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Evolutionary Game Dynamics for Crowd Behavior in Emergency Evacuations

Diego Marti Mason[1], Leonardo Stella[2] and Dario Bauso[3]

Abstract— **This paper studies the problem of a large group of individuals that has to get to a safety exit in the context of high-stress emergency evacuations. We model this problem as a discrete-state continuous-time game, where the players update their strategies to reach the exit within a defined time horizon, whilst avoiding undesirable situations such as congestion and being trampled. The proposed model builds on crowd dynamics in a two-strategy game theoretic context, which we extend to include aspects of crowd behavior originating in sociology and psychology, and in the analogous studies performed in immersive virtual environments. The main contribution of this paper is threefold: i) we propose a novel game formulation of the model in terms of the population distribution across three strategies, and provide a link with prospect theory; ii) we study the equilibria of the system and their stability via Lyapunov stability theory of nonlinear systems; iii) we extend the model to a multi-population setting, where each population represents the group of players at a certain distance from the exit.**

I. Introduction

Motivated by the study of crowd dynamics in emergency evacuations, see [1], we consider a population of individuals that have to choose the best strategy at any given time in order to get to safety, whilst avoiding congestion at bottlenecks such as corridors and doors. We reframe this problem within the framework of game theory, where we model each strategy through a payoff matrix. The dynamics of a crowd share similarities with many other social and biological systems, see [2] for an example of the impact of a strongly opinionated minority on uninformed agents. A good example of conflict-free consensus in decision-making is constituted by eusocial insects such as honeybees, see [3] and [4] for seminal works on the topic.

We consider a large population of players that have to choose the best strategy in order to get to safety in an emergency evacuation setting. We model the crowd dynamics in a continuous-time dynamic game framework, where each player controls their state using some optimality criteria. In a first approximation, we assume that players are homogeneous, meaning that they behave in the same way in the same situation. Additionally, we consider that each player's choices do not affect the evolution of the game, and are instead added to the mass, meaning that the players are indistinguishable and the game is symmetric with respect to permutation of players.

Highlights of contributions: The contribution of this paper is threefold. First, we provide an evolutionary perspective of the macroscopic dynamics that regulate crowd behavior. We extend a common model in the literature for crowd dynamics to include an extra state and another parameter: the addition of the *neutral* is motivated by the context of opinion dynamics where individuals are susceptible to one of the other two antithetical options, namely *patient* and *impatient*; the new parameter represents a gain for an orderly escape. We define what we call the *expected gain* to formulate the nonlinear discrete-state continuous-time macroscopic dynamics and study the evolution of the population distribution across the three strategies. To study this model, we consider a macroscopic parameter which averages across all agents the expected time to escape. Second, we identify the equilibrium points and carry out the stability analysis of the model via Lyapunov stability theory of nonlinear systems. Third, we extend the model to a multi-population setting, where we capture the microscopic parameter of expected evacuation time in each different population. In the multi-population model, we assume that players are homogeneous with respect to each population.

Related literature: Understanding crowd dynamics has an important impact in evacuation management and in escape training, e.g. for employees on a large variety of contexts. Some recent studies use an immersive environment to study this topic from an experimental viewpoint, see [1], whereas a vast literature has investigated these dynamics from a theoretical perspective, see [5] and plenty of references therein. The first attempt to use a game theoretic approach to model crowd behavior in an evacuation setting is due to Lo et al., see [6] and plenty of references therein. A common game model used in the literature includes two possible strategies and a cost of congestion, denoted by c, that weighs all the players choosing to be impatient and the corresponding interactions with other impatient agents. This game is also used to model a spatial scenario where players interact with their neighbors by choosing one of the two strategies, namely *patient* and *impatient*, see [7]. A cellular automaton model is proposed in [8] to study the same spatial game model. A particle-based approach to model pedestrians and crowd behavior is also common in the literature, see [9]. A mean-field game theoretic approach is proposed in [10] to study the evacuation dynamics in the scenario of a multi-level building.

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This paper is organized as follows. In Section II, the evolutionary model is introduced and the stability analysis is carried out. In Section III, the model is extended to a multi-population scenario. In Section IV, the numerical analysis is presented. In Section V, conclusions are drawn and future directions are discussed.

