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Chapter 3
Strong Structural Controllability and Zero Forcing

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Abstract In this chapter, we study controllability and output controllability of systems defined over graphs. Specifically, we consider a family of state-space systems, where the state matrix of each system has a zero/non-zero structure that is determined by a given directed graph. Within this setup, we investigate under which conditions all systems in this family are controllable, a property referred to as strong structural controllability. Moreover, we are interested in conditions for strong structural output controllability. We will show that the graph-theoretic concept of zero forcing is instrumental in these problems. In particular, as our first contribution, we prove necessary and sufficient conditions for strong structural controllability in terms of so-called zero forcing sets. Second, we show that zero forcing sets can also be used to state both a necessary and a sufficient condition for strong structural output controllability. In addition to these main results, we include interesting results on the controllability of subfamilies of systems and on the problem of leader selection.

3.1 Introduction

Structural controllability has been an active research area ever since its introduction in the early seventies of the previous century by Lin in [9]. Originally, the concept of structural controllability was introduced in order to deal with uncertainty in the

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state and input matrix representing a given linear input-state system. Instead of being known exactly, some of the entries in these matrices are assumed to be unknown, while the remaining entries are equal to zero. The unknown entries can take arbitrary (zero or non-zero) values. The system is then called weakly structurally controllable if there exists a choice of values for the unknown entries such that the corresponding numerical realization is a controllable pair in the classical sense of Kalman. In the context of structural controllability, the system matrices are no longer matrices with real entries, but are *pattern matrices*, i.e., matrices with some entries equal to zero, and the remaining ones free. With the given pattern matrices defining the system, or rather, the system structure, Lin [9] associated a *directed graph*, and subsequently established conditions for weak structural controllability in terms of topological properties of this graph. Later on, this work was extended to the multi-input case in [6, 14]. In all of the above references, the pattern matrices consist of entries equal to zero and entries whose values are unknown. These unknown entries are allowed to be either zero or non-zero. If we require the non-zero entries to take non-zero values only, then we can also ask the question: under what conditions are *all* numerical realizations controllable in the sense of Kalman. In this case, we call the system defined by the pattern matrices *strongly structurally controllable*. This notion was introduced in [10].

The last decade has witnessed a revival of research related to structural controllability. For a major part, this has been caused by the outburst of research on networks of systems, also referred to as multi-agent systems. The interaction between the agents in such a network is usually represented by a directed graph, where the vertices are identified with the agents, and the arcs correspond to the communication between these agents. In the context of controllability of networks, two types of vertices are distinguished: those that are influenced by inputs from outside the network, called *leaders*, and those that are influenced only by their neighbours, called *followers*. Controllability of such leader–follower networks deals with the issue whether it is possible to steer the states of all vertices in the network to any desired state by applying inputs to the leaders. Clearly, the interaction between the agents is represented by the network graph, including the weights of the arcs.

If one ignores the exact values of the arc weights, and instead focusses on the graph topology only, one arrives at the issue of structural controllability of such leader–follower network: it is called strongly structurally controllable if, roughly speaking, for all (non-zero) values of the arc weights the corresponding numerical realization of the network is controllable. It should be noted that a special role is played by the diagonal entries of the state matrix, which are allowed to take arbitrary (zero or non-zero) values, depending on whether or not the graph contains self-loops. The system is called weakly structurally controllable if there exists a choice of arc weights such that the corresponding network is controllable. The input matrix is determined by the choice of leader vertices. Thus, with a given network graph, a whole family of linear input-state systems is associated, and strong structural controllability deals with the question whether all members of this family are controllable in the classical sense. Weak structural controllability requires that at least one member of the family is controllable. Like in the classical literature on structural controllability, conditions for strong and weak structural controllability of networks are formulated in terms of properties of the underlying network graph, see, e.g., [3–5, 12, 15]. A topological condition for weak structural controllability of networks in terms of maximum matchings was established in [5]. Strong structural controllability was characterized in terms of constrained matchings in [4]. In [12, 15], necessary and sufficient conditions for strong structural controllability were given in terms of *zero forcing sets*.

More recently, research on the topic of strong structural controllability has been extended to strong *output* controllability, also called *strong targeted controllability*. Here, in addition to the subset of leader vertices, a subset of target vertices is given. Then, targeted controllability deals with the question whether the states of the target vertices can be steered to arbitrary desired states by applying appropriate inputs to the leader vertices. For a given network graph with leader set and target set, we have a family of *input-state-output* systems, and the network is called strongly targeted controllable if all members of this family are output controllable, see [16]. Also for strong targeted controllability, graph topological conditions have been obtained. In particular, in [17], such conditions were obtained for the subfamily of all input-state-output systems with distance-information preserving state matrices.

In the present paper, we will give an introduction to the concept of zero forcing and its application to strong structural controllability and strong targeted controllability. The outline of the paper is as follows. In Sect. 3.2, we will introduce preliminaries on the so-called colour change rule, and introduce the concept of zero forcing set. In Sect. 3.3, we introduce the notions of qualitative class associated with a given graph, and give a definition of strong structural controllability. We study the strong structurally reachable subspace associated with a given qualitative class and leader set. This leads to a necessary and sufficient condition for strong structural controllability. Subsequently, we discuss subclasses of the given qualitative class and give a definition of the notion of sufficient richness. Finally, in this section, we address the issue of leader selection, which is concerned with determining the minimal number of leaders required for strong structural controllability. It is shown that this number is equal to the zero forcing number of the graph. Our final section, Sect. 3.4, deals with strong targeted controllability. We define this concept starting from classical output controllability for linear input-state-output systems. Then, we give sufficient conditions for strong targeted controllability. Also, we strengthen these conditions by restricting ourselves to the important subclass of distance-information preserving state matrices. Finally, the paper closes with conclusions in Sect. 3.5.

3.2 Zero Forcing

In this section, we review the notion of zero forcing. Let $G = (V, E)$ be a simple directed graph with vertex (or node) set V and edge set $E \subseteq V \times V$. We say that $v \in V$ is an *out-neighbour* of vertex $u \in V$ if $(u, v) \in E$. Now suppose that the vertices in V are coloured either black or white. The *colour change rule* is defined in the following way. If $u \in V$ is a black vertex and *exactly one* out-neighbour $v \in V$ of u is white, then change the colour of v to black [8]. When the colour change rule is applied to node u to change the colour of v, we say *u forces v*, which we denote by $u \to v$.

