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Normal forms and internal regularization of nonlinear differential-algebraic control systems

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Abstract

In this article, we propose two normal forms for nonlinear differential-algebraic control systems (DACSs) under external feedback equivalence, using a notion called maximal controlled invariant submanifold. The two normal forms simplify the system structures and facilitate understanding the various roles of variables for nonlinear DACSs. Moreover, we study when a given nonlinear DACS is internally regularizable, that is, when there exists a state feedback transforming the DACS into a differential-algebraic equation (DAE) with internal regularity, the latter notion is closely related to the existence and uniqueness of solutions of DAEs. We also revise a commonly used method in DAE solution theory, called the geometric reduction method. We apply this method to DACSs and formulate it as an algorithm, which is used to construct maximal controlled invariant submanifolds and to find internal regularization feedbacks. Two examples of mechanical systems are used to illustrate the proposed normal forms and to show how to internally regularize DACSs.

KEYWORDS

differential-algebraic equations, external feedback equivalence, internal regularization, mechanical systems, nonlinear control systems, normal forms

1 | INTRODUCTION

Consider a nonlinear differential-algebraic control system DACS of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x \in X$ is the generalized state, with X an n-dimensional differentiable manifold (or an open subset of $\mathbb{R}^n$) and $u \in \mathbb{R}^m$ is the control vector. For the differentiable manifold X, we denote by TX the tangent bundle of X and by T_xX the tangent space of X at $x \in X$. The maps $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l\times m}$ are smooth and the word "smooth" will always mean C^∞-smooth throughout the article. For each $x \in X$, we have $E(x) : T_xX \to \mathbb{R}^l$, which is the linear map $\dot{x} \mapsto E(x)\dot{x}$. In particular, if X is an open subset of $\mathbb{R}^n$, then for each $x \in X$, we have $E(x) : \mathbb{R}^n \to \mathbb{R}^l$, that is, $E(x) \in \mathbb{R}^{l\times n}$. A DACS of the form (1) will be denoted by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, Ξ^u. A particular case of (1) is a linear DACS of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{2}$$

where $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{l\times m}$, denoted by $\Delta^u_{l,n,m} = (E, H, L)$. If $G(x) = 0$, that is, the control u is absent ($L = 0$ in the case of (2)), then we will speak about differential-algebraic equations (DAEs). The DAEs/DACSs are also called implicit, singular, generalized or descriptor systems. There are many practical applications of DAEs/DACSs, and as surveys and books using DAEs/DACSs to model physical systems, the reader may consult, for example, References 1-3 and chapter 1 of Reference 4. In particular, DAEs/DACSs are suitable tools to describe constrained mechanics,[1,5] electrical circuits,[2,6] and chemical processes.[7] The necessity of using DAEs/DACSs to model physical systems instead of ordinary differential equations (ODEs) is justified by the presence of constraints (e.g., nonholonomic and holonomic constraints for mechanical systems, see (3a)-(3c), and the algebraic constraints resulting from Kirchoff's laws and characterizations of nonlinear components for electric circuits). These constraints result in an implicit differential equation for which it is impossible to explicitly express the derivative $\dot{x}$ as a function of the state variables x, that is, as an ODE $\dot{x} = f(x)$.

For a DACS of the form (1), the map E is not necessarily square (i.e., in general, $l \neq n$) nor invertible (even if $l = n$) and, as a consequence, some variables of the generalized state x play different roles for the system. More specifically, the noninvertibility of E may imply the existence of algebraic constraints and some variables of x (also some u-variables) are constrained by those algebraic constraints. On the other hand, because of the nonsquareness of E, some other variables of x may enter into system statically (since there may not exist differential equations defining their evolutions) and we may call them free variables. One of results of this article (see Theorem 1) will reveal (under a suitable coordinate transformation) four different types of the generalized state-variables: the unconstrained state variables z_1, the unconstrained free variable z_2, the constrained state variables z_3 and the constrained free variables (or the algebraic variables) z_4. Note that although the free variables of x may perform "like" inputs of the system, throughout we will distinguish them from the original control inputs u. The variables u are predefined control inputs, such as external forces, and we can change them in order to act on the system. However, the free variables of x are predefined states which cannot be changed actively and arbitrarily. Such free variables may come from unknown constraint forces or some redundancies of mathematical modeling. For the behavioral approach to systems theory (see Reference 8), there is also a distinction between the latent/internal variable (i.e., x) and the manifest/external variables (i.e., the inputs and outputs (u, y)).

A typical example to illustrate that the control variables u and the free variables of x are different is the following DACS which represents the dynamics of a mechanical system under both nonholonomic and holonomic constraints (see, e.g., Reference 1 for the definitions of nonholonomic and holonomic constraints):

$$M(q)\ddot{q} + V(\dot{q}, q) = \tau + H^T(q)\lambda_n + N^T(q)\lambda_h, \tag{3a}$$

$$H(q)\dot{q} = 0, \tag{3b}$$

$$C(q) = 0, \tag{3c}$$

where q is the vector of position (configuration) variables, $M(q)$ is a matrix-valued function which is associated with masses (or inertia), and $V(\dot{q}, q)$ is a row vector function which characterizes the Coriolis, the centrifugal and the gravity forces, τ is a vector of external torques, $C(q)$ is a vector of scalar functions $c_i(q)$, $i = 1, \ldots, k$ and $N(q) = \frac{\partial C(q)}{\partial q}$, and $H(q)$ is matrix-valued function of appropriate size. Clearly, Equation (3b) defines nonholonomic constraints, which depend on both velocities and positions, Equation (3c) defines holonomic constraints, which depend on positions only. The variables λ_n and λ_h are the Lagrange multipliers with respect to the nonholonomic and holonomic constraints, respectively. We can regard system (3) as a DACS of the form (1), with the generalized state $x = (q, \dot{q}, \lambda_n, \lambda_h)$ and the control input $u = \tau$. Observe that the variables λ_h and λ_n are free variables since there are no equations for $\dot{\lambda}_n$ and $\dot{\lambda}_h$ but they are not active control inputs contrary to the external force τ. The latter can be realized by some actuators (e.g., electric and hydraulic motors) while λ_h and λ_n are variables related to unknown constrained forces and are sometimes called the constrained input variables.[9]

One purpose of this article is to find normal forms under the external feedback equivalence (see Definition 3). We will construct our normal form using a notion called maximal controlled invariant submanifold, which is, roughly speaking, the locus where the solutions of the DACSs exist and is defined by the constraints which the system should respect (for the precise definition, see Definition 2). For linear DACSs of the form (2), a canonical form, which consists of six independent subsystems, was proposed in Reference 10. One can easily conclude the roles of the variables (e.g., which variables are free and which are constrained) from the canonical structure of each subsystem. For nonlinear DACSs, although it is hard to find a fully decoupled normal form, we intend to simplify the system structures utmost such that the above mentioned various roles of variables can be explicitly and easily seen from our proposed form. The authors of Reference 11 offered a nonlinear generalization of the Kronecker canonical form using an algebraic inversion algorithm for differential-algebraic equations DAEs of the general form $F(\dot{x}, x, t) = 0$, while we intend to find normal forms for nonlinear DACSs using geometric methods. A zero dynamics form for DACSs with outputs was proposed in Reference 12 using the notion of maximal output zeroing submanifold introduced in Reference 13. Note that our system Ξ^u is different in two ways from the DACSs studied in References 12,13. First, in References 12,13, the distribution $\ker E(x)$ is assumed to be involutive while we consider any $E(x)$. Second, systems in References 12,13 are equipped with outputs. Calculating the zero dynamics of a DACS Ξ^u with zero output $y = h(x) = 0$ can be seen as studying an extended DACS $\Xi^u_{ext} : E(x)\dot{x} = F(x) + G(x)u,\ 0 = h(x)$, because the maximal output zeroing submanifold of Ξ^u (with the output $y = h(x)$) coincides with the maximal controlled invariant submanifold of Ξ^u_{ext}. Some differences of our proposed normal forms and the zero dynamics form in Reference 12 are explained in Remark 3(vii).

We also investigate the internal regularizability of DACSs, that is, given a DACS Ξ^u, when there exists a feedback $u = \alpha(x)$ such that the resulting DAE $E(x)\dot{x} = F(x) + G(x)\alpha(x)$ is *internally regular*. The latter notion characterizes the existence and uniqueness of solutions of DAEs, its formal definition will be given in Definition 4. Regularization problems of nonlinear DAEs and DACSs can be consulted in References 14-18, where both numerical and geometrical methods have appeared. The second aim of our article is to give a geometric characterization of the internal regularizability of nonlinear DACSs. For linear DACSs, some equivalent characterization of the internal regularizability are given in theorem 3.5 of Reference 19 using a geometric notion named the augmented Wong sequences (see Remark 1(iv)). Note that the internal regularizability is called autonomizability in Reference 19; the reason, for which we insist to use the word "internal", is to stress the difference between two cases. One case is to consider a DAE "internally" on its maximal invariant submanifold (i.e., on the set where the solutions exist). Another is to consider a DAE "externally" on a whole neighborhood, even although there exist no solutions for any initial point outside the maximal invariant submanifold, it is still meaningful to study how to steer the initial point toward the constraints via, for example, jumps and impulses. The reader may consultReferences 4,20-22 for the details of the differences between the internal and external analysis of DAEs.

The article is organized as follows. In Section 2, we recall the notion of maximal controlled invariant submanifold and discuss its relations with the solutions of DACSs. In Section 3, we define the external feedback equivalence of two DACSs and propose two normal forms. In Section 4, we discuss the internal regularization problem. In Section 5, we illustrate our results of Sections 3 and 4 by two examples of mechanical systems. In Section 6, we give the conclusions of the article. The Appendix contains an algorithm using which we can construct the maximal controlled invariant submanifold and the feedback which we need to internally regularize a DACS. We use the following notations. We use $\mathbb{R}^{n\times m}$ to denote the set of real valued matrices with n rows and m columns, $GL(n, \mathbb{R})$ to denote the group of nonsingular matrices of $\mathbb{R}^{n\times n}$ and I_n to denote the $n \times n$-identity matrix. We denote by C^k the class of k-times differentiable functions. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \dots, f_m]^T$, we denote its differential by $\mathrm{D}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume the reader is familiar with some basic notions from differential geometry as smooth manifolds, embedded submanifolds, tangent bundles, distributions; the reader may also consult, for example, the book[23] for definitions of those notions.

2 | PRELIMINARIES ON SOLUTIONS OF DIFFERENTIAL-ALGEBRAIC CONTROL SYSTEMS

We define a solution of a DACS as follows.

Definition 1 (solution). For a DACS $\Xi^u_{l,n,m} = (E, F, G)$, a curve $(x, u) : I \to X \times \mathbb{R}^m$ defined on an open interval $I \subseteq \mathbb{R}$ with $x(\cdot) \in C^1(I)$ and $u(\cdot) \in C^0(I)$ is called a solution of Ξ^u, if for all $t \in I$, $E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t)$.

We call a point $x_0 \in X$ an *admissible point* of Ξ^u if there exists at least one solution $(x(\cdot), u(\cdot))$ satisfying $x(t_0) = x_0$ for a certain $t_0 \in I$. We will denote admissible points by x_a and the set of all admissible points by S_a. Note that for any DACS Ξ^u, there may exist some free variables among the components of x. As a consequence, even for a fixed $u(\cdot)$ defined on $\mathbb{R}$, there is not a unique prolongation of a solution (x, u) defined on I to a maximal solution. For this reason, we will not use the concept of maximal solutions (although they can be defined, see, e.g., Reference 12) except for Section 4, where we can deal with maximal solutions due to an identification of free (algebraic) variables.

Definition 2 (controlled invariant submanifold). Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$. A smooth connected embedded submanifold M is called a controlled invariant submanifold of Ξ^u if for any point $x_0 \in M$, there exists a solution $(x, u) : I \to X \times \mathbb{R}^m$ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M$ for all $t \in I$.

We fix a point $x_p \in X$, a smooth embedded submanifold M containing x_p is *locally* controlled invariant (around x_p) if $\exists$ a neighborhood U of x_p in X such that $M \cap U$ is controlled invariant (and thus, by definition, connected). Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, let $N \subseteq X$ and fix a point $x_p \in N$; we introduce the following constant rank assumption:

(CR) there exists a neighborhood U in X of x_p such that $N \cap U$ is a smooth connected embedded submanifold, and such that $\dim E(x)T_xN = const.$ and $\dim(E(x)T_xN + \operatorname{Im} G(x)) = const.$ for $x \in N \cap U$.

The following characterization of local controlled invariance, under the constant rank assumption **(CR)** satisfied for M, was given as theorem 9 in Reference 13 for DACSs whose $\ker E(x)$ is an involutive distribution. The DACSs in Reference 13 is of the form $\frac{\mathrm{d}e(x(t))}{\mathrm{d}t} = f(x(t)) + g(x(t))u(t)$. Note that $e(x)$, denoted by $E(x)$ in Reference 13, is an $\mathbb{R}^l$-valued function, while $E(x)$ of our article is a matrix-valued function, whose rows, in the case of $\frac{\mathrm{d}e(x(t))}{\mathrm{d}t}$ of Reference 13, are $E^i(x) = \mathrm{d}e^i(x) = \sum_{i=1}^n \frac{\partial e^i(x)}{\partial x_j}\mathrm{d}x_j = [\frac{\partial e^i(x)}{\partial x_1}, \ldots, \frac{\partial e^i(x)}{\partial x_n}]$, for $1 \le i \le l$, and thus are exact 1-forms. Hence the distributions defined by $\ker E(x)$ in Reference 13 are involutive (by Frobenius theorem, see, e.g., Reference 23).

