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Brief paper

Empirical differential Gramians for nonlinear model reduction[☆]

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ABSTRACT

In this paper, we present an empirical balanced truncation method for nonlinear systems whose input vector fields are constants. First, we define differential reachability and observability Gramians. They are matrix valued functions of the state trajectory (i.e. the initial state and input trajectory), and it is difficult to find them as functions of the initial state and input. The main result of this paper is to show that for a fixed state trajectory, it is possible to compute the values of these Gramians by using impulse and initial state responses of the variational system. Therefore, balanced truncation is doable along the fixed state trajectory without solving nonlinear partial differential equations, differently from conventional nonlinear balancing methods. We further develop an approximation method, which only requires trajectories of the original nonlinear systems.

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1. Introduction

Along with the development of new technologies, control systems are becoming more complex and large-scale. To capture systems' components which are essential for analysis and controller design, model order reduction techniques have been established, see e.g. Antoulas (2005). In systems and control, typical methods are balanced truncation and moment matching (Antoulas, 2005), (Zhou, Doyle, & Glover, 1996, Chapters 7, 8). Both of them have been extended to nonlinear systems (Astolfi, 2010; Besselink, van de Wouw, Scherpen, & Nijmeijer, 2014; Fujimoto & Scherpen, 2005; Ionescu & Astolfi, 2016; Kawano & Scherpen, 2017b; Scherpen, 1993). In contrast to successive theoretical developments, nonlinear model reduction methods still have computational challenges, since they require solutions to nonlinear partial differential equations (PDEs) in general. There are few papers tackling this challenging problem such as Fujimoto and Tsubakino (2008), Kawano, Besselink, Scherpen, and Cao (2020), Newman and Krishnaprasad (2000), Sassano and Astolfi (2014) and Scarciotti and Astolfi (2017). As a data-driven model order reduction method, proper orthogonal decomposition (POD) (Antoulas, 2005; Holmes, Lumley, Berkooz, & Rowley, 2012) is often used in practice. However, POD is mainly proposed for non-control systems.

For linear time-invariant (LTI) systems, POD and balancing are connected based on the fact that the controllability and observability Gramians can be computed by using impulse and initial state responses, respectively. That is, balanced truncation of LTI systems can be performed by using empirical data. Applying linear empirical methods to nonlinear systems has attracted various research interests, see e.g., Condon and Ivanov (2004), Hahn and Edgar (2002a, 2002b), Himpe (2018), Lall, Marsden, and Glavaški (2002) and Willcox and Peraire (2002). Such methods are exploited to reduce the computational complexity of nonlinear controller design such as model predictive control (Choroszucha, Sun, & Butts, 2016; Hahn, Kruger, & Edgar, 2002).

However, these empirical methods have been proposed only around a steady-state because the aforementioned nonlinear balancing method gives the same reduced order model as the linear balancing method at a steady-state. For analysis and control of nonlinear systems, a steady-state is not always important. For instance, in a trajectory tracking control problem, a reduced order model around the trajectory could be useful. As another example, preserving a limit cycle under model order reduction could be interesting to research. To study such problems, it is worth developing empirical nonlinear model reduction methods, which are also applicable around transient states. Recently, a connection between POD and nonlinear controllability functions has been established by Kashima (2016) in a stochastic setting. Empirical nonlinear observability Gramians have also been proposed (Krener & Ide, 2009; Powel & Morgansen, 2015). Nevertheless,

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neither of these two methods deals with both controllability and observability Gramians, and there is no direct connection between these two works.

In this paper, we propose an empirical balancing method for nonlinear systems whose input vector fields are constants by utilizing its variational system. Since the variational system can be viewed as a linear time-varying (LTV) system along the trajectory of the nonlinear system, one can extend the concept of the controllability and observability Gramians of the LTV system (Kawano & Scherpen, 2019a; Verriest & Kailath, 1983). We call these extensions the differential reachability and observability Gramians, respectively. They depend on the state trajectory of the nonlinear system. In general, it is not easy to obtain them as functions of the trajectory. Nevertheless, we show that their values at each fixed trajectory can be computed from the impulse and initial state responses of the variational system along this fixed trajectory. These obtained trajectory-wise Gramians are constant matrices, and thus one can compute balanced coordinates and a reduced order model in a similar manner as in the LTI case.

The proposed empirical balancing method requires the variational system model. For large-scale systems, computing it may be challenging. Therefore, we further develop approximation methods, which do not require the variational model. Since the variational system is a state space representation of the Fréchet derivative of an operator defined by the nonlinear system, we use its discretization approximation. For the observability Gramian, similar approximation methods are found in Krener and Ide (2009), Powel and Morgansen (2015). However, there has been no corresponding controllability Gramian, which has been a bottleneck for developing the corresponding balancing method.

Similar nonlinear balanced realizations are found in flow balancing (Verriest, 2008; Verriest & Gray, 2000, 2004) and in differential balancing (Kawano & Scherpen, 2017b). However, they are not empirical methods and require solutions to nonlinear PDEs, and Kawano and Scherpen (2017b) do not give the concept of a Gramian. A preliminary version of our work is found in Kawano and Scherpen (2017a). In this paper, we further develop the discretization approximation methods of the variational systems. Moreover, we newly propose another differential balancing method for a class of nonlinear systems, which only requires the impulse responses of the variational system.

The remainder of this paper is organized as follows. In Section 2, we provide comprehensive background of linear balanced truncation to help understanding the whole picture of this paper. In Section 3, we define the differential reachability and observability Gramians and then a differentially balanced realization along a trajectory of the system. In Section 4, we show that the value of the differential reachability/observability Gramian can be computed by using the impulse/initial state responses of the variational system. Then, we develop approximation methods, which only require empirical data of the original nonlinear system. In Section 5, we propose another differential balancing method, which is further computationally oriented. In Section 6, an example demonstrates the proposed method. Finally in Section 7, we conclude the paper by summarizing our results.

2. Review of linear empirical balancing

In this section, we summarize the results for balanced truncation of linear time-invariant (LTI) systems (for more details, see, e.g. Antoulas, 2005; Willcox & Peraire, 2002) to help understanding the whole picture of this paper.

Consider the following SISO LTI system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

where $x(t) \in \mathbb{R}^n$ and $u(t), y(t) \in \mathbb{R}$; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C^\top \in \mathbb{R}^n$. Its general solution is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Be^{A(t-\tau)}u(\tau)d\tau. \tag{1}$$

From the general solution, the controllability and observability Gramians are defined by

$$G_c(t_0, t_f) := \int_{t_0}^{t_f} e^{A(t-t_0)}BB^\top e^{A^\top(t-t_0)}dt, \tag{2}$$

$$G_o(t_0, t_f) := \int_{t_0}^{t_f} e^{A^\top(t-t_0)}C^\top Ce^{A(t-t_0)}dt. \tag{3}$$

They both are positive definite for the finite interval $[t_0, t_f]$, $t_f > t_0$ if and only if the system is controllable and observable.