Suppose that we have a colouring of G, that is, a set $C \subseteq V$ of only black vertices, and a set $V \setminus C$ consisting of only white vertices. Then the *derived set* $D(C)$ is the set of black vertices obtained by applying the colour change rule until no more changes are possible [8]. It can be shown that for a given graph G and set C, the derived set $D(C)$ is unique [1]. However, note that the order in which forces occur in the colouring process is in general not unique.

The set C is called a *zero forcing set* of G if $D(C) = V$. Let $|C|$ denote the cardinality of C. Then, the *zero forcing number* $Z(G)$ of the graph G is the minimum of $|C|$ over all zero forcing sets C of G. Moreover, a zero forcing set $C \subseteq V$ is called a *minimum zero forcing set* if $|C|$ equals $Z(G)$.

3.3 Zero Forcing and Structural Controllability

A linear time-invariant input/state system of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if and only if the well-known Kalman rank condition holds:

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \text{ is of full row rank.}$$

In this chapter, we are interested in *structural controllability* properties that depend on the (zero) *structure* of the matrices A and B, rather than their numerical values.

To be more specific, let $G = (V, E)$ be a simple directed graph with vertex set $V = \{1, 2, \dots, n\}$ and edge set $E \subseteq V \times V$. Define the *qualitative class* of G, denoted by $Q(G)$, as

$$Q(G) := \{X \in \mathbb{R}^{n \times n} \mid \text{ for } i \neq j,\ X_{ij} \neq 0 \iff (j, i) \in E\}. \tag{3.1}$$

For $V' = \{v_1, v_2, \dots, v_k\} \subseteq V$, let $P(V; V')$ denote the $n \times k$ matrix whose ij-th entry is given by

$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Consider the family of linear time-invariant systems

$$\dot{x}(t) = Xx(t) + Uu(t), \tag{3.3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $X \in Q(G)$, and $U = P(V; V_L)$ where $V_L \subseteq V$ is the so-called *leader set*.

Examples of systems of the form (3.3) are encountered in [7] where X is the adjacency matrix, in [2] the normalized matrix, and in [11] the (in-degree or out-degree) Laplacian.

We call the system (3.3) *strongly structurally controllable* if the pair (X, U) is controllable for all $X \in Q(G)$. In that case, we write $(Q(G); V_L)$ is controllable. The term "strong" is used to distinguish with the case of "weak structural controllability" which amounts to the existence of a controllable pair (X, U) with $X \in Q(G)$.

3.3.1 Strong Structural Controllability

In this subsection, we aim at a topological characterization of strong structural controllability and investigate how the graph structure can determine the controllability of $(Q(G); V_L)$, given a leader set V_L. A related problem is *minimal leader selection*, where the goal is to choose a leader set V_L with minimum cardinality such that $(Q(G); V_L)$ is controllable. We will see that both strong structural controllability and minimal leader selections problems are intimately related to the colour change rule and zero forcing sets discussed in Sect. 3.2.

Throughout this chapter, we denote the image (range) of the matrix $P(V; V_L)$ by $\mathscr{V}_L$, and the image of $P(V; D(V_L))$ by $\mathscr{D}(V_L)$. The *reachable subspace* associated with the pair (X, U), denoted by $\langle X \mid \mathrm{im} U\rangle$, is defined as

$$\langle X \mid \mathrm{im} U\rangle = \mathrm{im} U + X \mathrm{im} U + \cdots + X^{n-1} \mathrm{im} U.$$

The subspace $\langle X \mid \mathrm{im} U\rangle$ is the smallest X-invariant subspace containing $\mathrm{im} U$. It is well known (see e.g., [16]) that (X, U) is controllable if and only if $\langle X \mid \mathrm{im} U\rangle = \mathbb{R}^n$.

In the following lemma, we state that the reachable subspace is not affected by the colour change rule.

Lemma 3.1 ([13]) *For any given $X \in Q(G)$ and leader set $V_L \subseteq V$, we have $\langle X \mid \mathscr{V}_L\rangle = \langle X \mid \mathscr{D}(V_L)\rangle$.*

Proof First, we prove that $\langle X \mid \mathscr{D}(V_L)\rangle \subseteq \langle X \mid \mathscr{V}_L\rangle$. This trivially holds in case $D(V_L) = V_L$, and thus $\mathscr{D}(V_L) = \mathscr{V}_L$. Now, suppose that $D(V_L) \neq V_L$, and vertex $v \in V_L$ forces vertex $w \notin V_L$. Then, we *claim* that

$$\mathrm{im} P(V; V_L \cup \{w\}) \subseteq \langle X \mid \mathscr{V}_L\rangle, \tag{3.4}$$

where P is given by (3.2). Clearly, the subspace inclusion (3.4) holds if and only if

$$\langle X \mid \mathscr{V}_L\rangle^{\perp} \subseteq P(V; V_L \cup \{w\})^{\perp}. \tag{3.5}$$

Without loss of generality, assume that $V_L = \{1, 2, \dots, m\}$, $v = m$ and $w = m + 1$. Then, the matrix X can be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}, \tag{3.6}$$

where $X_{11} \in \mathbb{R}^{(m-1)\times(m-1)}$, $X_{22} \in \mathbb{R}$, $X_{33} \in \mathbb{R}$, $X_{44} \in \mathbb{R}^{(n-m-1)\times(n-m-1)}$, and the rest of the matrices involved have compatible dimensions. Notice that the matrix $P(V; V_L \cup \{w\})$ now reads as

$$P(V; V_L \cup \{w\}) = \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\xi \in \mathbb{R}^n$ be a vector in $\langle X \mid \mathscr{V}_L \rangle^\perp$. Clearly, we have $\xi^T X^{k-1} P(V; V_L) = 0$ for each $k \in \mathbb{N}$. We write $\xi = \text{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ compatible with the partitioning of X. Setting $k = 1$ yields the equality $\xi^T P(V_L; V) = 0$, and hence $\xi_1 = 0$ and $\xi_2 = 0$. In addition, by setting $k = 2$, we obtain the equality $\xi^T X P(V_L; V) = 0$, which results in

$$\begin{bmatrix} \xi_3^T & \xi_4^T \end{bmatrix} \begin{bmatrix} X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{bmatrix} = 0. \tag{3.7}$$

Since $v \to w$, the vertex v has exactly one out-neighbour in $V \setminus V_L$, and thus we have $X_{32} \neq 0$ and $X_{42} = 0$. Therefore, noting (3.7), the scalar ξ_3 must be equal to zero. As $\xi = \text{col}(0, 0, 0, \xi_4)$ is orthogonal to the subspace $P(V; V_L \cup \{w\})$, the subspace inclusion (3.5) and consequently (3.4) holds. By repeating the argument above, we conclude that $\text{im} P(V; D(V_L)) = \mathscr{D}(V_L) \subseteq \langle X \mid \mathscr{V}_L \rangle$, which results in $\langle X \mid \mathscr{D}(V_L) \rangle \subseteq \langle X \mid \mathscr{V}_L \rangle$.

