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Distributed Model Predictive Control and Optimization for Linear Systems With Global Constraints and Time-Varying Communication

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***Abstract*—In the article, we study the distributed model predictive control (DMPC) problem for a network of linear discrete-time systems, where the system dynamics are decoupled, the system constraints are coupled, and the communication networks are described by time-varying directed graphs. A novel distributed optimization algorithm called the push-sum dual gradient (PSDG) algorithm is proposed to solve the dual problem of the DMPC optimization problem in a fully distributed way. We prove that the sequences of the primal, and dual variables converge to their optimal values. Furthermore, to solve the implementation issues, stopping criteria are designed to allow early termination of the PSDG Algorithm, and the gossip-based push-sum algorithm is proposed to check the stopping criteria in a distributed manner. It is shown that the optimization problem is iteratively feasible, and the closed-loop system is exponentially stable. Finally, the effectiveness of the proposed DMPC approach is verified via an example.**

***Index Terms*—Distributed model predictive control (DMPC), global constraints, gossip-based push-sum algorithm, push-sum dual gradient (PSDG) algorithm, time-varying directed graphs.**

I. Introduction

It is well known that model predictive control (MPC) can explicitly handle constraints and provide prescribed control performance, and major progress has been made in MPC both from theoretical research (such as stability and feasibility analysis) and industrial applications (such as process control). Recently, the demand on controlling large-scale and/or geographically isolated systems promotes the development of distributed MPC (DMPC), which requires less communication resources, and is more reliable compared to centralized MPC.

For systems with coupled constraints, the main challenge lies in the guarantee of exactly satisfying global coupled constraints in a decentralized and/or distributed manner. Some existing work [1]–[3] presented sequential DMPC methods, where the global problem is divided into some small subproblems with each subproblem being solved once at each time step. As an extension of [1], a parallel DMPC approach has been reported in [4], allowing subsystems to carry out optimizations simultaneously. However, the global optimality of the overall system is not necessarily guaranteed under these methods. Some other works [5], [6] presented DMPC schemes for linear systems with both coupled dynamics and constraints using inexact optimization and constraint tightening. In [7], a modified optimization problem was formulated inspired by robust MPC to provide stability despite inexact minimization. Compared to [5] and [6], this approach is fully distributed with no centralized operations. However, these works require communication among all coupled subsystems, which means that global communication is necessary when systems dynamics or constraints are global ones. In order to overcome these shortcomings, an iterative DMPC approach based on dual decomposition has been developed in [8]. An alternating direction method of multipliers (ADMM) algorithm is used to solve the consensus optimization problem in a fully distributed way. The method requires that all subsystems communicate with its neighbors, and that the communication networks are represented as a bidirectional time-invariant graph. More recently, Wang and Ong [9] and [8] by using a Nesterov-accelerated-gradient algorithm to increase convergence speed, where the needed global variables for executing the Nesterov-accelerated-gradient algorithm are obtained by using a finite-consensus algorithm at each iteration. Moreover, the work in [10] proposed a stochastic iterative DMPC approach to solve the dual form of the MPC optimization problem under communication noises. However, these results are only valid for fixed bidirectional communication network. In this article, we study the DMPC for linear systems with global constraints over time-varying directed communication networks.

It is worth noting that the problem over directed time-varying networks is significantly more challenging than the case of directed fixed networks in the following two aspects: 1) The distributed optimization algorithm should entail the time-varying communication links, but still achieve the global optima in such dynamic environments. In addition, the convergence error in terms of optimal cost value should be quantified at each iteration step to reduce the computational requirement. 2) The stopping criterion on terminating iteration in each step should be checked distributively, and further ensure the recursive feasibility of the optimization problem and the closed-loop stability when the PSDG algorithm is terminated in advance.

To resolve the first challenge, this article proposes a novel distributed optimization algorithm called the push-sum dual gradient (PSDG) algorithm. The primal optimization of DMPC problem is first converted into a dual problem with the dual variable being the global optimization variable. We have redesigned a new algorithm based on the constraint-free subgradient-push algorithm in [11] to solve the constrained dual problem. Compared with the proposed PSDG algorithm, most existing distributed optimization algorithms except [11], e.g., [12]–[17] are only applicable to fixed directed, undirected communication networks, or time-varying communication networks with some form of balancedness, reflected in the requirement of building a sequence of doubly stochastic matrices that are commensurate with the sequence of underlying communication graphs. Finally, we prove that the sequences

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of the primal and dual variables converge to their optima under the proposed PSDG algorithm.

To overcome the second challenge, a global stopping criterion is first proposed in terms of the accuracy of the computed cost and global constraint satisfaction, based on the convergence property of the PSDG algorithm. Then, we propose the gossip-based push-sum algorithm to enable the global stopping criterion being checked in a fully distributed manner. Under the proposed stopping criterion, the optimization problem is proved to be recursively feasible, and the closed-loop system is proved to be exponentially stable.

The rest of this article is organized as follows. In Section II, the preliminaries of DMPC and problem formulation are introduced. In Section III, a novel distributed PSDG algorithm is designed and convergence analysis is conducted. In Section IV, the distributed stopping criterion and the overall DMPC algorithm are designed. In Section V, the proof of the feasibility of the DMPC optimization problem and the closed-loop stability are presented. In Section VI, a case study is provided and the concluding remarks are summarized in Section VII.

The notations: Let $m, L, M > 0$ be some integers, x be a vector, A and Q be some matrices, and $\mathcal{N}$ be a finite set. Then, $\mathbb{Z}^M$ and $\mathbb{Z}_L^M$ stand for the sets $\{1, 2, \ldots, M\}$ and $\{L, L+1, \ldots, M\}$, respectively. $A > (\geq) 0$ denotes that A is positive definite (semidefinite). $\|x\|$ stands for the Euclidean norm, $||x||_\infty$ and $||x||_1$ stand for the infinity norm and 1-norm of x, respectively, and $\|x\|_Q$ is defined as $\|x\|_Q := x^T Q x$. $[x]_l$ denotes the lth element of x. $x > (\geq) 0$ means that $[x]_l > (\geq) 0$ for each l. $\mathbf{1}_m$ represents an m-dimensional unit vector. $|\mathcal{N}|$ represents the cardinality of $\mathcal{N}$.

II. Problem Formulation and Preliminaries

A. System Description

We consider M linear discrete-time subsystems, described by the following state equations:

$$x^i(t+1) = A^i x^i(t) + B^i u^i(t), i \in \mathbb{Z}^M \tag{1}$$

where $x^i(t) \in \mathbb{R}^{n_i}$ denotes the state of subsystem i at time t, and $u^i(t) \in \mathbb{R}^{m_i}$ denotes the input of subsystem i at time t.

Assumption 1: Suppose (A^i, B^i) is controllable and the state $x^i(t)$ is available to subsystem i for each $i \in \mathbb{Z}^M$.