Proposition 1. *Consider a DACS $\Xi^u = (E, F, G)$ and let M be a smooth embedded submanifold. Assume that M satisfies the above assumption* **(CR)** *around a point $x_p \in M$. Then M is a locally controlled invariant submanifold (around x_p) of Ξ^u if and only if there exists a neighborhood U of x_p in X such that*

$$F(x) \in E(x)T_xM + \operatorname{Im} G(x), \quad \forall x \in M \cap U. \tag{4}$$

A locally controlled invariant submanifold M^*, around a point $x_p \in M^*$, is called locally *maximal* if there exists a neighborhood U of x_p such that for any other locally controlled invariant submanifold M containing x_p, we have $M \cap U \subseteq M^* \cap U$. The following procedure is a geometric method to construct the locally maximal controlled invariant submanifold.

Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$ and let U_0 be an open connected subset of X containing x_p. Set $M_0 = X$, $M_0^c = U_0$. Suppose that there exist an open neighborhood U_{k-1} of x_p and a sequence of smooth connected embedded submanifolds $M^c_{k-1} \subsetneq \ldots \subsetneq M^c_0$ of U_{k-1} for a certain $k \ge 1$, has been constructed. Define recursively

$$M_k := \left\{x \in M^c_{k-1} \mid F(x) \in E(x)T_xM^c_{k-1} + \operatorname{Im} G(x)\right\}. \tag{5}$$

Then either $x_p \notin M_k$ or $x_p \in M_k$, and in the latter case, assume that there exists a neighborhood U_k of x_p such that $M^c_k = M_k \cap U_k$ is a smooth embedded submanifold (which can always be assumed connected by taking U_k sufficiently small).

Proposition 2. *In the above recursive procedure, there always exists $k^* \le n$ such that either k^* is the smallest integer for which $x_p \notin M_{k^*+1}$ (and then there is a neighborhood of x_p in which there does not exist any controlled invariant submanifold) or k^* is the smallest integer such that $x_p \in M^c_{k^*+1}$ and $M^c_{k^*+1} \cap U_{k^*+1} = M^c_{k^*} \cap U_{k^*+1}$. In the latter case, we assume that $M^* = M^c_{k^*+1}$ satisfies the constant rank condition* **(CR)** *in a neighborhood $U^* \subseteq U_{k^*+1}$ of x_p in X and then*

(i) x_p is an admissible point and M^* is a locally maximal controlled invariant submanifold on U^* (by taking a smaller U^*, if necessary);
(ii) M^* coincides locally with the admissible set S_a, that is, $M^* \cap U^* = S_a \cap U^*$.

Proof. Let k be the largest integer such that $M^c_0 \supsetneq M^c_1 \supsetneq \ldots \supsetneq M^c_k$ and $x_p \in M^c_k$, where M^c_i, $0 \le i \le k$ are connected embedded submanifolds, and then either $x_p \notin M_{k+1}$ or $x_p \in M_{k+1}$ and $M^c_{k+1} = M_{k+1} \cap U_{k+1}$ is a submanifold (by the recursive procedure assumptions) such that $\dim M^c_k = \dim M^c_{k+1}$. Then $k^* = k$ is the integer whose existence is indicated. The condition $k^* \le n$ follows from $\dim M^c_{i-1} > \dim M^c_i$, $1 \le i \le k^*$.

Claim. If an admissible point $x_a \in S_a \cap U_{k^*}$, then $x_a \in M_{k^*+1}$.

To prove the *Claim*, notice that if x_a is admissible, there exists a solution $(x(t), u(t))$ and $t_0 \in I$ such that $x(t_0) = x_a$. It follows that for all $t \in I$,

$$E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t). \tag{6}$$

So $F(x(t)) \in \mathrm{Im}\, E(x(t)) + \mathrm{Im}\, G(x(t))$, $\forall t \in I$. Thus by Equation (5), we have $x(t) \in M_1$, $\forall t \in I$. Suppose that for a certain $i > 1$, we have $x(t) \in M_{i-1}$, $\forall t \in I$. We then have that $\dot{x}(t) \in T_{x(t)} M_{i-1}$, $\forall t \in I$ (note that when restricted to U_{i-1}, the set M_{i-1} is a submanifold). Thus in $U_{k^*} \subseteq U_i$, Equation (6) implies $F(x(t)) \in E(x(t)) T_{x(t)} M^c_{i-1} + \mathrm{Im}\, G(x(t))$. It follows that $x(t) \in M_i \cap U_{i-1}$, for any $t \in I$, due to (5). By an induction argument, we conclude that $x(t) \in M_{k^*+1} \cap U_{k^*}$, and, in particular, we have $x_a = x(t_0) \in M_{k^*+1} \cap U_{k^*}$, which proves the *Claim*.

(i) If $x_p \in M_{k^*+1}$, we have $\dim M^c_{k^*+1} = \dim M^c_{k^*}$ and since $M^c_{k^*+1} \subseteq M^c_{k^*}$, it follows that there exists an open neighborhood U_{k^*+1} such that $M^c_{k^*+1} \cap U_{k^*+1} = M^c_{k^*} \cap U_{k^*+1}$. By assumption, $M^* = M^c_{k^*+1} \cap U^*$ satisfies **(CR)** in $U^* \subseteq U_{k^*+1}$. So, using Proposition 1, we conclude that M^* is a locally invariant submanifold on U^*. To prove that M^* is maximal in U^*, let M' be any controlled invariant submanifold, then any point $x_0 \in M' \cap U^*$ is admissible, so $x_0 \in S_c \cap U^*$ and thus by the above *Claim*, $x_0 \in M_{k^*+1} \cap U^* = M^* \cap U^*$ showing that M^* is maximal in U^*.

(ii) We now prove that M^* coincides with the admissible set S_a on U^*. Since $M^* \cap U^*$ is locally controlled invariant, for any point $x_0 \in M^* \cap U^*$, there exist at least one solution $(x(\cdot), u(\cdot))$ on I and $t_0 \in I$ such that $x(t_0) = x_0$ (by Definition 2), which implies that x_0 is admissible, that is, $x_0 \in S_a$. It follows that $M^* \cap U^* \subseteq S_a \cap U^*$. Conversely, consider any point $x_0 \in S_a \cap U^*$, using again the above *Claim*, we conclude that $x_0 \in M_{k^*+1} \cap U^* = M^* \cap U^*$, which implies $S_a \cap U^* \subseteq M^* \cap U^*$. Therefore, $M^* \cap U^* = S_a \cap U^*$. ∎

Note that the proof of Proposition 2(i) can be performed in a similar way as that of theorem 12(ii) of Reference 13 for proving that M^* is a maximal output zeroing submanifold with output function taken as zero. However, in order to show item (ii), we need the *Claim* that implies implicitly the maximality property in item (i), and therefore we provide a proof of both (i) and (ii) of Proposition 2.

Remark 1.

(i) Proposition 2 is a geometric method to construct the locally maximal controlled invariant submanifold M^*. Such an iterative way of identifying the admissible set of a DAE is called the geometric reduction method and has appeared frequently in the geometric analysis of nonlinear DAEs (see, e.g., References 6,24-26 and the recent papers[13,22]). We state a practical implementation of this geometric method as Algorithm 1 of the Appendix, where we also compare our Algorithm 1 with an existing geometric reduction method of section 3.4 of Reference 6. A preliminary version of Algorithm 1 for DAEs (without control u) can be consulted in Reference 27.

(ii) Item (ii) of Proposition 2 asserts that in the neighborhood U^* of an admissible point $x_a = x_p$, the solutions of Ξ^u exist on M^* only, which implies that for any point $x_0 \in U^* \backslash M^*$ there are no solutions passing through x_0. For practical systems, the initialization x_0 of the state x could be any point of the state space X. If $x_0 \in U^* \backslash M^*$ (i.e., x_0 is not admissible), we need an instantaneous change of x_0, that is, a jump at $t = t_0$, to steer the inadmissible point x_0 into an admissible one. The jump of x_0 at $t = t_0$ will cause a distributional term, the Dirac impulse δ, to be present in $\dot{x}$. For linear DAEs/DACSs, the distributional solution theory of linear DAEs/DACSs has been established to deal with the discontinuity caused by inadmissible initial points, see e.g., References 28,29. We will not discuss distributional solutions of nonlinear DAEs/DACSs since the purpose of the present article is just to propose normal forms to simplify system structures. Note, however, that the normal forms studied in Section 3 are external forms that hold on a whole neighborhood (not just on M^*) of a nominal point x_p. This is a useful tool for studying jumps and distributional solutions of DAEs/DACSs and is an ongoing research, c.f. our recently submitted conference contribution.[30]

(iii) If for a fixed x_p, we drop the requirements that $x_p \in M_k$ and that M^c_k are connected, then Proposition 2 allows to detect all admissible points x_a in U^* that form the union $\bigcup M^*_i$ of all locally maximal controlled invariant submanifolds in U^*. Notice that, first, that union $\bigcup M^*_i$ may have more than one connected components (each of them being a locally maximal controlled invariant submanifold), second, x_p may not be in $\bigcup M^*_i$ (implying that x_p is not admissible) and, third, $\bigcup M^*_i$ can be empty (implying that there are no admissible points in U^*).

(iv) The recursive procedure of Proposition 2 leads to the sequence of nested submanifolds

$$M^c_{k^*+1} = M^c_{k^*} \subsetneq M^c_{k^*-1} \subsetneq \dots \subsetneq M^c_0 = U_0.$$

At each step, we construct a submanifold M^c_{k+1} that is of a smaller dimension than M^c_k, except for the last step, where M_{k^*+1}, defined by Equation (5), coincides with $M^c_{k^*}$, although not on U_{k^*} but on a smaller neighborhood U_{k^*+1} and $M^c_{k^*+1}$ is actually $M^c_{k^*}$ restricted to U_{k^*+1}. The need to take a smaller neighborhood $U_{k^*+1} \subseteq U_{k^*}$ is a purely nonlinear phenomenon. Take, for example, the following nonlinear DACS: $x\dot{x} = f(x)$ defined on $X = \mathbb{R}$ for some $f : X \to \mathbb{R}$ such that $f(0) \neq 0$. Fix a point $x_p > 0$, we have $M_0^c = U_0 = \mathbb{R}$ and $M_1 = \{x \in M_0^c \mid f(x) \in \operatorname{Im} x\}$, it follows that $U_1 = \{x \in \mathbb{R} \mid x > 0\}$ and $M_1^c = M_1 \cap U_1 = U_1$, so it is clear that $k^* = 0$ since $\dim M_0^c = \dim M_1^c$. It is seen that $M_{k^*+1}^c = M_1^c$ coincides with $M_{k^*}^c = M_0^c$ on U_1 but not on U_0. Note that in the linear case, $\mathscr{M}^* = M^* = M_{k^*}^c$ with $\mathscr{M}^*$ defined in item (v).

(v) If we apply the above procedure of constructing M_k to a linear DACS $\Delta^u = (E, H, L)$, then we get a sequence of subspaces

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_k = H^{-1}(E\mathscr{V}_{k-1} + \operatorname{Im} L). \tag{7}$$

The sequence $\mathscr{V}_k$ is one of the *augmented Wong sequences* (see Reference 31), that play an important role in the geometric analysis of linear DACSs (see, e.g., Reference 32). In particular, it is shown in References 10,19 that the indices of the feedback canonical form of linear DACSs are closely related to these sequences. In the linear case, the submanifold M^* is the largest subspace such that $HM^* \subseteq EM^* + \operatorname{Im} L$, which we denote by $\mathscr{M}^*$. Clearly, $\mathscr{M}^* = \mathscr{V}^* = \mathscr{V}_{k^*}$, where k^* is the smallest integer k such that $\mathscr{V}_k = \mathscr{V}_{k+1}$.

3 | TWO NORMAL FORMS UNDER EXTERNAL FEEDBACK EQUIVALENCE

The canonical form of linear DACSs in Reference 10 is with respect to the equivalence relation: $(E, H, L) \sim (QEP^{-1}, Q(H + LF^u)P^{-1}, QLT^{-1})$, where Q, P, T are invertible real matrices and F^u defines a static state feedback. In the following definition, we generalize this equivalence relation to the nonlinear case.

Definition 3 (external feedback equivalence). Two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively, are called external feedback equivalent, shortly ex-fb-equivalent, if there exists a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to GL(l, \mathbb{R})$, $\alpha : X \to \mathbb{R}^m$, $\beta : X \to GL(m, \mathbb{R})$ such that

$$\begin{aligned}
\tilde{E}(\psi(x)) &= Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}, \\
\tilde{F}(\psi(x)) &= Q(x)\left(F(x) + G(x)\alpha(x)\right), \\
\tilde{G}(\psi(x)) &= Q(x)G(x)\beta(x).
\end{aligned} \tag{8}$$

The ex-fb-equivalence of two DACSs is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_0 and $\tilde{U}$ of $\tilde{x}_0$, and $Q(x)$, $\alpha(x)$, $\beta(x)$ are defined on U, we will talk about local ex-fb-equivalence.