Now, to prove the statement of the lemma, it remains to show that $\langle X \mid \mathscr{V}_L \rangle \subseteq \langle X \mid \mathscr{D}(V_L) \rangle$. The latter holds since $\mathscr{V}_L \subseteq \mathscr{D}(V_L)$, and the proof is complete.

Now that we have established the result of Lemma 3.1, an intriguing question is to characterize the subspace containing all the states that can be reached by applying appropriate input signals to the nodes in the leader set V_L, *independent* of the particular choice of $X \in Q(G)$. Geometrically, this subspace is given by $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle$, and provides a strong structural counterpart of the reachability subspace. Hence, we will refer to it as the *strongly structurally reachable subspace*. Consistent with the previous treatment, here we are after a topological characterization of this subspace.

From Lemma 3.1, we obtain that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle. \tag{3.8}$$

The subspace on the left is the strongly structurally reachable subspace. Interestingly, the one on the right simplifies to $\mathscr{D}(V_L)$, as stated in the theorem below.

Theorem 3.1 ([13]) *For any given leader set $V_L \subseteq V$, we have*

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathscr{D}(V_L). \tag{3.9}$$

Proof Given equality (3.8), and noting that

$$\mathscr{D}(V_L) \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle,$$

it suffices to show that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle \subseteq \mathscr{D}(V_L). \tag{3.10}$$

To this end, we define the set S as

$$S = \{s \in \mathbb{R}^n : s_i = 0 \Leftrightarrow i \in D(V_L)\}. \tag{3.11}$$

Let s be a vector in S. Without loss of generality, let $D(V_L) = \{1, 2, \ldots, d\}$. Then, s can be written as $\operatorname{col}(0_d, s_2)$ where each element of $s_2 \in \mathbb{R}^{n-d}$ is non-zero. Let the matrix X be partitioned accordingly as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Clearly, we have

$$s^T X = s_2^T \begin{bmatrix} X_{21} & X_{22} \end{bmatrix}.$$

Note that X_{21} corresponds to the arcs from a vertex $v \in D(V_L)$ to a vertex $w \notin D(V_L)$. Hence, by the colour change rule, each column of X_{21} is either identically zero or contains at least two non-zero elements. We choose these non-zero elements, if any, such that $s_2^T X_{21} = 0$. Since the diagonal elements of X_{22} are free parameters, we conclude that, for any vector $s \in S$, there exists a matrix $X \in Q(G)$ such that $s^T X = 0$. Therefore, we obtain that

$$s \in \langle X \mid \mathscr{D}(V_L) \rangle^{\perp}$$

for some matrix $X \in Q(G)$. Now, let $\xi \in \mathbb{R}^n$ be a vector in $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{D}(V_L) \rangle$. Hence, by definition, $\xi \in \langle X \mid \mathscr{D}(V_L) \rangle$ for all $X \in Q(G)$. Therefore, we have $s^T \xi = 0$ which yields $s_2^T \xi_2 = 0$, noting $\xi = \operatorname{col}(\xi_1, \xi_2)$. As this conclusion holds for any arbitrary choice of $s \in S$, we conclude that $\xi_2 = 0$. Consequently, we obtain $\xi \in \mathscr{D}(V_L)$, which results in (3.10), and completes the proof.

The most notable consequence of Theorem 3.1 is obtained by looking at the scenario where $D(V_L) = V$, which, by definition, is the case in which V_L is a zero forcing set. In this case, the result of Theorem 3.1 can be used to state necessary and sufficient conditions for strong structural controllability, as stated below.

Theorem 3.2 ([12]) *The system* (3.3) *is strongly structurally controllable, i.e.,* $(Q(G); V_L)$ *is controllable, if and only if* V_L *is a zero forcing set in* G.

Proof Suppose that $(Q(G); V_L)$ is controllable. This means that $\langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ for all $X \in Q(G)$. Therefore, $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$. By Theorem 3.1, $\mathscr{D}(V_L) = \mathbb{R}^n$. We conclude that V_L is a zero forcing set. Conversely, suppose that V_L is a zero forcing set. Hence, $\bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ by Theorem 3.1. We conclude that $\langle X \mid \mathscr{V}_L \rangle = \mathbb{R}^n$ for all $X \in Q(G)$, that is, $(Q(G); V_L)$ is controllable.

3.3.2 Leader Selection

Next, we discuss the *minimal leader selection* problem. In the context of strong structural controllability this amounts to selecting a leader set with minimum cardinality such that (3.3) is strongly structurally controllable. To make this more precise, we define $\ell_{\min}(Q(G))$ as follows:

$$\ell_{\min}(Q(G)) = \min_{V_L \subseteq V(G)} \{|V_L| : (Q(G); V_L) \text{ is controllable}\}. \tag{3.12}$$

An immediate consequence of Theorem 3.2 is

$$\ell_{\min}(Q(G)) = Z(G).$$

The equality above relates the minimal leader selection problem for strong structural controllability of networks to minimal zero forcing sets and the zero forcing number in graph theory [8]. While finding a minimal zero forcing set in general is a difficult combinatorial problem, such sets can be efficiently computed for several types of graphs including path, cycle, acyclic graphs and complete graphs [12].

3.3.3 Qualitative Subclasses

So far, we have investigated controllability of systems given by (3.3), where the matrix X belongs to the family of matrices given by $Q(G)$ in (3.1). In many examples, the state matrix may have more structure than the one captured by $Q(G)$. For instance, in the case that the graph $G = (V, E)$ is *symmetric* (i.e., $(i, j) \in E \iff (j, i) \in E$)

one might be interested in the class of symmetric state matrices only. This gives rise to a subclass of $Q(G)$, namely

$$Q_{\text{sym}}(G) = \{X \in Q(G) \mid X = X^T\}. \tag{3.13}$$

Given a leader set $V_L \subseteq V$ and a subclass of $Q_s(G) \subseteq Q(G)$, we say that $(Q_s(G); V_L)$ is controllable if (X, U) is controllable for all $X \in Q_s(G)$ and $U = P(V; V_L)$.

