Refining dichotomy convergence in vector-field guided path-following control

Yao, Weijia; Lin, Bohuan; Anderson, Brian D. O.; Cao, Ming

Published in:
European Control Conference (ECC)

DOI:
10.23919/ECC54610.2021.9654862

Refining dichotomy convergence in vector-field guided path-following control

Weijia Yao, Bohuan Lin, Brian D. O. Anderson, Ming Cao

Abstract— **In the vector-field guided path-following problem, the desired path is described by the zero-level set of a sufficiently smooth real-valued function and to follow this path, a (guiding) vector field is designed, which is not the gradient of any potential function. The value of the aforementioned real-valued function at any point in the ambient space is called the *level value* at this point. Under some broad conditions, a dichotomy convergence property has been proved in the literature: the integral curves of the vector field converge either to the desired path or the singular set, where the vector field attains a zero vector. In this paper, the property is further developed in two respects. We first show that the vanishing of the level value does not necessarily imply the convergence of a trajectory to the zero-level set, while additional conditions or assumptions identified in the paper are needed to make this implication hold. The second contribution is to show that under the condition of real-analyticity of the function whose zero-level set defines the desired path, the convergence to the singular set (assuming it is compact) implies the convergence to a single point of the set, dependent on the initial condition, i.e. limit cycles are precluded. These results, although obtained in the context of the vector-field guided path-following problem, are widely applicable in many control problems, where the desired sets to converge to (particularly, a singleton constituting a desired equilibrium point) form a zero-level set of a Lyapunov(-like) function, and the system is not necessarily a gradient system.**

I. Introduction

Although equilibrium points of a dynamical system have often been the subject of study in the control literature, it is important to recognize that the convergence of trajectories of a dynamical system to a *closed invariant set* is also of intense research interest in many control problems, which include the geometric path-following problem [1]–[3], the formation maneuvering problem [4] and the synchronization problem [5]. Note in particular that in the path-following problem, the trajectories of a system are required to converge to and traverse along a desired path, which is usually a geometric object such as a closed curve rather than an equilibrium point [6].

The closed invariant set can sometimes be described by the zero-level set of a continuous real-valued non-negative function, such as a Lyapunov(-like) function [7] or (the norm of) an error signal, while convergence of trajectories to the set is usually characterized by the distance of points on a trajectory to the set with respect to a metric (e.g., the Euclidean metric) [8]–[11]. For convenience, such a continuous non-negative function is referred to as the *level function* and its value at a point is called the point's *level value*. Therefore, one natural idea is to use the level value, instead of the distance to the set, along a system trajectory to characterize the convergence to the zero-level set. This idea is utilized in vector-field guided path-following algorithms [1], [3], [11]–[13], and in some applications of Barbalat's lemma (e.g., [8, Lemma 8.2, Theorem 8.4], [1, Theorem 1], [3, Theorem 1]). Now a central set-theoretic issue is whether the vanishing of the level value entails the convergence to the zero-level set of the level function: as clarified with examples later, a trajectory might diverge to infinity and the associated level value could still converge to zero. An associated issue arises from the fact that convergence with respect to a topology is a stronger notion than that with respect to a metric, while the former is relatively less studied in the control literature. This stronger notion is especially needed when a system evolves on some topological space rather than a Euclidean space, or when there are different metrics in a metric space but a metric-independent convergence result is required.

A quite separate issue arising with the dichotomy convergence property associated with path-following algorithms is that generally, convergence (e.g., with respect to a metric) to a closed invariant set does not automatically imply the convergence to a single point of the set, but it is known that this implication is true under some conditions for gradient flows [14], [15], while it is not yet completely clear for non-gradient flows. In particular, the guiding vector fields for path-following designed in [3], [11], [12], [16]–[19] are not gradients of any potential functions, but as shown in [11], [12], [16], under some conditions, the integral curves of the vector fields (i.e., the trajectories of the autonomous differential equation where the right-hand side is the vector field) have the dichotomy convergence property: they either converge to the desired path or the singular set, where the vector field attains a zero vector. As the desired path is a limit cycle (when the desired path is homeomorphic to the unit circle), it is obvious that trajectories do not converge to a single point in the desired path, but it is to this point unresolved whether trajectories converging to the singular set will converge to a single point in the singular set (where, in general, the point depends on the initial condition).

Contributions: In this paper, we discuss the two set-theoretic issues mentioned above. The first is related to the

Weijia Yao and Ming Cao are with the Engineering and Technology Institute Groningen (ENTEG), University of Groningen, the Netherlands. {w.yao,m.cao}@rug.nl. Bohuan Lin is with the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence (BI), University of Groningen, the Netherlands. b.lin@rug.nl. Brian D. O. Anderson is with the School of Engineering, Australian National University, Canberra, Australia. brian.anderson@anu.edu.au. The work of Cao was supported in part by the European Research Council (ERC-CoG-771687) and the Netherlands Organization for Scientific Research (NWO-vidi-14134). The work of Yao and Lin was supported in part by the China Scholarship Council.

relationship between vanishing of the level value and the convergence of trajectories to the zero-level set. This issue is motivated by but independent of the vector-field guided path-following scenario. We show that as the level value evaluated at an infinite sequence of points (which is more general than a continuous trajectory) converges to zero, this sequence might not converge to the (possibly compact) zero-level set in the Euclidean space. Specifically, we prove that the sequence converges (with respect to a topology) to the union of the zero-level set and infinity. This result is of interest in many control problems where the desired set forms the zero-level set of a Lyapunov(-like) function or (the norm of) an error signal. Additional conditions or assumptions are suggested such that the vanishing of the level value does imply the convergence to the zero-level set, which is the intuitive idea behind many of the results in the literature (e.g., [1], [3], [11], [12]).