Remark 2. If two DACSs are ex-fb-equivalent, the diffeomorphism $\tilde{x} = \psi(x)$ and the feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$ establish a one-to-one correspondence of solutions $(x(\cdot), u(\cdot))$ and $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ of the DACSs, that is, $\tilde{x}(\cdot) = \psi(x(\cdot))$ and $u(\cdot) = \alpha(x(\cdot)) + \beta(x(\cdot))\tilde{u}(\cdot)$. On the other hand, if the solutions of two DACSs correspond to each other via a diffeomorphism and a feedback transformation, then the two DACSs are not necessarily ex-fb-equivalent (since the diffeomorphism is defined on the whole neighborhood U but the solutions exist on the maximal controlled invariant submanifold M^* only), which is the main reason for us to distinguish the "external" and "internal" analysis of DACSs. As a simple example, we consider the following two DAEs $\Xi^u_{2,1,1} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{2,1,1} = (\tilde{E}, \tilde{F}, \tilde{G})$, where

$$E(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} (x-1)^2 \\ -x^2 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 \\ e^x \end{bmatrix}, \quad \tilde{E}(\tilde{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}, \quad \tilde{G}(\tilde{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is clear that $(x, u) = (1, e^{-1})$ and $(\tilde{x}, \tilde{u}) = (0, 0)$ are the unique solutions of the two DACSs and the diffeomorphism $\tilde{x} = \psi(x) = x - 1$ and the feedback transformation $\tilde{u} = -x^2 + e^x u$ map (x, u) to $(\tilde{x}, \tilde{u})$. However, the two DACSs cannot be ex-fb-equivalent since E and $\tilde{E}$ are not of the same rank (two ex-fb-equivalent DACSs should have E-matrices of the same pointwise rank).

Theorem 1 (normal forms). *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix a point $x_p \in X$. Let $M^* \subseteq X$ be a smooth connected embedded submanifold containing x_p. Assume that M^* is a locally maximal controlled invariant submanifold around x_p and that there exists a neighborhood V of x_p such that*

(A1) $\operatorname{rank} E(x) = const. = r$ *and* $\operatorname{rank}\begin{bmatrix} E(x) & G(x) \end{bmatrix} = const. = r + m_2$, $\forall x \in V$.
(A2) The submanifold M^* satisfies the constant rank assumption **(CR)**, *that is,* $\dim E(x)T_xM^* = const. = r_1$ *and* $\dim(E(x)T_xM^* + \operatorname{Im} G(x)) = const. = r_1 + m_1 + m_2$, $\forall x \in M^* \cap V$.

Then there exist a neighborhood $U \subseteq V$ of x_p such that Ξ^u is locally ex-fb-equivalent to a DACS represented in the following normal form

$$
(\mathbf{NF}):\begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & E_2^2(z) & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ G_2(z) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{9}
$$

where (z_1, z_2) are local coordinates on $M^* = \{z \mid z_3 = 0, z_4 = 0\}$ and $z = (z_1, z_2, z_3, z_4)$, where E_1^2, E_1^4, E_2^2, E_2^4 are smooth matrix-valued functions defined on U with values in $\mathbb{R}^{r_1\times(n_1-r_1)}$, $\mathbb{R}^{r_1\times(n_2-r_2)}$, $\mathbb{R}^{r_2\times(n_1-r_1)}$, $\mathbb{R}^{r_2\times(n_2-r_2)}$, respectively, where $r = r_1 + r_2$, $n_1 = \dim M^*$, $n = n_1 + n_2$ and $m \geq m_1 + m_2$. Moreover, for all $z \in M^*$, we have that $E_2^2(z) = 0$, $F_4(z) = 0$ and $\operatorname{rank} G_2(z) = m_1$.

Furthermore, if the above (A2) is replaced by the condition that there exist a neighborhood V of x_p and an involutive distribution $\mathcal{D}$ such that $\mathcal{D}(x) = T_xM^*$, $\forall x \in M^* \cap V$, satisfying

(A3) $\dim E(x)\mathcal{D}(x) = const. = r_1$ *and* $\dim(E(x)\mathcal{D}(x) + \operatorname{Im} G(x)) = const. = r_1 + m_1 + m_2$, $\forall x \in V$,

then there exists a neighborhood $U \subseteq V$ of x_p such that Ξ^u is locally ex-fb-equivalent to Equation (9), for which, additionally, $E_2^2(z) \equiv 0$ and $\operatorname{rank} G_2(z) = m_1$, $\forall z \in U$, *which we call the special normal form*

$$
(\mathbf{SNF}):\begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & 0 & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ G_2(z) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{10}
$$

Proof. Since M^* is a smooth connected embedded submanifold, there exist a neighborhood U_1 of x_p and local coordinates (ζ_1, ζ_2) in U_1 such that $M^* \cap U_1 = \{x \in U_1 \mid \zeta_2(x) = 0\}$, where $\dim M^* = n_1$ and $\zeta_1 : U_1 \to \mathbb{R}^{n_1}$, $\zeta_2 : U_1 \to \mathbb{R}^{n_2}$. In the local (ζ_1, ζ_2)-coordinates, defined by the local diffeomorphism $\zeta(x) = (\zeta_1(x), \zeta_2(x))$, the system Ξ^u is expressed as

$$
\begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) \end{bmatrix}\begin{bmatrix} \dot\zeta_1 \\ \dot\zeta_2 \end{bmatrix} = \tilde F(\zeta) + \tilde G(\zeta)u,
$$

where $\tilde E_1 : U_1 \to \mathbb{R}^{l\times n_1}$ and $\tilde E_2 : U_1 \to \mathbb{R}^{l\times n_2}$, and where $\begin{bmatrix} \tilde E_1(\zeta(x)) & \tilde E_2(\zeta(x)) \end{bmatrix} = E(x)\left(\frac{\partial\zeta(x)}{\partial x}\right)^{-1}$, $\tilde F(\zeta(x)) = F(x)$, $\tilde G(\zeta(x)) = G(x)$. Then, by assumption (A1), for all $\zeta \in U_2 = U_1 \cap V$, we have

$$
\operatorname{rank}\begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) \end{bmatrix} = const. = r, \quad \operatorname{rank}\begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) & \tilde G(\zeta) \end{bmatrix} = const. = r + m_2.
$$

Thus, by Dolezal's theorem (see Reference 33), there exists a smooth map $Q_1 : U_2 \to GL(l, \mathbb{R})$,

$$
Q_1(\zeta)\begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) & \tilde G(\zeta) \end{bmatrix} = \begin{bmatrix} \bar E_1(\zeta) & \bar E_2(\zeta) & \overline{G}_1(\zeta) \\ 0 & 0 & \overline{G}_2(\zeta) \\ 0 & 0 & 0 \end{bmatrix},
$$

where $\bar{E}_1 : U_2 \to \mathbb{R}^{r\times n_1}$, $\bar{E}_2 : U_2 \to \mathbb{R}^{r\times n_2}$, and $\overline{G}_2 : U_2 \to \mathbb{R}^{m_2\times m}$, such that the matrices $[\bar{E}_1(\zeta), \bar{E}_2(\zeta)]$ and $\overline{G}_2(\zeta)$ above are of full row rank.

By dim $E(x)T_xM^* = const. = r_1$ of assumption (A2), it is immediate to see that rank $\bar{E}_1(\zeta) = r_1$ for $\zeta \in M^*$. It follows from the smoothness of $\bar{E}_1(\zeta)$ that by taking a smaller U_2, if necessary, there exist r_1 columns of $\bar{E}_1(\zeta)$ that are linearly independent in U_2. Now we write the matrix

$$\left[\bar{E}_1(\zeta) \;\middle|\; \bar{E}_2(\zeta)\right] = \left[\begin{array}{cc|cc} E_1^1(\zeta) & E_1^2(\zeta) & E_2^1(\zeta) & E_2^2(\zeta) \\ E_1^3(\zeta) & E_1^4(\zeta) & E_2^3(\zeta) & E_2^4(\zeta) \end{array}\right],$$

where $E_1^1 : U_2 \to \mathbb{R}^{r_1\times r_1}$ and $E_2^3 : U_2 \to \mathbb{R}^{r_2\times r_2}$ and where $r_2 = r - r_1$ (and other matrices are of suitable dimensions). We can always permute the rows (by a constant Q-transformation) and the columns (by permuting the components of ζ_1) of the above matrix such that $E_1^1(\zeta)$ is invertible. Then by a suitable Q-transformation, $[\bar{E}_1, \bar{E}_2]$ admits the form

$$\left[\bar{E}_1(\zeta) \;\middle|\; \bar{E}_2(\zeta)\right] = \left[\begin{array}{cc|cc} I_{r_1} & E_1^2(\zeta) & E_2^1(\zeta) & E_2^2(\zeta) \\ 0 & E_1^4(\zeta) & E_2^3(\zeta) & E_2^4(\zeta) \end{array}\right].$$

Since rank $E(x) =$ rank $[\bar{E}_1(\zeta), \bar{E}_2(\zeta)] = r$, the matrix $[E_1^4, E_2^3, E_2^4]$ is of full row rank $r_2 = r - r_1$. Notice that $E_1^4(\zeta) = 0$ for $\zeta \in M^*$ (since rank $\bar{E}_1(\zeta) = r_1$ for $\zeta \in M^*$), so rank $[E_2^3(\zeta), E_2^4(\zeta)] = r_2$ for $\zeta \in M^*$. By the smoothness of $\bar{E}_2(\zeta)$, we have that $[E_2^3(\zeta), E_2^4(\zeta)]$ is of full row rank r_2 for $\zeta \in U_2$. Then we can always permute the columns (by permuting the components of ζ_2) of $\bar{E}_2$ such that E_2^3 is invertible. On the other hand, we write

$$\left[\begin{array}{c} \bar{G}_1(\zeta) \\ \hline \bar{G}_2(\zeta) \end{array}\right] = \left[\begin{array}{cc} G_1^1(\zeta) & G_1^2(\zeta) \\ G_1^3(\zeta) & G_1^4(\zeta) \\ \hline G_2^1(\zeta) & G_2^2(\zeta) \end{array}\right],$$

where $G_2^2(\zeta)$ is a $m_2 \times m_2$ matrix (and others are of suitable dimensions). Since $\overline{G}_2(\zeta)$ is of full row rank m_2 for $\zeta \in U_2$, we can permute the components of u (by a feedback transformation) such that $G_2^2(\zeta)$ is invertible. Since both $E_2^3(\zeta)$ and $G_2^2(\zeta)$ are invertible, we can set

$$Q_2(\zeta) = \begin{bmatrix} I_{r_1} & Q_1^2(\zeta) & Q_1^3(\zeta) & 0 \\ 0 & Q_2^2(\zeta) & Q_2^3(\zeta) & 0 \\ 0 & 0 & Q_3^3(\zeta) & 0 \\ 0 & 0 & 0 & I_{l-m-r} \end{bmatrix},$$

where $Q_1^2 = -E_2^1(E_2^3)^{-1}$, $Q_1^3 = -(G_1^2 - E_2^1(E_2^3)^{-1}G_1^4)(G_2^2)^{-1}$, $Q_2^2 = (E_2^3)^{-1}$, $Q_2^3 = -(E_2^3)^{-1}G_1^4(G_2^2)^{-1}$, $Q_3^3 = (G_2^2)^{-1}$, and then we have

$$Q_2(\zeta)Q_1(\zeta)\left[\bar{E}_1(\zeta) \quad \bar{E}_2(\zeta) \;\middle|\; \tilde{G}(\zeta)\right] = \left[\begin{array}{cccc|cc} I_{r_1} & \tilde{E}_1^2(\zeta) & 0 & \tilde{E}_2^2(\zeta) & \tilde{G}_1^1(\zeta) & 0 \\ 0 & \tilde{E}_1^4(\zeta) & I_{r_2} & \tilde{E}_2^4(\zeta) & \tilde{G}_1^3(\zeta) & 0 \\ 0 & 0 & 0 & 0 & \tilde{G}_2^1(\zeta) & I_{m_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where $\tilde{E}_1^2 = E_1^2 + Q_1^2E_1^4$, $\tilde{E}_2^2 = E_2^2 + Q_1^2E_2^4$, $\tilde{G}_1^1 = G_1^1 + Q_1^2E_1^4 + Q_1^3G_2^1$, $\tilde{E}_1^4 = Q_2^2E_1^4$, $\tilde{E}_2^4 = Q_2^2E_2^4$, $\tilde{G}_1^3 = Q_2^2G_1^3 + Q_2^3G_2^1$, $\tilde{G}_2^1 = Q_3^3G_2^1$. Denote $Q_2Q_1\tilde{F} = (F_1, F_2, F_3, F_4)$. Then by the feedback transformation

$$\begin{bmatrix} 0 \\ F_3(\zeta) \end{bmatrix} + \begin{bmatrix} I_{m_1} & 0 \\ \tilde{G}_2^1(\zeta) & I_{m_2} \end{bmatrix} u = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

both $\tilde{G}_2^1$ and F_3 become zero. Rewrite $z = \zeta$, $(z_1, z_2) = \zeta_1$, $(z_3, z_4) = \zeta_2$, $(u_1, u_2) = (\tilde{u}_1, \tilde{u}_2)$, $E_1^2 = \tilde{E}_1^2$, $E_2^2 = \tilde{E}_1^4$, $E_1^4 = \tilde{E}_2^2$, $E_2^4 = \tilde{E}_2^4$, $G_1 = \tilde{G}_1^1$, $G_2 = \tilde{G}_1^3$, then it is straightforward to see that Ξ^u is locally ex-fb-eq on $U = U_2$ to the normal form **(NF)**, given by (9).