Each subsystem i is subject to the following local state constraints and input constraints:

$$x^i(t) \in \mathcal{X}^i, u^i(t) \in \mathcal{U}^i. \tag{2}$$

In addition, p global constraints exist among all the subsystems

$$\sum_{i=1}^{M} (\Psi_x^i x^i(t) + \Psi_u^i u^i(t)) \leq \mathbf{1}_p \tag{3}$$

where $\Psi_x^i \in \mathbb{R}^{p \times n_i}$ and $\Psi_u^i \in \mathbb{R}^{p \times u_i}$ are some given matrices.

Assumption 2: $\mathcal{X}^i$ and $\mathcal{U}^i$ are bounded, and closed polytopes containing the origins in the respective interiors.

The communication networks among the M subsystems can be described by a sequence of time-varying directed graphs $G(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{V} := \{1, 2, \ldots, M\}$ is the vertex set, and $\mathcal{E}(t)$ is the edge set at time t. If $(i, j) \in \mathcal{E}(t)$, subsystem i can send message to subsystem j at time t; otherwise subsystem i cannot send message to subsystem j at time t. Furthermore, we define the out-neighbors and in-neighbors of node i as $\mathcal{N}_{\text{out}}^i(t) := \{j | (i, j) \in \mathcal{E}(t)\} \cup \{i\}$ and $\mathcal{N}_{in}^i(t) := \{j | (j, i) \in \mathcal{E}(t)\} \cup \{i\}$, respectively. Define $d^i(t) := |\mathcal{N}_{\text{out}}^i(t)|$ as the out-degree of node i.

Assumption 3: The sequence of graphs $\{G(t)\}$ is B-strongly connected, i.e., there exists some integer $B > 0$ such that the graph with edge set $\mathcal{E}_B(t) = \bigcup_{i=tB}^{(t+1)B-1} \mathcal{E}(i)$ is strongly connected for any $t > 0$.

B. Optimization Problem

In this section, we introduce the MPC optimization problem [8]–[10]. Consider the following MPC optimization problem:

$$\textbf{Problem1}: \min_{\{\boldsymbol{u}_p^i, i \in \mathbb{Z}^M\}} \sum_{i=1}^{M} J^i(x^i, \boldsymbol{u}_p^i) \tag{4}$$

$$\boldsymbol{u}_p^i \in \mathcal{U}_p^i(x^i) \quad \forall i \in \mathbb{Z}^M \tag{5}$$

$$\sum_{i=1}^{M} f^i(x^i, \boldsymbol{u}_p^i) \leq b(\epsilon). \tag{6}$$

where inequality (6) is a tightened global constraint, $b(\epsilon) := [(1 - M\epsilon)\mathbf{1}_p^T, \ldots, (1 - NM\epsilon)\mathbf{1}_p^T]^T$, $f^i(x^i, \boldsymbol{u}_p^i) := \begin{bmatrix} \sum_{i=1}^M (\Psi_x^i x_p^i(0) + \Psi_u^i u_p^i(0)) \\ \vdots \\ \sum_{i=1}^M (\Psi_x^i x_p^i(N-1) + \Psi_u^i u_p^i(N-1))) \end{bmatrix}$, $u_p^i(l)$ and $x_p^i(l)$ denote the input and state predictions for time $t + l$ at time t. Note that the time index t is omitted in these variables to simplify symbols when there is no ambiguity.

The local cost function is defined as $J^i(x^i, \boldsymbol{u}_p^i) := \sum_{l=0}^{N-1} (\|x_p^i(l)\|_{Q^i}^2 + \|u_p^i(l)\|_{R^i}^2) + \|x_p^i(N)\|_{P^i}^2$, x^i denotes the state of subsystem i at time t, $\boldsymbol{u}_p^i := \{u_p^i(0), \ldots, u_p^i(N-1)\}$ represents the predicted input sequence, respectively, N denotes the length of prediction horizon, and $Q^i > 0$, $R^i > 0$, and $P^i > 0$ are weight matrices.

The local constraint set $\mathcal{U}_p^i(x^i)$ in (5) is defined as

$$\begin{aligned} &\mathcal{U}_p^i(x^i) := \{\boldsymbol{u}_p^i \in \mathbb{R}^{m_i N} \\ &x_p^i(l) \in X^i, u_p^i(l) \in \mathcal{U}^i, x_p^i(N) \in \mathcal{X}_t^i \\ &x_p^i(0) = x^i, x_p^i(l+1) = A^i x_p^i(l) + B^i u_p^i(l), l \in Z_0^{N-1}\} \end{aligned} \tag{7}$$

where $\mathcal{X}_t^i$ is the maximal closed polytopes such that for every $x^i \in \mathcal{X}_t^i$, we have $x^i \in \mathcal{X}^i, K^i x^i \in \mathcal{U}^i, A_k^i x^i \in \mathcal{X}_t^i$. In addition, if $x^i \in \mathcal{X}_t^i$ for all $i \in \mathbb{Z}^M$, the following equation holds:

$$\sum_{i=1}^{M} (\Psi_x^i x^i + \Psi_u^i A_K^i x^i) \leq (1 - MN\epsilon)\mathbf{1}_p \tag{8}$$

with $A_k^i := A^i + B^i K^i$, (K^i, P^i) being the solution of the Algebraic Riccati Equation $(A_K^i)^T P^i A_K^i - P^i = -(Q^i + (K^i)^T R^i K^i)$, (8) being a tightened constraint, and $\epsilon > 0$ being some positive number associated with the required degree of the suboptimality of the solution when the proposed push-sum based dual gradient algorithm is terminated in advance and is chosen by the practitioner.

Remark 1: The tightening of constraints in (6) and (8) is to ensure the feasibility in the case that the designed PSDG Algorithm is terminated when convergence accuracy is met.

Remark 2: The proposed DMPC approach cannot be applied directly to deal with coupled dynamics and a coupled cost function. This is because in this case, the dual problem cannot be represented as the sum of several decoupled subproblems. One possible way is to consider the coupled states of neighbors as inputs and add some equality constraints. In this manner, a similar optimization problem might be obtained and then solved by the PSDG algorithm, but it requires extra efforts to ensure algorithm feasibility.

Assumption 4: Suppose for the initial state x_0, the Slater condition holds, i.e., there is a $(\boldsymbol{u}_p^1, \boldsymbol{u}_p^2, \ldots, \boldsymbol{u}_p^M) \in \mathcal{U}_p^1 \times \mathcal{U}_p^2 \times \cdots \times \mathcal{U}_p^M$ satisfying: $\sum_{i=1}^M f^i(x^i, \boldsymbol{u}_p^i) \leq b(\epsilon)$.

Also because the recursive feasibility holds as shown in Section V, the strong duality holds for Problem 1 at each time and problem (2) can be handled by solving the following dual problem [18].

C. Dual Form

Using $\lambda \in \mathbb{R}^{Np}$ denote the dual variable of constraint (6), the Lagrange dual problem of Problem 1 is as follows:

$$\max_{\lambda \geq 0} \min_{\boldsymbol{u}_p^i \in \mathcal{U}_p^i(x^i)} \mathcal{L}(\{\boldsymbol{u}_p^i\}_{i \in \mathbb{Z}^M}, \lambda)$$

where

$$\mathcal{L}(\{\boldsymbol{u}_p^i\}_{i \in \mathbb{Z}^M}, \lambda) := \sum_{i=1}^M J^i\left(x^i, \boldsymbol{u}_p^i\right) + \lambda^T \left(\sum_{i=1}^M f^i\left(x^i, \boldsymbol{u}_p^i\right) - b\left(\epsilon\right)\right) \quad (9)$$

is the Lagrangian.