The second issue is pertinent to the relationship between convergence of trajectories of a non-gradient system to a set and convergence to a single point of the set. Under the condition of real analyticity of the level functions, we obtain a refined version of the dichotomy convergence: the convergence to the singular set entails the convergence to a single point of the set. This result not only is relevant to the specialized path-following problem, but also extends the results in [14], [15] (using proof techniques suggested by those works and appealing to the Łojasiewicz inequality [20]) to some non-gradient flows.

The rest of the paper is organized as follows. Section II introduces the vector-field guided path-following problem and raises two set-theoretic questions. Then the main results are presented in Section III, including different convergence notions and answers to the two questions raised in Section II. Finally, Section IV concludes the paper.

II. Background and Problem Formulation

In the vector field guided path-following problem, the desired path $\mathcal{P}$ is a set-theoretic object in $\mathbb{R}^n$, and it is the intersection of several hyper-surfaces described by the zero-level sets of sufficiently smooth functions [3], [12], [17], [21]–[24]:

$$\mathcal{P} = \{\xi \in \mathbb{R}^n : \phi_i(\xi) = 0, i = 1, \dots, n-1\}, \qquad (1)$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions. Some conditions will subsequently be adopted to ensure that the ϕ_i functions define a genuine path (see Remark 1). Let $f = \|(\phi_1, \dots, \phi_{n-1})\|$, then $\mathcal{P}$ is the zero-level set of f; i.e., $\mathcal{P} = f^{-1}(0)$. For convenience, we call the non-negative real-valued function f the *level function*, and for any point $\xi \in \mathbb{R}^n$, the value $f(\xi)$ is called the *level value* of f at the point ξ. Since $f(\xi) = 0 \iff \big(\phi_1(\xi), \dots, \phi_{n-1}(\xi)\big) = \mathbf{0} \iff \xi \in \mathcal{P}$ for a point $\xi \in \mathbb{R}^n$, one may use $f(\xi) = \|(\phi_1(\xi), \dots, \phi_{n-1}(\xi))\|$ to roughly represent the distance from a point ξ to the desired path $\mathcal{P}$. The following question arises naturally:

Q1. If $f(\xi(t)) = \|(\phi_1(\xi(t)), \dots, \phi_{n-1}(\xi(t)))\| \to 0$ as $t \to \infty$ along a continuous trajectory $\xi(t)$ defined on $[0, \infty)$, which can be an arbitrary continuous function or a trajectory of an autonomous system, is it true that the trajectory $\xi(t)$ will converge to the set $\mathcal{P}$ with respect to a metric or a topology (called *metrical convergence* and *topological* convergence respectively, and to be discussed later)?

Note that this question **Q1** does *not* depend on the path-following setting, but is relevant to any problem where a set is described by the zero-level set of a level function, and the convergence to the set is an indispensable requirement of the problem. However, the second question **Q2** that we will formulate shortly is closely related to the vector field guided path-following problem, as discussed below.

The guiding vector field $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ for path following in the 2D case $\mathbb{R}^2$ is [25]–[27]:

$$\chi(\xi) = E\nabla\phi(\xi) - k\phi(\xi)\nabla\phi(\xi), \qquad (2)$$

where $\nabla\phi$ is the gradient vector of the function ϕ, $E = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is a $90°$ rotation matrix, and $k > 0$ is a constant. In higher dimensions, the vector field $\chi : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathbb{R}^n$ for $n \geq 3$ is studied in [12], [16], [17]. The level function can be defined as $f = \phi^2$. Note that the vector field in (2) is *not* the gradient of any potential function. It consists of two terms: the second term is a weighted sum of the gradients (pointing towards the desired path $\mathcal{P}$) and the first term is *orthogonal* to the second term (and pointing tangentially to the desired path $\mathcal{P}$). The integral curves of the vector field, i.e., the trajectories of the autonomous system described by the differential equation $\dot{\xi}(t) = \chi(\xi(t))$ for $\xi \in \mathbb{R}^n$, converge to the desired path under some conditions, and the desired path $\mathcal{P}$ turns out to be a limit cycle of the aforementioned autonomous system if the desired path is homeomorphic to the unit circle. However, trajectories may also converge to the singular set $\mathcal{C}$ defined as $\mathcal{C} = \{\xi \in \mathbb{R}^n : \chi(\xi) = \mathbf{0}\}$, and its elements are called *singular points*.