Let us assume that the system is exponentially stable, controllable, and observable. When $t_0 = 0$ and $t_f \to \infty$, it is known that the eigenvalues of the product $G_o(0,\infty)G_c(0,\infty)$ correspond to the Hankel singular values of the linear system. Furthermore, there is a change of coordinates $z = Tx$ such that

$$\begin{aligned} TG_c(0,\infty)T^\top &= T^{-\top}G_o(0,\infty)T^{-1} \\ &= \mathrm{diag}\{\sigma_1, \dots, \sigma_n\},\ \ \sigma_i \ge \sigma_{i+1}, \end{aligned}$$

where $TG_c(0,\infty)T^\top$ and $T^{-\top}G_o(0,\infty)T^{-1}$ are the controllability and observability Gramians in the z-coordinates; see e.g. Antoulas (2005, Lemma 7.3). In these coordinates, the elements z_i are sorted in descending order corresponding to the Hankel singular values σ_i without loss of generality. If $\sigma_i > \sigma_{i+1}$, z_i is more important in capturing the input–output behavior than z_{i+1}. Therefore, a reduced order model is constructed by truncating the state variables corresponding to small Hankel singular values.

It is possible to compute the controllability/observability Gramian based on the impulse/initial state responses. From (1), the impulse response of the linear system is $x_{\mathrm{Imp}}(t) = e^{A(t-t_0)}B$. From (2), one notices that

$$G_c(t_0, t_f) = \int_{t_0}^{t_f} x_{\mathrm{Imp}}(t)x_{\mathrm{Imp}}^\top(t)dt.$$

Next, let $e_i^n \in \mathbb{R}^n$, $i = 1, \dots, n$ denote the standard basis, i.e., whose ith element is 1, and the other elements are zero, and let $y_{\mathrm{Is},i}(t)$ denote the corresponding output response. Then, we have

$$y_{\mathrm{Is}}(t) := \begin{bmatrix} y_{\mathrm{Is},1}(t) & \cdots & y_{\mathrm{Is},n}(t) \end{bmatrix} = Ce^{A(t-t_0)}.$$

Moreover, from (3), one notices that

$$G_c(t_0, t_f) = \int_{t_0}^{t_f} y_{\mathrm{Is}}^\top(t)y_{\mathrm{Is}}(t)dt.$$

Therefore, balanced truncation can be achieved based on empirical data. In this paper, we consider to extend these empirical results to nonlinear systems.

3. Differential balancing along a trajectory

We present an empirical balancing method for a nonlinear system whose input vector fields are constants by using its variational system; the reason considering such a vector field is elaborated in Remark 4.1 later. The proposed empirical balancing method is based on two Gramians, which we call differential reachability and observability Gramians. They can be viewed as extensions of Gramians for linear time-varying (LTV) systems (Kawano & Scherpen, 2019a; Verriest & Kailath, 1983) because the variational system can be viewed as an LTV system along a trajectory of the nonlinear system.

3.1. Preliminaries

Consider the following nonlinear system whose input vector fields are constants:

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + Bu(t), \\ y(t) = h(x(t)), \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and $B \in \mathbb{R}^{n\times m}$. Functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are of class C^2. Let $\varphi_{t-t_0}(x_0, u)$ denote the state trajectory $x(t)$ of the system Σ starting from $x(t_0) = x_0 \in \mathbb{R}^n$ for each choice of $u \in L_2^m[t_0, \infty)$. Note that since f is of class C^2, if u is also of class C^2, then the solution $\varphi_{t-t_0}(x_0, u)$ is a class C^2 function of (t, x_0) as long as it exists. Throughout the paper, we assume that $(\varphi_{t-t_0}(x_0, u), u(t))$ are of class C^2 in a finite time interval $[t_0, t_f]$.

In our method, we use the prolonged system (Cortés, van der Schaft, & Crouch, 2005) of the system Σ, which consists of the original system Σ and its variational system $d\Sigma$ along $x(t) = \varphi_{t-t_0}(x_0, u)$,

$$d\Sigma : \begin{cases} \delta\dot{x}(t) := \dfrac{d\delta x(t)}{dt} = \dfrac{\partial f(\varphi_{t-t_0})}{\partial \varphi_{t-t_0}}\delta x(t) + B\delta u(t), \\ \delta y(t) = \dfrac{\partial h(\varphi_{t-t_0})}{\partial \varphi_{t-t_0}}\delta x(t), \end{cases}$$

where $\delta x(t) \in \mathbb{R}^n$, $\delta u(t) \in \mathbb{R}^m$ and $\delta y(t) \in \mathbb{R}^p$. In the time interval $[t_0, t_f]$, the solution $\delta x(t)$ exists for any bounded input $\delta u(t)$ because the variational system $d\Sigma$ is an LTV system along $\varphi_{t-t_0}(x_0, u)$.

Since the variational system is an LTV system, it is possible to extend the aforementioned linear empirical balancing method to a nonlinear system via the variational system. To this end, we compute the solution $\delta x(t)$ of $d\Sigma$. It follows from the chain rule that

$$\begin{aligned} \frac{d}{dt}\frac{\partial \varphi_{t-\tau}(x_\tau, u)}{\partial x_\tau} &= \frac{\partial}{\partial x_\tau}\frac{d\varphi_{t-\tau}(x_\tau, u)}{dt} \\ &= \frac{\partial f(\varphi_{t-\tau}(x_\tau, u))}{\partial x_\tau} \\ &= \frac{\partial f(\varphi_{t-\tau}(x_\tau, u))}{\partial \varphi_{t-\tau}}\frac{\partial \varphi_{t-\tau}(x_\tau, u)}{\partial x_\tau}, \end{aligned} \tag{4}$$

where in the first equality, the orders of derivatives are commutative because $\varphi_{t-\tau}(x_\tau, u)$ is a class C^2 function of (t, x_τ). Therefore, $\partial\varphi_{t-\tau}(x_\tau, u)/\partial x_\tau$ is the transition matrix of $\partial f(\varphi_{t-\tau})/\partial\varphi_{t-\tau}$ as an LTV system. From the general solution of an LTV system, the solution $\delta x(t)$ to the variational system $d\Sigma$ starting from $\delta x(t_0) = \delta x_0$ with input $\delta u(t)$ along the trajectory $\varphi_{t-t_0}(x_0, u)$ is obtained as

$$\delta x(t) = \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x}\delta x_0 + \int_{t_0}^{t}\frac{\partial \varphi_{t-\tau}(x(\tau), u)}{\partial x}B\delta u(\tau)d\tau. \tag{5}$$