Obviously, $(Q_s(G); V_L)$ is controllable if $(Q(G); V_L)$ is controllable. The natural question arises here is whether the converse result holds for certain subclasses of $Q(G)$. This motivates the following definition.

Definition 3.1 A subclass $Q_s(G) \subseteq Q(G)$ is called *sufficiently rich* if for any $V_L \subseteq V$ such that $(Q_s(G); V_L)$ is controllable we have that $(Q(G); V_L)$ is controllable as well.

A useful sufficient algebraic condition for a subclass $Q_s(G)$ to be sufficiently rich is provided next.

Lemma 3.2 ([12]) *Let $Q_s(G) \subseteq Q(G)$. Then, $Q_s(G)$ is sufficiently rich if the following implication holds:*

$$z \in \mathbb{R}^n,\ z^T X = 0 \textit{ for some } X \in Q(G) \implies \exists\, X_s \in Q_s(G) \textit{ such that } z^T X_s = 0.$$

Notably, for any symmetric graph G, the algebraic condition in Lemma 3.2 holds for the subclass $Q_{\text{sym}}(G)$ (see Proposition IV.9 of [12]). This implies that $Q_{\text{sym}}(G)$ is a sufficiently rich subclass of $Q(G)$. Hence, Theorem 3.2 can be used to characterize controllability of $(Q_{\text{sym}}(G); V_L)$. Specifically, $(Q_{\text{sym}}(G); V_L)$ is controllable if and only if V_L is a zero forcing set.

Another important subclass of matrices is the class of so-called *distance-information preserving* matrices. To define this class of matrices, we need some terminology first. We define the *distance* $d(u, v)$ between two vertices $u, v \in V$ as the length of the shortest path from u to v. If there does not exist a path from vertex u to v, the distance $d(u, v)$ is defined as infinite. Moreover, the distance from a vertex to itself is equal to zero. For a non-empty subset $S \subseteq V$ and a vertex $j \in V$, the distance from S to j is defined as

$$d(S, j) := \min_{i \in S} d(i, j). \tag{3.14}$$

With this in mind, we state the following definition.

Definition 3.2 Consider a directed graph $G = (V, E)$. A matrix $X \in Q(G)$ is called *distance-information preserving* if for any two distinct vertices $i, j \in V$ we have that $d(j, i) = k$ implies $(X^k)_{ij} \neq 0$.

Although the distance-information preserving property does not hold for all matrices $X \in Q(G)$, it does hold for the adjacency and Laplacian matrices. Because these matrices are often used to describe network dynamics, distance-information preserving matrices form an important subclass of $Q(G)$. We will denote the subclass of distance-information preserving matrices by $Q_d(G)$. It turns out the $Q_d(G)$ is sufficiently rich, as asserted in the following lemma.

Lemma 3.3 ([17]) *The subclass $Q_d(G) \subseteq Q(G)$ is sufficiently rich.*

3.4 Targeted Controllability

In case (3.3) fails to be structurally controllable, it is worthwhile to investigate whether it is “partially” controllable. To elaborate, let $G = (V, E)$ be a simple directed graph where $V = \{1, 2, \ldots, n\}$ is the vertex set and $E \subseteq V \times V$ is the edge set. Consider the following linear time-invariant input/state/output system:

$$\dot{x}(t) = Xx(t) + Uu(t) \tag{3.15a}$$

$$y(t) = Hx(t), \tag{3.15b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. Here, $X \in Q(G)$, $U = P(V; V_L)$ for some given *leader set* $V_L \subseteq V$, and $H = P^T(V; V_T)$ for some given $V_T \subseteq V$ called the *target set*.

In what follows we will investigate the structural output controllability problem for systems of the form (3.15). We therefore first review output controllability for linear systems.

3.4.1 Output Controllability

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.16}$$

$$y(t) = Cx(t) \tag{3.17}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Denote the output trajectory corresponding to the initial state x_0 and input u by $y_u(t, x_0)$. The system (3.16) is then called *output controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input u and a $T > 0$ such that $y_u(T, x_0) = y_1$. We sometimes say that the triple (A, B, C) is output controllable meaning that the system (3.16) is output controllable.

It is well known (see, e.g., [16, Exc. 3.22]) that (A, B, C) is output controllable if and only if

$$\text{rank}\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} = p.$$

In turn, this is equivalent to the condition

$$C\langle A \mid \text{im} B\rangle = \mathbb{R}^p.$$

If C has full row rank, the condition above is equivalent to

$$\ker C + \langle A \mid \operatorname{im} B\rangle = \mathbb{R}^n.$$

Finally, by taking orthogonal complements, the latter holds if and only if

$$\operatorname{im} C^T \cap \langle A \mid \operatorname{im} B\rangle^{\perp} = \{0\}.$$

3.4.2 Problem Formulation

Let $G = (V, E)$ be a simple directed graph. Also, let $V_L \subseteq V$ be a leader set and let $V_T \subseteq V$ be a target set. We say that the system (3.15) is *strongly targeted controllable with respect to* $Q' \subseteq Q(G)$ if the system (3.15) is output controllable for all $X \in Q'$. For brevity, we will say $(Q'; V_L; V_T)$ is targeted controllable meaning that (3.15) is strongly targeted controllable with respect to $Q' \subseteq Q(G)$.

The following proposition translates the output controllability results mentioned in Sect. 3.4.1 to targeted controllability.

Proposition 3.1 *The following statements are equivalent:*

(a) $(Q'; V_L; V_T)$ *is targeted controllable.*
(b) $\operatorname{rank}\begin{bmatrix} HU & HXU & HX^2U & \cdots & HX^{n-1}U\end{bmatrix} = p$ *for all* $X \in Q'$.
(c) $H\langle X \mid \mathscr{V}_L\rangle = \mathbb{R}^p$ *for all* $X \in Q'$.
(d) $\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L\rangle^{\perp} = \{0\}$ *for all* $X \in Q'$.
(e) $\ker H + \langle X \mid \mathscr{V}_L\rangle = \mathbb{R}^n$ *for all* $X \in Q'$.

The main goal of this section is to use Proposition 3.1 to establish conditions for targeted controllability of $(Q'; V_L; V_T)$ in terms of *zero forcing sets*. We will first focus on the case that $Q' = Q(G)$ in Sect. 3.4.3. Subsequently, we discuss the case that $Q' = Q_d(G)$ in Sect. 3.4.4.