Remark 1. In the path-following problem setting, we assume (reasonably) that there are no singular points on the desired path $\mathcal{P}$ (see Assumption 2 later). Namely, the gradients $\nabla\phi_i(\xi)$, $i = 1, \dots, n-1$, are linear independent $\forall \xi \in \mathcal{P}$. Consequently, $\mathcal{P}$ is a regular submanifold in $\mathbb{R}^2$ [28, Corollary 5.14], [12], [13]. The desired path is also assumed to be one-dimensional, and so is homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Note that these assumptions are not required for **Q1**. ◁

The second question **Q2** is:

Q2. It has been known in the literature [16], [17], [25] that under some mild assumptions, the desired path is an asymptotically stable limit cycle when it is homeomorphic to the unit circle, and trajectories "spiral" and converge to the desired path but do not converge to any single point on the desired path. Nevertheless, the answer to the following question is not yet clear: when the trajectories converge to the singular set rather than the desired path, will they converge to a singular point, or might they also "spiral" towards the singular set and not converge to any single point of it?

For **Q1**, one might be inclined to give a positive answer based on intuition, but as shown later, the answer is negative even if the set $\mathcal{P}$ is compact. For **Q2**, we will prove that real analyticity of the level function f is a sufficient condition for trajectories converging to the singular set to actually converge to a single point in this set.

III. Main Results

A. Preliminaries

We first recall some basic concepts [28]–[30]. Suppose $(\mathcal{M}, d)$ is a metric space with a metric d and $\mathcal{A}$ is a subset in $\mathcal{M}$. The distance between a point $p \in \mathcal{M}$ and the set $\mathcal{A}$ is $\operatorname{dist}(p, \mathcal{A}) := \inf\{d(p,q) : q \in \mathcal{A}\}$, and if $\mathcal{A} = \emptyset$, then $\operatorname{dist}(p, \mathcal{A}) = \inf\{\emptyset\} = +\infty$. The distance between two subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ is $\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \inf\{d(a,b) : a \in \mathcal{A}, b \in \mathcal{B}\}$. If we consider the n-dimensional Euclidean space $\mathcal{M} = \mathbb{R}^n$, then we use the Euclidean metric by default unless otherwise mentioned[1]; i.e., $\operatorname{dist}(p, \mathcal{A}) = \inf\{d_{l^2}(p,q) : q \in \mathcal{A}\}$, where $d_{l^2}(p,q) = \|p - q\|$ and $\|\cdot\|$ is the Euclidean norm. An (open) neighborhood of $\mathcal{A} \subseteq \mathcal{M}$ is an open set $\mathcal{U} \subseteq \mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{U}$. An ϵ-neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{A} \subseteq \mathcal{M}$, where $\epsilon > 0$ is a constant, is an open neighborhood of $\mathcal{A}$ defined by $\mathcal{U}_\epsilon := \{p \in \mathcal{M} : \operatorname{dist}(p, \mathcal{A}) < \epsilon\}$. Note that an ϵ-neighborhood is an open neighborhood, but the converse is not necessarily true. In particular, there can exist an open neighborhood $\mathcal{U}$ such that no ϵ-neighborhood $\mathcal{U}_\epsilon$ is a subset of $\mathcal{U}$. For example, let $\mathcal{A}$ be the x-axis in the plane; i.e., $\mathcal{A} = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and choose an open neighborhood of $\mathcal{A}$ as $\mathcal{U} = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, |y| < \exp(-x)\}$ (see Fig. 1). Intuitively, the neighborhood $\mathcal{U}$ is "shrinking" infinitely close to the set $\mathcal{A}$ as x increases. Then there does not exist an $\epsilon > 0$ such that $\mathcal{U}_\epsilon \subseteq \mathcal{U}$. However, as will be shown in Lemma 1, if $\mathcal{A}$ is compact, then (unsurprisingly perhaps) for any open neighborhood of $\mathcal{A}$, there always exists an epsilon neighborhood $\mathcal{U}_\epsilon$ that is a subset of $\mathcal{U}$. The space $\mathcal{M}$ is *locally compact* at $x \in \mathcal{M}$ if there is a compact subspace $\mathcal{N} \subseteq \mathcal{M}$ that contains a neighborhood of x. If $\mathcal{M}$ is locally compact at every point, then $\mathcal{M}$ is said to be locally compact. Since the *one-point compactification* [30, p. 185] of $\mathcal{M}$ is used in the proofs of the subsequent results, we assume throughout the paper that $\mathcal{M}$ is locally compact to ensure the existence of the one-point compactification. This assumption is satisfied if $\mathcal{M}$ is a smooth manifold or a Euclidean space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We use $\mathbb{R}_{\geq 0}$ to denote the non-negative reals, and the notation ":=" means "defined to be".