The abovementioned Lagrange problem can be rewritten as [8]–[10]

$$\textbf{Problem 2}: \max_{\lambda \geq 0} \sum_{i=1}^M g^i\left(\lambda\right) \quad (10)$$

where $g^i(\lambda)$ is defined as follows:

$$g^i\left(\lambda\right) := \min_{\boldsymbol{u}_p^i \in \mathcal{U}_p^i(x^i)} J^i\left(x^i, \boldsymbol{u}_p^i\right) + \lambda^T \left(f^i\left(x^i, \boldsymbol{u}_p^i\right) - \frac{b\left(\epsilon\right)}{M}\right).$$

Lemma 1. (see [19]): $g^i(\lambda)$ is a concave, differentiable function, and the gradient of $g^i(\lambda)$ is $f^i(x^i, \boldsymbol{u}_{po}^i(\lambda)) - \frac{b(\epsilon)}{M}$, where $\boldsymbol{u}_{po}^i(\lambda) := \arg\min_{\boldsymbol{u}_p^i \in \mathcal{U}_p^i(x^i)} g^i(\lambda)$.

III. PSDG Algorithm

A. Algorithm Description

In this section, we develop a distributed optimization algorithm over time-varying directed graphs, called PSDG algorithm, to solve Problem 2. Each subsystem i maintains vector variables $z^{i,k}, \omega^{i,k}, y^{i,k}, \boldsymbol{u}_p^{i,k+1}$, and $\lambda^{i,k}$ for all $i \in \mathbb{Z}^M$ and $k \geq 0$, which are updated according to the following laws:

$$\omega^{i,k+1} = \sum_{j \in \mathcal{N}_{in}^{i,k}} \frac{z^{j,k}}{d^{j,k}} \quad (11)$$

$$y^{i,k+1} = \sum_{j \in \mathcal{N}_{in}^{j,k}} \frac{y^{j,k}}{d^{j,k}} \quad (12)$$

$$\lambda^{i,k+1} = \frac{\omega^{i,k+1}}{y^{i,k+1}} \quad (13)$$

$$\boldsymbol{u}_p^{i,k+1} = \arg\min_{\boldsymbol{u}_p^i \in \mathcal{U}_p^i} J^i(x^i, \boldsymbol{u}_p^i) + \lambda^{i,k+1}\left(f^i(x^i, \boldsymbol{u}_p^i) - \frac{b(\epsilon)}{M}\right) \quad (14)$$

$$z^{i,k+1} = \left[\omega^{i,k+1} + \alpha^{k+1} M \left(f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - \frac{b(\epsilon)}{M}\right)\right]^+ \quad (15)$$

where $\mathcal{N}_{in}^{i,k}$ and $d^{i,k}$ denote the in-neighbor set and out-degree of subsystem i at iteration k, respectively, α^k represents the step size at iteration k, $[x]^+ : \mathbb{R}^n \to \mathbb{R}^n$ denotes a projection operator, defined as

$$[[x]^+]_i = \begin{cases} 0, & [x]_i < 0 \\ [x]_i, & [x]_i \geq 0 \end{cases}$$

for $i \in \mathbb{Z}^n$ with x being an n-dimensional vector. The PSDG Algorithm is summarized in the Algorithm 1.

Algorithm 1: Push-Sum Dual Gradient Algorithm.

1: Set $k = 0$, $z^{i,0} = 0$, $\lambda^{i,0} = 0$ and $y^{i,0} = 1$ for all $i \in \mathbb{Z}^M$
2: Subsystem i exchanges $\frac{z^{i,k}}{d^{j,k}}$ and $y^{i,k}$ with its neighbors for all $i \in \mathbb{Z}^M$
3: Subsystem i calculates $\omega^{i,k+1}$ from (11) for all $i \in \mathbb{Z}^M$
4: Subsystem i calculates $y^{i,k+1}$ from (12) for all $i \in \mathbb{Z}^M$
5: Subsystem i calculates $\lambda^{i,k+1}$ from (13) for all $i \in \mathbb{Z}^M$
6: Subsystem i calculates $\boldsymbol{u}_p^{i,k+1}$ from (14) for all $i \in \mathbb{Z}^M$
7: Subsystem i calculates $z^{i,k+1}$ from (15) for all $i \in \mathbb{Z}^M$;
8: $k \leftarrow k + 1$
9: Return to Step 2

Remark 3: The main idea underlying the PSDG Algorithm is that the variable $\lambda^{i,k}$ for every i converges to a common point by means of the interactions (11) and (12), while (14) and (15) steer the common point to its optimal value. The proposed PSDG Algorithm is a generalization of the push-sum protocol [20]–[22] in the framework of the dual gradient method to accommodate time-varying directed graphs.

Remark 4: Note that only neighbor-to-neighbor communication is required for each step of Algorithm 1. In steps 3 and 4 of Algorithm 1, it is required that each node j knows its out-degree $d^{j,k}$ at time k. This number can be computed correctly for communication networks without packet dropouts and/or time delays.

B. Convergence Results

Assumption 5: The step size sequence $\{\alpha_k\}$ is nonincreasing and satisfies $\sum_{k=1}^\infty \alpha_k = \infty$ and $\sum_{k=1}^\infty (\alpha_k)^2 < \infty$.

In general, α_k has many choices, and one common choice can be $c\frac{1}{k}$ with c being any positive constant. The main convergence result for the PSDG Algorithm is shown in the following theorem.

Theorem 1: Suppose Assumptions 1–5 hold. Let $\boldsymbol{u}_p^{i*}$, $i \in \mathbb{Z}^M$ be the optimal solution of Problem 1, λ^* be an optimal dual variable of Problem 2, $\{\boldsymbol{u}_p^{i,k}\}$ and $\{\lambda^{i,k}\}$, $i \in \mathbb{Z}^M$ be the sequences generated by the PSDG Algorithm. Then, for any state x^i, $i \in \mathbb{Z}^M$ in the feasible set of Problem 1, the sequences $\{\boldsymbol{u}_p^{i,k}\}$ and $\{\lambda^{i,k}\}$ converge to $\boldsymbol{u}_p^{i*}$ and λ^* for all $i \in \mathbb{Z}^M$, respectively.

Algorithm 1 is actually a specific form of the perturbed push-sum algorithm in [11]. Since $\mathcal{U}_p^i$ is bounded and $f^i(\cdot)$ is continuous, the following lemma can be given.

Lemma 2. (see [11]): Suppose Assumptions 1–5 hold. Let $\{\lambda^{i,k}\}$ and $\{z^{i,k}\}$, $i \in \mathbb{Z}^M$ be the sequences generated by Algorithm 1. Then

a) $\lim_{k\to\infty} ||\lambda^{i,k+1} - \frac{\mathbf{1}^T z^{i,k}}{M}|| = 0$;
b) $\sum_{k=0}^\infty \alpha^{k+1} ||\lambda^{i,k+1} - \frac{\mathbf{1}^T z^{i,k}}{M}|| < \infty$.

