying (44). Recall (12) and we write $\bar{q}^\star(x) := q^\star(x) - q_0(v)$, $\bar{q}^\star(x) = (\bar{q}_1^\star(x), \ldots, \bar{q}_n^\star(x))^T$, $\psi^\star(x) = (\psi_1^\star(x), \ldots, \psi_n^\star(x))^T$. For each $i = 1, \ldots, n$, denote

$$\begin{aligned} \Pi_i(x) &= \begin{bmatrix} \bar{q}_i^\star(x) & \frac{d\bar{q}_i^\star(x)}{dt} & \ldots & \frac{d^{(\ell_i-1)}\bar{q}_i^\star(x)}{dt^{(\ell_i-1)}} \end{bmatrix}^T \\ \Upsilon_i(x) &= \begin{bmatrix} \psi_i^\star(x) & \frac{d\psi_i^\star(x)}{dt} & \ldots & \frac{d^{(\ell_i-1)}\psi_i^\star(x)}{dt^{(\ell_i-1)}} \end{bmatrix}^T. \end{aligned} \tag{51}$$

Since $\bar{q}_i^\star(x)$, $\xi_i^\star(x)$ and $\psi_i^\star(x)$ satisfying (12) have common frequency parameters, using $\Xi_i(x)$ and $\Phi_i(b_i)$ in (43), we have

$$\Pi_i(x) = \Phi_i^{-1}(b_i)\Xi_i(x), \quad \Upsilon_i(x) = \Phi_i(b_i)\Xi_i(x). \tag{52}$$

By generator (44) and [49, Proof of Theorem 3.1], it assures an invertible matrix $T_{ai}(b_i)$ satisfying

$$T_{ai}^{-1}(b_i)\Phi_i(b_i)T_{ai}(b_i) = \Phi_i(b_i) \tag{53}$$

and

$$\Psi_i T_{ai}^{-1}(b_i) = [m_{1i} - \langle b_i \rangle]^T[\Phi_i(b_i) + I] \tag{54}$$

such that

$$\theta_{ia}(x) = T_{ai}(b_i)\Xi_i(x). \tag{55}$$

Substituting (55) into (52) gives

$$\begin{aligned} \Pi_i(x) &= \underbrace{\Phi_i^{-1}(b_i)T_{ai}^{-1}(b_i)}_{\text{using (53)}}\theta_{ia}(x) = T_{ai}^{-1}(b_i)\Phi_i^{-1}(b_i)\theta_{ia}(x) \\ \Upsilon_i(x) &= \underbrace{\Phi_i(b_i)T_{ai}^{-1}(b_i)}_{\text{using (53)}}\theta_{ia}(x) = T_{ai}^{-1}(b_i)\Phi_i(b_i)\theta_{ia}(x). \end{aligned}$$

This together with (51) and Ψ_i in (43) yields

$$\begin{aligned} \bar{q}_i^\star(x) = \Psi_i\Pi_i(x) &= \underbrace{\Psi_i T_{ai}^{-1}(b_i)}_{\text{using (54)}}\Phi_i^{-1}(b_i)\theta_{ia}(x) \\ &= [m_{1i} - \langle b_i \rangle]^T[\Phi_i(b_i) + I]\Phi_i^{-1}(b_i)\theta_{ia}(x) \\ \psi_i^\star(x) = \Psi_i\Upsilon_i(x) &= \underbrace{\Psi_i T_{ai}^{-1}(b_i)}_{\text{using (54)}}\Phi_i(b_i)\theta_{ia}(x) \\ &= [m_{1i} - \langle b_i \rangle]^T[\Phi_i(b_i) + I]\Phi_i(b_i)\theta_{ia}(x). \end{aligned}$$