B. Metrical convergence and topological convergence

We can regard $\mathcal{M}$ as a topological space with the topology induced by its metric d. Suppose a set $\mathcal{A} \subseteq \mathcal{M}$, called the *desired set*, is a level set of a function $g : \mathcal{M} \to \mathbb{R}^n$; that is, $\mathcal{A} = g^{-1}(\boldsymbol{c})$ for some constant $\boldsymbol{c} \in \mathbb{R}^n$. We can define a (non-negative) *level function* $e(\cdot) = \|g(\cdot) - \boldsymbol{c}\|$, where $\|\cdot\| = \sqrt{d(\cdot,\cdot)}$, such that $\mathcal{A} = e^{-1}(0)$. Namely, $\mathcal{A}$ is the zero-level set of the level function e. Therefore, every point in the desired set $\mathcal{A}$ renders the level value $e = 0$. When we consider convergence to a set, it is important to clarify if this convergence is with respect to a metric or a topology,

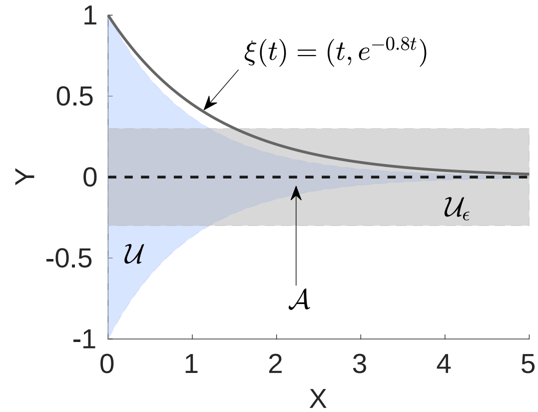


Fig. 1. The non-compact desired set $\mathcal{A} \subseteq \mathbb{R}^2$ is the x-axis, $\mathcal{U} = \{(x,y) \in \mathbb{R}^2 : |y| < \exp(-x)\}$ is an open neighborhood of $\mathcal{A}$ and $\mathcal{U}_\epsilon$ is an ϵ-neighborhood of $\mathcal{A}$ for $\epsilon = 0.3$. It is obvious that there does not exist an $\epsilon > 0$ such that $\mathcal{U}_\epsilon \subseteq \mathcal{U}$. Also note that the continuous trajectory $\xi(t) = (t, \exp(-0.8t))$ converges metrically but *not* topologically to the desired set $\mathcal{A}$, since $\operatorname{dist}(\xi(t), \mathcal{A}) \to 0$ as $t \to \infty$ but $\xi(t) \notin \mathcal{U}$ for sufficiently large $t > 0$.

which correspond to the notions *metrical convergence* and *topological convergence* respectively defined below.

Definition 1 (Metrical and topological convergence)**.** Consider a metric space $(\mathcal{M}, d)$ and the topology induced by the metric d. Suppose $\mathcal{A} \subseteq \mathcal{M}$ is a closed and non-empty set, and let $(\xi_i)_{i=0}^\infty \in \mathcal{M}$ be an infinite sequence of points. The sequence converges to $\mathcal{A}$ *metrically* if for any $\epsilon > 0$, there exists $I > 0$ such that $\xi_i(i \geq I) \subseteq \mathcal{U}_\epsilon$ (or equivalently, $\operatorname{dist}(\xi_i, \mathcal{A}) \leq \epsilon$ for $i \geq I$), where $\xi_i(i \geq I) := \{\xi_i \in \mathcal{M} : i \geq I\}$. The sequence $(\xi_i)_{i=0}^\infty$ converges to $\mathcal{A}$ *topologically* if for any open neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exists $I' > 0$ such that $\xi_i(i \geq I') \subseteq \mathcal{U}$.

In the sequel, we will clarify the relationship between level value convergence (to a constant), metrical convergence (to a set) and topological convergence (to a set). If we consider a Euclidean space, then metrical convergence suffices for many purposes. Indeed, this notion has been used in many, if not most, of the control-related textbooks (e.g., [8]–[10], [32]). However, the notion of topological convergence is more general and is necessary when a topological space is considered, or when there are different metrics to choose but one wants the convergence results to be independent of which metric to use. From the definition, if a trajectory converges topologically to the desired set $\mathcal{A}$, then it also converges to $\mathcal{A}$ metrically, but the converse is not true in general (see Fig. 1). Nevertheless, if the set $\mathcal{A}$ is compact, then metrical convergence also implies topological convergence. To prove this, we first present the following lemma, which is a standard result in topology (see [30, p.177, Exercise 2(d)].

Lemma 1. *Let $\mathcal{A}$ be non-empty and compact in the metric space $(\mathcal{M}, d)$. For any open neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exists an ϵ-neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{A}$, such that $\mathcal{U}_\epsilon \subseteq \mathcal{U}$.*

We can now prove the following proposition.

Proposition 1. *Suppose the desired set $\mathcal{A}$ is non-empty*

[1] Other metrics in $\mathbb{R}^n$ include but not limit to the taxi-cab metric and the sup norm metric [31, Examples 1.1.7, 1.1.9].

and compact. Then an infinite sequence of points converges metrically to the desired set $\mathcal{A}$ if and only if it converges topologically to $\mathcal{A}$.

Proof. Due to the page limit, the proof is omitted. □

In much of the literature, an isolated equilibrium point of a system is studied, often taken as the origin for convenience, and thus in these cases, the desired set $\mathcal{A} = \{\mathbf{0}\}$ is a singleton, which is obviously compact in the Euclidean space $\mathbb{R}^n$. Therefore, metrical convergence automatically implies the stronger notion of topological convergence, and the existing results about convergence can directly be applied to general topological spaces. However, in the study of, e.g., path-following control, the desired set is usually not a singleton. If the desired set is non-compact, then it is necessary to clarify which convergence notions are used[2]. For simplicity, we mostly consider Euclidean space in the sequel (but the notion of topological convergence will still be used wherever this stronger notion is applicable).