Consider Equation (9), then the condition $\dim E(x)T_xM^* = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1$ for all $z \in M^*$, of assumption (A2), implies $E_2^2(z) = 0$, for all $z \in M^*$, and the condition $\dim(E(x)T_xM^* + \operatorname{Im} G(x)) = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) & G_1(z) & 0 \\ 0 & E_2^2(z) & G_2(z) & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} = r_1 + m_1 + m_2$ for all $z \in M^*$, of assumption(A2), implies $\operatorname{rank} G_2(z) = m_1$, for all $z \in M^*$. Moreover, by the fact that M^* is a controlled invariant submanifold and due to condition (4) of Proposition 1, it follows that $F_4(z) = 0$ for all $z \in M^*$.

Now we prove that under assumptions (A1) and (A3), Ξ^u is locally ex-fb-equivalent to the special normal form **(SNF)**, given by (10). The construction of the **(SNF)** is similar to the above construction of the **(NF)** given by (9) but we choose coordinates $\zeta = (\zeta_1, \zeta_2)$ differently. By the involutivity of $\mathcal{D}$ of assumption (A3) and Frobenius theorem (see, e.g., Reference 23), there exist a neighborhood U_1 of x^0 and two vector-valued functions $\zeta_1 : U_1 \to \mathbb{R}^{n_1}$ and $\zeta_2 : U_1 \to \mathbb{R}^{n_2}$ such that $\operatorname{span}\{d\zeta_1^1, \dots, d\zeta_1^{n_1}\} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ denotes the annihilator of the distribution $\mathcal{D}$, and the differentials $d\zeta_1^i$ and $d\zeta_2^j$ are linearly independent. Since $\mathcal{D}(x) = T_xM^*$ locally for $x \in M^*$, we still have $M^* \cap U_1 = \{x \mid \zeta_2(x) = 0\}$. Observe that assumption (A3) implies (A2), so we may transform Ξ^u into **(NF)**, given by (9), using the construction described above. But now by assumption (A3), we have

$$\dim\ E(z)\mathcal{D}(z) = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1,$$

$$\dim\ (E(z)\mathcal{D}(z) + \operatorname{Im} G(z)) = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) & G_1(z) & 0 \\ 0 & E_2^2(z) & G_2(z) & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} = r_1 + m_1 + m_2,$$

for all $z \in U_2$, which, respectively, implies $E_2^2(z) \equiv 0$ and $\operatorname{rank} G_2(z) = m_1$ on U_2. Therefore, under assumptions (A1) and (A3), the DACS Ξ^u is locally ex-fb-equivalent to the **(SNF)** given by (10). ■

The following observations are crucial.

Remark 3.

(i) If the submanifold M^* exists and Ξ^u satisfies the constant rank assumptions (A1) and (A2), which are regularity assumptions, then Ξ^u is locally ex-fb-equivalent to the **(NF)**, given by (9). If Ξ^u satisfies the constant rank and *involutivity* assumptions (A1) and (A3), then it is locally ex-fb-equivalent the **(SNF)**, given by (10), in which, additionally compared to (9), we have $E_2^2(z) \equiv 0$ and $\operatorname{rank} G_2(z) = m_1$ for all $z \in U$. Note that if M^* is replaced by M being any controlled invariant submanifold (not necessarily maximal) and satisfying (A1) and (A2), or (A1) and (A3), we may still transform Ξ into form (9) or (10) since we do not use the maximality of M^* to construct the two normal forms as shown in the above of proof. However, if M^* is not locally maximal, we can neither conclude that $M^* = \{z | z_3 = z_4 = 0\}$ nor that (z_1, z_2) are the local coordinates on the admissible set $S_a = M^*$.

(ii) By a suitable feedback transformation introducing new controls (u_1^1, u_1^2) (possibly also by a permutation of z_3-variables), the second equation $\begin{bmatrix} I_{r_2} & E_2^4(z) \end{bmatrix}\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = F_2(z) + G_2(z)u_1$ of (10) can be further simplified as

$$\begin{bmatrix} I_{r_2 - m_1} & 0 & \tilde{E}_2^4(z) \\ 0 & I_{m_1} & \bar{E}_2^4(z) \end{bmatrix}\begin{bmatrix} \dot{z}_3^1 \\ \dot{z}_3^2 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \overline{F}_2(z) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \overline{G_2}(z) \\ 0 & I_{m_1} \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix},$$

where $u_1^1 \in \mathbb{R}^{r_2 - m_1}$, $u_1^2 \in \mathbb{R}^{m_1}$ and $\overline{F}_2(z) = 0$ for $z \in M^*$.

(iii) The forms **(NF)** and **(SNF)** are two normal forms under the external feedback equivalence, meaning that both hold locally everywhere around x_p, not just on the maximal controlled invariant manifold M^* passing through x_p. For any point $x_0 \notin M^*$ around x_p, the system does not have solutions passing through x_0 (see item (ii) of Remark 1), but the system admits the above normal forms, which can be useful if we want to steer x_0 toward M^*.

(iv) Note that $M^* = \{z \mid z_3 = 0, z_4 = 0\}$. If we consider Ξ^u "internally," that is, locally on M^*, by setting z_3 and z_4 to be zero, we get from Equation (9) the following system (we may do the same for Equation 10):

$$\begin{bmatrix} I_r & E_1^2(z_1,z_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} F_1(z_1,z_2) \\ F_2(z_1,z_2) \end{bmatrix} + \begin{bmatrix} G_1(z_1,z_2) \\ G_2(z_1,z_2) \end{bmatrix} u_1.$$

Since rank $G_2(z) = m_1$ for $z \in M^*$, via a suitable feedback transformation introducing new controls (u_1^1, u_1^2), and a $Q(x)$-transformation (defined on M^* but it can be extended to U^* that is open in X), the above DACS can be transformed into

$$\begin{bmatrix} I_r & E_1^2(z_1,z_2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(z_1,z_2) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(z_1,z_2) & 0 \\ 0 & 0 \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix},$$

for some maps $\tilde{F}_1$ and $\tilde{G}_1^1$. It can be seen from item (i) of Theorem 2 that Ξ^u has solutions isomorphic with those of the first subsystem $\dot{z}_1 + E_1^2(z_1,z_2)\dot{z}_2 = \tilde{F}_1(z_1,z_2) + \tilde{G}_1^1(z_1,z_2)u_1^1$, which we denote by $\Xi^u|_{M^*}$ and call the restriction of Ξ^u to M^*; the latter can be regarded as an ODE control system with controls $w = \dot{z}_2$ and u_1^1:

$$\begin{cases} \dot{z}_1 = \tilde{F}_1(z_1,z_2) + \tilde{G}_1^1(z_1,z_2)u_1^1 - E_1^2(z_1,z_2)w, \\ \dot{z}_2 = w. \end{cases}$$

(This is a particular case of a general procedure proposed in Reference 22 under the name of (Q,w)-explicitation). From the above analysis, it is seen that for a fixed control u, the original Ξ^u has a unique maximal solution (see the definition of maximal solution in Section 4) if and only if $n_1 = r_1$ (since in this case, the z_2-variables are absent).

(v) The above two normal forms **(NF)** and **(SNF)** facilitate understanding the actual roles of the variables in the nonlinear DACS Ξ^u. As a result, some generalized states, namely (z_1, z_3), behave like state variables of differential equations and some generalized states, namely (z_2, z_4), are free variables since their derivatives $(\dot{z}_2, \dot{z}_4)$ are not constrained and can be seen as extra inputs which are different from u. Moreover, some generalized states, namely (z_3, z_4) are constrained and some controls, namely u_1^2 and u_2 are also not free to be chosen (since they are forced to be 0 by the constraints) when the DACS is considered internally on M^*. The generalized state z_2 and the control u_1^1 are the truly free variables and are not constrained.

(vi) It is worth to mention that the behavioral approach of system theory (see Reference 8) does not a priori distinguish the roles of the variables (which is also the case of our variables x consisting of all components of the generalized state; we distinguish, however, the control u from the generalized state x) and only the analysis of the system reveals the nature of those variables. The observations of item (v) above could be regarded as an instruction for the reinterpretation of the meaning of those variables, the latter has already been addressed in Reference 14,19 for the regulation problems. For instance, the free generalized states z_2 could be reinterpreted as a new input (but they should be distinguished from the true controls u considering the physical meanings of the generalized state variables, see Section 1) and the constrained generalized states z_3 and z_4 could be redefined as zeroing outputs of the system. Consider the DACS Ξ (which describes a 3-link manipulator with a free end-joint) and its **(SNF)** of Example 1. It is seen that F_f (the friction force at the end joint) is a free generalized state, which, however, should be distinguished from the real active input $u = (F_x, F_y)$ since the physical meaning of regarding F_f as a new active control is that we add a motor/actuator to the free joint and consider instead a fully actuated manipulator.

(vii) Our forms **(NF)** and **(SNF)** are different from the zero dynamics form **(ZDF)** proposed in Reference 12 in many ways. First, the feedback transformations, which play important roles for our normal forms, are not used for the **(ZDF)**. Second, it is assumed for the **(ZDF)** that

$$\dim E(x)T_xM^* + \operatorname{Im} G(x) = \dim M^* + m, \tag{11}$$

while we only assume that $\dim E(x)T_xM^* + \operatorname{Im} G(x) = const.$, which is more general since assumption (11) excludes the existence of free generalized states and control inputs in the internal dynamics. Third, the utilization of the involutive distribution $\mathcal{D}$, not present in **(ZDF)**, shows a possibility to further simplify the structure of the matrix $E(x)$ in **(SNF)**.

4 INTERNAL REGULARIZATION OF NONLINEAR DACSS

In this section, we first consider the uncontrolled case of (1), that is, nonlinear DAEs, which are of the form

$$\Xi : E(x)\dot{x} = F(x),$$

and are denoted by $\Xi_{l,n} = (E, F)$ or, equivalently, by $\Xi^u_{l,n,0} = (E, F, 0)$. If we apply Definition 2 to a DAE Ξ, then M^* is called a locally maximal invariant submanifold. It is well known (see, e.g., References 6,20,22,25,26) that the solutions of a DAE Ξ exist locally on its maximal invariant submanifold M^* only and that the uniqueness of solutions can be characterized by the notion of local *internal regularity*, which is defined below. We will say that a solution $x : I \to M^*$ satisfying $x(t_0) = x_0$, where $t_0 \in I$ and $x_0 \in M^*$, is maximal if for any solution $\tilde{x} : \tilde{I} \to M^*$ such that $t_0 \in \tilde{I}$, $\tilde{x}(t_0) = x_0$ and $x(t) = \tilde{x}(t)$ for all $t \in I \cap \tilde{I}$, we have $\tilde{I} \subseteq I$.

Definition 4 (local internal regularity). Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be a locally maximal invariant submanifold around a point $x_p \in M^*$. Then Ξ is called locally internally regular (around x_p) if there exists a neighborhood $U \subseteq X$ of x_p such that for any point $x_0 \in M^* \cap U$, there exists only one maximal solution $x : I \to M^* \cap U$ satisfying $t_0 \in I$ and $x(t_0) = x_0$.

Remark 4. Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be a locally maximal invariant submanifold around a point $x_p \in M^*$. Assume that there exists a neighborhood U of x_p such that $\dim E(x)T_xM^* = const.$, $\forall x \in M^* \cap U$. Then the following conditions are equivalent (see theorem 4.3.14 of Reference 4 or 22):

(i) Ξ is internally regular around x_p;
(ii) $\dim E(x)T_xM^* = \dim M^*$, $\forall x \in M^* \cap U$;
(iii) Via a Q-transformation defined on M^* around x_p, the system $\Xi|_{M^*}$ can be transformed into an ODE $\dot{z}^* = f(z^*)$, where z^* are local coordinates on M^*, given by ψ, and $\Xi|_{M^*}$ denotes Ξ restricted to the submanifold M^* (compare item (iv) of Remark 3).

Definition 5 (local internal regularizability). A DACS $\Xi^u = (E, F, G)$ is called locally internally regularizable (around x_p) if there exist a neighborhood U of x_p and a smooth map $\gamma : U \to \mathbb{R}^m$ such that the DAE $\Xi_{l,n} = (E, F + G\gamma)$ is internally regular around x_p.

Now we use Algorithm 1 in the appendix to study the problems of when is a DACS locally internally regularizable and how to design internal regularization feedback laws. Note that Algorithm 1 is a practical implementation of the recursive procedure of Proposition 2, see Remark 1(i), with additional Assumptions 1 and 2. At every step of Algorithm 1, we construct a submanifold M^c_k and a local form, given by (A1), under the external feedback equivalence, based on which we give an explicit expression of the restricted/reduced system defined by Equation (A2). Moreover, at every k-step, we show in details how to construct the coordinate transformations ψ_k and the feedback transformations $(u_k, \bar{u}_k) = a_k + b_k u_{k-1}$, which lead to the local form. In the statement of Theorem 2, we refer to the submanifold $M^* = M^c_{k^*+1}$, and to the open neighborhood $U^* = U_{k^*+1}$ (in X) of Step $k^* + 1$ of Algorithm 1.