II. Evolutionary Game Dynamics

Given a large population of players, we model the evolutionary crowd dynamics where players have to choose within a set of three pure strategies, namely *patient*, *impatient* and *neutral*. Due to the structure of the game under consideration, we will use the terms *strategy*, *state* and *decision* interchangeably. We consider the frequency of each strategy $i \in \mathcal{I}^3$, where $\mathcal{I}^3$ represents the set of states. We denote the probability distribution at time index $t > 0$ with $x(t) = [x_1, x_2, x_3]^T \in \mathcal{S}^3$, where $\mathcal{S}^3$ is the probability simplex in $\mathbb{R}^3$. Parameter g represents a gain for an orderly escape, c is a cost usually referred to as *cost of congestion* in the literature, and $\Delta u(\tilde{\tau})$ is a fixed macroscopic parameter that models the interactions between a patient agent and an impatient agent. The notation for parameter $\Delta u(\tilde{\tau})$ is left the same as in the literature to reflect the difference between the cost for the two interacting players i and j in the corresponding microscopic dynamics, cost that is indicated by $u(T_i)$ and $u(T_j)$, respectively.

Let us now derive the payoff matrix corresponding to the case where players cannot switch directly from strategy patient to impatient. For simplicity, we include the payoffs only for the row player. Let $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$ be the payoff matrix defined as follows:

$$A = \begin{pmatrix} g & -\Delta u(\tilde{\tau}) & 0 \\ \Delta u(\tilde{\tau}) & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

where g and $\Delta u(\tilde{\tau})$ are strictly positive parameters and c is nonnegative.

In the above game, the row player earns g and loses $\Delta u(\tilde{\tau})$ for matching strategy 1, namely patient, and for playing strategy 1 against a player whose strategy is impatient, respectively. The row player incurs in a cost of c when matching strategy 2, namely impatient, or earns $\Delta u(\tilde{\tau})$ if playing strategy 2 against the column player's strategy 1. In random-matching, players that choose to stay neutral neither gain nor lose anything. The neutral state accounts for those players that are usually referred to as *susceptible* in opinion dynamics, and the presence of this additional state is essential to mimic those individuals who are undecided at the time of the emergency. Let ρ_{ij} be the transition rate from state i to state j. To model the evolution of the frequency of each strategy, we consider the following game dynamics which are in accordance with innovative dynamics as in [11]:

$$\dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij}. \tag{2}$$

Before we can present the macroscopic dynamics for the problem originating in the context of emergency evacuations,

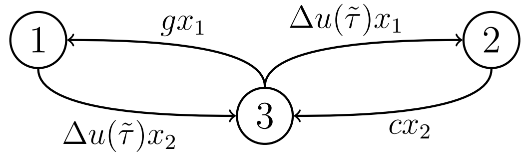


Fig. 1: The transition rates between each pair of states is described by this Markov chain of system (4).

we introduce the following definition of *expected gain* given x for our game dynamics. This constitutes the first contribution of the paper, and takes inspiration from the model developed in the context of swarm behavior, see [12].

Definition 1: **(Expected gain)** Let A be a payoff matrix. The expected gain from strategy j to strategy i is defined as:

$$E_{ji}(x) = \sum_{k=1}^{n} (a_{ik} - a_{jk})_+ x_k, \tag{3}$$

where $(a_{ik} - a_{jk})_+$ denotes the positive part of $a_{ik} - a_{jk}$.

The above definition models a player's expected gain by taking into account only the increase in payoff when the player changes strategy. In a risk-seeking scenario, Kahneman and Tversky's prospect theory can be linked to the above equation if we design a weighting function that gives weight zero to the probability of unfavorable events, see [13]. Furthermore, prospect theory can be used to model behavioral dynamics that come from sociology such as leadership or stress in evacuation contexts.

By assuming that each transition rate depends on the expected gain defined above as $\rho_{ij} = E_{ij}(x)$, we can now calculate the transition rates as in the following: $\rho_{31} = gx_1$, $\rho_{13} = \Delta u(\tilde{\tau})x_2$, $\rho_{32} = \Delta u(\tilde{\tau})x_1$, $\rho_{23} = cx_2$.