Perhaps surprisingly, the convergence of the level value to zero for an infinite sequence of points in $\mathcal{M}$ does *not* imply that the sequence converges (metrically or topologically) to the desired set $\mathcal{A}$. As shown later, the sequence may even *converge to infinity*, even if the desired set $\mathcal{A}$ is compact in $\mathcal{M}$.

Theorem 1. *Define the (closed) set $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is a continuous function and $m \in \mathbb{N}$. If $(\xi_i)_{i=0}^{\infty} \in \mathcal{M}$ is an infinite sequence of points such that $\|\phi(\xi_i)\| \to 0$ as $i \to \infty$, then the sequence converges topologically to the set $\mathcal{B} := \mathcal{A} \cup \{\infty\}$ as $i \to \infty$.*

Proof. Due to the page limit, the proof is omitted. □

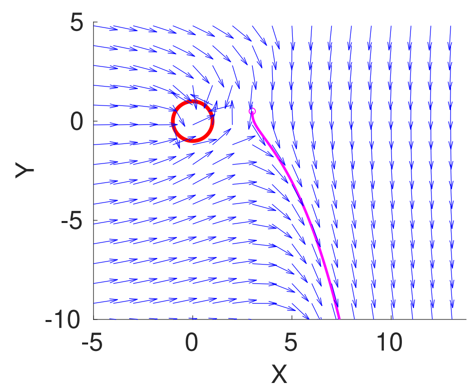
Note that the sequence converging topologically to the set $\mathcal{B} := \mathcal{A} \cup \{\infty\}$ implies four mutually exclusive possibilities: 1) The sequence converges to $\mathcal{A}$; 2) The sequence converges to ∞; 3) The sequence converges to both $\mathcal{A}$ and ∞ (in which case the set $\mathcal{A}$ is unbounded); 4) The sequence converges neither to $\mathcal{A}$ nor ∞. The fourth case happens if the sequence has a subsequence converging to $\mathcal{A}$ and another subsequence converging to ∞, but the whole sequence is not convergent. However, if the set $\mathcal{A}$ is compact and a continuous trajectory is considered, then only the first two cases are possible, as shown in the following theorem.

Theorem 2. *Define the (closed) set $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is continuous and $m \in \mathbb{N}$. If $\mathcal{A}$ is compact, and $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}$ is continuous and $\|\phi(\xi(t))\| \to 0$ as $t \to \infty$, then $\xi(t)$ converges topologically to the set $\mathcal{A}$ or to ∞ exclusively as $t \to \infty$.*

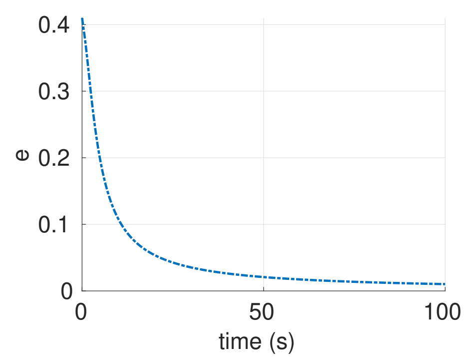
Proof. Due to the page limit, the proof is omitted. □

Note that Theorem 1 is independent of whether the desired set $\mathcal{A}$ is compact or not, and it does *not* depend on the path-following setting either, but for convenience, we use path-following examples to illustrate the result of convergence to ∞ permitted in Theorem 1. One example is presented in [12, Section IV.B] where the desired set $\mathcal{A}$ (i.e., the desired path $\mathcal{P}$) is non-compact and a trajectory converges to infinity even when the level value converges to 0. A perhaps more surprising example is when the desired set $\mathcal{A}$ is a compact set as in the following example.

Example 1. Suppose the desired set $\mathcal{A}$ (i.e., the desired path $\mathcal{P}$) is a unit circle, which is obviously compact. The ϕ function to describe the desired set $\mathcal{A} = \mathcal{P}$ is chosen as $\phi(x, y) = (x^2+y^2-1)\exp(-x)$ in (1), and the vector field is constructed as in (2). As illustrated in Fig. 2, even though the level value $e = \phi$ converges to 0, a trajectory may not converge to the circle but rather escape to infinity. This undesirable behavior does not appear if $\exp(-x)$ is removed from ϕ. See Remarks 3 and 4 for a "good" choice of ϕ. ◁



(a)



(b)

Fig. 2. The desired set $\mathcal{A}$ is a unit circle illustrated by a red curve in (a), and in this subfigure, the arrows represent the *normalized* vector field computed by (2). Although the level value $e = \phi = (x^2+y^2-1)\exp(-x)$ converges to 0 in (b), the trajectory given by the magenta curve in (a) escapes to infinity.