3.4.3 Targeted Controllability for $Q(G)$

In this section, we discuss conditions for targeted controllability with respect to the entire qualitative class. That is, we let $Q' = Q(G)$ and investigate under which conditions $(Q(G); V_L; V_T)$ is targeted controllable. We start with a sufficient condition from [13].

Theorem 3.3 *Let* $G = (V, E)$ *be a directed graph with leader set* $V_L \subseteq V$ *and target set* $V_T \subseteq V$. *Then* $(Q(G); V_L; V_T)$ *is targeted controllable if* $V_T \subseteq D(V_L)$.

Proof Assume that $V_T \subseteq D(V_L)$, and thus $\mathscr{V}_T \subseteq \mathscr{D}(V_L)$. By Theorem 3.1, this is equivalent to

$$\mathscr{V}_T \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathscr{V}_L \rangle. \tag{3.18}$$

Therefore, it is easy to observe that

$$\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L \rangle^{\perp} = \{0\} \tag{3.19}$$

for all $X \in Q(G)$, which results in targeted controllability of $(Q(G); V_L; V_T)$ by Proposition 3.1(d).

Theorem 3.3 provides a sufficient condition for targeted controllability. In particular, targeted controllability is guaranteed provided that the target nodes belong to the derived set of V_L.

As an example, consider the graph depicted in Fig. 3.1, and let $V_L = \{1, 2\}$. It is easy to observe that the derived set of V_L is obtained as $D(V_L) = \{1, 2, 3, 4\}$. By Theorem 3.3, we have that $(Q(G); V_L; V_T)$ is targeted controllable for any

$$V_T \subseteq \{1, 2, 3, 4\}. \tag{3.20}$$

However, this is not necessary as one can show that $(Q(G); V_L; V_T)$ is also targeted controllable with

$$V_T = \{1, 2, 3, 4, 5, 6, 7\}. \tag{3.21}$$

Next, we show that the sufficient condition provided by Theorem 3.3 can be sharpened by extending the derived set of V_L. To this end, we define the subgraph $G' = (V, E')$, where E' is defined as

$$E' := \{(i, j) \mid i \in D(V_L) \text{ and } j \in V_T\}. \tag{3.22}$$

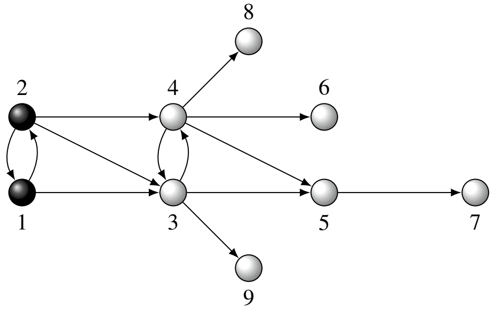

Fig. 3.1 The graph $G = (V, E)$

We define $D'(V_L)$ as the derived set of $D(V_L)$ in the subgraph G'. The following theorem extends the result of Theorem 3.3. In particular, it states that $(Q(G); V_L; V_T)$ is targeted controllable if $V_T \subseteq D'(V_L)$.

Theorem 3.4 *Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Then $(Q(G); V_L; V_T)$ is targeted controllable if $V_T \subseteq D'(V_L)$.*

Proof Assume that $V_T \subseteq D'(V_L)$. If $D'(V_L) = D(V_L)$ the claim follows immediately from Theorem 3.3. It remains to be shown that the theorem holds if $V_E := D'(V_L) \setminus D(V_L) \neq \emptyset$. In this case, we write

$$\mathscr{V}_T \subseteq \mathscr{D}'(V_L) = \mathscr{D}(V_L) \oplus \mathscr{V}_E, \tag{3.23}$$

where $\mathscr{D}'(V_L) = \operatorname{im} P(V; D'(V_L))$, $\mathscr{V}_E = \operatorname{im} P(V; V_E)$ and $\oplus$ denotes direct sum. Without loss of generality, assume that

$$D(V_L) = \{1, 2, \dots, d\}$$

and

$$V_E = \{d + 1, d + 2, \dots, d + e\}.$$

Note that $V_E \subseteq V_T$ and therefore the nodes in V can be relabeled such that

$$V_T = \{d - t, d - t + 1, \dots, d, d + 1, \dots, d + e\}$$

for some $t < d$. Consider the fourth statement in Proposition 3.1. Let $X \in Q(G)$ and ξ be a vector in the subspace $\mathscr{V}_T \cap \langle X \mid \mathscr{V}_L\rangle^{\perp}$. Hence, $\xi \in \mathscr{V}_T \cap \langle X \mid \mathscr{D}(V_L)\rangle^{\perp}$ by Lemma 3.1. We write $\xi \in \mathbb{R}^n$ as $\xi = \operatorname{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ by partitioning the vertices into the subsets $D(V_L) \setminus V_T$, $D(V_L) \cap V_T$, V_E, and $V \setminus D'(V_L)$, respectively. Now, compatible with ξ, let the matrix X be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}. \tag{3.24}$$

By (3.23) we have $\xi \in \mathscr{D}'(V_L)$. This implies that $\xi_4 = 0$. Moreover, we have

$$\xi^T X^{k-1} P(V; D(V_L) = 0 \tag{3.25}$$

for each $k \in \mathbb{N}$. The equality $\xi^T P(V; D(V_L)) = 0$ yields $\xi_1 = \xi_2 = 0$. Then, from $\xi^T X P(V; D(V_L)) = 0$, we obtain that

$$\xi_3^T \begin{bmatrix} X_{31} & X_{32} \end{bmatrix} = 0. \tag{3.26}$$

Note that X_{21}, X_{22}, X_{31} and X_{32} correspond to the arcs from the vertices in the derived set to those in the target set V_T. Therefore, the matrix

$$X' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & X_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

belongs to the qualitative class $Q(G')$. By Lemma 3.1, we have

$$\mathscr{D}'(V_L) \subseteq \langle X' \mid \mathscr{D}(V_L)\rangle. \tag{3.27}$$

It is not difficult to see that the subspace in the right-hand side of (3.27) is computed as

$$\langle X' \mid \mathscr{D}(V_L)\rangle = \text{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, (3.27) yields

$$\mathscr{D}'(V_L) = \text{im} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \subseteq \text{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that $\begin{bmatrix} X_{31} & X_{32} \end{bmatrix}$ is full row rank. Consequently, (3.26) results in $\xi_3 = 0$, that is, $\xi = 0$. We conclude that $(Q(G); V_L; V_T)$ is targeted controllability by the fourth statement of Proposition 3.1.