In the above, using the fact $b_i(\sigma) = \mathrm{odd}[m_{1i}] - \theta_{ib}(\sigma) =: \varrho_i(\theta_{ib}(\sigma))$ and by substitutions, it gives, for all $x \in \mathbb{X}$,

$$\begin{aligned} \bar{q}_i^\star(x) &= [m_{1i} - \langle \varrho_i(\theta_{ib}) \rangle]^T[\Phi_i \circ \varrho_i(\theta_{ib}) + I] \\ &\quad \cdot [\Phi_i \circ \varrho_i(\theta_{ib})]^{-1}\theta_{ia}(x) =: L_{1i} \circ \theta_i(x) \end{aligned} \tag{56}$$

$$\begin{aligned} \psi_i^\star(x) &= [m_{1i} - \langle \varrho_i(\theta_{ib}) \rangle]^T[\Phi_i \circ \varrho_i(\theta_{ib}) + I] \\ &\quad \cdot [\Phi_i \circ \varrho_i(\theta_{ib})]\theta_{ia}(x) =: L_{2i} \circ \theta_i(x). \end{aligned} \tag{57}$$

Finally, the smooth mappings $L_1(\cdot)$ and $L_2(\cdot)$ in (23) can be obtained as $L_1(\theta) = (L_{11}(\theta_1), \ldots, L_{1n}(\theta_n))^T$ and $L_2(\theta) = (L_{21}(\theta_1), \ldots, L_{2n}(\theta_n))^T$, respectively. The proof is complete.

Appendix D
Proof of Proposition 3.1

Before proceeding, we recall a convenient and useful technical lemma whose proof can be found in [59, Lemma A.1].

Lemma D.1: Suppose that $F : \mathbb{R}^n \times \mathbb{D}$ is continuous in (e, x) for a compact set $\mathbb{D}$ and satisfies: (i) $F(0, x) = 0, \forall x \in \mathbb{D}$; (ii) $F(e, x)$ is locally Lipschitz at $e = 0$ uniformly on $\mathbb{D}$. Then there is a function $\gamma \in \mathcal{K} \cap \mathcal{O}(Id)$ such that

$$|F(e, x)|^2 \le \gamma(|e|^2), \quad \forall (e, x) \in \mathbb{R}^n \times \mathbb{D}. \tag{58}$$

In addition to (i) and (ii), suppose that (iii) $f(e, x)$ is bounded for all $(e, x) \in \mathbb{R}^n \times \mathbb{D}$. Then we have $\gamma \in \mathcal{K}_o \cap \mathcal{O}(Id)$ satisfying (58).

The internal model property in Proposition 3.1 can be easily shown by combining Lemmas 3.1, 3.2, and 3.3, and thus is omitted. In the rest of the proof, we only show the TiISS property by a Lyapunov function approach.

Toward that end, we first write up the vector field of (28) explicitly. Using (46), (50), and (27), we write

$$F(z + \bar{N}\nu, x, \tilde{u}) = \begin{bmatrix} M_1 z_{1a} + N_1(z_{1c} + \nu_1) \\ -\Theta(x)z_{1b} + F_{1b}(z_1 + N_{\mathrm{a}}\nu_1, x) \\ -(z_{1c} + \nu_1) + \tilde{u}_1 \\ M_{2a}z_{2a} + F_{2a}(z_1 + N_{\mathrm{a}}\nu_1, x) \\ M_2(z_{2b} + N_2\nu_2) + N_2\tilde{u}_2 \end{bmatrix}$$

with $\Theta(x) = \Lambda\chi(\theta_a(x))\chi^T(\theta_a(x))$, $z_1 = (z_{1a}, z_{1b}, z_{1c})^T$, $z_2 = (z_{2a}, z_{2b})^T$, $N_{\mathrm{a}} = [0 \quad 0 \quad I_n]^T$, $N_{\mathrm{b}} = [0 \quad N_2^T]^T$, $F_{1b} := F_{1b}(z_1 + N_{\mathrm{a}}\nu_1, x)$, $F_{2a} := F_{2a}(z_1 + N_{\mathrm{a}}\nu_1, x)$ and