By taking into account the conservation of mass, namely $\dot{x}_3 = -\dot{x}_1 - \dot{x}_2$, we can now substitute the above transition rates into the game dynamics in (2) to obtain the following set of Kolmogorov equations, which describe the macroscopic dynamics of the crowd:

$$\begin{cases} \dot{x}_1 = gx_1x_3 - \Delta u(\tilde{\tau})x_1x_2, \\ \dot{x}_2 = \Delta u(\tilde{\tau})x_1x_3 - cx_2^2, \\ \dot{x}_3 = -(\Delta u(\tilde{\tau}) + g)x_1x_3 + (\Delta u(\tilde{\tau})x_1 + cx_2)x_2. \end{cases} \tag{4}$$

By taking into account the conservation of mass, namely $\dot{x}_3 = -\dot{x}_1 - \dot{x}_2$, we formulate the corresponding bi-dimensional system as follows:

$$\begin{cases} \dot{x}_1 = gx_1(1 - x_1 - x_2) - \Delta u(\tilde{\tau})x_1x_2, \\ \dot{x}_2 = \Delta u(\tilde{\tau})x_1(1 - x_1 - x_2) - cx_2^2. \end{cases} \tag{5}$$

The above system has an initial condition for the distribution, namely $x(0) = x_0$. Figure 1 depicts the Markov chain corresponding to system (4).

A. Equilibria and Stability Analysis

In this section, we carry out the stability analysis for the three-state model proposed in the previous section, namely system (4). First, we find all the equilibrium points of the system and then we study the stability property of each equilibrium point via Lyapunov stability theory of nonlinear systems.

Theorem 1: System (4) admits the following equilibrium points $x^* = (x_1^*, x_2^*, x_3^*)$:

- **Case 1**: When $x_1 = 1$ and $x_2 = 0$, $x^* = (1, 0, 0)$.
- **Case 2**: When $x_1 = 0$ and $x_2 = 0$, $x^* = (0, 0, 1)$.
- **Case 3**: In general, by substitution,

$$x^* = \Big(\frac{cg^2}{p}, \frac{\Delta u(\tilde{\tau})^2 g}{p}, \frac{\Delta u(\tilde{\tau})^3}{p}\Big),$$

where $p = \Delta u(\tilde{\tau})^3 + \Delta u(\tilde{\tau})^2 g + cg^2$ for brevity.

Corollary 1: Let us consider $x_1(g - \Delta u(\tilde{\tau}))(1 - x_1) = x_2(gx_1 - cx_2)$, by equating the two equations of the bi-dimensional system (5) to zero. When $x_1 = c/gx_2$ and $g = \Delta u(\tilde{\tau})$, the equilibrium point in *Case 3* reduces to:

$$x^* = \Big(\frac{c}{2g + c}, \frac{g}{2g + c}, \frac{g}{2g + c}\Big).$$

Additionally, when all three parameters are equal, namely $g = c = \Delta u(\tilde{\tau})$, the equilibrium point reduces to $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which represents the case where the population is evenly distributed across the three states.

Corollary 2: When $c \to 0$, the equilibrium point in *Case 3* reduces to $x^* = (0, \frac{1}{2}, \frac{1}{2})$. When $c \to \infty$, the equilibrium point in *Case 3* reduces to $x^* = (1, 0, 0)$, namely the equilibrium point in *Case 1*.

We now present the next result, which establishes the stability properties of the fixed points in Theorem 1.

Theorem 2: Given an initial condition $x(0) = x_0$, the fixed point $x^* = (1, 0, 0)$ is a saddle point, $x^* = (0, 0, 1)$ is unstable, and the fixed point in *Case 3* is asymptotically stable. Additionally, when the first condition in Corollary 2 holds true, namely $c \to 0$, the fixed point in *Case 3* is asymptotically stable.