Remark 2. Besides the theoretical interest in its own right, the importance of Theorem 1 is also due to its close relevance to many control problems where an error signal $e : \mathbb{R}^n \to \mathbb{R}^m$ is defined and the system's desired states correspond to $\|e\| = 0$; namely, if $f(x) = \|e(x)\|$, then the system's desired states form the zero-level set $f^{-1}(0)$. Often, a Lyapunov or Lyapunov-like function V which takes the error signal as the argument is involved, and a typical case is the quadratic

[2] One can similarly define stability with respect to a metric or a topology, but the development of these notions is omitted here.

form $V(e) = e^\top P e$, where $P \in \mathbb{R}^{m\times m}$ is a positive definite matrix. Therefore, the desired states (e.g., an equilibrium of the system) form the zero-level set $V^{-1}(0)$ of V. In general, as shown by Theorem 1, the Lyapunov function value $V \to 0 \implies \|e\| \to 0$ along the system trajectory does not necessarily mean that the trajectory will converge to the desired states $V^{-1}(0) = e^{-1}(\mathbf{0})$, since the trajectory might also diverge to infinity. Nevertheless, as shown in many control textbooks (e.g., [8], [32]), the desired state is often an equilibrium point (i.e., $V^{-1}(0) = e^{-1}(\mathbf{0}) = \{\mathbf{0}\}$), and extra detailed analysis (e.g., [8, Theorem 4.1]) guarantees that once a trajectory starts close enough to the equilibrium point, the trajectory will stay in a compact set containing the equilibrium point, and thus the possibility of divergence to infinity is excluded. However, if the desired states form a non-compact set, then it is more involved to exclude this divergence possibility, or extra assumptions are necessary.

Theorem 1 is also relevant when the desired set convergence is proved by using Barbalat's lemma (e.g., [8, Lemma 8.2], [33, Lemma 4.2]); Take Theorem 8.4 in [8] as an example, which is an invariance-like theorem for non-autonomous systems. This theorem states that under some conditions, we have $W(x(t)) \to 0$ and hence $x(t) \to W^{-1}(0)$, where $x(t)$ is a trajectory of a non-autonomous system $\dot{x}(t) = f(t,x)$ and $W(\cdot)$ is a continuous positive semidefinite function. This does not contradict Theorem 1 as the assumptions in Theorem 8.4 in [8] guarantee that the trajectory $x(t)$ is bounded. $\triangleleft$

Remark 3. Theorem 1 gives a negative answer to **Q1**. If the desired set $\mathcal{A}$ is compact, to exclude the possibility of trajectories escaping to infinity such that $\|\phi(\xi_i)\| \to 0$ implies topological convergence to $\mathcal{A}$, one may retreat to one of the following two ways:

1) Prove that trajectories are bounded. For example, one can find a Lyapunov-like function V and a compact set $\Omega_\alpha := \{x : V(x) \le \alpha\}$, and prove that $\dot{V} \le 0$ in this compact set Ω_α. One might also retreat to the LaSalle's invariance principle [8, Theorem 4.4].

2) Modify $\phi(\cdot)$, if feasible, such that $\|\phi(x)\|$ tends to a non-zero constant (possibly infinity) as $\|x\|$ tends to infinity. In other words, $\phi(\cdot)$ is modified to be *radially non-vanishing*.

Furthermore, regardless of whether the desired set $\mathcal{A}$ is compact or not, one could impose the verifiable assumption introduced in Lemma 2 below. $\triangleleft$

C. Convergence characterized by different level functions

The following result is a generalization of [17, Lemma 5].

Lemma 2. *Suppose there are two non-negative continuous functions $M_i : \mathcal{M} \to \mathbb{R}_{\ge 0}$, $i = 1, 2$. If for any given constant $\kappa > 0$, it holds that*

$$\inf\{M_1(p) : p \in \mathcal{M}, M_2(p) \ge \kappa\} > 0, \tag{3}$$

then there holds

$$\lim_{k\to\infty} M_1(p_k) = 0 \implies \lim_{k\to\infty} M_2(p_k) = 0,$$

where $(p_k)_{k=1}^\infty$ is an infinite sequence of points in $\mathcal{M}$.

Proof. Due to the page limit, the proof is omitted. $\square$

Based on Lemma 2 and Proposition 1, we have the following result as a specialization of Theorem 1.

Corollary 1. *Suppose $\mathcal{A} := \{\xi \in \mathcal{M} : \|\phi(\xi)\| = 0\}$, where $\phi : \mathcal{M} \to \mathbb{R}^m$ is a continuous function. Let $M_1(\cdot) = \|\phi(\cdot)\|$ and $M_2 = \mathrm{dist}(\cdot, \mathcal{A})$ in Lemma 2, and suppose the condition (3) holds. If $(\xi_i)_{i=0}^\infty$ is a sequence of points $\xi_i \in \mathcal{M}$ such that $\|\phi(\xi_i)\| \to 0$ as $i \to \infty$, then the sequence converges metrically to $\mathcal{A}$ (i.e., $\mathrm{dist}(\xi_i, \mathcal{A}) \to 0$). Moreover, if $\mathcal{A}$ is compact, then the convergence is also topological.*

Remark 4. One can verify that the ϕ function in Example 1 does not satisfy the condition in (3) with M_1 and M_2 defined as in Corollary 1, but the condition is met if the ϕ function is changed to $\phi(x,y) = x^2 + y^2 - 1$, and thus Corollary 1 holds. This modification also renders ϕ radially non-vanishing. $\triangleleft$

D. Refined dichotomy convergence

The result in this subsection is related to the vector field defined in (2). According to the discussions above, we first present the following assumption.