$$\begin{aligned} F_{1b} &= \chi_1(\theta_a)\chi_1(\theta_a)^T(z_{1b} + \theta_b) + \chi_1(\theta_a)\chi_2(\theta_a)^T\mathrm{even}(m_1) \\ &\quad - \Lambda[\chi_1(z_{1a} + \theta_a)\chi_1(z_{1a} + \theta_a)(z_{1b} + \theta_b) \\ &\quad + \chi_1(z_{1a} + \theta_a)(z_{1c} + \nu_1 + \theta_c) - \chi_1(\theta_a)\theta_c \\ &\quad - \chi_1(z_{1a} + \theta_a)\chi_2(z_{1a} + \theta_a)^T\mathrm{even}(m_1)] \\ F_{2a} &= N_{2a}\mathrm{vec}\big(\bar{Y}(L(z_1 + N_{\mathrm{a}}\nu_1 + \theta)) - \bar{Y}(L(\theta))\big). \end{aligned}$$

The proof is divided in two parts.

Part I. The task of this part is to construct a smooth iISS Lyapunov function $V_0(t,z)$ for system (28) such that

$$\begin{aligned}&\underline{\alpha}_0(|z|) \le V_0(t,z) \le \bar{\alpha}_0(|z|)\\ &\dot{V}_0|_{(28)} \le -\alpha_0(V_0) + c_0|\nu|^2 + c_0|\tilde{u}|^2 \end{aligned} \tag{59}$$

for constant $c_0 > 0$ and $\underline{\alpha}_0, \bar{\alpha}_0 \in \mathcal{K}_\infty$, $\alpha_0 \in \mathcal{K}\cap\mathcal{O}(Id)$. Based on that, the 0-TUES property can be assured.

Consider the z_{1b}-subsystem of system (28). By the assumption and using [50, Theorem 6.14], the induced linear system $\dot{z}_{1b} = -\Theta(x)z_{1b}$ is exponentially stable at origin. Thus, using the converse theorem [35, Theorem 4.14], there is a smooth Lyapunov function $W(t,z_{1b})$ satisfying $c_{b1}|z_{1b}|^2 \le W(t,z_{1b}) \le c_{b2}|z_{1b}|^2$, $\frac{\partial W}{\partial t} - \frac{\partial W}{\partial z_{1b}}\Theta(x)z_{1b} \le -c_{b3}|z_{1b}|^2$, $|\partial W/\partial z_{1b}| \le c_{b4}|z_{1b}|$ for constants $c_{b1}, c_{b2}, c_{b3}, c_{b4} > 0$. Let

$$V_{1b}(t,z_{1b}) = \ln(1 + W(t,z_{1b})) \tag{60}$$

which satisfies, $\underline{\alpha}_{1b}(|z_{1b}|) \le V_{1b}(t,z_{1b}) \le \bar{\alpha}_{1b}(|z_{1b}|), \forall (t,z_{1b})$ with $\underline{\alpha}_{1b}(s) = \ln(1 + c_{b1}s^2)$, $\bar{\alpha}_{1b}(s) = c_{b2}s^2$, $s \ge 0$, and

$$\begin{aligned}\dot{V}_{1b}|_{(28)} &= \frac{1}{1+W(t,z_{1b})}\left[\frac{\partial W}{\partial t} - \frac{\partial W}{\partial z_{1b}}\big(\Theta(x)z_{1b} - F_{1b}\big)\right]\\ &\le -\frac{c_{b3}|z_{1b}|^2}{1+W(t,z_{1b})} + \frac{c_{b4}|z_{1b}|}{1+W(t,z_{1b})}|F_{1b}|.\end{aligned}$$

In the above, note that, there is a constant $c_b > 0$ such that $c_{b4}|z_{1b}||F_{1b}| \le \frac{c_{b3}}{2}|z_{1b}|^2 + c_b(1 + c_{b1}|z_{1b}|^2)(|z_{1a}|^2 + |z_{1c}|^2 + |\nu_1|^2)$ and consequently,

$$\begin{aligned}\dot{V}_{1b}|_{(28)} &\le -\frac{c_{b3}}{2}\frac{|z_{1b}|^2}{1+c_{b2}|z_{1b}|^2} + c_b(|z_{1a}|^2 + |z_{1c}|^2 + |\nu_1|^2)\\ &\le -\alpha_b(|z_{1b}|^2) + c_b(|z_{1a}|^2 + |z_{1c}|^2 + |\nu_1|^2)\end{aligned}$$

where $\alpha_b(s) = \bar{c}_b\frac{s}{1+s} \in \mathcal{K}_o\cap\mathcal{O}(Id)$ for some constant $\bar{c}_b > 0$.