As an example, consider the graph in Fig. 3.1 with $V_L = \{1, 2\}$. Recall that the derived set of V_L is given by $D(V_L) = \{1, 2, 3, 4\}$. Suppose that V_T is given by

$$V_T = \{1, 2, 3, 4, 5, 6\}. \tag{3.28}$$

Then, Fig. 3.2 shows the subgraph $G' = (V, E')$ with E' given by (3.22). It is straightforward to show that the derived set of $D(V_L)$ in G' is equal to $D'(V_L) = \{1, 2, 3, 4, 5, 6\}$. Therefore, noting that $V_T = D'(V_L)$, we conclude that $(Q(G); V_L; V_T)$ is targeted controllable by Theorem 3.4. Observe that Theorem 3.4 extends Theorem 3.3. Indeed, the condition $V_T \subseteq D(V_L)$ has been replaced by a less conservative condition $V_T \subseteq D'(V_L)$. However, the sufficient condition provided by Theorem 3.4 is not necessary. Indeed, as previously mentioned, $(Q(G); V_L; V_T)$ is targeted controllable for V_T given in (3.21). However, note that node 7 is not contained in the set $D'(V_L)$. Consequently, the conditions of Theorem 3.4 are not satisfied in this case.

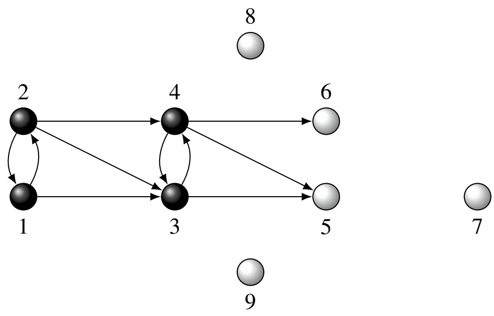

Fig. 3.2 The subgraph $G' = (V, E')$

3.4.4 Targeted Controllability for $Q_d(G)$

In the previous section, we have established sufficient conditions for targeted controllability in the case that state matrices are contained in the qualitative class $Q(G)$. We saw that it is possible to assess targeted controllability even if target nodes are not contained in $D(V_L)$ (but are incident to nodes in $D(V_L)$). However, it turns out to be difficult to assess targeted controllability if target nodes have distance larger than 1 with respect to $D(V_L)$. In this section, we restrict the state matrices to be *distance-information preserving*. We will show that for the class $Q_d(G)$ it is possible to assess targeted controllability even if the distance from $D(V_L)$ to nodes in V_T is arbitrary.

Before we start, we introduce some terminology that will become useful in the rest of this section. A directed graph $G = (V, E)$ is called *bipartite* if there exist disjoint sets of vertices V^- and V^+ such that $V = V^- \cup V^+$ and $(u, v) \in E$ only if $u \in V^-$ and $v \in V^+$. We denote such a bipartite graph by $G = (V^-, V^+, E)$, to indicate the partition of the vertex set. Suppose that the vertex sets V^- and V^+ are given by

$$\begin{aligned} V^- &= \{r_1, r_2, ..., r_s\} \\ V^+ &= \{q_1, q_2, ..., q_t\}. \end{aligned} \tag{3.29}$$

Then, the *pattern class* $\mathscr{P}(G)$ of the bipartite graph G is defined as

$$\mathscr{P}(G) = \{M \in \mathbb{R}^{t \times s} \mid M_{ij} \neq 0 \iff (r_j, q_i) \in E\}. \tag{3.30}$$

Note that the cardinalities of V^- and V^+ can differ; hence, the matrices in the pattern class $\mathscr{P}(G)$ are not necessarily square.

With these definitions in place, we continue our discussion on targeted controllability. Consider any directed graph $G = (V, E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. We assume that all target nodes have finite distance with respect to V_L. This assumption is necessary, as it can be easily shown that $(Q_d(G); V_L; V_T)$ is not targeted controllable if a target node $v \in V_T$ cannot be reached from any leader.

Let $V_S \subseteq V \setminus D(V_L)$ be a subset. We partition the set V_S according to the distance of its nodes with respect to $D(V_L)$, that is,

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{3.31}$$

where for each $i = 1, 2, ..., d$ and $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$. With each of the sets $V_1, V_2, ..., V_d$ we associate a bipartite graph $G_i = (D(V_L), V_i, E_i)$, where for $j \in D(V_L)$ and $k \in V_i$ we have $(j, k) \in E_i$ if and only if $d(j, k) = i$ in the network graph G.

Example 3.1 We consider the network graph $G = (V, E)$ as depicted in Fig. 3.3. The set of leaders is $V_L = \{1, 2\}$, which implies that $D(V_L) = \{1, 2, 3\}$, see Fig. 3.4.

In this example, we define the subset $V_S \subseteq V \setminus D(V_L)$ as $V_S := \{4, 5, 6, 7, 8\}$. Note that V_S can be partitioned according to the distance of its nodes with respect to $D(V_L)$ as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. Then, the bipartite graphs G_1, G_2 and G_3 are given in Figs. 3.5, 3.6 and 3.7, respectively.

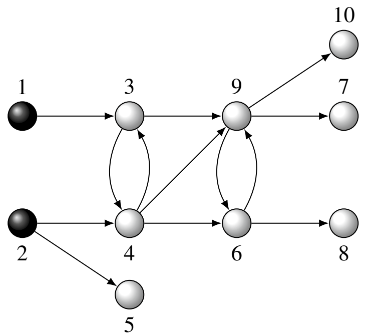

Fig. 3.3 Graph G with $V_L = \{1, 2\}$

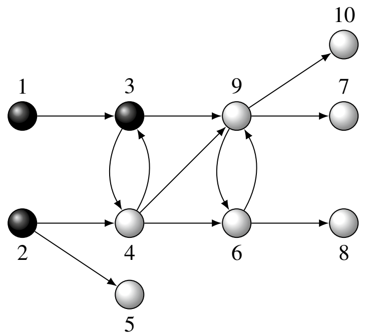

Fig. 3.4 $D(V_L) = \{1, 2, 3\}$

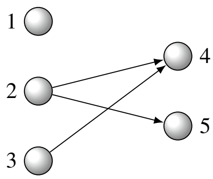

Fig. 3.5 Graph G_1

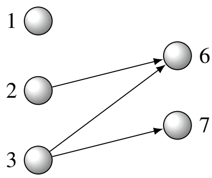

Fig. 3.6 Graph G_2

The next result provides a sufficient graph-theoretic condition for targeted controllability of $(Q_d(G); V_L; V_T)$.