Theorem 2. *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix a point $x_p \in X$. Suppose that Assumptions 1 and 2 of Algorithm 1 are satisfied. Then M^c_k, for $k = 0, \dots, k^* + 1$, of the recursive procedure given just before Proposition 2 are smooth connected embedded submanifolds and $M^* = M^c_{k^*+1}$ satisfies the constant rank condition* **(CR)** *in U^* and thus by that proposition, x_p is an admissible point and M^* is a locally maximal controlled invariant submanifold around x_p, given by*

$$M^* = \left\{ x \mid \bar{z}_1(x) = 0, \dots, \bar{z}_{k^*}(x) = 0 \right\}.$$

Moreover,

(i) *locally, around* x_p *there exist a diffeomorphism* $\hat{z} = \Psi(x)$ *and an invertible feedback* $u = \alpha(x) + \beta(x)\hat{u}$*, where* $\hat{z} = (z^*, \bar{z}) = (z^*, \bar{z}_1, \dots, \bar{z}_{k^*})$ *and* $\hat{u} = (u^*, \bar{u}) = (u^*, \bar{u}_1, \dots, \bar{u}_{k^*+1})$*, transforming the set of all solutions of* $\Xi^u_{l,n,m}$ *into that of* $\hat{\Xi}^{\hat{u}}_{\hat{l},\hat{n},\hat{m}} = (\hat{E}, \hat{F}, \hat{G})$*, where* $\hat{l} = r^* + (n - n^*) + (m - m^*)$*,* $\hat{n} = n$*,* $\hat{m} = m$*, and*

$$\hat{E}(\hat{z}) = \begin{bmatrix} E^*(z^*) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{F}(\hat{z}) = \begin{bmatrix} F^*(z^*) \\ \bar{z} \\ 0 \end{bmatrix}, \quad \hat{G}(\hat{z}) = \begin{bmatrix} G^*(z^*) & 0 \\ 0 & 0 \\ 0 & I_{m-m^*} \end{bmatrix},$$

or, equivalently,

$$\hat{\Xi}^{\hat{u}} : \begin{cases} E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)u^*, \\ \bar{z}_1 = 0, \dots, \bar{z}_{k^*} = 0, \\ \bar{u}_1 = 0, \dots, \bar{u}_{k^*} = 0, \bar{u}_{k^*+1} = 0, \end{cases} \tag{12}$$

where $E^* = E_{k^*+1} : M^* \to \mathbb{R}^{r^* \times n^*}$*,* $F^* = F_{k^*+1} : M^* \to \mathbb{R}^{r^*}$*,* $G^* = G_{k^*+1} : M^* \to \mathbb{R}^{r^* \times m^*}$ *and* $n^* = n_{k^*} = n_{k^*+1}$*,* $r^* = r_{k^*+1}$*,* $m^* = m_{k^*+1}$ *come from Step* $k^* + 1$ *of Algorithm 1, and where* z^* *are local coordinates on* M^**, and* $\operatorname{rank} E^*(z^*) = r^*$*,* $\forall z^* \in M^*$*, that is,* $E^*(z^*)$ *is of full row rank.*

(ii) *The DACS* Ξ^u *is internally regularizable around* x_p *if and only if* $r^* + \overline{m} \geq n^*$*, where* $\overline{m} = m - m^*$ *or, equivalently, for any point* $x \in M^* \cap U$*, where* $U \subseteq X$ *is an open neighborhood of* x_p*, we have*

$$\dim(E(x)T_xM^* + \operatorname{Im} G(x)) \geq \dim M^*. \tag{13}$$

(iii) *Since* $E^*(z^*)$ *of system (12) is of full row rank* r^**, we assume that the first* r^* *columns of* $E^*(z^*)$ *are linearly independent (if not, we can always permute the components of* z^**). Rewrite* $E^*(z^*)\dot{z}^* = [E_1^*(z^*)\ E_2^*(z^*)] \begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \end{bmatrix}$ *such that* $E_1^*(z^*)$ *is invertible. If (13) holds, then define the following feedback law for (12):*

$$\hat{u} = \begin{bmatrix} u^* \\ \bar{u}_1 \\ \vdots \\ \bar{u}_{k^*+1} \end{bmatrix} = \begin{bmatrix} \gamma^*(z_1^*, z_2^*) \\ z_2^* \\ 0 \end{bmatrix} = \hat{\gamma}(z^*). \tag{14}$$

where $u^* = \gamma^*(z_1^*, z_2^*)$*,* $z_2^* \in \mathbb{R}^{n^* - r^*}$*, and* $\gamma^* : M^* \to \mathbb{R}^{m^*}$ *is an arbitrary smooth map. Then the feedback law* $u = \gamma(x) = \alpha(x) + \beta(x)\hat{\gamma}(\psi(x))$*, where the diffeomorphism* $\Psi(x) = (z^*, \bar{z})$ *and invertible feedback* $u = \alpha(x) + \beta(x)\hat{u}$ *are those of item (i) and* $\psi(x) = \Psi^{-1}(z^*, 0)$*, internally regularizes the original system* Ξ^u*.*

Proof. (i) At the general Step k of Algorithm 1, consider the DACSs $\tilde{\Xi}^{\tilde{u}}_k = \Xi^u_{k-1} = (E_{k-1}, F_{k-1}, G_{k-1})$ and $\hat{\Xi}^{\hat{u}}_k = (\hat{E}_k, \hat{F}_k, \hat{G}_k)$, the latter given by (A1). Then we show that the following items are equivalent.

(a) $(z_{k-1}(\cdot), u_{k-1}(\cdot))$, where $z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), \bar{z}_k(\cdot))$ and $u_{k-1}(\cdot) = \alpha_k(z_{k-1}(\cdot)) + \beta_k((z_{k-1}(\cdot))(u_k(\cdot), \bar{u}_k(\cdot))$, is a solution of Ξ^u_{k-1};
(b) $(z_k(\cdot), \bar{z}_k(\cdot), u_k(\cdot), \bar{u}_k(\cdot))$ is a solution of $\hat{\Xi}^{\hat{u}}_k$;
(c) $\bar{z}_k(\cdot) = 0$, $\bar{u}_k(\cdot) = 0$ and $(z_k(\cdot), u_k(\cdot))$ is a solution of

$$\Xi^u_k : E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)u_k,$$

where $E_k = \hat{E}^1_k$, $F_k = \hat{F}^1_k$, $G_k = \hat{G}^1_k$, and where $\hat{E}^1_k$, $\hat{F}^1_k$, $\hat{G}^1_k$ are defined by formula (A2).

Since $\tilde{\Xi}_k^{\tilde{u}} = \Xi_{k-1}^u$ is locally ex-fb-equivalent to $\hat{\Xi}_k^{\hat{u}}$ via Q_k, ψ_k, α_k and β_k, we have that item (a) and item (b) above are equivalent (see Remark 2). The equivalence of item (b) and item (c) follows from the fact that the solutions exist on M_k only and should respect the constraints $\bar{z}_k = 0$ and $\bar{u}_k = 0$.

Then by the equivalence of (c) and (a), we have at the first step of Algorithm 1 that $(z_1(\cdot), 0, u_1(\cdot), 0)$ is a solution of $E_1(z_1)\dot{z}_1 = F_1(z_1) + G_1(z_1)u_1, \bar{z}_1 = 0, \bar{u}_1 = 0$, if and only if $(z_0(\cdot), u_0(\cdot))$ is a solution of $\Xi_0^u = \Xi^u = (E, F, G)$, where $z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), \bar{z}_1(\cdot))$ and $u_0(\cdot) = \alpha_1(z_0(\cdot)) + \beta_1(z_0(\cdot))(u_1(\cdot), \bar{u}_1(\cdot))$. In general, by an induction argument, we can prove that $(z_k(\cdot), 0, \dots, 0, u_k(\cdot), 0, \dots, 0)$ is a solution of

$$E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)u_k, \quad \bar{z}_1 = 0, \dots, \bar{z}_k = 0, \quad \bar{u}_1 = 0, \dots, \bar{u}_k = 0,$$

if and only if $(x(\cdot), u(\cdot))$ is a solution of Ξ^u, where $x(\cdot)$ and $u(\cdot)$ are given by the following iterative formula

$$x(\cdot) = z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), 0),\ z_1(\cdot) = \psi_2^{-1}(z_2(\cdot), 0),\ \dots,\ z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), 0) \tag{15}$$

and

$$\begin{cases} u(\cdot) &= \alpha_1(x(\cdot)) + \beta_1(x(\cdot))(u_1(\cdot), 0), \\ u_1(\cdot) &= \alpha_2(z_1(\cdot)) + \beta_2(z_1(\cdot))(u_2(\cdot), 0), \\ \vdots & \\ u_{k-1}(\cdot) &= \alpha_k(z_{k-1}(\cdot)) + \beta_k(z_{k-1}(\cdot))(u_k(\cdot), 0). \end{cases} \tag{16}$$

Each diffeomorphism ψ_k and feedback transformation (α_k, β_k) are defined on W_k, and to extend it to U_k, we put $\Psi_k = (\psi_k, \bar{z}_k, \dots, \bar{z}_1)$. Then we first extend $(\alpha_k, \beta_k)(\Psi_{k-1}^{-1} \circ \cdots \circ \Psi_1^{-1})$ arbitrarily to U_k (keeping their values on W_k) and, second, we enlarge the extended (α_k, β_k) to a feedback transformation (A_k, B_k) acting on the whole control vector $u = (u_{k-1}, \bar{u}_{k-1}, \dots, \bar{u}_1)$ by acting on u_{k-1} via (α_k, β_k) and keeping $(\bar{u}_{k-1}, \dots, \bar{u}_1)$ unchanged. Now we define the local diffeomorphism $\Psi := \Psi_{k^*} \circ \cdots \circ \Psi_2 \circ \Psi_1 : U_{k^*+1} \to \mathbb{R}^n$ (note that $\Psi_{k^*+1} = \Psi_{k^*}$) and the feedback transformation (α, β) as the composition of $(A_1, B_1), (A_2, B_2), \dots, (A_{k^*+1}, B_{k^*+1})$ with $\alpha : U_{k^*+1} \to \mathbb{R}^m$ and $\beta : U_{k^*+1} \to \mathbb{R}^{m \times n}$. To show that the local diffeomorphism $\hat{z} = \Psi(x)$, where $\hat{z} = (z^*, \bar{z})$, and the feedback transformation $u = \alpha(x) + \beta(x)\hat{u}$ transform solutions of Ξ^u into those of $\hat{\Xi}^{\hat{u}}$, it is enough to observe that Ψ and $u = \alpha(x) + \beta(x)\hat{u}$ satisfy (15) and (16), for $k = k^* + 1$.

Now we prove that $E^*(z^*)$, for $z^* \in M^*$, is of full row rank. Consider Step $k^* + 1$ of Algorithm 1, note that the Q_{k^*+1}-transformation ensures that $\tilde{E}_{k^*+1}^1(z_{k^*})$ is of full row rank. By $M_{k^*+1}^c = \left\{ z_{k^*} \in M_{k^*}^c \cap U_{k^*+1} \mid \tilde{F}_{k^*+1}^3(z_{k^*}) = 0 \right\}$ and the fact that $\dim M_{k^*}^c = n_{k^*} = n_{k^*+1} = \dim M_{k^*+1}^c$, we have $\tilde{F}_{k^*+1}^3(z_{k^*}) = 0$, $\forall z_{k^*} \in M_{k^*}^c \cap U_{k^*+1}$. As a consequence, the $\bar{z}_{k^*+1}$-coordinates are not present, so there is no equation $\bar{z}_{k^*+1} = 0$ in (12). Moreover, we have $M_{k^*+1}^c = M_{k^*}^c$ in U_{k^*+1}, implying that $z_{k^*+1} = z_{k^*}$. Finally, it is seen from $E^*(z^*) = E_{k^*+1}(z_{k^*+1}) = \hat{E}_{k^*+1}^1(z_{k^*}) = \tilde{E}_{k^*+1}^1(z_{k^*})$ that $E^*(z^*)$ is of full row rank for all $z^* = z_{k^*+1} \in M^* = M_{k^*+1}^c$.

(ii) To begin with, we prove that Ξ^u is internally regularizable if and only if $\hat{\Xi}^{\hat{u}}$, given by (12), is internally regularizable. Observe that Ξ^u is internally regularizable, that is, there exists a feedback $u = \gamma(x)$ such that $\Xi = (E, F + G\gamma)$ is internally regular if and only if the algebraic constraint $0 = u - \gamma(x)$ is such that the DAE $\Xi^\gamma : \begin{cases} E(x)\dot{x} &= F(x) + G(x)u \\ 0 &= u - \gamma(x) \end{cases}$ has a unique maximal solution $(x(\cdot), u(\cdot))$ satisfying $x(t_0) = x_0$ and $u(t_0) = \gamma(x_0)$ for any $x_0 \in M_\gamma^* \cap U$, where M_γ^* is a locally maximal invariant submanifold of Ξ^γ and U is a neighborhood of x_p. By item (i) of Theorem 2, there is a one-to-one correspondence, given by a local diffeomorphism $\hat{z} = \Psi(x)$ and a feedback transformation $u = \alpha(x) + \beta(x)\hat{u}$, between the solutions of Ξ^u and those of $\hat{\Xi}^{\hat{u}}$. As a consequence, Ξ^u is internally regularizable if and only if there exists $\gamma : M^* \to \mathbb{R}^n$ such that the DAE

$$\begin{cases} \hat{E}(\hat{z})\dot{\hat{z}} &= \hat{F}(\hat{z}) + \hat{G}(\hat{z})\hat{u} \\ \quad 0 &= \hat{u} - \hat{\gamma}(\hat{z}), \end{cases}$$

where $\hat{\gamma}(\hat{z}) = \beta^{-1}(\gamma(\Psi^{-1}(\hat{z})) - \alpha(\Psi^{-1}(\hat{z})))$, has a unique maximal solution $(\hat{z}(\cdot), \hat{u}(\cdot))$ satisfying $\hat{z}(t_0) = \hat{z}_0$, where $\hat{z}_0 = \Psi(x_0)$, and $\hat{u}(t_0) = \beta^{-1}(x_0)(\gamma(x_0) - \alpha(x_0))$, for any $x_0 \in M_\gamma^* \cap U$, that is, $\hat{\Xi}^{\hat{u}}$ is internally regularizable. Now we will show that

$\hat{\Xi}^{\hat{u}}$ is internally regularizable if and only if (13) holds. Since $E^*(z^*)$ of (12) is of full row rank, we may view the first equation of (12) as an ODE control system with extra free variables. More precisely, assume that the first r^* columns of $E^*(z^*)$ are linearly independent (if not, we can always permute the components of z^*), then we can rewrite $E^*(z^*)\dot{z}^*$ as $\begin{bmatrix} E_1^*(z^*) & E_2^*(z^*) \end{bmatrix} \begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \end{bmatrix}$, where $z^* = (z_1^*, z_2^*)$ and $E_1^* : M^* \to R^{r^* \times r^*}$ is invertible. Thus we can rewrite the first equation of (12) as

$$\begin{cases} \dot{z}_1^* = (E_1^*)^{-1}F^*(z^*) + (E_1^*)^{-1}G^*(z^*)u^* - (E_1^*)^{-1}E_2^*(z^*)w \\ \dot{z}_2^* = w. \end{cases} \tag{17}$$

It follows that $\hat{\Xi}^{\hat{u}}$ is internally regularizable if and only if the free variables z_2^* can be fixed via the constraints $\bar{u} = 0$, which is equivalent to the fact that the number of constrained inputs $\bar{u}$ (there are $\overline{m} = m - m^*$ of them) is not less than the number of components of z_2^* (which is $n^* - r^*$) and thus equivalent to (13).