Next, using (60), let

$$\begin{aligned}V_0(t,z) = \epsilon_1^{-1}z_{1a}^TP_1z_{1a} + \epsilon_2^{-1}V_{1b}(t,z_{1b}) + \frac{\epsilon_3^{-1}}{2}z_{1c}^Tz_{1c}\\ + z_{2a}^TP_{2a}z_{2a} + z_{2b}^TP_{2b}z_{2b}\end{aligned} \tag{61}$$

where P_1, P_{2a} and P_{2b} are positive definite matrices solved from $M_1^TP_1 + P_1M_1 = -I$, $M_{2a}^TP_{2a} + P_{2a}M_{2a} = -I$ and $M_2^TP_{2b} + P_{2b}M_2 = -I$, and $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ to be specified by (66). For (61), it manifests the 1st condition of (59) with

$$\underline{\alpha}_0(s) = \underline{c}_0\ln(1+s^2), \quad \bar{\alpha}_0(s) = \bar{c}_0s^2, \quad s > 0. \tag{62}$$

To show the second condition of (59), we have

$$\begin{aligned}\dot{V}_0|_{(28)} \le &-\Big(\frac{\epsilon_1^{-1}}{2} - \epsilon_2^{-1}c_b\Big)|z_{1a}|^2 - \epsilon_2^{-1}\alpha_b(|z_{1b}|^2) - \frac{1}{2}|z_{2a}|^2\\ &-\Big(\frac{\epsilon_3^{-1}}{2} - 4\epsilon_1^{-1}|P_1N_1|^2 - \epsilon_2^{-1}c_b\Big)|z_{1c}|^2 - \frac{1}{2}|z_{2b}|^2\\ &+\Big(4\epsilon_1^{-1}|P_1N_1|^2 + \epsilon_2^{-1}c_b + \epsilon_3^{-1}\Big)|\nu_1|^2 + \epsilon_3^{-1}|\tilde{u}_1|^2\\ &+4|P_{2b}M_2N_2|^2|\nu_2|^2 + 4|P_{2b}N_2|^2|\tilde{u}_2|^2\\ &+2|P_{2a}|^2|F_{2a}(z_1 + N_{\rm a}\nu_1, x)|^2.\end{aligned} \tag{63}$$

In (63), since function L is smooth and compactly supported, by Lemma D.1, we have, for all $x \in \mathbb{X}$,

$$|F_{2a}(z_1+N_{\rm a}\nu_1,x)|^2 \le c_{2a}(|z_{1a}|^2+\alpha_b(|z_{1b}|^2)+|z_{1c}|^2+|\nu_1|^2) \tag{64}$$

for a constant $c_{2a} > 0$. Substituting (64) into (63) gives

$$\begin{aligned}\dot{V}_0|_{(28)} \le &-\Big(\frac{\epsilon_1^{-1}}{2} - \epsilon_2^{-1}c_b - c_{2a}\Big)|z_{1a}|^2\\ &-\Big(\epsilon_2^{-1} - 2|P_{2a}|^2c_{2a}\Big)\alpha_b(|z_{1b}|^2)\\ &-\Big(\frac{\epsilon_3^{-1}}{2} - 4\epsilon_1^{-1}|P_1N_1|^2 - \epsilon_2^{-1}c_b - c_{2a}\Big)|z_{1c}|^2\\ &-\frac{1}{2}|z_{2a}|^2 - \frac{1}{2}|z_{2b}|^2 + c_0(|\nu|^2 + |\tilde{u}|^2)\end{aligned} \tag{65}$$

with $c_0 = \max\{4|P_{2b}M_2N_2|^2, 4|P_{2b}N_2|^2, 2|P_{2a}|^2c_{2a} + \epsilon_3^{-1} + \epsilon_2^{-1}c_b + 4\epsilon_1^{-1}|P_1N_1|^2\}$.