C. Proof of Theorem 1

Proof: $\lambda^{i,k}$ is an (Np)-dimension vector. We will prove the convergence of the lth element of $\lambda^{i,k}$ with l be any positive constant satisfying: $l \in \mathbb{Z}^M$. We first consider the case when $[\lambda^*]_l > 0$ while the other case when $[\lambda^*]_l = 0$ can be coped with by using the same method.

According to Lemma 1, $-g^i(\lambda)$ is a convex function and $-[f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - \frac{b(\epsilon)}{M}]$ is the gradient of $-g^i(\lambda)$ at $\lambda^{i,k+1}$. By Defining $\hat{\lambda}_l(\epsilon)$ for any constant ϵ as: $\hat{\lambda}_l(\epsilon) := [0, \ldots, 0, \epsilon, 0, \ldots, 0] \in \mathbb{R}^{Np}$, we have $\lambda^{i,k+1} \geq \hat{\lambda}_l([\lambda^{i,k+1}]_l)$ since $[\lambda^{i,k+1}]_j \geq 0, \forall j \in \mathbb{Z}^M$. Then, the following equation holds:

$$-[f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - b(\epsilon)] = -\nabla g^i(\lambda^{i,k+1}) \geq -\nabla g^i(\hat{\lambda}_l([\lambda^{i,k+1}]_l)). \quad (16)$$

Let ϵ_0 be an arbitrary small constant. According to Lemma 2, given any positive constant $\epsilon_1 < \epsilon_0$, there exists a finite iteration k_1 such that: $|[\lambda^{i,k+1}]_l - [\bar{z}^k]_l| < \epsilon_1, \forall i \in \mathbb{Z}^M, \forall l \in \mathbb{Z}^{Np}$.

Then, for every $[\bar{z}^k]_l \geq [\lambda^* + \epsilon_0]$ with $k > k_1$, we have

$$\begin{aligned} -[f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - b(\epsilon)] &= -\nabla g^i(\lambda^{i,k+1}) \\ &\geq -\nabla g^i(\hat{\lambda}_l([\lambda^{i,k+1}]_l)) \\ &\geq -\nabla g^i(\hat{\lambda}_l([\lambda^*]_l + \epsilon_0 - \epsilon_1)). \end{aligned} \quad (17)$$

Since $-g^i(\lambda)$ is convex about λ, $-g^i(\lambda)$ is convex about $[\lambda]_l$. Also because $[\lambda^*]_l + \epsilon_0 - \epsilon_1 > [\lambda^*]_l$, $-[f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - b(\epsilon)]_l \geq -[\nabla g^i(\hat{\lambda}_l([\lambda^*]_l + \epsilon_0 - \epsilon_1))]_l > 0$ for every $[\bar{z}^k]_l \geq [\lambda^* + \epsilon_0]$ with $k > k_1$.

According to Assumption 3, there exists a positive constant y_0, such that: $y^{i,k} > y_0$ for each $k \geq 0$ and every $i \in \mathbb{Z}^M$. Then, for every $[\bar{z}^k]_l \geq [\lambda^* + \epsilon_0]$ with $k > k_1$, $\omega^{i,k} > y_0([\lambda^*]_l + \epsilon_0 - \epsilon_1)$. Since $\mathcal{U}_p^i$ is bounded and $\{\alpha_k\}_k$ is a decreasing sequence according to Assumption 5, there exists a finite iteration k_2, such that

$$\alpha^{k+1} M |[f^i(x^i, \boldsymbol{u}_p^{i,k+1})]_l| < \epsilon_4 \quad \forall k > k_2 \quad (18)$$

where ϵ_4 is an arbitrary small positive constant satisfy $\epsilon_4 < y_0([\lambda^*]_l + \epsilon_0 - \epsilon_1)$. Therefore, the projection operator in (15) actually does not work for any $k > k_3$ with $k_3 := \max\{k_1, k_2\}$.

According to (11) and the definition of the out-degree of node i, we have: $\sum_{i=1}^M \omega^{i,k+1} = \sum_{i=1}^M z^{i,k}$. Furthermore, we have

$$\bar{z}^{i,k+1} = \left[\bar{z}^{i,k} + \alpha^{k+1}\left(\sum_{i=1}^M f^i\left(x^i, \boldsymbol{u}_p^{i,k+1}\right) - b(\epsilon)\right)\right]^+$$

where $\bar{z}^{i,k} := \frac{1}{M}\sum_{i=1}^M z^{i,k}$ for any $k > 0$.

Assume that $[\bar{z}^k]_l \geq [\lambda^* + \epsilon_0]$ for each $k \geq k_3$, we have

$$\begin{aligned} [\bar{z}^{k+1}]_l &= [\bar{z}^k]_l + \alpha^{k+1}\left[\sum_{i=1}^M f^i(x^i, \boldsymbol{u}_p^{i,k+1}) - b(\epsilon)\right]_l \\ &\leq [\bar{z}^k]_l + \alpha^{k+1}\nabla g(\hat{\lambda}_l[\lambda^*]_l + \alpha^{k+1}\epsilon_0 - \epsilon_1). \end{aligned} \quad (19)$$

Adding it from k_3 to any $k > k_3$, we have

$$[\bar{z}^{k+1}]_l \leq [\bar{z}^{k_3}]_l + \sum_{j=k_3}^{k} \alpha^{j+1}\nabla g(\hat{\lambda}_l[\lambda^*]_l + \epsilon_0 - \epsilon_1). \quad (20)$$

Let k approaches to ∞, we have $\bar{z}^k$ diminishes to $-\infty$. That is to say, any $[\bar{z}^k]_l$ satisfying $[\bar{z}^k]_l \geq [\lambda^* + \epsilon_0]$ with $k \geq k_3$ will diminish until $[\bar{z}^k]_l < [\lambda^* + \epsilon_0]$. Once $[\bar{z}^{k_0}]_l < [\lambda^* + \epsilon_0]$ for some k_0, it will never leave the area $\{x | x < [\lambda^*]_l + \epsilon_0 + \epsilon_4\}$ according to Equation (18). Similarly, it can be obtained that given an arbitrary small ϵ_5, there exists an iteration k_5 such that $[\bar{z}]_l$ enters the area $\{x | x > [\lambda^*]_l - \epsilon_5\}$. Therefore, given any small neighborhood of $[\lambda^*]$, there exist a finite iteration when $[\bar{z}]_l$ enters it and stays inside it forever.