For the analysis, furthermore, we use the corresponding output to $\delta u \equiv 0$. Substituting (5) with $\delta u \equiv 0$ into the output equation of $d\Sigma$ yields

$$\delta y(t) = \frac{\partial h(\varphi_{t-t_0}(x_0, u))}{\partial \varphi_{t-t_0}}\frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x}\delta x_0. \tag{6}$$

The variational system is used to evaluate the sensitivity of the system Σ. Consider a pair of initial states $(x_0, x_0') \in \mathbb{R}^n \times \mathbb{R}^n$ and a pair of inputs $(u(t), u'(t))$. For the paths $\gamma(s) = (1-s)x_0 + sx_0'$ and $\nu(t, s) = (1-s)u(t) + su'(t)$, one notices that $\delta x(t) = \partial\phi_{t-t_0}(\gamma(s), \nu(t, s))/\partial s$ and $\delta u(t) = \partial\nu(t, s)/\partial s$ satisfy the dynamics of the variational system. Note that $\partial\phi_{t-t_0}(\gamma(s), \nu(t, s))/\partial s$ evaluates the sensitivity of $\phi_{t-t_0}(\gamma(s), \nu(t, s))$ with respect to s. Especially for $s = 0$, the sensitivity of $\phi_{t-t_0}(x_0, u)$ can be evaluated. The motivation of using the variational system is to develop a model reduction method based on the sensitivity.

3.2. Differential balanced realization

Inspired by results for LTI or LTV systems (Kawano & Scherpen, 2019a; Verriest & Kailath, 1983), we define the differential reachability and observability Gramians from the variational systems as follows.

Definition 3.1. For given $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$, the differential reachability Gramian is defined as

$$G_{\mathcal{R}}(t_0, t_f, x_0, u) := \int_{t_0}^{t_f}\frac{\partial \varphi_{t-t_0}}{\partial x}B\left(\frac{\partial \varphi_{t-t_0}}{\partial x}B\right)^{\top}dt, \tag{7}$$

where the arguments of φ_{t-t_0} are (x_0, u). ◁

Definition 3.2. For given $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$, the differential observability Gramian is defined as

$$G_{\mathcal{O}}(t_0, t_f, x_0, u) := \int_{t_0}^{t_f}\left(\frac{\partial h(\varphi_{t-t_0})}{\partial \varphi_{t-t_0}}\frac{\partial \varphi_{t-t_0}}{\partial x}\right)^{\top}\frac{\partial h(\varphi_{t-t_0})}{\partial \varphi_{t-t_0}}\frac{\partial \varphi_{t-t_0}}{\partial x}dt, \tag{8}$$

where the arguments of φ_{t-t_0} are (x_0, u). ◁

The differential Gramians exist in $[t_0, t_f]$ from the assumption that the solution $\varphi_{t-t_0}(x_0, u)$ exists and is of class C^2 in $[t_0, t_f]$. In the LTI case, the Gramians defined by (7) and (8) respectively are the controllability Gramian (2) and observability Gramian (3).

In a similar manner as a standard procedure, one can define a balanced realization between the differential reachability and observability Gramians. Since these differential Gramians are defined as functions of $\varphi(x_0, u)$, we define our balanced realization trajectory-wise as follows.

Definition 3.3. Let the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n\times n}$ and differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u) \in \mathbb{R}^{n\times n}$ at fixed $\varphi_{t-t_0}(x_0, u)$ be positive definite. A realization of the system Σ is said to be a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ if there exists a constant diagonal matrix

$$\Lambda = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}, \;\; \sigma_1 \geq \cdots \geq \sigma_n > 0$$

such that $G_{\mathcal{R}}(t_0, t_f, x, u) = G_{\mathcal{O}}(t_0, t_f, x, u) = \Lambda$. ◁

It is possible to show that there always exists a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ if the differential Gramians are positive definite. The positive definiteness of the differential reachability Gramian and differential observability Gramian have relations with local strong accessibility and local observability of the nonlinear system Σ, respectively; for more details, see the arXiv version of this paper (Kawano & Scherpen, 2019b).

Theorem 3.4. *Suppose that the differential Gramians $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ and $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ are positive definite at fixed $\varphi_{t-t_0}(x_0, u)$. Then, there exists a non-singular matrix $T_\varphi \in \mathbb{R}^{n\times n}$ which achieves*

$$T_\varphi G_{\mathcal{R}}(t_0, t_f, x_0, u)T_\varphi^{\top} = T_\varphi^{-\top}G_{\mathcal{O}}(t_0, t_f, x_0, u)T_\varphi^{-1} = \Lambda. \tag{9}$$

That is, a differentially balanced realization along $\varphi_{t-t_0}(x_0, u)$ is obtained after a coordinate transformation $z = T_\varphi x$.

Proof. Since the values $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ and $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ are constants and positive definite, one can find T_φ and Λ satisfying (9) in a similar manner as Antoulas (2005, Lemma 7.3). After the change of coordinates $z = T_\varphi x$, it follows that

$$\begin{cases} \dot{z}(t) = T_\varphi f(T_\varphi^{-1}z(t)) + T_\varphi Bu(t), \\ y(t) = h(T_\varphi^{-1}z(t)), \end{cases}$$

and its solution is $z(t) = T_\varphi \varphi_{t-t_0}(T_\varphi^{-1} z_0, u)$. From their definitions in (7) and (8), the differential reachability and observability Gramians in the new coordinates are obtained as $T_\varphi G_{\mathcal{R}}(t_0, t_f, T_\varphi^{-1} z_0, u) T_\varphi^\top$ and $T_\varphi^{-\top} G_{\mathcal{O}}(t_0, t_f, T_\varphi^{-1} z_0, u) T_\varphi^{-1}$, respectively. From $z_0 = T_\varphi x_0$ and (9), the system is differentially balanced in the z-coordinates. □

Now, we consider the balanced coordinates $z = T_\varphi x$ under the assumption $\sigma_r > \sigma_{r+1}$. Correspondingly, we divide the system in the z-coordinates into

$$\begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{f}_1(z_1(t), z_2(t)) \\ \bar{f}_2(z_1(t), z_2(t)) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u, \\ y(t) = \bar{h}(z_1(t), z_2(t)), \end{cases}$$

where $z_1(t) \in \mathbb{R}^r$ and $z_2(t) \in \mathbb{R}^{n-r}$. This system is differentially balanced according to Theorem 3.4. A reduced order model is constructed as

$$\begin{cases} \dot{z}_r(t) = \bar{f}_1(z_r(t), 0) + \bar{B}_1 u, \\ y_r(t) = \bar{h}(z_r(t), 0). \end{cases}$$

Clearly, a reduced order model depends on a trajectory φ_{t-t_0} and time interval $[t_0, t_f]$.