Remark 1: The importance of the above result lies in the fact that the consensus on option 1, namely patient, and on option 3, namely neutral, are not stable equilibria. Therefore, the only stable fixed point for the system is when players choose different strategies according to the parameters g, c and $\Delta u(\tilde{\tau})$. To add depth to the model, we recall that in the microscopic formulation of the model $\Delta u(\tilde{\tau})$ represents the perceived time to escape of each agent. In our model, however, we approximate it with an average value for the whole population.

III. Multi-Population Model

In this section, we want to capture the scenario where players are no longer homogenous, namely react in different ways: our approach is to study a multi-population formulation of the model to account for different behaviors. Let $P(k)$ be the probability distribution of players in population k, and let $\eta_k = \frac{d_k}{d_{max}}$ be the parameter capturing the distance from the exit, where d_k is the distance for class k and d_{max} is the value corresponding to the maximum distance. We indicate with $\langle d \rangle$ the mean value of d_k across all populations. Let $\theta_i = \frac{1}{\langle d \rangle} \sum_k kP(k)x_{i,k}$, where $x_{i,k}$ is the population described by distance d_k using strategy i. For each population $k \in \mathbb{Z}$, the bi-dimensional model (5) becomes:

$$\begin{cases} \dot{x}_{1,k} = (1 - x_{1,k} - x_{2,k})g\theta_1 - x_{1,k}\eta_k \Delta u(\tilde{\tau})\theta_2, \\ \dot{x}_{2,k} = (1 - x_{1,k} - x_{2,k})\eta_k \Delta u(\tilde{\tau})\theta_1 - x_{2,k}c\theta_2. \end{cases} \tag{6}$$

We use this model to investigate the impact of players' distance from the exit, but likewise it can capture a variety of different behavioral aspects such as anxiety or stress, to mention a few. The role of parameter η_k is to influence the players choosing diametrically opposed strategies, namely patient and impatient, and thus it affects the macroscopic parameter $\Delta u(\tilde{\tau})$.

A. Equilibria and Stability Analysis

To investigate the stability in the proposed multi-population model, we study the mean-field response for a given population by assuming that the distribution of the remaining part of the population is fixed. First, we put system (6) in matrix form as:

$$\begin{bmatrix} \dot{x}_{1,k} \\ \dot{x}_{2,k} \end{bmatrix} = \overbrace{\begin{bmatrix} -g\theta_1 - \eta_k \Delta u(\tilde{\tau})\theta_2 & -g\theta_1 \\ -\eta_k \Delta u(\tilde{\tau})\theta_1 & -\eta_k \Delta u(\tilde{\tau})\theta_1 - c\theta_2 \end{bmatrix}}^{A_k(\theta)} \cdot \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{\begin{bmatrix} g\theta_1 \\ \eta_k \Delta u(\tilde{\tau})\theta_1 \end{bmatrix}}_{c_k(\theta)}, \tag{7}$$

where $A_k(\theta)$ indicates the system matrix, and for simplicity we denote the dependence on both θ_1 and θ_2 through θ, and $c_k(\theta)$ is a vector of coefficients that do not depend on the state. The above system can be rewritten in compact form as in the following:

$$\begin{bmatrix} \dot{x}_{1,k} \\ \dot{x}_{2,k} \end{bmatrix} = A_k(\theta) \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + c_k(\theta).$$

We are now ready to establish the next result, assessing the stability of the multi-population system (7).

Theorem 3: Given an initial state $x_{i,k}(0)$ for $i = 1, 2$ and all populations k, system (7) is locally asymptotically stable. Furthermore, the eigenvalues of system (7) for the minimum distance $\eta_k = 0$ and maximum distance $\eta_k = 1$ are: $\Big(\frac{-q + i\sqrt{3cg\theta_1\theta_2}}{2}, \frac{-q - i\sqrt{3cg\theta_1\theta_2}}{2}\Big)$ for $\eta_k = 0$ and $\Big(\frac{-\Delta u(\tilde{\tau})(\theta_1+\theta_2) - q + \sqrt{\Delta_{\eta_k=1}}}{2}, \frac{-\Delta u(\tilde{\tau})(\theta_1+\theta_2) - q - \sqrt{\Delta_{\eta_k=1}}}{2}\Big)$ for $\eta_k = 1$, where $q = g\theta_1 + c\theta_2$ for brevity, and $\Delta_{\eta_k=1} = \Delta u(\tilde{\tau})^2(\theta_1^2 + \theta_2^2 - 3\theta_1\theta_2) + \Delta u(\tilde{\tau}(g\theta_1\theta_2 + g\theta_1^2 + c\theta_1\theta_2 - 3\theta_2^2) - 3cg\theta_1\theta_2$.