For $[\lambda^*]_l = 0$, the same conclusion can also be obtained. Therefore, $\bar{z}^k$ converges to λ^*. According to Lemma 2, $\lambda^{i,k}, \forall i \in \mathbb{Z}^M$ converge to λ^*. ■

IV. Algorithm Implementation

A. Stopping Criterion

In practical implementation, one needs to terminate Algorithm 1 after certain steps, as it cannot iterate for infinite steps. In this section, we design the stopping criteria for Algorithm 1 to facilitate practical implementation while ensuring constraint satisfaction and optimal performance. Consider the following two stopping criteria:

$$\sum_{i=1}^M f^i(x^i, \boldsymbol{u}_p^{i,k}) - b(\epsilon) \leq \epsilon M \mathbf{1}_{pN} \quad (21)$$

$$\sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i,k}) - \sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i*}) \leq \sigma \quad (22)$$

with $\sigma > 0$ being a parameter to de designed by the user. Inequality (21) guarantees that the solution of Algorithm 1 satisfies the global constraint (3), and inequality (22) ensures the prescribed optimality of the solution. In [8], it has been proved that Problem 2 is iteratively feasible and the closed-loop system is exponentially stable when (21) and (22) are chosen as the stopping criteria for Algorithm 1. However, $\boldsymbol{u}_p^{i*}$ in (22) is not available, then we have to find a new stopping criterion to replace (22).

Consider the following stopping criterion:

$$\sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i,k}) - \sum_{i=1}^M g^i(\lambda^{i,k}) + b_g \sum_{i=1}^M ||\bar{z}^{k-1} - \lambda^{i,k}|| \leq \sigma \quad (23)$$

with $\bar{z}^{-1}$ is chosen as 0.

Lemma 3: Suppose Assumptions 1–5 hold. If the input $\boldsymbol{u}_p^i$ satisfies (23), it also fulfills (22).

Proof: We know that for any $\lambda \geq 0$, $\sum_{i=1}^M g^i(\lambda)$ is a lower bound of $\sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i*})$. Then, the following inequality holds:

$$\sum_{i=1}^M g^i(\bar{z}^k) \leq \sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i*}). \quad (24)$$

In Section II, we obtain that the gradient of $g^i(\lambda)$ is $f^i(x^i, \boldsymbol{u}_{po}^i(\lambda)) - \frac{b(\epsilon)}{M}$. According to Assumption 2, there exists a positive constant b_g such that $||f^i(x^i, \boldsymbol{u}_{po}^i(\lambda)) - \frac{b(\epsilon)}{M}|| \leq b_g, \forall \boldsymbol{u}_{po}^i(\lambda) \in \mathcal{U}_p^i$, for all $i \in \mathbb{Z}^M$. Hence, we have that

$$|\sum_{i=1}^M g^i(\lambda^{i,k+1}) - \sum_{i=1}^M g^i(\bar{z}^k)| \leq b_g \sum_{i=1}^M ||\bar{z}^k - \lambda^{i,k+1}||. \quad (25)$$

Incorporating (24) with (25), the following inequality holds:

$$\sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i*}) \geq \sum_{i=1}^M g^i(\lambda^{i,k+1}) - b_g \sum_{i=1}^M ||\bar{z}^k - \lambda^{i,k+1}||.$$

■

In what follows, we show that (21) and (23) are attainable. The result is summarized in the following lemma.

Lemma 4: There exists at least one iteration k, at which (21) and (23) hold.

Proof: According to Theorem 1, we have that $\lim_{k\to\infty} \sum_{i=1}^M f^i(x^i, \boldsymbol{u}_p^{i,k}) - b(\epsilon) \leq 0$ and $\lim_{k\to\infty} \sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i,k}) - \sum_{i=1}^M g^i(\lambda^{i,k}) = 0$. Incorporating the second equality with the fact that $\lim_{k\to\infty} ||\bar{z}^{k-1} - \lambda^{i,k}|| = 0$ for all $i \in \mathbb{Z}^M$ from Lemma 2, we have that $\lim_{k\to\infty} \sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i,k}) - \sum_{i=1}^M g^i(\lambda^{i,k}) + b_g \sum_{i=1}^M ||\bar{z}^{k-1} - \lambda^{i,k}|| = 0$. Then, there exists an iteration k, at which (21) and (23) hold. ■

Remark 5: Note that $\sum_{i=1}^{M} f^i(x^i, \boldsymbol{u}_p^{i,k})$ in (21), $\sum_{i=1}^{M} J^i(x^i, \boldsymbol{u}_p^{i,k})$, $\sum_{i=1}^{M} g^i(\lambda^{i,k})$, $\sum_{i=1}^{M} \|\bar{z}^{k-1} - \lambda^{i,k}\|$, and $\bar{z}^{k-1}$ in (23) are global variables, which are the sum of corresponding local variables. In order to check the stopping criteria in a fully distributed way, we present a consensus average algorithm, called gossip-based push-sum algorithm, to calculate these global variables in a fully distributed way.

B. Gossip-Based Push-Sum Algorithm

Definition 1: Let z, $\hat{z}$, and $\rho > 0$ be some scalars. We call $\hat{z}$ a ρ-approximation of z if $|z - \hat{z}| \leq \rho$.

The main idea of the gossip-based push-sum algorithm is given as follows. Consider the network $G(t)$ described in Section II and a vector $\boldsymbol{\pi} = [\pi^1, \pi^2, \ldots, \pi^M]^T$ with π^i being some scalars for all $i \in \mathbb{Z}^M$, where π^i is only available to subsystem i for all $i \in \mathbb{Z}^M$, and every subsystem can only communicate with its neighbors. Given some scalar $\rho > 0$, the function of the gossip-based push-sum algorithm is to obtain a ρ-approximation of the average $\bar{\pi} := \sum_{i=1}^{M} \pi^i$ in a distributed manner.

Before designing the gossip-based push-sum algorithm, we need to make use of the standard push-sum algorithm [22]. The iteration steps are given as follows:

$$p^{i,k+1} = \sum_{j \in \mathcal{N}_{\text{in}}^{i,k}} \frac{p^{j,k}}{d^{j,k}} \tag{26}$$

$$y^{i,k+1} = \sum_{j \in \mathcal{N}_{\text{in}}^{j,k}} \frac{y^{j,k}}{d^{j,k}} \tag{27}$$

$$c^{i,k+1} = \frac{p^{i,k+1}}{y^{i,k+1}} \tag{28}$$

where $y^{i,0} = 1$, and $p^{i,0} = \pi^i$ for all $i \in \mathbb{Z}^M$.

To calculate the ρ-approximation of the average $\bar{\pi}$, we fix the iteration step lower bounded by some k_0 in the standard push-sum algorithm.

Choose $k_0 = \log_r \frac{\rho\delta}{8M\|\boldsymbol{\pi}\|_\infty} + 1$ with $\delta = \frac{1}{M^{MB}}$, and $\gamma = (1 - \frac{1}{M^{MB}})^{\frac{1}{B}}$. Then, we have that [11]

$$|c^{i,k} - \bar{\pi}| \leq \rho \tag{29}$$

for any $k > k_0$. That is, for any $k \geq k_0$ and $i \in \mathbb{Z}^M$ $c^{i,k}$ is a ρ-approximation of $\bar{\pi}$.

However, k_0 is a global variable because $\boldsymbol{\pi}$ is not available to any subsystem. In the following, we will propose a way to calculate k_0 in a distributed way.