4. Empirical methods

4.1. Empirical differential gramians

In the previous section, we defined a differentially balanced realization along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$. For computing the differential Gramians as functions of $\varphi_{t-t_0}(x_0, u)$, or equivalently (x_0, u), one needs to solve nonlinear partial differential equations (PDEs) as for similar nonlinear balancing methods (Kawano & Scherpen, 2017b; Verriest, 2008; Verriest & Gray, 2000, 2004) in general. Hereafter, we focus on computing the values of the differential Gramians trajectory-wise.

First, we show that the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$ can be computed by using an impulse response of the variational system $d\Sigma$. Let $\delta_D(\cdot)$ be Dirac's delta function, and let $\delta x_{\mathrm{Imp,i}}(t)$ be the impulse response of the variational system $d\Sigma$ along the trajectory $\varphi_{t-t_0}(x_0, u)$ with $\delta u(t) = e_i^m \delta_D(t - t_0)$, where $e_i^m \in \mathbb{R}^m$ is the standard basis. Then, substituting $\delta x_0 = 0$ and $u(t) = e_i^m \delta_D(t - t_0)$ into (5) yields

$$\delta x_{\mathrm{Imp},i}(t) = \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} B_i, \tag{10}$$

where B_i is the ith column vector of B. Note that $\delta x_{\mathrm{Imp},i}(t)$ exists as long as $\varphi_{t-t_0}(x_0, u)$ exists. From (7), we obtain

$$G_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \delta x_{\mathrm{Imp}}(t) \delta x_{\mathrm{Imp}}^\top(t) dt, \tag{11}$$
$$\delta x_{\mathrm{Imp}}(t) := \begin{bmatrix} \delta x_{\mathrm{Imp},1}(t) & \cdots & \delta x_{\mathrm{Imp},m}(t) \end{bmatrix}.$$

Therefore, for each $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$, the value of the differential reachability Gramian $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ is obtained by using the impulse response of $d\Sigma$.

Remark 4.1. The equality (10) does not hold if B is not constant. Indeed, for the system $\dot{x} = f(x, u)$ and its trajectory $\psi_{t-t_0}(x_0, u)$, the differential reachability Gramian is

$$\bar{G}_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \frac{\partial \psi_{t-t_0}}{\partial x} \frac{\partial f(\psi_{t-t_0}, u)}{\partial u} \left(\frac{\partial \psi_{t-t_0}}{\partial x} \frac{\partial f(\psi_{t-t_0}, u)}{\partial u} \right)^\top dt.$$

However, the impulse response of the corresponding variational system is

$$\begin{aligned} \delta \bar{x}_{\mathrm{Imp}}(t) &= \int_{t_0}^{t} \frac{\partial \psi_{t-\tau}(x(\tau), u)}{\partial x} \frac{\partial f(\psi_{\tau-t_0}(x_0, u), u)}{\partial u} \delta_D(\tau - t_0) d\tau \\ &= \frac{\partial \psi_{t-t_0}(x_0, u)}{\partial x} \frac{\partial f(x_0, u(t_0))}{\partial u}. \end{aligned}$$

The reachability Gramian and impulse response do not coincide with each other for non-constant B. ◁

Next, we show that the differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ along a fixed trajectory $\varphi_{t-t_0}(x_0, u)$ can be computed by using initial state responses. By substituting $\delta x_0 = e_i^n$ and $\delta u = 0$ into (6), one obtains the initial output response of $d\Sigma$ along $\varphi_{t-t_0}(x_0, u)$ as

$$\delta y_{\mathrm{Is},i}(t) = \frac{\partial h(\varphi_{t-t_0}(x_0, u))}{\partial x} \frac{\partial \varphi_{t-t_0}(x_0, u)}{\partial x} e_i^n, \tag{12}$$

From (8), we obtain

$$G_{\mathcal{O}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} \delta y_{\mathrm{Is}}^\top(t) \delta y_{\mathrm{Is}}(t) dt,$$
$$\delta y_{\mathrm{Is}}(t) := \begin{bmatrix} \delta y_{\mathrm{Is},1}(t) & \cdots & \delta y_{\mathrm{Is},n}(t) \end{bmatrix}.$$

Thus, for each $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$, the value of the differential observability Gramian $G_{\mathcal{O}}(t_0, t_f, x_0, u)$ is obtained by using the initial state responses of $d\Sigma$.

In summary, the value of the differential reachability/observability Gramian for given x_0 and u is obtained by computing impulse/initial state responses of a variational system $d\Sigma$ along the trajectory $\varphi_{t-t_0}(x_0, u)$. Therefore, trajectory-wise differential balanced truncation is doable based on empirical data.

4.2. Approximation of the Fréchet Derivative

The empirical approach in the previous subsection requires the variational system model in addition to the original system model. If the original nonlinear systems are large-scale, computing the variational system model may need an effort. Therefore, we present approximation methods not requiring the variational system model.

To be self-contained, we first introduce the Fréchet derivative of a nonlinear operator. The system Σ induces an operator from $(x_0, u) \in \mathbb{R}^n \times L_2[t_0, t_f]$ to $(x_f, y) \in \mathbb{R}^n \times L_2[t_0, t_f]$. With some abuse of notation, Σ denotes the operator $(x_f, y) = \Sigma(x_0, u)$. A linear operator $d\Sigma_{(x_0,u)}$ is said to be the Fréchet derivative of the operator Σ if for each $x_0 \in \mathbb{R}^n$ and $u \in L_2[t_0, t_f]$, the following limit exists

$$d\Sigma_{(x_0,u)}(\delta x_0, \delta u) := \lim_{s \to 0} \frac{\Sigma(x_0 + s\delta x_0, u + s\delta u) - \Sigma(x_0, u)}{s}$$

for all $\delta x_0 \in \mathbb{R}^n$ and $\delta u \in L_2[t_0, t_f]$. From its definition, the Fréchet derivative of the nonlinear operator Σ is given by the variational system $d\Sigma$. Therefore, by using an approximation of the Fréchet derivative, one can approximately compute the impulse or initial state responses of the variational systems. A simple approximation is

$$d\Sigma_{(x_0,u)}(\delta x_0, \delta u) \approx d\Sigma_{(x_0,u)}^{\mathrm{app}}(\delta x_0, \delta u) := \frac{\Sigma(x_0 + s\delta x_0, u + s\delta u) - \Sigma(x_0, u)}{s}.$$

Since the nonlinear operator $\Sigma(x_0, u)$ is given by the system Σ, a state space representation of the discretized approximation $d\Sigma_{(x_0,u)}^{\mathrm{app}}(\delta x_0, \delta u)$ is obtained as follows.