Remark 2: The above result provides an interesting insight on the impact of the distance on the players' choice to be patient or impatient. Specifically, when $\Delta_{\eta_k=1} < 0$, the eigenvalues shift towards the negative $\Re(\lambda)$ and thus the higher the distance, the faster the convergence. When $\Delta_{\eta_k=1} > 0$, depending on its absolute value and on the absolute value of the trace, the system can converge faster as before or it can turn unstable due to the eigenvalues having different sign.

Theorem 4: Given an initial state $x_{i,k}(0)$ for $i = 1, 2$ and all populations k, when $cg\theta_1\theta_2 + \eta_k \Delta u(\tilde{\tau})g\theta_1^2 >$

$\eta_k^2 \Delta u(\tilde{\tau})^2 \theta_1^2$ and $\eta_k^2 \Delta u(\tilde{\tau})^2 \theta_1 \theta_2 + \eta_k \Delta u(\tilde{\tau}) g \theta_1^2 > g^2 \theta_1^2$, system (7) admits the following fixed points: $[x_{1,k}^*, x_{2,k}^*]^T = -A_k^{-1}(\theta) c_k(\theta)$.

IV. Numerical Analysis

In this section, we provide a numerical analysis to corroborate the theoretical results. We provide two sets of simulations. In the first set, we show the evolution of the population distribution over time, In the second set, we model the spatial dynamics for a finite number of agents, namely $n = 18$, randomly distributed in a 3D environment.

Equilibria and Stability. In the first set, we present the evolution of the population distribution over time. We set the constant parameters to $\Delta_u(\tilde{\tau}) = g = 1$ and we vary parameter c as $c = 1$ in the first scenario, and $c = 2$ in the second scenario.

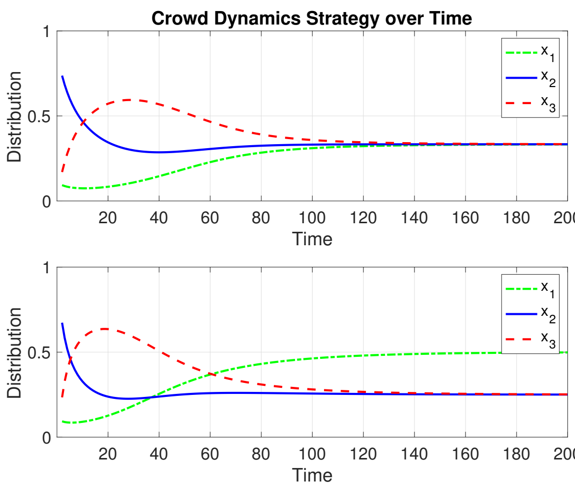


Fig. 2: Plot describing the population evolution over time of system (4) for $c = 1$ (top) and $c = 2$ (bottom).

Figure 2 shows how the population distribution evolves over time across the three strategies, namely patient, impatient and neutral. In accordance with Theorem 2, the population distribution converges towards the equilibrium point, explicitly calculated as $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is asymptotically stable. In Fig. 2 (top), it can be observed that players choosing strategy impatient decrease proportionally to the cost of congestion c in favour of strategy neutral. Thereafter, a number of these players proportional to g, parameter that represents the gain of escaping in an orderly manner, shifts to strategy patient. At the equilibrium, the population is evenly distributed across the three strategies. Given the same initial condition, we now set $c = 2$. The new equilibrium point can be calculated as $x^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. To give a physical interpretation, one can look at the cost of congestion c as a cost that each player has to pay for choosing strategy impatient. As an example, consider a room with many obstacles that can harm the people during their rush to escape if they start to exhibit an impatient behavior. As a result, when c increases, the number of players choosing strategy impatient decreases. Therefore, players are more willing to choose strategy patient, and we can see an increased number at equilibrium. This scenario is depicted in Fig. 2 (bottom).