In (65), choosing parameters ϵ_1, ϵ_2, ϵ_3 such that

$$\begin{aligned}&\epsilon_2^{-1} - 2|P_{2a}|^2c_{2a} \ge c_{z_1}, \quad \epsilon_1^{-1}/2 - \epsilon_2^{-1}c_b - c_{2a} \ge c_{z_1}\\ &\epsilon_3^{-1}/2 - 4\epsilon_1^{-1}|P_1N_1|^2 - \epsilon_2^{-1}c_b - c_{2a} \ge c_{z_1}\end{aligned} \tag{66}$$

for a constant $c_{z_1} > 0$, yields

$$\begin{aligned}\dot{V}_0|_{(28)} \le &-c_{z_1}|z_{1a}|^2 - c_{z_1}\alpha_b(|z_{1b}|^2) - c_{z_1}|z_{1c}|^2\\ &-\frac{1}{2}|z_{2a}|^2 - \frac{1}{2}|z_{2b}|^2 + c_0(|\nu|^2 + |\tilde{u}|^2).\end{aligned}$$

Further, using the inequalities $s \ge s/(1+s)$, $\forall s \ge 0$, and $\sum_{i=1}^N \alpha_b(s_i) \ge \alpha_b(\sum_{i=1}^N s_i)$, $\forall s_i \ge 0$, $i = 1,\ldots,N$ for any positive integer N, there is a constant $\bar{c}_0 > 0$ such that the second condition of (59) is confirmed with gain function

$$\alpha_0(s) = \bar{c}_0s/(1+s), \quad s \ge 0. \tag{67}$$

In the rest of this part, on the basis of (62) and (67), we show the 0-TUES property of system (28). In fact, note that there are constants s_0, $\underline{c}_{01}$, $\bar{c}_{01}$, c_{01} such that $\underline{\alpha}_0(s) \ge \underline{c}_{01}s^2, \bar{\alpha}_0(s) \le \bar{c}_{01}s^2, \alpha_0(s) \ge c_{01}s, \forall s \in [0,s_0)$. Thus, using [3, Theorem 4.10], it confirms the exponential stability of system (28) at the origin with $(\tilde{u},\nu) = (0,0)$. Immediately, there are constants $c_{z1}, c_{z2} > 0$ such that every trajectory $z(t) = z(z(0),x(0),0,t)$ starting from $z(0)$ in a neighborhood of $z = 0$ and $x(0) \in \mathbb{X}$ satisfies $|z(z(0),x(0),0,t)| \le c_{z1}|z(0)|\exp(-c_{z2}t)$, $t \ge 0$.

Part II. This part is to show the TiISS property of system (28). Consider (59) with the specified gain function $\alpha_0(\cdot)$ in (67). Using [48, Lemma IV.1], one has functions $\bar{\rho}_1 \in \mathcal{K}_\infty$ and $\bar{\rho}_2 \in \mathcal{L}$ so that $\alpha_0(s^2) \ge \bar{\rho}_1(s)\bar{\rho}_2(s)$, $\forall s \ge 0$. Then, it is possible to choose a Lipschitz continuous and positive definite function $\bar{\rho}(\cdot)$ so that $\bar{\rho}(s) \le \big(\bar{\rho}_1 \circ \bar{\alpha}_0^{-1}(s)\big)\big(\bar{\rho}_2 \circ \underline{\alpha}_0^{-1}(s)\big)$, $\forall s \ge 0$. Substituting these inequalities in (59) gives

$$\begin{aligned}\dot{V}_0|_{(28)} &\le -\bar{\rho}_1(|z|)\bar{\rho}_2(|z|) + c_{01}(|\nu|^2 + |\tilde{u}|^2)\\ &\le -\bar{\rho}(V_0(t,z)) + c_{01}(|\nu|^2 + |\tilde{u}|^2).\end{aligned}$$