Assumption 1. For any constant $\kappa > 0$, there holds $\inf\{|e(\xi)| : \xi \in \mathbb{R}^n, \mathrm{dist}(\xi, \mathcal{P}) \ge \kappa\} > 0$, where $e(\cdot) = \phi(\cdot)$ in (2).

Another natural assumption is that there are no singular points on the desired path.

Assumption 2. There holds $\mathrm{dist}(\mathcal{P}, \mathcal{C}) > 0$.

In this subsection, we show that if a trajectory of $\dot{\xi}(t) = \chi(\xi(t))$ converges to the singular set $\mathcal{C}$, then under some conditions, it converges to a point in $\mathcal{C}$. This result depends on a property of real analytic functions stated below.

Lemma 3 (Łojasiewicz gradient inequality [20])**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighborhood of $\xi^* \in \mathbb{R}^n$. Then there are constants $c > 0$ and $\mu \in [0,1)$ such that $\|\nabla V(\xi)\| \ge c|V(\xi) - V(\xi^*)|^\mu$ in some neighborhood $\mathcal{U}$ of ξ^*.*

Inspired by [14], [15], we have the following result.

Theorem 3 (Refined Dichotomy Convergence)**.** *Let $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined in (2). Then the trajectory of $\dot{\xi}(t) = \chi(\xi(t))$ converges metrically either to the desired path $\mathcal{P}$ or the set $\mathcal{C}$ if the initial path-following error $|e(t_0)|$ is sufficiently small. Moreover, suppose ϕ in (1) is real analytic and the set $\mathcal{C}$ is bounded (hence compact). If a trajectory $\xi(t)$ converges metrically to the set $\mathcal{C}$, then the trajectory converges to a point in $\mathcal{C}$.*

Proof. Due to the page limit, the proof is omitted. $\square$

The same conclusion applies for the n-dimensional vector field in [12], [16], [17], and the proof will be presented in an extended version not subject to the page limit.

Remark 5. It is shown in [14], [15] that single limit-point convergence of a bounded solution of a *gradient flow* cannot be proved in general for smooth but *non-analytic* cost functions, whereas the real analyticity of the cost function can guarantee the single limit-point convergence. Note that these results *cannot* be directly applied here since the vector field in (2) contains an orthogonal term, and thus it is not the gradient of any cost functions. Nevertheless, we reach the same conclusion under the condition regarding the real-analyticity of ϕ. Therefore, Theorem 3 can be regarded as an extension of the results in [14], [15]. $\triangleleft$

IV. Conclusions and future work

This paper is motivated by the recent interest in the vector-field guided path-following control problem, where one important issue is the convergence with respect to a metric or a topology to a compact or non-compact desired set. The desired set is a zero-level set of a non-negative continuous level function. We first show that the convergence of the level value to zero does not necessarily imply the convergence of an infinite sequence of points (which is more general than a continuous trajectory) to the compact or non-compact desired set. This result is closely related to many control problems, where the desired set is the zero-level set of a Lyapunov(-like) function. We then turn to the more specific path-following problem, and give a refined dichotomy convergence result. In particular, we show that real analyticity of the level function leads to the refined conclusion that converging of a trajectory to a singular set implies converging to a point in this set. This is in contrast with the convergence to the desired path, where a trajectory spirals towards the set without converging to any single point of the set. Although the guiding vector field is *not* the gradient of any potential function, this result is consistent with [14], [15] where only gradient flows are considered.

For future work, we are interested in finding out whether Theorem 3 can be applied to guiding vector fields defined on a smooth manifold [13]. We are also interested to find an example where the level function is not real-analytic and a trajectory converging to the desired set does not converge to a point in the desired set to further support Theorem 3.

References

[1] D. A. Lawrence, E. W. Frew, and W. J. Pisano, "Lyapunov vector fields for autonomous unmanned aircraft flight control," *Journal of Guidance, Control, and Dynamics*, vol. 31, no. 5, pp. 1220–1229, 2008.

[2] D. R. Nelson, D. B. Barber, T. W. McLain, and R. W. Beard, "Vector field path following for miniature air vehicles," *IEEE Transactions on Robotics*, vol. 23, no. 3, pp. 519–529, 2007.

[3] V. M. Goncalves, L. C. A. Pimenta, C. A. Maia, B. C. O. Dutra, and G. A. S. Pereira, "Vector fields for robot navigation along time-varying curves in n-dimensions," *IEEE Transactions on Robotics*, vol. 26, no. 4, pp. 647–659, Aug 2010.

[4] M. Deghat, B. D. Anderson, and Z. Lin, "Combined flocking and distance-based shape control of multi-agent formations," *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1824–1837, 2015.

[5] Y. Qin, Y. Kawano, and M. Cao, "Stability of remote synchronization in star networks of kuramoto oscillators," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 5209–5214.