Denote

$$k_0^i := \log_r \frac{\rho\delta}{8M|\pi^i|} + 1 \tag{30}$$

for all $i \in \mathbb{Z}^M$. According to the definition of the ∞-norm of a vector, we have that $k_0 = \max_{i \in \mathbb{Z}^M} k_0^i$. Hence, the stopping criterion $k \geq k_0$ is equal to $k \geq k_0^i, \forall i \in \mathbb{Z}^M$. Then, the key is to make each subsystem i know the information whether $k \geq k_0^j$ or not, for all $j \in \mathbb{Z}^M$. We present a global way to check it: Assign each subsystem a one-bit binary variable $D^{i,t}$ at each iteration t with $D^{i,t} = 0, \forall i \in \mathbb{Z}^M$. $D^{i,t}$ for each $i \in \mathbb{Z}^M$ is updated as the following law if $k < k_0^i$ or there exists at least one $j \in \mathcal{N}_{in}^{i,t}$ satisfying $D^{j,t} = 1$, $D^{i,t+1} = 1$; otherwise, $D^{i,t+1} = 0$. Since the communication network is B-strongly connected, $D^{i,t}$ for all $i \in \mathbb{Z}^M$ will reach consensus at iteration $t_s = BM$. $D^{i,t_s} = 0$ means $k \geq k_0^j, \forall j \in \mathbb{Z}^M$ holds, $D^{i,t_s} = 1$ means $k \geq k_0^j, \forall j \in \mathbb{Z}^M$ does not hold. Then, the gossip-based push-sum algorithm can be summarized in Algorithm 2.

Algorithm 2: Gossip-Based Push-Sum Algorithm.

1: Set $\rho > 0$, $k = 0$, $y^{i,0} = 1$, $p^{i,0} = \pi^i$, $j \in \mathbb{Z}^M$
2: Calculate k_0^i from (30);
3: **repeat**
4: Subsystem i sends $p^{i,k}$ and $y^{i,k}$ to its neighbors
5: Subsystem i calculates $p^{i,k+1}$ from (26)
6: Subsystem i calculates $y^{i,k+1}$ from (12)
7: Subsystem i calculates $c^{i,k+1}$ from (28)
8: **until** $k \geq k_0^i$ for all $i \in \mathbb{Z}^M$
9: **Output** $c^{i,\hat{k}^i}$ for all $i \in \mathbb{Z}^M$
10: $k \leftarrow k + 1$

C. Distributed Stopping Criteria

In this section, we will utilize Algorithm 2 to design fully distributed stopping criteria to terminate the PSDG Algorithm. To simplify the presentation, denote $\frac{1}{M}[\sum_{i=1}^{M} f^i(x^i, \boldsymbol{u}_p^{i,k}) - b(\epsilon)]$ in (21) and $\frac{1}{M}[\sum_{i=1}^{M} J^i(x^i, \boldsymbol{u}_p^{i,k}) - \sum_{i=1}^{M} g^i(\lambda^{i,k}) + b_g \sum_{i=1}^{M} ||\bar{z}^{k-1} - \lambda^{i,k}||]$ in (23) by $\bar{C}^k$ and $\bar{O}^k$, respectively. Then, (21) and (23) can be rewritten as $\sum_{i=1}^{M} C^{i,k} \leq \epsilon M \mathbf{1}_{pM}$ and $M\bar{O}^k \leq \sigma$, respectively. Given any $\epsilon_1 > 0$ and $\sigma_1 > 0$, subsystem i can obtain an ϵ_1-approximation of $\bar{C}^k$, denoted by $\hat{C}^{i,k}$, and a σ_1-approximation of $\bar{O}^k$, denoted by $\hat{O}^{i,k}$ by means of Algorithm 2, respectively.

Choose the following inequalities as new stopping criteria for subsystem i:

$$M\hat{C}^{i,k} \leq \epsilon_2 M \mathbf{1}_{pN} \tag{31}$$

$$M\hat{O}^{i,k} \leq \sigma_2 \tag{32}$$

with $\epsilon_2 > 0$ and $\sigma_2 > 0$ being a constant, satisfying $\epsilon_1 + \epsilon_2 \leq \epsilon$ and $M\sigma_1 + \sigma_2 \leq \sigma$.

Remark 6: Denote the iteration when (31) and (32) are satisfied by $\bar{k}$. Note that (31) and (32) are just the sufficient conditions of (21) and (23), and the resulting $\bar{k}$ might be conservative, which depends on the communication network. For example, if the weighted matrices are doubly stochastic, the standard push-sum algorithm will converge faster [11], which means the accuracy of the approximation of an average is higher within the same number of iterations. If the communication network is fixed, bidirectional and connected, the exact average can be obtained within finite iteration steps by using the finite-time average consensus algorithm [8] and there is no conservatism in this case.

Lemma 5: Assume $\epsilon_1 + \epsilon_2 \leq \epsilon$. Then, if (31) holds, (21) is fulfilled. Assume $M\sigma_1 + \sigma_2 \leq \sigma$. Then, if (32) holds, (23) is satisfied. Choose $\epsilon_1 \leq \epsilon_2$ and $M\sigma_1 \leq \sigma_2$. Then, (31) and (32) are attainable for at least one iteration.

The proof can be easily proved according to the Definition of 1 and Theorem 1 and is omitted here. This lemma indicates that (31) and (32) are qualified to be the stooping criteria to replace (21) and (23), respectively.

D. DMPC Algorithm

In this section, we present the overall DMPC controller, which is given in Algorithm 3.

Remark 7: Note that Step 8 in Algorithm 3 requires that each subsystem knows the global information about whether (31) and (32) are satisfied for all $i \in \mathbb{Z}^M$ at some iteration. This process can be implemented in a distributed manner by using Algorithm 2, and the detailed procedure is omitted since it is the same as the case for checking whether $k \geq k_0^j$ for all $j \in \mathbb{Z}^M$ in Section IV-B.

Algorithm 3: DMPC Algorithm.

1: Initialize the parameters $\sigma, \sigma_1, \sigma_2, \epsilon, \epsilon_1$, and ϵ_2
2: All the M subsystems obtain its state $x(t)$
3: Set $k = 0$
4: **repeat**
5: All the M subsystems execute (11)–(15)
6: $k \leftarrow k + 1$
7: All the M subsystems check the stopping criteria (31) and (32) using Algorithm 2
8: **until** The stopping criteria (31) and (32) are satisfied for all $i \in \mathbb{Z}^M$ at $\bar{k}$
9: Output the input $\boldsymbol{u}_p^{i,\bar{k}}, i \in \mathbb{Z}^M$
10: Wait for the next sample time, let $t \leftarrow t + 1$, and return to Step 1

V. Feasibility and Stability Analysis

Before we prove the feasibility of the designed optimization problem and stability of the closed-loop system, a necessary lemma is given first.