$d\Sigma_{(x_0,u)}^{\mathrm{app}}(\delta x_0, \delta u)$:

$$\begin{aligned}&\mathbb{R}^n \times L_2^m[t_0,t_f] \times \mathbb{R}^n \times L_2^m[t_0,t_f] \to \mathbb{R}^n \times L_2^p[t_0,t_f],\\&(x_0,u,\delta x_0,\delta u) \mapsto (x_{vf},y_v),\\&\begin{cases}\dot{x}(t)=f(x(t))+Bu(t),\\ \qquad\qquad x(t_0)=x_0,\ u(t)=u(t)\\ \dot{x}'(t)=f(x'(t))+Bu'(t),\\ \qquad\qquad x'(t_0)=x_0+s\delta x_0,\ u'(t)=u(t)+s\delta u(t)\\ x_{vf}=\dfrac{x'(t_f)-x(t_f)}{s},\ y_v(t)=\dfrac{h(x'(t))-h(x(t))}{s}.\end{cases}\end{aligned}$$

Therefore, $\delta x(t)$ and $\delta y(t)$ can be approximately computed as $\delta x(t) \approx (x'(t)-x(t))/s$ and $\delta y(t) \approx y_v(t)$, where δx_0 and δu coincide with the differences of a pair of the initial states $(x_0'-x_0)/s$ and a pair of inputs $(u'-u)/s$, respectively.

From the above discussion, an approximation of the impulse response (10) is obtained as

$$\delta x_{\mathrm{Imp,i}}(t) \approx \frac{x'(t)-x(t)}{s},\ \delta x_0=0, \delta u=e_i^m\delta_D(t-t_0),\quad i=1,2,\dots,m.$$

Similar to the reachability Gramian, we need $m+1$ trajectories of the original nonlinear system in this computation by changing $x'(t)$ depending on the choice of δu. Differently from the variational system $d\Sigma$, the solution $x'(t)$ to the impulse input may not exist. According to Orlov (2000), by generalizing the concept of solutions, impulse responses for nonlinear systems can be analyzed. The discussions on the existence of impulse responses can be avoided in practice and numerical simulations if the impulse input is approximated by a bounded input such as

$$\delta u(t)=\begin{cases}1/\Delta t, & t\in[t_0,t_0+\Delta t]\\ 0, & t\notin[t_0,t_0+\Delta t]\end{cases},\ \Delta t>0. \tag{13}$$

For a bounded input, the solution $x'(t)$ exists at least in a short time interval.

Next, an approximation of the initial state response (12) is

$$\delta y_{\mathrm{Is},i}(t) \approx y_v(t),\ \delta x_0=e_i^n, \delta u=0,\ i=1,\dots,n.$$

In this computation, we need $n+1$ trajectories of the original nonlinear system. In summary, the differential reachability and observability Gramians can be approximately computed by using $n+m+1$ trajectories of the original nonlinear system, where $x(t)$ is the same for the approximations of both differential reachability and observability Gramians. An advantage of the empirical method is that even if one does not have an exact model of a real-life system, one only needs the impulse and initial state responses. In other words, it may be possible to compute an approximation of a differentially balanced realization along $\varphi_{t-t_0}(x_0,u)$ by empirical data. In this paper, the forward approximation of the Fréchet derivative is used for ease of explanation. However, other approximation methods are also applicable, and performance may differ per approximation method. Therefore, it may be relevant to compare a few approximation methods.

By applying our empirical methods, a change of coordinates $z=T_\varphi x$ for the balanced realization is obtained, and T_φ depends on $\varphi_{t-t_0}(x_0,u)$. Still it is challenging to construct a reduced order model, which gives a good approximation for all trajectories because this essentially requires solving nonlinear PDEs. A potential solution to this problem is to apply function fitting techniques, see e.g. Rojas (2013). We first generate T_φ for different choices of $\varphi_{t-t_0}(x_i,u_i)$, $i=1,\dots,r$ and then collect pairs of data $(z_i,x_i)=(T_{\varphi_{t-t_0}(x_i,u_i)}x_i,x_i)$. By using a function fitting technique, we construct a function ψ approximately satisfying $z_i=\psi(x_i)$ for all $i=1,\dots,r$. One can use the constructed $z=\psi(x)$ as a change of nonlinear coordinates and then achieve model reduction in the new coordinates. The obtained reduced order model is expected to give a good approximation at least around $\varphi_{t-t_0}(x_i,u_i)$ used for the computation of T_φ. We can take arbitrary many trajectories, thus resulting in an approximate global method for model reduction. Arbitrary existing function fitting techniques can be used for constructing a function ψ. However, the obtained reduced order model can depend on the used function fitting technique, since there is an approximation error of a function in general. Investigating the effect of function fitting techniques on the model reduction is a topic for future research.

Another potential solution to avoid solving nonlinear PDEs is to employ the basic idea of proper orthogonal decomposition (Antoulas, 2005; Holmes et al., 2012). First, we compute the summation of differential reachability and observability Gramians,

$$G_{\mathcal{R}}(t_0,t_f):=\frac{1}{r}\sum_{i=1}^r G_{\mathcal{R}}(t_0,t_f,x_i,u_i),$$

$$G_{\mathcal{O}}(t_0,t_f):=\frac{1}{r}\sum_{i=1}^r G_{\mathcal{O}}(t_0,t_f,x_i,u_i)$$

for different choices of $\varphi_{t-t_0}(x_i,u_i)$, $i=1,\dots,r$. Second, we construct a linear change of coordinates which simultaneously diagonalizes $G_{\mathcal{R}}(t_0,t_f)$ and $G_{\mathcal{O}}(t_0,t_f)$, where both are supposed to be positive definite. We then can apply truncation based on the simultaneous diagonalization.

4.3. Literature comparison

In literature, there are similar nonlinear balancing methods. We compare our methods with them.

First, another type of differential balancing method (Kawano & Scherpen, 2017b) employs the following differential controllability and observability functions $L_{\mathcal{C}}$ and $L_{\mathcal{O}}$.

$$L_{\mathcal{C}}(x_0,u,\delta x_0):=\inf_{\substack{\delta u\in L_2^m(-\infty,t_0]\\ x(t_0)=x_0,\ u\in L_2^m(-\infty,t_0]\\ \delta x(t_0)=\delta x_0,\ \delta x(-\infty)=0}} \frac{1}{2}\int_{-\infty}^{t_0}\|\delta u(t)\|^2dt, \tag{14}$$

and

$$L_{\mathcal{O}}(x_0,\delta x_0):=\frac{1}{2}\int_{t_0}^{\infty}\|\delta y(t)\|^2dt,$$

where $x(t_0)=x_0\in\mathbb{R}^n$, $u(t)\equiv 0$, $\delta x(t_0)=\delta x_0\in\mathbb{R}^n$, $\delta x(\infty)=0$, and $\delta u(t)\equiv 0$. Note that the differential controllability function gives the minimum energy to transfer the state of the prolonged system from $\delta x(-\infty)=0$ to $\delta x(t_0)=\delta x_0$ for given $x(t_0)=x_0$ and u. Therefore, it depends on x_0, u, and δx_0. A similar discussion holds for the differential observability function.