Spatial Dynamics. In the second set of simulations, we use the population distribution at an equilibrium as input of an evacuation scenario where a finite number of agents have to escape from a room. For this scenario, we generate the environment and define the players' dynamics through Unity, one of the most widely renowned game engines, extensively used in industry for a variety of uses, spanning from games to virtual reality (VR) applications. The simulation environment consists of a simple square room and a single exit. We consider $n = 18$ agents, and each agent i moves on the y-z plane towards the exit following these dynamics:

$$\begin{cases} \dot{y}_i = c_{y,i} + \sigma \mathrm{d}\mathcal{B}, \\ \dot{z}_i = c_{z,i}, \end{cases} \tag{8}$$

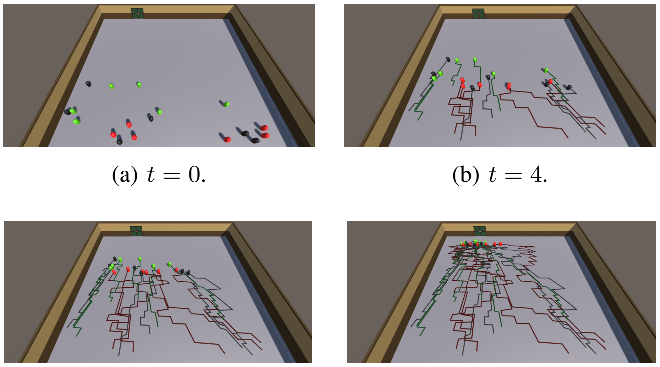
where we use y in place of x to avoid confusion with the distribution x from previous sections, $c_{y,i}$ and $c_{y,i}$ describe the line passing between the agent's initial position and the exit, $\mathrm{d}\mathcal{B}$ is the Brownian motion and σ is its strength.

We represent agents as capsules in the 3D environment, with different colors, one for each strategy: we use green to denote players choosing strategy patient, red for impatient and black for neutral. The initial distribution of players follow the equilibrium point in Theorem 2 when $c = g = \Delta u(\tilde{\tau}) = 1$. Players' strategies influence the strength of the noise as $\sigma = 2$ for patient and $\sigma = 4$ for impatient players, respectively, and we assume that the speed is the same for all players. We treat neutral players as *susceptible*, similarly to the context of opinion dynamics, see [19], meaning that they will mimic the behavior of the closest agents. We randomly apply the noise at each time step with probability 0.5.

Figure 4 depicts the scenario described so far in the context of an evacuation situation. We show four time instants: at $t = 0$, agents are randomly placed at the back of the room and they are assigned to one of the three strategies; at $t = 4$, it can be observed that agents start to move towards the door, and that neutral agents mimic the surrounding players' behavior, e.g. look at the black trajectory on the right-hand side. In the last two time steps, it can be seen that agents move towards the door, at $t = 6$, and finally reach the exit, at $t = 8$, respectively.

We now provide a physical interpretation that links the spatial dynamics to the original formulation of the game, see matrix (1). Due to the higher values of the noise for strategy impatient, players choosing this strategy are more likely to deviate from the line pointing at the exit. This would cause them to hit other agents or obstacles (if any are present). Therefore, the number of collisions would be much higher for these agents than for agents choosing strategy patient. This behavior can be interpreted in terms of the cost of congestion in matrix (1) as in the following: consider the number of impatient players increasing, the number of potential collisions would increase as a result. This would mean a higher probability for two players choosing this strategy to play in random matching, in turn both losing c.

This also explains why, at the equilibrium, the number of impatient agents decreases when the cost of congestion c increases.



(a) $t = 0$. (b) $t = 4$. (c) $t = 6$. (d) $t = 8$.

Fig. 3: Spatial dynamics: at first, agents are randomly positioned at the back of the room (a), then they slowly move towards the door (b) (c), until they approach the exit (d).