Theorem 3.5 *Consider a directed graph $G = (V, E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $V_T \setminus D(V_L)$ be partitioned as in (3.31). Then $(Q_d(G); V_L; V_T)$ is targeted controllable if $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$.*

In the special case of a single leader, i.e., $|V_L| = 1$, the condition of Theorem 3.5 can be simplified. In this case, $(Q_d(G); V_L; V_T)$ is targeted controllable if no pair of target nodes has the same distance with respect to the leader. This is formulated in the following corollary.

Corollary 3.1 *Consider a directed graph $G = (V, E)$ with singleton leader set $V_L = \{v\} \subseteq V$ and target set $V_T \subseteq V$. Then $(Q_d(G); V_L; V_T)$ is targeted controllable if $d(v, i) \neq d(v, j)$ for all distinct $i, j \in V_T$.*

Note that the condition of Corollary 3.1 is similar to the "k-walk theory" for (weak) targeted controllability established in Theorem 2 of [5]. However, it is worth mentioning that k-walk theory [5] was only proven for directed tree networks with a single leader. On the other hand, Theorem 3.5 establishes a condition for strong targeted controllability that is applicable to general directed networks with multiple leaders.

It is interesting to note that the conditions of Theorem 3.5 are the same as the conditions of Theorem 3.4 in the case that all target nodes have a distance of at most one from $D(V_L)$. However, the advantage of Theorem 3.5 lies in the fact that it can be applied to target nodes that have arbitrary distance with respect to $D(V_L)$.

Example 3.2 Once again, consider the network graph depicted in Fig. 3.3, with leader set $V_L = \{1, 2\}$ and assume the target set is given by $V_T = \{1, 2, ..., 8\}$. The

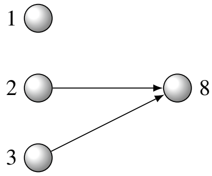

Fig. 3.7 Graph G_3

goal of this example is to prove that $(Q_d(G); V_L; V_T)$. Note that $V_S := V_T \setminus D(V_L)$ is given by $V_S = \{4, 5, 6, 7, 8\}$, which is partitioned as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. The graphs G_1, G_2 and G_3 have been computed in Example 3.1. Note that $D(V_L) = \{1, 2, 3\}$ is a zero forcing set in all three graphs (see Figs. 3.5, 3.6 and 3.7). We conclude by Theorem 3.5 that $(Q_d(G); V_L; V_T)$ is targeted controllable.

Before proving Theorem 3.5, we need two auxiliary lemmas. The following lemma states that $(X^k)_{ij} = 0$ if the distance from j to i is greater than k.

Lemma 3.4 *Consider a directed graph $G = (V, E)$ and two distinct vertices $i, j \in V$. Moreover, let k be a positive integer and $X \in Q(G)$. If $d(j, i) > k$ then $(X^k)_{ij} = 0$.*

The proof of Lemma 3.4 follows simply from induction on k, and is therefore omitted. The next lemma gives conditions under which all matrices in the pattern class $\mathscr{P}(G)$ of a bipartite graph G have full row rank.

Lemma 3.5 *Let $G = (V^-, V^+, E)$ be a bipartite graph and assume V^- is a zero forcing set in G. Then all matrices in $\mathscr{P}(G)$ have full row rank.*

For the proof of Lemma 3.5, we refer to Lemma 7 of [17]. With these results in place, we can now prove Theorem 3.5.

Proof of Theorem 3.5 Suppose that $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, ..., d$. We want to prove that $(Q_d(G); V_L; V_T)$ is targeted controllable. By Proposition 3.1(c) and Lemma 3.1, $(Q_d(G); V_L; V_T)$ is targeted controllable if and only if $(Q_d(G); D(V_L); V_T)$ is targeted controllable. Therefore, our goal is to prove that $(Q_d(G); D(V_L); V_T)$ is targeted controllable. Relabel the nodes in V such that $D(V_L) = \{1, 2, ..., m\}$, and let the matrix $U = P(V; D(V_L))$ be given by $U = \begin{pmatrix} I & 0 \end{pmatrix}^T$. Furthermore, we let $V_S := V_T \setminus D(V_L)$ be given by $\{m + 1, m + 2, ..., p\}$, where the vertices are ordered in non-decreasing distance with respect to $D(V_L)$. Partition V_S according to the distance of its nodes with respect to $D(V_L)$ as

$$V_S = V_1 \cup V_2 \cup \cdots \cup V_d, \tag{3.32}$$

where for $i = 1, 2, ..., d$ and $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$. We define $\check{V}_i$ and $\hat{V}_i$ to be the sets of vertices in V_S that have distance less than i (respectively, greater than i) from $D(V_L)$. More precisely,

$$\begin{aligned}\check{V}_i &:= V_1 \cup \cdots \cup V_{i-1} \text{ for } i = 2, \ldots, d \\ \hat{V}_i &:= V_{i+1} \cup \cdots \cup V_d \text{ for } i = 1, \ldots, d-1.\end{aligned} \tag{3.33}$$

By convention $\check{V}_1 := \varnothing$ and $\hat{V}_d := \varnothing$. In addition, we assume without loss of generality that the target set V_T contains all nodes in the derived set $D(V_L)$. This implies that the matrix $H = P(V; V_T)^T$ is of the form $H = \begin{pmatrix} I & 0 \end{pmatrix}$. Note that by the structure of H and U, the matrix HX^iU is simply the $p \times m$ upper left corner submatrix of X^i. We now claim that HX^iU can be written as

$$HX^iU = \begin{pmatrix} * \\ M_i \\ 0 \end{pmatrix}, \tag{3.34}$$

where $M_i \in P(G_i)$ is a $|V_i| \times m$ matrix in the pattern class of G_i, the $\left(m + |\check{V}_i|\right) \times m$ matrix $*$ contains elements of less interest and 0 denotes a zero matrix of dimension $|\hat{V}_i| \times m$.

We proceed as follows: first, we prove that the bottom submatrix of (3.34) contains zeros only. Second, we prove that $M_i \in P(G_i)$. From this, we will conclude that Eq. (3.34) holds.

Note that for $k \in D(V_L)$ and $j \in \hat{V}_i$, we have $d(k, j) > i$ and by Lemma 3.4 it follows that $(X^i)_{jk} = 0$. This means that the bottom $|\hat{V}_i| \times m$ submatrix of HX^iU is a zero matrix.