In fact, by using (6) and (8), the differential observability function and our differential observability Gramian are directly related as

$$L_{\mathcal{O}}(x_0,u,\delta x_0)=\lim_{t_f\to\infty}\frac{1}{2}\delta x_0^\top G_{\mathcal{O}}(t_0,t_f,x_0,u)\delta x_0.$$

However, the differential reachability Gramian in (7) and the differential controllability function in (14) are different. This corresponds to the difference between reachability and controllability of LTV systems (Verriest & Kailath, 1983). Reachability is the property to transfer the state from zero to an arbitrary terminal state, and controllability is the property to transfer the state from an arbitrary initial state to zero. Based on the controllability Gramian of LTV systems, we define the differential controllability Gramian as

$$G_{\mathcal{C}}(t_0,x_0,u):=\int_{-\infty}^{t_0}\frac{\partial\varphi_{t_0-\tau}}{\partial x}B\left(\frac{\partial\varphi_{t_0-\tau}}{\partial x}B\right)^\top d\tau,$$

where the arguments of $\varphi_{t_0-\tau}$ are $(x(\tau),u)$. If this differential controllability Gramian $G_{\mathcal{C}}(t_0,x_0,u)$ exists and is positive definite, the differential controllability function $L_C(x_0, u, \delta x_0)$ can be described as

$$L_C(x_0, u, \delta x_0) = \frac{1}{2}\delta x_0^\top G_C^{-1}(t_0, x_0, u)\delta x_0.$$

The differential controllability Gramian is defined by using a backward trajectory of the nonlinear system Σ. In contrast, the differential reachability Gramian is based on a forward trajectory and is computationally oriented.

Relating with differential balancing, flow balancing is proposed by Verriest (2008) and Verriest and Gray (2000, 2004). For flow balancing, the reachability and observability Gramians are defined on different time intervals, and the input is fixed for any initial state. Thus, the Gramians for flow balancing are defined as functions of the initial states. In contrast, our differential Gramians also depend on the input trajectory in addition to the initial state. Moreover, to achieve flow balancing, solutions to PDEs are required. Our methods may be applicable to develop empirical methods for flow balancing, which is included in future work.

The papers Condon and Ivanov (2004), Hahn and Edgar (2002a, 2002b), Himpe (2018), Lall et al. (2002), Willcox and Peraire (2002) extend linear empirical balancing methods to nonlinear systems by focusing on a steady-state and attract a lot of research interests as computationally tractable nonlinear model reduction methods. Except Condon and Ivanov (2004), these methods can be viewed as our method with an approximation of the Fréchet derivative at a steady-state, and Condon and Ivanov (2004) give an empirical method with differential controllability (not reachability) and observability Gramians. In other words, we provide interpretations of those methods in terms of the variational system and an approximation of the Fréchet derivative. For observability, similar Gramians as ours are found for non-control systems (Krener & Ide, 2009) and control systems (Powel & Morgansen, 2015). However, those papers do not provide the explicit description of the Gramians by using the solution of the original system or an interpretation in terms of the Fréchet derivative and do not establish the corresponding controllability Gramian.

This is the first paper to develop empirical nonlinear balancing methods, which releases the requirement of $\varphi_{t-t_0}(x_0, u)$ being a steady state. This relaxation is beneficial to enlarge the class of applications such as analysis and stabilization of a limit cycle and reducing computational complexity of trajectory tracking controller design for an arbitrary trajectory. On the other hand, as in Condon and Ivanov (2004), Hahn and Edgar (2002a, 2002b), Himpe (2018), Lall et al. (2002) and Willcox and Peraire (2002), one may use non-impulse or non-initial state responses for model reduction. These different choices of inputs or initial states enable us to deal with wider classes of model reduction problems such as in Heinkenschloss, Reis, and Antoulas (2011) although such methods may not be interpreted in terms of Gramians.

5. Another differential balancing method for variationally symmetric systems

Balancing methods including the proposed ones require computing two Gramians in general. One is for controllability, and the other is for observability. However, for linear systems, there is a class of systems for which one Gramian is constructed from the other. Such systems are called symmetric (Antoulas, 2005; Kawano & Scherpen, 2019a; Sorensen & Antoulas, 2002). Motivated by the results for symmetric systems, we develop another differential balancing method.

The symmetry concept is extended to nonlinear systems (Ionescu, Fujimoto, & Scherpen, 2011) and variational systems (Kawano & Scherpen, 2016). We further extend the latter symmetry concept.

Definition 5.1. The system Σ is said to be variationally symmetric if there exist a class C^1 and non-singular $S : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that

$$\sum_{i=1}^n \frac{\partial S(x)}{\partial x_i} f_i(x) + S(x)\frac{\partial f(x)}{\partial x} = \left(\frac{\partial f(x)}{\partial x}\right)^\top S(x), \tag{15}$$

$$S(x)B = \left(\frac{\partial h(x)}{\partial x}\right)^\top \tag{16}$$

hold. ◃

Even though B is constant, a variationally symmetric system can have a nonlinear output because S is a function. If S is constant, the output function should be linear for a system being variationally symmetric. Variational symmetry implies that after a change of coordinates $\delta z = S(x)\delta x$, the variational system becomes

$$\begin{cases} \delta \dot z(t) = \left(\dfrac{\partial f(x(t))}{\partial x}\right)^\top \delta x(t) + \left(\dfrac{\partial h(x(t))}{\partial x}\right)^\top \delta u(t), \\ \delta y(t) = B^\top \delta z(t). \end{cases} \tag{17}$$

In the LTI case, the system (17) is called the dual system of the system Σ, and the variational symmetry property is called symmetry. Many physical systems such as mechanical systems and RL circuits have this property; see e.g. van der Schaft (2011).

For an LTI system, the observability Gramian of a system is the controllability Gramian of its dual system, and if a system is symmetric, the controllability Gramian of the system yields that of its dual system (i.e. the observability Gramian of the system itself) (Antoulas, 2005; Kawano & Scherpen, 2019a; Sorensen & Antoulas, 2002). Therefore, to achieve balanced truncation of an LTI symmetric system, one only has to compute the controllability Gramian of the system. Motivated by this fact, we establish the following connection between the differential reachability Gramians of the system Σ and the system (17) in the variationally symmetric case. A similar relation holds between the differential observability Gramians of these two systems. Due to limitations of space, we leave checking the relation for observability to the reader.