Further by [48, Corollary IV.3], one has $\beta_0 \in \mathcal{KL}$ such that

$$\begin{aligned}V_0(t,z(t)) \le &\beta_0(V_0(0,z(0)),t)\\ &+ 2c_{01}\int_0^t (c_{01}|\nu(s)|^2 + |\tilde{u}(s)|^2)ds, \quad t \ge 0.\end{aligned}$$

The above together with the first condition of (59) gives

$$\begin{aligned}\underline{\alpha}_0(|z(t)|) &\le V_0(t,z(t))\\ &\le \beta_0\Big(\bar{\alpha}_0\big(|(z(0),x(0))|\big),t\Big)\\ &\quad + 2c_{01}\int_0^t (|\nu(s)|^2 + |\tilde{u}(s)|^2)ds, \quad t \ge 0.\end{aligned}$$

which manifests the TiISS property of system (28) in the sense of Definition 2.3. The proof is complete.

Appendix E
Proof of Lemma 3.4

Consider (39) with constants $\tilde{\kappa}_1, \tilde{\kappa}_2$ to be specified. Using the dissipation inequalities (36), we have

$$\begin{aligned}\dot{V}|_{(35)} &\le \sum_{i=1}^{2} \tilde{\kappa}_i \big\{ -\alpha_i(V_i) + \gamma_i(V_{i+1}) + r_i(|d_0|) \big\} \\ &\le -(\tilde{\kappa}_1 - \tilde{\kappa}_2\kappa_2)\alpha_1(V_1) - (\tilde{\kappa}_2 - \tilde{\kappa}_1\kappa_1)\alpha_2(V_2) + r(|d_0|)\end{aligned}$$

with $r(\cdot) := \tilde{\kappa}_1 r_1(\cdot) + \tilde{\kappa}_2 r_2(\cdot)$. In the above, it is possible to select parameters $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ such that $\tilde{\kappa}_1 - \tilde{\kappa}_2\kappa_2 \ge 1$ and $\tilde{\kappa}_2 - \tilde{\kappa}_1\kappa_1 \ge 1$. In fact, using the gain condition $\kappa_1\kappa_2 < 1$, we can set $\tilde{\kappa}_1 = \frac{1+\kappa_2}{1-\kappa_1\kappa_2}$ and $\tilde{\kappa}_2 = \frac{1+\kappa_1}{1-\kappa_1\kappa_2}$ which assures

$$\dot{V}|_{(35)} \le -\alpha_1(V_1) - \alpha_2(V_2) + r(|d_0|). \tag{68}$$

Hence, all the relevant conditions in [48, Definition II.2] are verifiable that ends the proof.

Appendix F
Proof of Proposition 3.2

To carry out the proof, we need the expressions of the vector fields of (35) described by

$$\begin{aligned}\mathbf{f}_1(\mathbf{z}_1, \mathbf{z}_2, x, d_0) &= F(\mathbf{z}_1 + \bar{N}\nu, x, \tilde{u}) \qquad (69)\\ \mathbf{f}_2(\mathbf{z}_1, \mathbf{z}_2, x, d_0) &= \begin{bmatrix} -k_0(e) + \delta + \Delta_1(\mathbf{z}_1, e, x) \\ H^{-1}\big(-k(\delta) + \Delta_2(\mathbf{z}_1, \mathbf{z}_2, x) + d_0 \big) \end{bmatrix}\end{aligned}$$

where $\mathbf{z}_{11} = z_1 - N_{\rm a} e$, $\mathbf{z}_{12} = z_2 - N_{\rm b} H(e + Q(v), w)\delta$, $\nu_1 = e$, $\nu_2 = H(e + Q(v), w)\delta$, $\tilde{u}_1 = 0$, $\tilde{u}_2 = \Delta_3(\mathbf{z}_{11}, \mathbf{z}_2, x, d_0)$ and