V. Conclusion

In this paper, we have formulated a discrete-state continuous-time evolutionary game for a game model originating in the context of high-stress emergency evacuations. To model this problem, we extended the most common game formulation in the literature to include a gain when the players choose the strategy patient. This accounts for the situation where all the players would benefit from an orderly escape. We have determined the equilibria of the system and carried out the stability analysis for each equilibrium point. Finally, we have studied a multi-population model and the impact of the distance on the players' interactions. Further directions of research include: i) the study of crowd dynamics in immersive virtual environments (XR), and ii) the closed loop analysis of the system where micro-macro dynamics are joined together in a single system.

Appendix

Proof of Theorem 1. To study the equilibrium points of the system, we take into account the conservation of mass and consider system (5). We impose the condition $\dot{x}_1 = \dot{x}_2 = 0$ and, after simple algebraic calculations, obtain the following:

$$x_1(g - \Delta u(\tilde{\tau}))(1 - x_1) + x_2(cx_2 - gx_1) = 0.$$

The above equation admits the following solutions $x_1 = 1$ and $x_2 = 0$, and $x_1 = 0$ and $x_2 = 0$, which lead to the equilibria that can be trivially calculated in **Case 1** and in **Case 2**, respectively. In **Case 3**, the calculation requires more effort. The most important steps are detailed in the following. First, we take the first equation in (5) and we obtain $x_1x_2(-\Delta u(\tilde{\tau}) - g) + gx_1(1 - x_1) = 0$, and then we solve for x_2 as:

$$x_2 = -\frac{gx_1(1 - x_1)}{x_1(\Delta u(\tilde{\tau}) + g)} = -\frac{g(1 - x_1)}{(\Delta u(\tilde{\tau}) + g)}.$$

Now, we substitute the calculated x_2 in the second equation of (5) as:

$$\begin{aligned} 0 &= -cx_2^2 + \Delta u(\tilde{\tau})x_1(1 - x_1 - x_2) \\ &= -c\Big(- \tfrac{g(1-x_1)}{d+g}\Big)^2 + \Delta u(\tilde{\tau})x_1\Big(1 - x_1 - - \tfrac{g(1-x_1)}{d+g}\Big). \end{aligned}$$

Then, let us use $\Delta := \Delta u(\tilde{\tau})$ for the sake of brevity, and let us expand the square in the above equation as:

$$\begin{aligned} 0 &= -c\tfrac{g^2x_1^2-2g^2x_1+g^2}{\Delta^2+2\Delta g+\Delta^2} + \Delta x_1\tfrac{-\Delta x_1-g}{\Delta+g} + \Delta x_1 \\ &= \tfrac{-g^2cx_1^2+2g^2cx_1-g^2c-\Delta^3x_1^2-\Delta^2gx_1^2-\Delta^2gx_1+\Delta^3x_1+2\Delta^2gx_1}{\Delta^2+2\Delta g+\Delta^2}. \end{aligned}$$

Next, let us rearrange the equation in the standard second-order form $ax^2 + bx + c = 0$:

$$\tfrac{-g^2c-\Delta^3-\Delta^2g}{\Delta^2+2\Delta g+\Delta^2}x_1^2 + \tfrac{2g^2c+\Delta^2g+\Delta^3}{\Delta^2+2\Delta g+\Delta^2}x_1 - \tfrac{g^2c}{\Delta^2+2\Delta g+\Delta^2} = 0.$$

By solving the above second-order equation, we obtain the following solutions: $x_1 = 1$ and $x_1 = \frac{g^2c}{\Delta^3+\Delta^2g+g^2c}$. The first solution would lead to the same equilibrium point in **Case 1**, so we take the second solution and substitute it in the first equation of (5). After a long calculation, which we omit for the sake of brevity, we have the following:

$$x_2 = \tfrac{\Delta^2g}{(\Delta^3+\Delta^2g+g^2c)}.$$

Having the solutions for the first two states, x_1 and x_2, we can finally calculate the value of x_3 at the equilibrium for (4), by using the conservation of mass:

$$x_3 = 1 - x_1 - x_2 = \tfrac{\Delta^3}{(\Delta^3+\Delta^2g+g^2c)}.$$

Therefore, the corresponding equilibrium point is the one as in **Case 3**. This concludes our proof. ■