Theorem 5.2. *For the variationally symmetric system Σ with respect to symmetric S, the differential reachability Gramian of the system (17) satisfies*

$$G^*_{\mathcal{R}}(t_0, t_f, x_0, u) = \int_{t_0}^{t_f} S(\varphi_{t-t_0})\frac{\partial \varphi_{t-t_0}}{\partial x}B\left(\frac{\partial \varphi_{t-t_0}}{\partial x}B\right)^\top S^\top(\varphi_{t-t_0})dt.$$

for any $x_0 \in \mathbb{R}^n$ and $u \in L_2^m[t_0, t_f]$ if it exists, where the arguments of φ_{t-t_0} are (x_0, u).

Proof. Throughout this proof, we omit the arguments of f, h and S, which are all $\varphi_{t-\tau}(x_\tau, u)$. By using (4) and (15), compute

$$\frac{d}{dt}\left(S\frac{\partial \varphi_{t-\tau}}{\partial x_\tau}S^{-1}\right) = \left(\sum_{i=1}^n \frac{\partial S}{\partial x_i}f_i + S\frac{\partial f}{\partial x}\right)\frac{\varphi_{t-\tau}}{\partial x_\tau}S^{-1} = \left(\frac{\partial f}{\partial x}\right)^\top S\frac{\partial \varphi_{t-\tau}}{\partial x_\tau}S^{-1},$$

where $dS^{-1}/dt = 0$ follows from $S = S^\top$ and (15). Thus, $S(\partial\varphi_{t-\tau}/\partial x_\tau)S^{-1}$ is the transition matrix of (17).

From (16), it follows that

$$S\frac{\partial \varphi_{t-t_0}}{\partial x_\tau}S^{-1}\left(\frac{\partial h}{\partial \varphi_{t-t_0}}\right)^\top = S\frac{\partial \varphi_{t-\tau}}{\partial x_\tau}B.$$

Therefore, the differential reachability Gramian of (17) satisfies the statement of the theorem. □

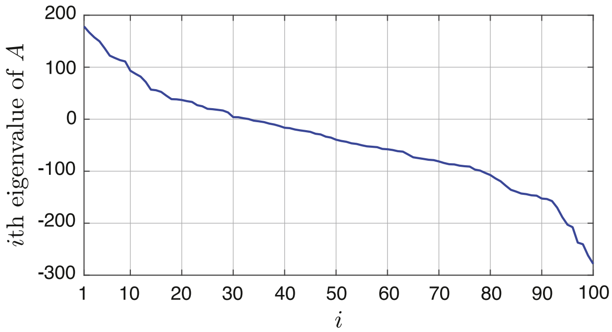

Fig. 1. Eigenvalues of A.

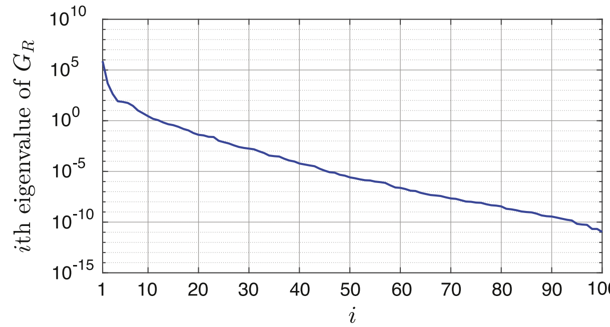

Fig. 2. Eigenvalues of the differential reachability Gramian.

Theorem 5.2 implies for variationally symmetric systems that $G^*_{\mathcal{R}}(t_0, t_f, x_0, u)$ can be computed only by using the impulse responses (10) of Σ. Therefore, differential balanced truncation based on two differential reachability Gramians $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ and $G^*_{\mathcal{R}}(t_0, t_f, x_0, u)$ can require less computational effort than having to compute the differential reachability and observability Gramians. In addition, if $G_{\mathcal{R}}(t_0, t_f, x_0, u)$ is diagonalized by an orthogonal matrix T as

$$TG_{\mathcal{R}}(t_0, t_f, x_0, u)T^\top = \Lambda := \mathrm{diag}\{\sigma_1, \dots, \sigma_n\},$$

then in the new coordinates $z = Tx$, $G^*_{\mathcal{R}}(t_0, t_f, x_0, u)$ is also diagonalized as $T^{-\top}G^*_{\mathcal{R}}(t_0, t_f, x_0, u)T^{-1} = \Lambda$, where $T^\top = T^{-1}$. For model order reduction, we truncate z_i corresponding to smaller σ_i as in Section 3.2.

The above procedure for balanced truncation can be viewed as an extension of that for symmetric LTI systems (Antoulas, 2005; Kawano & Scherpen, 2019a; Sorensen & Antoulas, 2002). In the linear case, the cross Gramian (Himpe, 2018; Kawano & Scherpen, 2019a) is useful for analysis of symmetric systems. However, the concept of a cross Gramian is missing in the differential case.

An advantage of developing a balanced truncation method based on only differential reachability Gramians is that an efficient empirical method for computing the linear controllability Gramian (Willcox & Peraire, 2002) can be used; there is no counterpart for the observability Gramian. This method can be extended to the computation of the differential reachability Gramians by using snapshots of the impulse response of the variational system. First, we construct the kernel from snapshots. Then, based on the obtained kernel, we can compute a low rank approximation of the differential reachability Gramian similar to the linear controllability Gramian.

6. Example

Consider the following system

$$\dot{x}_i = \sin\left(\sum_{j=1}^{n} a_{i,j}x_j\right) + b_i u, \ i = 1, \dots, n,$$
$$y = x_1,$$

where $n = 100$, $b_1 = 1$, and $b_i = 0$, $i = 2, \dots, n$. The matrix $A = (a_{i,j})$ is generated to be symmetric for making the system variationally symmetric with respect to $S = I_{100}$. In this example, A is generated as follows. First a square matrix $\bar{A} = (\bar{a}_{i,j})$ and diagonal matrix D are generated by the Matlab commands `Ā=rand(n,n)` and `D=diag(20*rand(n))`, respectively. Then, A is defined by $A := (\bar{A}+\bar{A}^\top)/2 - D$. The generated A has 33 positive real eigenvalues and 67 negative real eigenvalues. Fig. 1 shows the eigenvalues of the matrix A.