$$\begin{aligned}\Delta_1 &= \Gamma(\mathbf{z}_{11} + \theta + N_{\rm a} e) - \Gamma(\theta)\\ \Delta_2 &= \rho(\mathbf{z}_{12} + \vartheta + N_{\rm b} H\delta, L(\mathbf{z}_{11} + \theta + N_{\rm a} e), x) + D(w)\\ &\quad - G\big(e + Q(v), \delta - k_0(e) + \Gamma(\mathbf{z}_{11} + \theta + N_{\rm a} e), w\big)\\ &\quad - H(e + Q(v), w)[d\Gamma(\mathbf{z}_{11} + \theta + N_{\rm a} e)/dt]\\ &\quad + H(e + Q(v), w)[dk_0(e)/dt]\\ \Delta_3 &= G(Q(v), \Gamma(\theta), w) + H(Q(v), w)L_2(\theta)\\ &\quad - G\big(e + Q(v), \delta - k_0(e) + \Gamma(\mathbf{z}_{11} + \theta + N_{\rm a} e), w\big)\\ &\quad - H(e + Q(v), w)[d\Gamma(\mathbf{z}_{11} + \theta + N_{\rm a} e)/dt]\\ &\quad + H(e + Q(v), w)[dk_0(e)/dt] + H'\delta + d_0\\ &\quad \text{with } H' = dH(e + Q(v), w)/dt.\end{aligned}$$

To proceed, we first set the Lyapunov function candidates. Recalling (59), let

$$V_1(t, \mathbf{z}_1) = V_0(t, \mathbf{z}_1), \quad V_2(t, \mathbf{z}_2) = \frac{1}{2}e^T e + \frac{1}{2}\delta^T H\delta. \tag{70}$$

The 1st condition of (36) is manifest. The rest of the proof is divided in two parts.

Part I. This part is to verify the condition **H0** for Lemma 3.4 for (35b) and (35c). Note that system (35b) is written in line with system (28) with specified $(\tilde{u}, \nu)$ given in (69). According to (59), we have a constant $c_1 > 0$ so that

$$\begin{aligned}\dot{V}_1|_{(35b)} &\le -\alpha_0(V_1) + \underbrace{c_0(|e|^2 + |H\delta|^2 + |\Delta_3|^2)}_{\text{using } \mathbf{H1} \text{ \& Lemma D.1}}\\ &\le -\alpha_0(V_1) + c_1\big(\alpha_b(|\mathbf{z}_{11}|) + V_2 + |d_0|^2\big).\end{aligned}$$

Again, choosing ϵ_1, ϵ_2, ϵ_3 in (66) such that $c_{z_1} - c_1 > 1$, it assures the 2nd inequality of (36) with

$$\alpha_1(s) = c_\epsilon s/(1+s), \quad \gamma_1(s) = c_1 s, \quad r_1(s) = c_1 s^2 \tag{71}$$

for a constant $c_\epsilon > 0$. Similarly, we have

$$\begin{aligned}&\dot{V}_2|_{(35c)}\\ &= e^T(-k_0(e) + \delta + \Delta_1) + \delta^T(-k(\delta) + \Delta_2 + d_0 + H'\delta)\\ &= -e^T k_0(e) - \delta^T k(\delta) + \underbrace{e^T\Delta_1 + \delta^T(\Delta_2 + d_0 + H'\delta)}_{\text{using Young's inequality \& Lemma D.1}}\\ &\le -e^T k_0(e) - \delta^T k(\delta) + c_2\big(\alpha_0(V_1) + V_2 + |d_0|^2\big) \qquad (72)\end{aligned}$$

for a constant $c_2 > 0$. Thus, by selecting k_0 and k, we obtain the 2nd inequality of (36) with

$$\alpha_2(s) = c_k s, \quad \gamma_2(s) = c_2 s/(1+s), \quad r_2(s) = c_2 s^2 \tag{73}$$

for some constant $c_k > 0$ relying on k_0 and k.

Part II. Let us now verify the condition (38) in Lemma 3.4 and construct the iISS-Lyapunov function for the composite $(\mathbf{z}_1, \mathbf{z}_2)$-subsystem.