(iii) If $\overline{m} = m - m^* \geq n^* - r^*$, then there are enough components of constrained inputs $\bar{u} = 0$ that can be used to fix the free variables z_2^*. Namely, denote $\hat{u} = (u^*, \bar{u}', \bar{u}'') \in \mathbb{R}^{m^*} \times \mathbb{R}^{\overline{m}'} \times \mathbb{R}^{\overline{m}''}$, where $\overline{m}' = n^* - r^*$ and $\overline{m}'' = \overline{m} - (n^* - r^*)$, then we impose $z_2^* = 0$ by setting $\bar{u}' = z_2^* = 0$ and the remaining components $\bar{u}'' = 0$ to construct a controlled invariant submanifold. We can choose $u^* = \gamma^*(z^*)$ arbitrarily and then $M_\alpha^* = \left\{ z^* \in M^* \mid z_2^* = 0 \right\}$ is an invariant submanifold of the closed-loop system $\hat{\Xi}^{\hat{\gamma}}$, obtained from $\hat{\Xi}^{\hat{u}}$ via $\hat{u} = \hat{\gamma}(z^*)$, where

$$\hat{u} = (u^*, \bar{u}', \bar{u}'') = (\gamma^*(z^*), z_2^*, 0) = \hat{\gamma}(z^*) \tag{18}$$

is thus the feedback law (14). Now using the diffeomorphism $x = \Psi^{-1}(z^*, \bar{z})$ and invertible feedback $u = \alpha(x) + \beta(x)\hat{u}$ that transform solutions of Ξ^u into those of $\hat{\Xi}^{\hat{u}}$ (see item (i)), we conclude that the feedback law $u = \alpha(x) = \alpha(x) + \beta(x)\hat{\gamma}(\psi(x))$, where $\hat{\gamma}$ is given by (18) and $\psi(x) = \Psi^{-1}(z^*, 0)$, internally regularizes the original system Ξ^u, which completes the proof. ∎

Remark 5.

(i) Note that we perform $k^* + 1$ steps of Algorithm 1. Actually, we get M^* already at the step k^*, however, we need to perform one step more to know that Algorithm 1 stops (because $n_{k^*+1} = n_{k^*}$) but also in order to normalize the system $\Xi_{k^*}^u$ and obtain $\Xi^{u^*} = \Xi_{k^*+1}^u = (E^*, F^*, G^*)$.

(ii) The first equation of (12), that is, $E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)u^*$, which we denote by $\Xi^u|_{M^*}$, has isomorphic solutions with Ξ^u and can be seen as the "internal" dynamics of Ξ^u. Since $E^*(z^*)$ is of full row rank, we may view $\Xi^u|_{M^*}$ as an ODE control system (given by the first equation of (17), see also item (iv) of Remark 3) with two kinds of inputs, namely u^* and w.

(iii) The procedure of internal regularization, leading to Theorem 2(iii), that we propose is not unique at two stages. First, by setting $\bar{u}' = \overline{\gamma}(z^*)$ for any $\overline{\gamma}$ such that $\frac{\partial \overline{\gamma}(z^*)}{\partial z_2^*}$ is invertible, we can find $z_2^* = \gamma(z_1^*)$ satisfying $\overline{\gamma}(z_1^*, \gamma(z_1^*)) = 0$ and thus we constrain the z_2^*-variables via $\hat{u}' = \overline{\gamma}(z^*) = 0$. Second, we can choose $u^* = \gamma^*(z^*)$ arbitrarily and that choice does not affect internal regularity of Ξ^u (nor the invariant submanifold M_α^*) since the feedback law $u^* = \gamma^*(z^*)$ does not influence the constraints $\bar{u}' = \overline{\gamma}(z^*) = 0$.

(iv) A linear DACS $\Xi = (E, H, L)$, given by (2), is internally regularizable/autonomizable (see theorem 3.5 of Reference 19) if and only if

$$\dim(E\mathscr{V}^* + \operatorname{Im} L) \geq \dim \mathscr{V}^*,$$

where $\mathscr{V}^*$ is the limit of the augmented Wong sequence $\mathscr{V}_k$ of (7), which is, clearly, a linear counterpart of M^* (denoted $\mathscr{M}^*$, compare Remark 1(v)). Thus item (ii) of Theorem 2 is a nonlinear generalization of the above result for linear DACSs.

(v) Combining the results of Theorem 1 and Theorem 2, it is seen that if a DACS Ξ^u is ex-fb-equivalent to the **(NF)** or the **(SNF)**, then Ξ^u is internally regularizable if and only if $r_1 + m_1 + m_2 \geq n_1$.

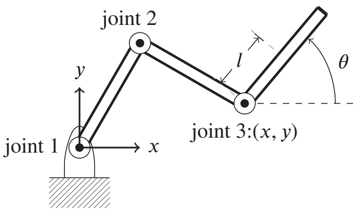


FIGURE 1 A 3-link manipulator with a free joint

5 | EXAMPLES

In this section, we give two examples to illustrate the results of Theorems 1 and 2. In particular, Example 1 shows how to use Algorithm 1 to find a feedback to internally regularize a given DACS, while Example 2 puts emphasis on finding a normal form and demonstrates that an internal regularization feedback can be constructed based on the obtained normal form.

Example 1. Consider the model of a 3-link manipulator taken from Reference 34 as shown in Figure 1, where joint 1 and joint 2 are active, and joint 3 is passive and called a free joint.

The dynamic equations of the system are given by:

$$\begin{cases} m\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta & = F_x \\ m\ddot{y} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta & = F_y \\ -ml\sin\theta\ddot{x} + ml\cos\theta\ddot{y} + ml^2\ddot{\theta} & = \tau_\theta + F_f, \end{cases} \tag{19}$$

where m and l are constants representing the mass and the half length of the free-link, respectively, x and y are the position variables of the free joint, and θ is the angle between the base frame (attached to joint 1) and the link frame, F_x and F_y are the translational force at the free joint, τ_θ is the torque applied to joint 3 (and we take $\tau_\theta = 0$ implying that joint 3 is free), F_f is the friction force caused by the rotation of the free link. We regard F_x and F_y as the active control inputs to the system. Note that the friction F_f is a generalized state variable rather than a control input since it cannot be changed actively. We require the trajectories of system (19) to respect the following constraint:

$$x - y = 0. \tag{20}$$

Denote $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$, $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$ and choose the generalized state $z_0 = (x_1, x_2, y_1, y_2, \theta_1, \theta_2, F_f)$. Rewrite (19) and (20) together as a DACS $\Xi^u_{7,7,2} = (E_1, F_1, G_1)$, given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\theta_1 & 0 & \cos\theta_1 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ F_f/ml \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

Consider Ξ^u around a point $z_{0p} = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, F_{fp})$, where

$$x_{1p} = x_{2p} = y_{1p} = y_{2p} = \theta_{1p} = \theta_{2p} = F_{fp} = 0.$$

We assume that $\theta_1 \neq \pm\frac{\pi}{2}$, so we do not work on $X = S^1 \times \mathbb{R}^6$ but on $U_0 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^6$. We now apply Algorithm 1 to Ξ^u. Step 0: Set $M_0 = X$, $M_0^c = U_0$.

Step 1: We have $\operatorname{rank} E_1(z_0) = 5$ and $\operatorname{rank} \begin{bmatrix} E_1(z_0) & G_1(z_0) \end{bmatrix} = 6$ in the neighborhood $U_1 = (-\frac{\pi}{4}, \frac{\pi}{2}) \times \mathbb{R}^6$ of z_{0p}. Set $Q_1(z_0) = \begin{bmatrix} I_5 & 0 & 0 \\ q(\theta_1) & m & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $q(\theta_1) = \begin{bmatrix} 0 & \sin\theta_1 & 0 & -\cos\theta_1 & 0 \end{bmatrix}$. We get

$$M_1 = \left\{ z_0 \in M_0^c \mid Q_1 F_1(z_0) \in \operatorname{Im} Q_1 E_1(z_0) + \operatorname{Im} Q_1 G_1(z_0) \right\} = \left\{ z_0 \in M_0^c \mid x_1 - y_1 = 0 \right\}.$$

Clearly, $z_{0p} \in M_1$. Then choose a new coordinate $\bar{z}_1 = x_1 - y_1$ and keep the remaining coordinates $z_1 = (x_2, y_1, y_2, \theta_1, \theta_2, F_f)$ unchanged, and set

$$\begin{bmatrix} u_1 \\ \bar{u}_1 \end{bmatrix} = a(z_2) + b(z_2) \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}. \tag{21}$$

It is seen that the DACS Ξ^u is ex-fb-equivalent to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & ml\cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \\ \dot{\bar{z}}_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 + \sec\theta_1 F_f/l \\ \theta_2 \\ 0 \\ \bar{z}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ \tan\theta_1 & -\sec\theta_1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \bar{u}_1 \end{bmatrix}.$$

Thus Ξ^u restricted to $M_1^c = \left\{ z_0 \in U_1 \mid \bar{z}_1 = 0 \right\}$ is $\Xi^u|_{M_1^c} = \Xi_2^{u_1} = (E_2, F_2, G_2)$, given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 + \sec\theta_1 F_f/l \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \tan\theta_1 \\ 0 \end{bmatrix} u_1.$$

Step 2: We have that $\operatorname{rank} E_2(z_1) = \operatorname{rank} \begin{bmatrix} E_2(z_1) & G_2(z_1) \end{bmatrix} = 4$ on $W_2 = U_2 \cap M_1^c$ with $U_2 = U_1$. Set $Q_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}$, we get (recall that $z_1 = (x_2, y_1, y_2, \theta_1, \theta_2, F_f)$)

$$M_2 = \left\{ z_1 \in M_1^c \mid Q_2 F_2(z_1) \in \operatorname{Im} Q_2 E_2(z_1) + \operatorname{Im} Q_2 G_2(z_1) \right\} = \left\{ z_1 \in M_1^c \mid x_2 - y_2 = 0 \right\}.$$

Clearly, $z_{0p} \in M_2$. Set $\bar{z}_2 = x_2 - y_2$ and keep the remaining coordinates unchanged, and the system Ξ_2^u is ex-fb-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 & m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \\ \dot{\bar{z}}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 + \sec\theta_1 F_f/l \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1.$$

So Ξ^u represented in new coordinates and restricted to $M_2^c = \{z_1 \in M_1^c \mid \bar{z}_2 = 0\}$ is $\hat{\Xi}^u|_{M_2^c} = \Xi_3^{u_1} = (E_3, F_3, G_3)$, given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2 \sin\theta_1 + \sec\theta_1 F_f/l \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \end{bmatrix} u_1.$$

Step 3: we have that rank $E_3(z_2) =$ rank $[E_3(z_2) \quad G_3(z_2)] = 4$ on $W_3 = U_3 \cap M_2^c$ with $U_3 = U_2 = U_1$. It follows that $k^* = 2$ and $M^* = M_3^c = M_3 \cap U_3$ is a controlled invariant submanifold since

$$M_3 = \{z_2 \in M_2^c \mid F_3(z_2) \in \operatorname{Im} E_3(z_2) + \operatorname{Im} G_3(z_2)\} = M_2^c.$$

Thus $z_{0p} \in M^*$ is an admissible point. Hence we get $z^* = z_2 = (y_1, y_2, \theta_1, \theta_2, F_f)$ and $\Xi^u|_{M^*} = \Xi_3^{u_1} = (E^*, F^*, G^*)$, where

$$E^*(z^*) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \end{bmatrix}, \quad F^*(z^*) = \begin{bmatrix} y_2 \\ ml\theta_2^2 \sin\theta_1 + \sec\theta_1 F_f/l \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \end{bmatrix}, \quad G^*(z^*) = \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \end{bmatrix}.$$

So $r^* = 4$ and $m^* = 1$, by item (ii) of Theorem 2, $r^* + (m - m^*) = 4 + 1 = n^* = 5$ implies that our system Ξ^u is internally regularizable. A feedback that internally regularizes Ξ^u can be deduced, by item (iii) of Theorem 2, from

$$\bar{u}_1 = F_f \Rightarrow F_f/l + \sin\theta_1 F_x - \cos\theta_1 F_y = F_f.$$