Lemma 6: Suppose $\epsilon > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\sigma > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\epsilon_1 + \epsilon_2 \leq \epsilon$, $M\sigma_1 + \sigma_2 \leq \sigma$, $\epsilon_1 \leq \epsilon_2$, $M\sigma_1 \leq \sigma_2$. Then, for any $x^i \in \mathcal{X}_t^i$, the solution of Algorithm 1 with the stopping criteria (31) and (32) is $\boldsymbol{u}_p^{i,\bar{k}} := \{u_p^{i,\bar{k}}(0), \ldots, u_p^{i,\bar{k}}(N-1)\} := \{K^i x^i, K^i A_K^i x^i, \ldots, K^i (A_K^i)^{N-1} x^i\} := K_A^i x^i$ with the stopping iteration $\bar{k} = 0$.

Proof: We know that $\{K_A^i x^i\}_{i \in \mathbb{Z}^M}$ is the optimal solution of the unconstrained counterpart of Problem 1. Also because for any $\{x^i \in \mathcal{X}^i\}_{i\in\mathbb{Z}^M}$, $\{K_A^i x^i\}_{i\in\mathbb{Z}^M}$ satisfies the local constraints and the global constraints of Problem 1 according to the definition of $\mathcal{X}_t^i$, $\{K_A^i x^i\}_{i\in\mathbb{Z}^M}$ is the optimal solution of Problem 1. That is to say, for any $\{x^i \in \mathcal{X}^i\}_{i\in\mathbb{Z}^M}$, Problem 1 is equivalent to its unconstrained counterpart. Since $\lambda^{i,0} = z^{i,0} = 0, \forall i \in \mathbb{Z}^M$, we have: $\sum_{i=1}^M f^i(x^i, \boldsymbol{u}_p^{i,0}) - b(\epsilon) \leq 0$ and $\sum_{i=1}^M J^i(x^i, \boldsymbol{u}_p^{i,0}) - \sum_{i=1}^M g^i(\lambda^{i,0}) + b_g \sum_{i=1}^M ||\bar{z}^{-1} - \lambda^{i,0}|| = 0$ for all $i \in \mathbb{Z}^M$. Then, the stopping criteria (31) and (32) are satisfied at $\bar{k} = 0$. ■

Theorem 2: Suppose Assumptions 1–5 hold. Choose (31) and (32) as the stopping criteria for Algorithm 1 with $\epsilon > 0, \epsilon_1 > 0, \epsilon_2 > 0, \sigma > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\epsilon_1 + \epsilon_2 \leq \epsilon$, $M\sigma_1 + \sigma_2 \leq \sigma$, $\epsilon_1 \leq \epsilon_2$, $M\sigma_1 \leq \sigma_2$. Then, for any initial state in the feasible set, 1) Problem 2 has a feasible solution at any time t; 2) furthermore, choose the parameter σ such that $\{x^i : ||x^i||_{Q^i}^2 \leq \sigma\} \subset \text{int}(\mathcal{X}_t^i)$ for all $i \in \mathbb{Z}^M$, then the closed-loop system under Algorithm 3 is exponentially stable.

Proof: Proof of 1): Denote the solution of Algorithm 1 at time t by: $\boldsymbol{u}_p^i(t) = \{u_p^i(0|t), u_p^i(1|t), \ldots, u_p^i(N-1|t)\}$, and the corresponding state sequence is represented by: $\boldsymbol{x}_p^i(t) = \{x_p^i(0|t), x_p^i(1|t), \ldots, x_p^i(N|t)\}$, for $i \in \mathbb{Z}^M$. Since $\boldsymbol{u}_p^i(t)$, $i \in \mathbb{Z}^M$ satisfies the designed stopping conditions, the following inequality holds: $\forall l \in \mathbb{Z}_0^{N-1}$

$$\sum_{i=1}^M \Psi_x^i x_p^i(l|t) + \Psi_u^i u_p^i(l|t) \leq (1 - \epsilon M l)\, \mathbf{1}_p. \tag{33}$$

Consider the following input sequence at time $t+1$

$$\begin{aligned}\hat{\boldsymbol{u}}_p^i(t+1) &:= \{\hat{u}_p^i(0|t+1), \ldots, \hat{u}_p^i(N-1|t+1)\} \\ &= \{u_p^i(1|t), \ldots, u_p^i(N-1|t), K^i x_p^i(N|t)\}\end{aligned} \tag{34}$$

and the corresponding state sequence $\hat{\boldsymbol{x}}_p^i(t+1) := \{\hat{x}_p^i(0|t+1), \ldots, \hat{x}_p^i(N|t+1)\} = \{x_p^i(1|t), \ldots, x_p^i(N|t), (A^i + B^i K^i) x_p^i(N|t)\}$.

According to (33), we have $\sum_{i=1}^M (\Psi_x^i\ \hat{x}_p^i\ (l|t+1) + \Psi_u^i\ \hat{u}_p^i\ (l|t+1)) = \sum_{i=1}^M (\Psi_x^i\ x_p^i(l+1|t) + \Psi_u^i\ u_p^i\ (l+1|t)) \leq (1 - \epsilon M(l+1))\mathbf{1}_p, \forall l \in \mathbb{Z}_0^{N-2}$. Furthermore, since $x_{N|t}^i \in \mathcal{X}_t^i$, it follows that $\sum_{i=1}^M (\Psi_x^i\ \hat{x}_p^i\ (N-1|t+1) + \Psi_u^i\ \hat{u}_p^i\ (N-1|t+1)) = \sum_{i=1}^M (\Psi_x^i\ x_p^i(N|t) + \Psi_u^i K^i\ x_p^i(N|t)) \leq (1 - \epsilon MN)\ \mathbf{1}_p$, and $\hat{x}_p^i(N|t+1) \in \mathcal{X}_t^i$. Hence, $\hat{\boldsymbol{u}}_p^i(t+1)$ is a feasible solution at time $t+1$.

Proof of 2): Define the Lyapunov function $V(x(t)) := \sum_{i=1}^M J^i(x^i(t), \boldsymbol{u}_p^{i*}(t))$, where $\boldsymbol{u}_p^{i*}(t)$, $i \in \mathbb{Z}^M$ is the optimal solution of Problem 1 at time t.

Since $\boldsymbol{u}_p^i(t)$ satisfies the designed stopping condition (22), we have

$$\sum_{i=1}^M J^i(x^i(t), \boldsymbol{u}_p^i(t)) - V(x(t)) \leq \sigma. \tag{35}$$

According to (34), $\hat{\boldsymbol{u}}_p^i(t+1)$ is a feasible solution of Problem 1. Hence, we have $J^i(x^i(t+1), \hat{\boldsymbol{u}}_p^i(t+1)) - J^i(x^i(t), \boldsymbol{u}_p^i(t)) = -\|x^i(t)\|_{Q^i}^2 - \|u_p^i(t)\|_{R^i}^2$, where the equality is obtained from the fact that (K^i, P^i) is the solution of the Algebraic Riccati equation $(A_K^i)^T P^i A_K^i - P^i = -(Q^i + (K^i)^T R^i K^i)$. Considering $\hat{\boldsymbol{u}}_p^i(t+1)$ may be not optimal, we have $V(x(t+1)) \leq \sum_{i=1}^M J^i(x^i(t+1), \hat{\boldsymbol{u}}_p^i(t+1)) = \sum_{i=1}^M (J^i(x^i(t), \hat{\boldsymbol{u}}_p^i(t)) - \|x^i(t)\|_{Q^i}^2)$. Utilizing (35), we have $V(x(t+1)) \leq V(x(t)) + \sigma - \sum_{i=1}^M \|x^i(t)\|_{Q^i}^2 := V(x(t)) - \theta(t)$. Then, the state $x^i(t), i \in \mathbb{Z}^M$ enter the set $\{x^i : \sum_{i=1}^M \|x^i(t)\|_{Q^i}^2 \leq \sigma\}$ in a finite time. Choose σ such that $\{x^i : \|x^i(t)\|_{Q^i}^2 \leq \sigma\} \subseteq \mathcal{X}_t^i$. After that, the closed-loop system will become $x^i(t+1) = A_K^i x^i(t)$ according to Lemma 6. Hence, the closed-loop system is asymptotically stable. ■