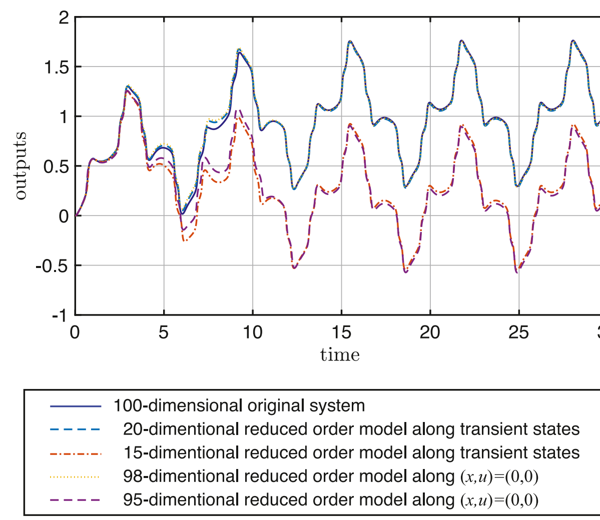

Fig. 3. Output trajectories of the original system and reduced-order models.

From Theorem 5.2, $G_{\mathcal{R}} = G^*_{\mathcal{R}}$ holds. For model reduction, we now only have to find an orthogonal matrix T which diagonalizes $G_{\mathcal{R}}$ as mentioned in Section 5. We compute the value of $G_{\mathcal{R}}$ numerically based on the method in Section 4.2 with $s = 0.01$ (but an arbitrary discretization method can be used). For this computation, we need snapshots of the trajectories of the system. As a numerical computational method of snapshots, we use the forward Euler method with the step size $\Delta t = 0.01$, and the considered time interval is $[t_0, t_f] = [0, 100]$ (but an arbitrary ODE solver can be used). Since the system is a single input system, we need snapshots of two trajectories; one is $x(t)$ around which a reduced oder model is constructed, and the other is $x'(t)$ needed for the approximate computation of the impulse response of the variational system. For instance, we choose $x(t)$ as the trajectory starting from $x(0) = 0$ with input $u(t) = \sin(t) + \sin(3t)$. Next, to approximately compute $x'(t)$, we approximate the impulse input as in (13). Then, the differential reachability Gramian $G_{\mathcal{R}}$ is computed numerically as follows

$$G_{\mathcal{R}} \approx \bar{G}_{\mathcal{R}} := \sum_{k=0}^{100/\Delta t} \delta\bar{x}(k\Delta t)\delta\bar{x}^\top(k\Delta t)\Delta t,$$
$$\delta\bar{x}(k\Delta t) := \frac{x'(k\Delta t) - x(k\Delta t)}{s}.$$

By using $\bar{G}_{\mathcal{R}}$, we compute reduced order models by the procedure mentioned in Section 5. Fig. 2 shows eigenvalues of $\bar{G}_{\mathcal{R}}$. Fig. 3 shows the output trajectories of the original system, 20-and 15-dimensional reduced order models along transient states. It can be observed that the trajectory of the 20th order model follows the trajectory of the original model well, but there is an off-set for the trajectory of the 15th order model from that of the original system.

As a comparison, other reduced order models are computed based on the impulse response $x''(t)$ along the steady-state $(x, u) = (0, 0)$, where we approximate the impulse input as in (13). Again, we use the forward Euler method with the step size $\Delta t = 0.01$, and the considered time interval is $[t_0, t_f] = [0, 100]$. Namely, the following Gramian is computed as

$$\bar{G}_{\mathcal{R}}^{(0,0)} = \sum_{k=0}^{100/\Delta t} x''(k\Delta t)(x'')^\top(k\Delta t)\Delta t.$$

Fig. 3 shows the output trajectories of the 98-and 95-dimensional reduced order models along $(x, u) = (0, 0)$; the reduced order models are constructed in a similar manner as those based on $\bar{G}_{\mathcal{R}}$. Each trajectory has a similar behavior as the 20-and 15-dimensional reduced order models along transient states. That is, the trajectory of the 95-dimensional reduced order model along $(x, u) = (0, 0)$ has an off-set from that of the original model, which illustrates the importance of developing model reduction methods along transient states.

Every process here is conducted by using Matlab 2019a on macOS Catalina, version 10.15.3, MacBook Pro (13-inch, 2018, Four Thunderbolt 3 Ports), Processor 2.7 GHz Quad-Core Intel Core i7, Memory 16 GB 2133 MHz LPDDR3, and Graphics Intel Iris Plus Graphics 655 1536 MB.

7. Conclusion and future work

7.1. Conclusion

In this paper, we have proposed a nonlinear empirical differential balancing method along a fixed state trajectory for nonlinear systems. The proposed method is based on the differential reachability and observability Gramians, which are functions of the state trajectory. The values of these Gramians at each trajectory are computable by using impulse and initial state responses of the variational system along the trajectory. We have also developed approximation methods for computing them, which only requires empirical data of the original nonlinear systems. Constructed reduced models depend on the choice of discretization methods, approximation methods of impulse responses and so on. The numerics are important, and many choices can be made. It requires additional research work to investigate which choices are most beneficial.

7.2. Possible application

In Choroszucha et al. (2016) and Hahn et al. (2002), empirical balancing at a steady-state is used to reduce the computational complexity of nonlinear model predictive control (MPC) (Camacho & Alba, 2013; Grüne & Pannek, 2011). Our proposed empirical differential balancing method along a fixed state trajectory can be used to reduce the computational complexity around transient states. In MPC, we repeatedly solve the following nonlinear optimal control problem.

$$J = \varphi(x(t + T), u(t + T)) + \int_t^{t+T} L(x(\tau), u(\tau))d\tau. \tag{18}$$

If the optimal control input u in the time interval $[t_0, t_0 + \Delta t]$ is obtained, one can compute the state trajectory of the controlled system in this time interval. Along this trajectory, it is possible to achieve the proposed empirical differential balanced truncation. Then, we have a reduced order model. To compute the optimal control input in the next time interval $[t_0 + \Delta t, t_0 + 2\Delta t]$, one can use the reduced order model. For this reduced order model, one can compute the corresponding cost function to (18). By solving the reduced order optimal control problem, one has the optimal control input for the reduced order model, which is an approximation of the optimal control input for the original system in the time interval $[t_0 + \Delta t, t_0 + 2\Delta t]$. Thus, one can use this input for controlling the original system and have the state trajectory of the controlled original system in the time interval $[t_0 + \Delta t, t_0 + 2\Delta t]$. Then, one can again exploit our empirical model reduction method for obtaining a reduced order model. By repeating this procedure, one can compute an approximation of the optimal control input in each time interval recursively.

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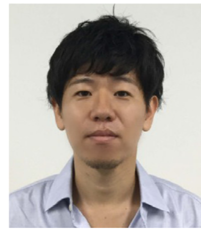

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