The above equation has a infinite number of solutions. The control $u_1 = u^* = \gamma^*(z^*)$ can be chosen arbitrarily (see the proof of Theorem 2(iii)). So we can chose $\gamma(z^*)$, for instance, to stabilize $\Xi^u|_{M^*}$, that is, $\Xi^{u^*} = (E^*, F^*, G^*)$ on M^* (which can be viewed as an ODE since E^* is of full row rank). Set $\alpha = -b^{-1}a$ and $\beta = b^{-1}$, where a, b are given in (21), and define $\hat{\gamma}(z_0) := \begin{bmatrix} \gamma^*(z^*) \\ F_f \end{bmatrix}$. Then the feedback which internally regularizes and stabilizes Ξ^u can be uniquely solved. The solution is $u = \gamma(z_0) = \alpha(z_0) + \beta(z_0)\hat{\gamma}(z_0)$, that is,

$$u = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma^*(z^*) \\ F_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma^*(z^*) \\ (1 - 1/l)F_f \end{bmatrix}.$$

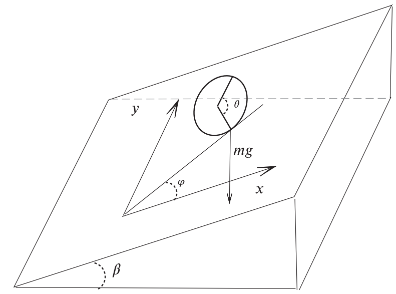
Note that our system Ξ^u satisfies assumptions (A1) and (A3) of Theorem 1 since rank $E(z_0) = 6$ and rank $[E(z_0) \quad G(z_0)] = 6$, and the distribution

$$\mathcal{D}(z_0) = \operatorname{span}\left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2} \right\}$$

satisfies $\mathcal{D}(z_0) = T_{z_0}M^*$ locally for all $z_0 \in M^*$, and $\dim E(z_0)\mathcal{D}(z_0) = 4$, $\dim(E(z_0)\mathcal{D}(z_0) + \operatorname{Im} G(z_0)) = 5$. In fact, Ξ^u is locally ex-fb-equivalent to

$$\textbf{(SNF)}: \begin{bmatrix} I_4 & 0 & 0 & E_1^4(\zeta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \dot{\zeta}_3 \\ \dot{\zeta}_4 \end{bmatrix} = \begin{bmatrix} F_1(\zeta) \\ F_2(\zeta) \\ 0 \\ F_4(\zeta) \end{bmatrix} + \begin{bmatrix} G_1(\zeta) & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

FIGURE 2 A rolling disk on an inclined plane



where $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ (we use ζ for **(SNF)** since z are already used as coordinates of the system obtained via Algorithm 1), $\zeta_1 = (y_1, y_2, \theta_1, \theta_2)$, $\zeta_2 = F_f$, $\zeta_3 = \bar{z}_1 = x_1 - y_1$, $\zeta_4 = \bar{z}_2 = x_2 - y_2$,

$$E_1^4(\zeta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{l(\cos\theta_1+\sin\theta_1)} \end{bmatrix}, F_1(\zeta) = \begin{bmatrix} y_2 \\ \frac{F_f \tan\theta_1}{ml(\cos\theta_1+\sin\theta_1)} + \frac{l\theta_2^2}{\cos\theta_1+\sin\theta_1} \\ \theta_2 \\ \frac{F_f}{ml^2(\cos\theta_1+\sin\theta_1)} - \frac{\theta_2^2(\cos\theta_1-\sin\theta_1)}{\cos\theta_1+\sin\theta_1} \end{bmatrix}, \ G_1(\zeta) = \begin{bmatrix} 0 \\ l \\ 0 \\ \sin\theta_1 - \cos\theta_1 \end{bmatrix}, \ F_2(\zeta) = \bar{z}_2, \ F_4(\zeta) = \bar{z}_1.$$

Note that for the system **(SNF)** represented in the z-coordinates, we have $M^* = \{\zeta \mid \zeta_3 = \zeta_4 = 0\}$. The variables $\zeta_1 = (y_1, y_2, \theta_1, \theta_2)$ and ζ_3 perform as the states of differential equations (there are differential equations for $\dot{\zeta}_1$ and $\dot{\zeta}_3$), but ζ_3 are constrained and equal to 0. Moreover, $\zeta_2 = F_f$ is a truly free variable, ζ_4 is a constrained free variable, and u_1 is a constrained control input.

Example 2. Consider a rolling disk on an inclined plane as shown in Figure 2. We denote the position of the disk by (x, y), the angles θ and φ describe the orientation of the disk with respect to the inclined plane, β is the angle between the horizontal plane and the inclined plane. If there are no external forces acting on the system, the Lagrangian of the system is given by

$$\mathcal{L} = -mgx\sin\beta + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2,$$

where m is the mass, J is the moment of inertia and throughout we assume $m = 1$ and $J = 1$, for simplicity. The following nonholonomic constraints represent the kinematic equations of the system

$$\begin{cases} 0 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi \\ 0 = \dot{x}\cos\varphi + \dot{y}\sin\varphi - \dot{\theta} \end{cases} \tag{22}$$

and we can derive the dynamic equations of the system as

$$\begin{cases} \ddot{x} = -g\sin\beta - \lambda_1\sin\varphi + \lambda_2\cos\varphi \\ \ddot{y} = \lambda_1\cos\varphi + \lambda_2\sin\varphi \\ \ddot{\theta} = -\lambda_2 \\ \ddot{\varphi} = 0, \end{cases} \tag{23}$$

where λ_1 and λ_2 are constraint forces (Lagrange multipliers). We have $X = \mathbb{R}^8 \times \mathbb{T}^2$, where $\mathbb{T}^2 = S^1 \times S^1$, and choose control inputs as (τ_1, τ_2, τ_2), where $\tau_1 = \sin\beta$ (so we control the slope of the plane), and τ_2 and τ_3 are external torques in the directions of $\dot{\theta}$ and $\dot{\varphi}$, respectively. We study the problem whether we can find an input force such that the trajectories of the system, besides fulling constraints (22), respect also the following constraint:

$$\varphi + \beta = \frac{\pi}{2}. \tag{24}$$

To this end, we will transform the system into the normal form **(SNF)**. The constraint (24) is equivalent to $0 = \sin\beta - \cos\varphi$ or to $\tau_1 = \cos\varphi$. Now considering (22), (23), and (24) together with the controls (τ_1, τ_2, τ_3), we get a DACS $\Xi^u_{11,10,3} = (E, F, G)$, given by

$$\Xi^u : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\varphi_1 & 0 & \sin\varphi_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\lambda_1 \sin\varphi_1 + \lambda_2 \cos\varphi_1 \\ y_2 \\ \lambda_1 \cos\varphi_1 + \lambda_2 \sin\varphi_1 \\ \theta_2 \\ -\lambda_2 \\ \varphi_2 \\ 0 \\ 0 \\ 0 \\ -\cos\varphi_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix},$$

where $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$, $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$, $\varphi_1 = \varphi$, $\varphi_2 = \dot{\varphi}$. The generalized state is $\xi = (x, \dot{x}, y, \dot{y}, \varphi, \dot{\varphi}, \theta, \dot{\theta}, \lambda_1, \lambda_2)$. We consider Ξ^u around a point $\xi_p = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, \varphi_{1p}, \varphi_{2p}, \lambda_{1p}, \lambda_{2p}) = 0$. Consider $\varphi \in (-\pi/2, \pi/2)$ (thus $\beta \in (0, \pi)$) and $\theta \in (-\pi/2, \pi/2)$. Applying Algorithm 1 to Ξ^u, we get

$$M_0^c = \mathbb{R}^8 \times (-\pi/2, \pi/2) \times (-\pi/2, \pi/2), \quad M_1^c = \left\{ \xi \in M_0^c \mid x_2 \sin\varphi_1 - y_2 \cos\varphi_1 = -x_2 \cos\varphi_1 - y_2 \sin\varphi_1 + \theta_2 = 0 \right\},$$
$$M_2^c = \left\{ \xi \in M_1^c \mid \varphi_2\theta_2 - \lambda_1 - \tfrac{g}{2} \sin 2\varphi_1 = 0 \right\}, \quad M^* = M_3^c = M_2^c.$$

It follows that $\xi_p \in M^*$ and that locally around ξ_p

$$\operatorname{rank} E(\xi) = r = 8, \quad \operatorname{rank} \begin{bmatrix} E(\xi) & G(\xi) \end{bmatrix} = r + m_2 = 9.$$

The distribution $\mathcal{D}(\xi) = \operatorname{span}\{g_1, g_2, g_3, g_4\} + \operatorname{span}\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \lambda_2} \right\}$, where

$$g_1 = \cos\varphi_1 \frac{\partial}{\partial x_2} + \sin\varphi_1 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial \theta_2}, \qquad g_2 = -y_2\varphi_2 \frac{\partial}{\partial x_2} + x_2\varphi_2 \frac{\partial}{\partial y_2} + g \cos 2\varphi_1 \frac{\partial}{\partial \theta_1},$$
$$g_3 = -\theta_1 \frac{\partial}{\partial \theta_1} + \varphi_2 \frac{\partial}{\partial \varphi_2}, \qquad g_4 = \varphi_2 \frac{\partial}{\partial \lambda_1} + \frac{\partial}{\partial \theta_1},$$

satisfies $\mathcal{D}(\xi) = T_\xi M^*$, locally for all $\xi \in M^*$, and

$$\dim E(\xi)\mathcal{D}(\xi) = r_1 = 6, \quad \dim(E(\xi)\mathcal{D}(\xi) + \operatorname{Im} G(\xi)) = r_1 + m_1 + m_2 = 8.$$

Thus assumptions (A1) and (A3) of Theorem 1 are satisfied. Now set

$$z_1 = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ \theta_1 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}, \; z_2 = \lambda_2, \; z_3 = \begin{bmatrix} \tilde{y}_2 \\ \tilde{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_2 \sin\varphi_1 - y_2 \cos\varphi_1 \\ -x_2 \cos\varphi_1 - y_2 \sin\varphi_1 + \theta_2 \end{bmatrix}, \; z_4 = \tilde{\lambda}_1 = \theta_2\varphi_2 - \lambda_1 - \frac{g}{2} \sin 2\varphi,$$

$$
Q(\xi)=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
\sin\varphi_1 & 0 & -\cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-\cos\varphi_1 & 0 & -\sin\varphi_1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},\quad
\alpha(\xi)=\begin{bmatrix}\cos\varphi_1\\ \varphi_2\tilde{y}_2-2\lambda_2\\ 0\end{bmatrix},\quad
\beta(\xi)=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.
$$

Then, via $Q(\xi)$, $\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}=\alpha(\xi)+\beta(\xi)\begin{bmatrix}u_1^1\\ u_1^2\\ u_2\end{bmatrix}$ and $z=(z_1,z_2,z_3,z_4)$, Ξ^u is, locally around ξ_p, ex-fb-equivalent to

$$
\textbf{(SNF)}:\begin{bmatrix}I_6&0&0&0\\0&0&I_2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix}=\begin{bmatrix}F_1(z)\\ F_2(z)\\ 0\\ F_4(z)\end{bmatrix}+\begin{bmatrix}G_1^1(z)&0&0\\0&G_2^2(z)&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}u_1^1\\ u_1^2\\ u_2\end{bmatrix},
$$

where

$$
F_1(z)=\begin{bmatrix}x_2\\ (\tilde{\lambda}_1+\frac{g}{2}\sin 2\varphi_1-\theta_2\varphi_2)\sin\varphi_1+(\lambda_2-g)\cos\varphi_1\\ x_2\tan\varphi_1-\frac{\tilde{y}_2}{\cos\varphi_1}\\ \tilde{\theta}_2-\tilde{y}_2\tan\varphi_1+\frac{x_2}{\cos\varphi_1}\\ \varphi_2\\ 0\end{bmatrix},\quad
F_2(z)=\begin{bmatrix}\tilde{\lambda}_1\\0\end{bmatrix},\quad
F_4(z)=\begin{bmatrix}\tilde{y}_2\\ \tilde{\theta}_2\end{bmatrix},\quad
G_1^1(z)=\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix},\quad
G_2^2(z)=\begin{bmatrix}0\\1\end{bmatrix}.
$$

Therefore, by item (v) of Remark 3, the free variables are $z_2=\lambda_2$ and $z_4=\tilde{\lambda}_1$. The variables z_2 and $u_1^1=\tau_3$ are truly free variables and they are *not* constrained by any constraints. The generalized states $z_3=(\tilde{y}_2,\tilde{\theta}_2)$, $z_4=\tilde{\lambda}_1$ and the controls (u_1^2,u_2) are constrained and required to be 0 by the algebraic constraints. In fact, $u_2=0$ assures the constraint $\tau_1=\cos\varphi_1$, while $\tilde{\lambda}_1=0$ and $u_1^2=2\lambda_2+\varphi_2\tilde{y}_2=0$ assure nonholonomic constraints (22). Moreover, by item (iv) of Remark 3, we have

$$
\Xi^u|_{M^*}:\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{y}_1\\ \dot{\theta}_1\\ \dot{\varphi}_1\\ \dot{\varphi}_2\end{bmatrix}=\begin{bmatrix}x_2\\ (\frac{g}{2}\sin 2\varphi_1-\theta_2\varphi_2)\sin\varphi_1+(\lambda_2-g)\cos\varphi_1\\ x_2\tan\varphi_1\\ \frac{x_2}{\cos\varphi_1}\\ \varphi_2\\ 0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}u_1^1,
$$

which is an ODE control system with one free variable λ_2 and one control $u_1^1=\tau_3$. We can see from item (i) of Theorem 2 that $\Xi^u|_{M^*}$ has isomorphic trajectories with those of Ξ^u. Moreover, since $\dim(E(\xi)T_\xi M^*+\mathrm{Im}\ G(\xi))=8>\dim M^*=7$, by item (ii) of Theorem 2, the system Ξ^u is internally regularizable, for example, a feedback which internally regularizes the system is given by $u_2=\lambda_2=0$ (implying that $\tau_1=\cos\varphi_1+u_2=\cos\varphi_1+\lambda_2$ assures the constraint (24) on $\{\lambda_2=0\}$), $u_1^2=0$ (implying $\lambda_2=0$ on $\{\tilde{y}_2=0\}$) and $\tau_3=u_1^1=\gamma^*(z_1)$, where $z_1=(x_1,x_2,y_1,\theta_1,\varphi_1,\varphi_2)$, for some smooth function γ^*. Notice that the choice of $\gamma^*(z_1)$ is made in order to reach designed control properties of the system $\Xi^u|_{M^*}$, like stabilizability (note that by substituting $\lambda_2=u_2=0$, the system $\Xi^u|_{M^*}$ becomes a single-input ODE control system).