VI. Numerical Example

Consider an example of four linear time-invariant subsystems. The edge sets of the communication graph is $\{(1,2),(3,4)\}$ when the iteration counter is odd; the edge sets of the communication graph is $\{(2,3),(4,1)\}$ when the iteration counter is even. Then, the communication graphs are B-strongly connected with $B = 2$. For subsystems 1 and 3, the state and input matrices are $A^i = [1, 1; 0, 1]$ and $B^i = [1; 1]$, respectively. For subsystems 2 and 4, the state and input matrices are $A^i = [2, 1; 0, 1]$ and $B^i = [1; 1]$, respectively. The initial states of subsystem $1-4$ are $[0.2; 0.18]$, $[0.17; 0.16]$, $[0.19; 0.18]$, and $[0.22; 0.21]$. For all the four subsystems, the local state and constraint sets are $\mathcal{X}^i = \{x^i : \|x^i\|_\infty \leq 1\}$ and $\mathcal{U}^i = \{u^i : \|u^i\|_\infty \leq 0.3\}$, respectively. The global constraint is: $|u^1 + u^2 + u^3 + u^4| \leq 1$. The state and input weight matrices are chosen as $Q^i = I$ and $R^i = 0.1$, respectively. The horizon length is chosen as $N = 5$. The sequence of the step sizes of Algorithm 1 is chosen as $\frac{50}{k}$ with k being the iteration counter of Algorithm 1.

We first illustrate the effectiveness of Algorithm 1. Fig. 1 shows the curves about the input sequence $\boldsymbol{u}_p^{1,k}$ of subsystem 1 over iteration k. Fig. 2 shows the curves about the dual variable $\lambda^{1,k}$ of subsystem 1 over iteration k. From these two figures, we can see that the primal and dual variables converge to their optimal values. From Fig. 2, we can also observe that some components of $\lambda^{1,k}$ converge to 0, which implies that the constraint $\lambda^1 \geq 0$ is active. Fig. 3 shows the summation of the calculated control input sequence of all agents with length $N = 5$, i.e., $S^k := \sum_{i=1}^4 u_p^{i,k}$. It can been seen that $S^k[1]$ converges to -1, which indicates that the global constraint $|u^1 + u^2 + u^3 + u^4| \leq 1$ is active. Fig. 4 shows the curve about the consensus error, defined as $e^k := \sum_{i=1}^4 ||\lambda^{i,k} - \bar{\lambda}^k||_2$ with $\bar{\lambda}^k := \sum_{i=1}^4 \lambda^{i,k}$, over iteration k. This figure indicates that the local dual variables $\lambda^{i,k}$ for all $i \in \mathbb{Z}^4$

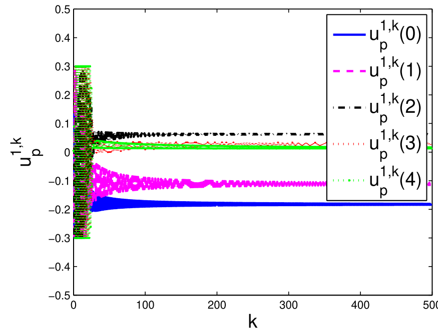

Fig. 1. Curves about the primal variable $\boldsymbol{u}_p^{1,k}$ under Algorithm 1.

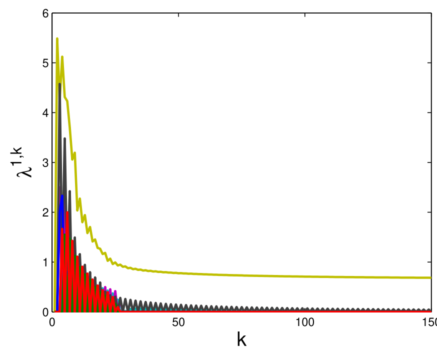

Fig. 2. Curves about the dual variable $\boldsymbol{\lambda}^{1,k}$ under Algorithm 1.

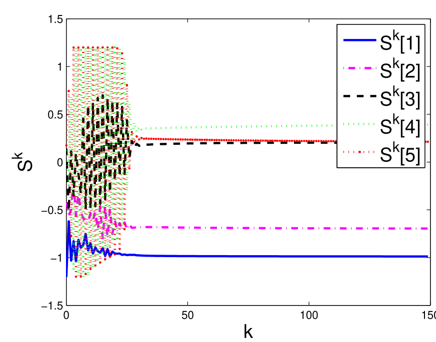

Fig. 3. Curves about the global constraint S^k under Algorithm 1.

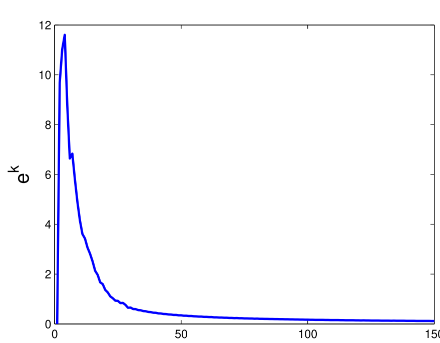

Fig. 4. Curves about the consensus error e^k under Algorithm 1.

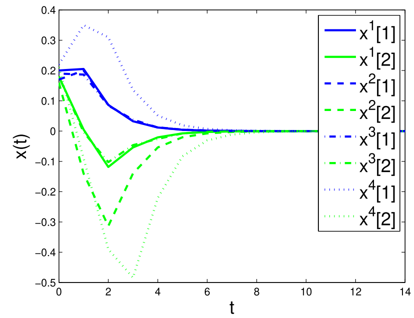

Fig. 5. State trajectory of the four subsystems over time t.

achieve consensus over iteration k. Fig. 5 illustrates the effectiveness of the overall DMPC controller. From the figure, we can observe that the four closed-loop systems are exponentially stable.

VII. Conclusion

In this article, a DMPC approach has been developed for a group of decoupled constrained linear discrete-time subsystems with both local and global constraints when the communication networks are directed time-varying graphs. The primal optimization problem is transformed into a consensus optimization problem, and the PSDG Algorithm is proposed to solve the consensus optimization problem. Moreover, distributed stopping criteria are designed to provide early termination for the PSDG algorithm. We have proved that the DMPC optimization is iteratively feasible, and the closed-loop system is exponentially stable.

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