In view of (71) and (73), the condition of (37) is verifiable with constants $\kappa_1 = c_1/c_k$ and $\kappa_2 = c_2/c_\epsilon$. Recall (72). It is possible to choose k_0 and k such that $c_k > c_1 c_2/c_\epsilon$, and (38) is verified. Consequently, we have $\tilde{\kappa}_1 = \frac{c_k c_\epsilon + c_k c_2}{c_k c_\epsilon - c_1 c_2} > 0$, $\tilde{\kappa}_2 = \frac{c_k c_\epsilon + c_1 c_\epsilon}{c_k c_\epsilon - c_1 c_2} > 0$ so that the function of (39) satisfies (68). Finally, the TiISS and 0-TUES properties for system (35) can be assured using the Lyapunov function (39) in the same manner as that has been done in Appendix D. The proof is complete.

Appendix G
Proof of Theorem 3.2

The proof is routine in the theory of direct adaptive control. Suppose that $[0, t_{\max})$ for $t_{\max} \ge 0$ or $t_{\max} = \infty$ is the argument of the maximal time interval such that the trajectory of the closed-loop system (11) and (40) in question is well defined. Using the same function $V(t, \mathbf{z}_1, \mathbf{z}_2)$ as specified in (39), let

$$U(t, \mathbf{z}_1, \mathbf{z}_2, \tilde{\Omega}) = V(t, \mathbf{z}_1, \mathbf{z}_2) + \frac{1}{2}\tilde{\Omega}^T\lambda^{-1}\tilde{\Omega}$$

with $\tilde{\Omega}(t) = \widehat{\Omega}(t) - \Omega$ be a Lyapunov function candidate. It satisfies, with Ω in (35) replaced by $\hat{\Omega}(t)$,

$$\dot{U}|_{(35)+(40)} \le -\alpha_1(V_1) - \alpha_2(V_2) + r(|d_0|), \quad t \in [0, t_{\max}).$$

By integration in both sides of the above, we have

$$\begin{aligned}&\int_0^t \big[\alpha_1(V_1(s, \mathbf{z}_1(s))) + \alpha_2(V_2(s, \mathbf{z}_2(s)))\big]ds\\ &\le c_3\int_0^t |d_0(s)|^2 ds + U(0, \mathbf{z}_1(0), \mathbf{z}_2(0), \tilde{\Omega}(0))\\ &\quad - U(t, \mathbf{z}_1(t), \mathbf{z}_2(t), \tilde{\Omega}(t)), \quad \forall t \in [0, t_{\max}).\end{aligned}$$

Suppose $d_0(t) \in \mathcal{L}_2^n$. Then $U(t, \mathbf{z}_1(t), \mathbf{z}_2(t), \tilde{\Omega}(t))$ is bounded over $[0, t_{\max})$ with $t_{\max} = \infty$, and further by the continuity, $(\mathbf{z}_1(t), \mathbf{z}_2(t), \tilde{\Omega}(t))$ and $(\eta(t), \zeta(t), q(t), \dot{q}(t))$ are all bounded over the time interval $[0, \infty)$.

In the rest, we show the convergence of $e(t)$. Let $E(t) = \frac{1}{2}\int_0^t e^T(s)e(s)ds,\ t \geq 0$. This together with $V_1(t, \mathbf{z}_1)$, $V_2(t, \mathbf{z}_2)$ in (70) and the dissipation gain α_1 and α_2 in (71) and (73) respectively implies, for $t \geq 0$, $E(t) \leq \int_0^t \left[\alpha_1(V_1(s, \mathbf{z}_1(s))) + \alpha_2(V_2(s, \mathbf{z}_2(s)))\right] ds < \infty$ and $\ddot{E}(t) = e^T(-k_0(e)e + \delta + \Delta_1(\mathbf{z}_1, e, x)) < \infty$. Hence, $\dot{E}(t) = |e(t)|^2/2$ is uniformly continuous. Using Barbalat's Lemma [5, pp. 123], it implies $\lim_{t\to\infty} \dot{E}(t) = 0$, and immediately, $\lim_{t\to\infty} e(t) = 0$. The proof is complete.

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