6 | CONCLUSIONS

In this article, we proposed two normal forms for nonlinear DACSs under external feedback equivalence to simplify the structure of systems and to clarify different roles of variables, which is our first main result. One normal form requires only the existence of a maximal controlled invariant submanifold and some constant rank assumptions of system matrices while another requires additionally involutivity of a certain distribution. Moreover, we give a necessary and sufficient geometric condition for a nonlinear DACS to be internally regularizable (second main result), we also formulate an algorithm to calculate the maximal invariant submanifold and a feedback which internally regularizes the system. Two examples of mechanical systems are given to illustrate the proposed normal forms and the internal regularization algorithm.

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CONFLICT OF INTEREST

The author(s) declared that there is no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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APPENDIX

In the appendix, we illustrate the internal regularization algorithm for nonlinear DACSs and give some remarks on the algorithm.

Remark 6.

(i) Our Algorithm 1 is related to the geometric reduction method used in section 3.4 of Reference 6. In both, one constructs a sequence of submanifolds recursively and then reduces/restricts the DACS to the constructed submanifolds. The main difference is that Algorithm 1 deals with DAEs with an extra control u, that is, DACSs, while in Reference 6 only DAEs are discussed and no feedback transformations are involved. Moreover, we relate our Algorithm 1 with the recursive procedure given before Proposition 2. Actually, Step k of Algorithm 1 provides an explicit construction of the manifolds M_k^c of the procedure.

(ii) The dimensions r_k, n_k, m_k satisfy

$$\begin{cases} r_0 \geq \cdots \geq r_k \geq \cdots \geq 0, & n_0 \geq \cdots \geq n_k \geq \cdots \geq 0, \quad m_0 \geq \cdots \geq m_k \geq \cdots \geq 0, \\ n_{k-1} \geq r_k, & r_{k-1} - r_k - (m_{k-1} - m_k) \geq n_{k-1} - n_k. \end{cases}$$

The integers r_k, n_k, m_k indicate the values of $\dim E(x)T_xM_k$, $\dim M_k$, and that the vector u_k is m_k-dimensional, respectively, and illustrate well the evolution of the reduction procedure.

(iii) Assumption 1 that rank $\tilde{E}_k(z_{k-1}) = const.$ and rank $[\tilde{E}_k(z_{k-1}), \tilde{G}(z_{k-1})] = const.$ is made to produce the full row rank matrices $\tilde{E}_k^1$ and $\tilde{G}_k^2$ and the zero-level set $M_k = \left\{ z_{k-1} \in W_k \mid \tilde{F}_k^3(z_{k-1}) = 0 \right\}$. Assumption 2 that rank $\mathrm{D}\tilde{F}_k^3(z_{k-1}) = const.$ makes it possible to use the components of $\tilde{F}_k^3$ with linearly independent differentials as a part of new local coordinates. Those two assumptions are somewhat related to but different from the two assumptions in Reference 6, for example, in order to produce a smooth embedded submanifold, the author of Reference 6 assumes that $H_k(x, \dot{x}) = \tilde{E}_k(x)\dot{x} - \tilde{F}_k(x)$ is a submersion.

Algorithm 1. Internal regularization algorithm for nonlinear DACSs

Initialization: Consider $\Xi^u_{l,n.m} = (E, F, G)$, fix $x_p \in X$ and let $U_0 \subseteq X$ be an open connected subset containing x_p. Below all sets U_k are open in X and W_k are open in M^c_{k-1}.

Step 0: Set $z_0 = x$, $u_0 = u$, $E_0(z_0) = E(x)$, $F_0(z_0) = F(x)$, $G_0(z_0) = G(x)$, $M_0 = X$, $M^c_0 = U_0$, $r_0 = l$, $n_0 = n$, $m_0 = m$.

Step k**:**

1: Suppose that we have defined at Step $k-1$: an open neighborhood $U_{k-1} \subseteq X$ of x_p, a smooth embedded connected submanifold M^c_{k-1} of U_{k-1} and a DACS $\Xi^u_{k-1} = (E_{k-1}, F_{k-1}, G_{k-1})$ given by smooth matrix-valued maps

$$E_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}\times n_{k-1}}, \quad F_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}}, \quad G_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}\times m_{k-1}},$$

whose arguments are denoted $z_{k-1} \in M^c_{k-1}$.

2: Rename the maps as $\tilde{E}_k = E_{k-1}$, $\tilde{F}_k = F_{k-1}$, $\tilde{G}_k = G_{k-1}$ and define $\tilde{\Xi}^{\tilde{u}}_k := (\tilde{E}_k, \tilde{F}_k, \tilde{G}_k)$.

Assumption 1: There exists an open neighborhood $U_k \subseteq U_{k-1} \subseteq X$ of x_p such that rank $\tilde{E}_k(z_{k-1}) = const. = r_k \leq n_{k-1}$ and rank $[\tilde{E}_k(z_{k-1}), \tilde{G}_k(z_{k-1})] = const. = r_k + m_{k-1} - m_k$, $\forall z_{k-1} \in W_k = U_k \cap M^c_{k-1}$.

3: Find a smooth map $Q_k : W_k \to GL(r_{k-1}, \mathbb{R})$, such that $\tilde{E}^1_k$ and $\tilde{G}^2_k$ of

$$Q_k\tilde{E}_k = \begin{bmatrix} \tilde{E}^1_k \\ 0 \\ 0 \end{bmatrix}, \quad Q_k\tilde{F}_k = \begin{bmatrix} \tilde{F}^1_k \\ \tilde{F}^2_k \\ \tilde{F}^3_k \end{bmatrix}, \quad Q_k\tilde{G}_k = \begin{bmatrix} \tilde{G}^1_k \\ \tilde{G}^2_k \\ 0 \end{bmatrix}$$

are of full row rank, where $\tilde{E}^1_k : W_k \to \mathbb{R}^{r_k\times n_{k-1}}$, $\tilde{G}^2_k : W_k \to \mathbb{R}^{(m_{k-1}-m_k)\times m_{k-1}}$, $\tilde{F}^3_k : W_k \to \mathbb{R}^{r_{k-1}-r_k-m_{k-1}+m_k}$ (so all the matrices depend on z_{k-1}).

4: Following (5), define $M_k = \left\{ z_{k-1} \in W_k \mid \tilde{F}^3_k(z_{k-1}) = 0 \right\}$.

Assumption 2: $x_p \in M_k$ and rank $\mathrm{D}\tilde{F}^3_k(z_{k-1}) = const. = n_{k-1} - n_k$ for $z_{k-1} \in M_k \cap U_k$, by taking a smaller U_k (if necessary).

5: By Assumption 2, $M_k \cap U_k$ is a smooth embedded submanifold and by taking again a smaller U_k, we may assume that $M^c_k = M_k \cap U_k$ is connected and choose new coordinates $(\bar{z}_k, z_k) = \psi_k(z_{k-1})$ on W_k, where $\bar{z}_k = (\bar{\varphi}^1_k(z_{k-1}), \dots, \bar{\varphi}^{n_{k-1}-n_k}_k(z_{k-1}))$, with $\mathrm{d}\bar{\varphi}^1_k(z_{k-1}), \dots, \mathrm{d}\bar{\varphi}^{n_{k-1}-n_k}_k(z_{k-1})$ being all independent rows of $\mathrm{D}\tilde{F}^3_k(z_{k-1})$, and $z_k = (\varphi^1_k(z_{k-1}), \dots, \varphi^{n_k}_k(z_{k-1}))$ are any complementary coordinates such that ψ_k is a local diffeomorphism.

6: Choose new control inputs $\begin{bmatrix} u_k \\ \bar{u}_k \end{bmatrix} = a_k(z_{k-1}) + b_k(z_{k-1})u_{k-1}$, where$a_k = \begin{bmatrix} 0 \\ \tilde{F}^2_k \end{bmatrix}$, $b_k = \begin{bmatrix} \tilde{b}_k \\ \tilde{G}^2_k \end{bmatrix}$, and where $\tilde{b}_k : W_k \to \mathbb{R}^{m_{k-1}\times m_k}$ is chosen such that $b_k(z_{k-1})$ is invertible $\forall z_{k-1} \in W_k$ (by taking again a smaller U_k, if necessary).

7: Set $\hat{E}_k = Q_k\tilde{E}_k \left(\frac{\partial \psi_k}{\partial z_{k-1}} \right)^{-1}$, $\hat{F}_k = Q_k(\tilde{F}_k + G_k\alpha_k)$, $\hat{G}_k = Q_k\tilde{G}_k\beta_k$, $\alpha_k = -b_k^{-1}a_k$ and $\beta_k = b_k^{-1}$.

8: By Definition 3, $\tilde{\Xi}^{\tilde{u}}_k \overset{ex\text{-}fb}{\sim} \hat{\Xi}^{\hat{u}}_k = (\hat{E}_k, \hat{F}_k, \hat{G}_k)$ via Q_k, ψ_k, and (α_k, β_k), where

$$\hat{\Xi}^{\hat{u}}_k : \begin{bmatrix} \hat{E}^1_k(z_k, \bar{z}_k) & \bar{E}^1_k(z_k, \bar{z}_k) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_k \\ \dot{\bar{z}}_k \end{bmatrix} = \begin{bmatrix} \hat{F}^1_k(z_k, \bar{z}_k) \\ 0 \\ \hat{F}^3_k(z_k, \bar{z}_k) \end{bmatrix} + \begin{bmatrix} \hat{G}^1_k(z_k, \bar{z}_k) & \bar{G}^1_k(z_k, \bar{z}_k) \\ 0 & I_{m_{k-1}-m_k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ \bar{u}_k \end{bmatrix}, \tag{A1}$$

with $\hat{E}^1_k : W_k \to \mathbb{R}^{r_k\times n_k}$, $\hat{F}^1_k : W_k \to \mathbb{R}^{r_k}$, $\hat{G}^1_k : W_k \to \mathbb{R}^{r_k\times m_k}$, and $\left[\hat{E}^1_k \circ\psi_k \quad \bar{E}^1_k \circ\psi_k \right] = \tilde{E}^1_k \left(\frac{\partial \psi_k}{\partial z_{k-1}} \right)^{-1}$, $\hat{F}^1_k \circ\psi_k = \tilde{F}^1_k\alpha_k$, $\hat{F}^3_k \circ\psi_k = \tilde{F}^3_k$ and $\left[\hat{G}^1_k \circ\psi_k \quad \bar{G}^1_k \circ\psi_k \right] = \tilde{G}^1_k\beta_k$.

9: Set $\bar{z}_k = 0$ and $\bar{u}_k = 0$ to define the restricted DACS on $M^c_k = \{z_{k-1} \in W_k \mid \bar{z}_k = 0\}$ as

$$\hat{\Xi}^{\hat{u}}_k|_{M^c_k} : \hat{E}^1_k(z_k, 0)\dot{z}_k = \hat{F}^1_k(z_k, 0) + \hat{G}^1_k(z_k, 0)u_k. \tag{A2}$$

10: On M^c_k, define a system

$$\Xi^u_k : E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)u_k,$$

where $E_k(z_k) = \hat{E}^1_k(z_k, 0)$, $F_k(z_k) = \hat{F}^1_k(z_k, 0)$, $G_k(z_k) = \hat{G}^1_k(z_k, 0)$ are matrix-valued maps and $E_k : M^c_k \to \mathbb{R}^{r_k\times n_k}$, $F_k : M^c_k \to \mathbb{R}^{r_k}$, $G_k : M^c_k \to \mathbb{R}^{r_k\times m_k}$.

Repeat: Step k for $k = 1, 2, 3, \dots$, **until** $n_{k+1} = n_k$, set $k^* = k$.

Result: Set $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$, $m^* = m_{k^*+1}$, $M^* = M^c_{k^*+1}$, $U^* = U_{k^*+1}$, $z^* = z_{k^*+1} = z_{k^*}$, $u^* = u_{k^*+1}$ and $\Xi^{u^*} = (E^*, F^*, G^*)$ with $E^* = E_{k^*+1}$, $F^* = F_{k^*+1}$, $G^* = G_{k^*+1}$.