Subsequently, we want to prove that M_i, the middle block of (3.34), is an element of the pattern class $P(G_i)$. Note that the j-th row of M_i corresponds to the element $l := m + |\check{V}_i| + j \in V_i$.

Suppose $(M_i)_{jk} \neq 0$ for a $k \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., |V_i|\}$. As M_i is a submatrix of HX^iU, this implies $(HX^iU)_{lk} \neq 0$. Recall that HX^iU is the $p \times m$ upper left corner submatrix of X^i; therefore, it holds that $(X^i)_{lk} \neq 0$. Note that for the vertices $k \in D(V_L)$ and $l \in V_i$ we have $d(k, l) \geq i$ by the partition of V_S. However, as $(X^i)_{lk} \neq 0$ it follows from Lemma 3.4 that $d(k, l) = i$. Therefore, by the definition of G_i, there is an arc $(k, l) \in E_i$.

Conversely, suppose there is an arc $(k, l) \in E_i$ for $l \in V_i$ and $k \in D(V_L)$. This implies $d(k, l) = i$ in the network graph G. By the distance-information preserving property of X, we consequently have $(X^i)_{lk} \neq 0$. We conclude that $(M_i)_{jk} \neq 0$ and hence $M_i \in P(G_i)$. This implies that Eq. (3.34) holds.

The previous discussion shows that we can write

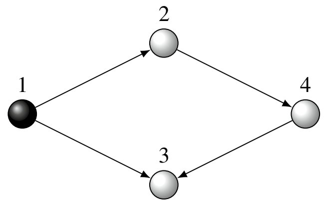

Fig. 3.8 Example showing that Theorem 3.5 not necessary

$$\begin{pmatrix} HU & HXU & HX^2U & \cdots & HX^dU \end{pmatrix} = \begin{pmatrix} I & * & * & \dots & * & * \\ 0 & M_1 & * & \dots & * & * \\ 0 & 0 & M_2 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & * & * \\ \vdots & \vdots & \vdots & \ddots & M_{d-1} & * \\ 0 & 0 & 0 & \dots & 0 & M_d \end{pmatrix}, \tag{3.35}$$

where asterisks denote matrices of less interest. As $D(V_L)$ is a zero forcing set in G_i for $i = 1, 2, ..., d$, the matrices $M_1, M_2, ..., M_d$ have full row rank by Lemma 3.5. We see that the matrix (3.35) has full row rank, and consequently $(Q_d(G); D(V_L); V_T)$ is targeted controllable by Proposition 3.1. We conclude that $(Q_d(G); V_L; V_T)$ is targeted controllable, which proves the theorem.

Note that the condition given in Theorem 3.5 is sufficient, but not necessary. Indeed, one can verify that the graph in Fig. 3.8 with leader set $V_L = \{1\}$ and target set $V_T = \{2, 3\}$ is such that $(Q_d(G); V_L; V_T)$ is targeted controllable. However, this graph does not satisfy the conditions of Theorem 3.5.

In addition to the previously established sufficient condition for targeted controllability, we also give a *necessary* condition in terms of zero forcing sets.

Theorem 3.6 *Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. If $(Q_d(G); V_L; V_T)$ is targeted controllable, then $V_L \cup (V \setminus V_T)$ is a zero forcing set in G.*

Proof Assume without loss of generality that $V_L \cap V_T = \varnothing$. Hence, $V_L \cup (V \setminus V_T) = V \setminus V_T$. We partition the vertex set V into the sets V_L, $V \setminus (V_L \cup V_T)$ and V_T. We label the vertices in V such that $V_L = \{1, 2, \dots, m\}$ and $V_T = \{n - p + 1, n - p + 2, \dots, n\}$. Accordingly, the input and output matrices $U = P(V; V_L)$ and $H = P^T(V; V_T)$ satisfy

$$U = \begin{pmatrix} I & 0 & 0 \end{pmatrix}^T, \tag{3.36}$$

and

$$H = \begin{pmatrix} 0 & 0 & I \end{pmatrix}. \tag{3.37}$$

Note that $\ker H = \operatorname{im} R$, where $R := P(V; (V \setminus V_T))$ is given by

$$R = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}^T. \tag{3.38}$$

By hypothesis, $(Q_d(G); V_L; V_T)$ is targeted controllable. By Proposition 3.1(e), we have

$$\ker H + \langle X \mid \operatorname{im} U \rangle = \mathbb{R}^n \tag{3.39}$$

for all $X \in Q_d(G)$. Equivalently,

$$\operatorname{im} R + \langle X \mid \operatorname{im} U \rangle = \mathbb{R}^n. \tag{3.40}$$

We therefore see that

$$\langle X \mid \operatorname{im} \begin{pmatrix} U & R \end{pmatrix} \rangle = \mathbb{R}^n. \tag{3.41}$$

As $\operatorname{im} U \subseteq \operatorname{im} R$, Eq. (3.41) implies $\langle X \mid \operatorname{im} R \rangle = \mathbb{R}^n$ for all $X \in Q_d(G)$. In other words, the pair (X, R) is controllable for all $X \in Q_d(G)$. Furthermore, by sufficient richness of $Q_d(G)$, it follows that (X, R) is controllable for all $X \in Q(G)$ (see Lemma 3.3). We conclude from Theorem 3.2 that $V \setminus V_T$ is a zero forcing set.

3.5 Conclusions

In this chapter, we have studied controllability and output controllability of systems defined over graphs. We have considered a family of state-space systems, where the state matrix of each system has a zero/non-zero structure that is determined by a given directed graph. In this context, we have investigated under which conditions all systems in the family are controllable, in other words, conditions under which the graph is strongly structurally controllable. We have shown that the strongly structurally reachable subspace can be obtained by a graph colouring rule called zero forcing. This yields neat necessary and sufficient conditions for strong structural controllability in terms of zero forcing sets. In addition, we have investigated controllability of certain subfamilies of systems via the notion of sufficient richness. For specific graph structures, we have developed leader selection strategies to find input sets of minimum cardinality that guarantee strong structural controllability. In addition, we have discussed sufficient conditions for strong structural output controllability in terms of zero forcing. We have shown that these results can be strengthened if we restrict the class of state matrices to be distance-information preserving. In the latter case, we have also shown that zero forcing sets can be used to establish a necessary condition for output controllability. Finding necessary and sufficient graph-theoretic conditions for strong structural output controllability is still an open problem that can be considered for future work.

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