[6] P. B. Sujit, S. Saripalli, and J. B. Sousa, "Unmanned aerial vehicle path following: A survey and analysis of algorithms for fixed-wing unmanned aerial vehicless," *IEEE Control Systems*, vol. 34, no. 1, pp. 42–59, Feb 2014.

[7] A. Iggidr, B. Kalitine, and R. Outbib, "Semidefinite lyapunov functions stability and stabilization," *Mathematics of Control, Signals and Systems*, vol. 9, no. 2, pp. 95–106, 1996.

[8] H. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[9] N. P. Bhatia and G. P. Szegö, *Stability theory of dynamical systems*. Springer Science & Business Media, 2002.

[10] W. Hahn, *Stability of motion*. Springer, 1967, vol. 138.

[11] Y. A. Kapitanyuk, A. V. Proskurnikov, and M. Cao, "A guiding vector-field algorithm for path-following control of nonholonomic mobile robots," *IEEE Transactions on Control Systems Technology*, vol. 26, no. 4, pp. 1372–1385, July 2018.

[12] W. Yao, Y. A. Kapitanyuk, and M. Cao, "Robotic path following in 3D using a guiding vector field," in *IEEE Conference on Decision and Control*, 2018, pp. 4475–4480.

[13] W. Yao, B. Lin, B. D. O. Anderson, and M. Cao, "Topological analysis of vector-field guided path following on manifolds," 2020, submitted. [Online]. Available: http://tiny.cc/yao_topo

[14] P.-A. Absil, R. Mahony, and B. Andrews, "Convergence of the iterates of descent methods for analytic cost functions," *SIAM Journal on Optimization*, vol. 16, no. 2, pp. 531–547, 2005.

[15] P.-A. Absil and K. Kurdyka, "On the stable equilibrium points of gradient systems," *Systems & control letters*, vol. 55, no. 7, pp. 573–577, 2006.

[16] W. Yao, H. G. de Marina, B. Lin, and M. Cao, "Singularity-free guiding vector field for robot navigation," *IEEE Transactions on Robotics*, vol. 37, no. 4, 2021, arXiv 2012.01826.

[17] W. Yao and M. Cao, "Path following control in 3D using a vector field," *Automatica*, vol. 117, p. 108957, 2020.

[18] W. Yao, H. G. de Marina, Z. Sun, and M. Cao, "Distributed coordinated path following using guiding vector fields," in *IEEE International Conference on Robotics and Automation (ICRA)*, 2021, to appear.

[19] W. Yao, H. G. de Marina, and M. Cao, "Vector field guided path following control: Singularity elimination and global convergence," in *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE, 2020, pp. 1543–1549.

[20] S. Łojasiewicz, "Sur la géométrie semi-et sous-analytique," in *Annales de l'institut Fourier*, vol. 43, no. 5, 1993, pp. 1575–1595.

[21] A. M. Rezende, V. M. Gonçalves, G. V. Raffo, and L. C. Pimenta, "Robust fixed-wing UAV guidance with circulating artificial vector fields," in *2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 5892–5899.

[22] M. M. Michałek and T. Gawron, "VFO path following control with guarantees of positionally constrained transients for unicycle-like robots with constrained control input," *Journal of Intelligent & Robotic Systems*, vol. 89, no. 1-2, pp. 191–210, 2018.

[23] A. Morro, A. Sgorbissa, and R. Zaccaria, "Path following for unicycle robots with an arbitrary path curvature," *IEEE Transactions on Robotics*, vol. 27, no. 5, pp. 1016–1023, 2011.

[24] K. D. Do, "Global output-feedback path-following control of unicycle-type mobile robots: A level curve approach," *Robotics and Autonomous Systems*, vol. 74, pp. 229–242, 2015.

[25] Y. A. Kapitanyuk, A. V. Proskurnikov, and M. Cao, "A guiding vector-field algorithm for path-following control of nonholonomic mobile robots," *IEEE Transactions on Control Systems Technology*, vol. PP, no. 99, pp. 1–14, 2017.

[26] W. Yao, B. Lin, and M. Cao, "Integrated path following and collision avoidance using a composite vector field," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 250–255.

[27] W. Yao, B. Lin, B. D. O. Anderson, and M. Cao, "Guiding vector fields for following occluded paths," 2021, submitted. [Online]. Available: http://tiny.cc/yao_colliext

[28] J. Lee, *Introduction to Smooth Manifolds: Second Edition*, ser. Graduate texts in mathematics. Springer, 2015.

[29] ——, *Introduction to topological manifolds*. Springer Science & Business Media, 2010, vol. 202.

[30] J. Munkres, *Topology: Pearson New International Edition*, 2nd ed. Pearson Education Limited, 2013.

[31] T. Tao, *Analysis II*. Springer Singapore, 2016.

[32] S. Sastry, *Nonlinear systems: analysis, stability, and control*. Springer Science & Business Media, 2013, vol. 10.

[33] J.-J. E. Slotine, W. Li *et al.*, *Applied nonlinear control*. Prentice hall Englewood Cliffs, NJ, 1991, vol. 199, no. 1.