Proof of Corollary 1. In *Case 3*, we consider the equilibrium point $(\frac{c}{g}\bar{x}, \bar{x}, \bar{x})$, where $\bar{x}$ is the value to be calculated. By applying the conservation of mass, we compute $\bar{x} = 1/(2 + \frac{c}{g})$, and, by substituting $\bar{x}$ in the previous equation, we obtain the following: $x^* = \left(\frac{c}{g\left(2+\frac{c}{g}\right)}, \frac{1}{2+\frac{c}{g}}, \frac{1}{2+\frac{c}{g}}\right)$, which corresponds, after simple algebra, to the equilibrium point calculated in *Case 3* when $x_1 = c/gx_2$ and $g = \Delta u(\tilde{\tau})$. This concludes our proof. ■

Proof of Theorem 2. To carry out the stability analysis, we apply Lyapunov linearisation method about each equilibrium point. First, we calculate the Jacobian matrix for system (5) for a generic equilibrium point x^* as in the following:

$$J(x^*) = \begin{bmatrix} -\Delta u(\tilde{\tau})x_2^* + g - 2gx_1^* - gx_2^* & -\Delta u(\tilde{\tau})x_2^* - gx_1^* \\ \Delta u(\tilde{\tau}) - 2\Delta u(\tilde{\tau})x_1^* - \Delta u(\tilde{\tau})x_2^* & -2cx_2^* - \Delta u(\tilde{\tau})x_2^* \end{bmatrix}.$$

By linearising about each equilibrium point in Theorem 1, we calculate the corresponding trace and determinant. In *Case 1*, the trace is calculated as $\mathrm{Tr}(J(1, 0, 0)) = -g$ and the determinant $\det(J(1, 0, 0)) = -\Delta u(\tilde{\tau})g$; since the trace is negative but the determinant is also negative, the equilibrium (1,0,0) is a saddle point. When we linearise about the equilibrium point in *Case 2*, we obtain $\mathrm{Tr}(J(0, 0, 1)) = g$ and $\det(J(0, 0, 1)) = 0$; in this case, the trace is positive and the determinant is zero, so the equilibrium (0,0,1) is an unstable fixed point. As for *Case 3*, let us use $\Delta := \Delta u(\tilde{\tau})$

for the sake of brevity, and let us calculate each entry of the Jacobian matrix, denoted as J_3, about the equilibrium point:

$$\begin{aligned} j_{11} &= \frac{-\Delta^3 g - 2cg^2 - \Delta^2 g^2 + \Delta^3 g + \Delta^2 g^2 + cg^3}{\Delta^3+\Delta^2 g+g^2 c} = -\frac{cg^3}{\Delta^3+\Delta^2 g+g^2 c}, \\ j_{12} &= -\frac{\Delta(\Delta^2 g)}{\Delta^3+\Delta^2 g+g^2 c} - \frac{g(cg^2)}{\Delta^3+\Delta^2 g+g^2 c} = -\frac{\Delta^3 g+cg^3}{\Delta^3+\Delta^2 g+g^2 c}, \\ j_{21} &= \frac{-\Delta^4+\Delta^3 g+\Delta cg^2-2\Delta cg^2-\Delta^3 g}{\Delta^3+\Delta^2 g+g^2 c} = \frac{\Delta^4-\Delta cg^2}{\Delta^3+\Delta^2 g+g^2 c}, \\ j_{22} &= -\frac{2c(\Delta^2 g)}{\Delta^3+\Delta^2 g+g^2 c} - \frac{\Delta(\Delta^2 g)}{\Delta^3+\Delta^2 g+g^2 c} = -\frac{2\Delta^2 cg+\Delta^3 g}{\Delta^3+\Delta^2 g+g^2 c}. \end{aligned}$$

Therefore, the trace and the determinant of J_3 are calculated as:

$$\begin{aligned} \mathrm{Tr}(J_3) &= -\frac{cg^3+2\Delta^2 cg+\Delta^3 g}{\Delta^3+\Delta^2 g+g^2 c}, \\ \det(J_3) &= \frac{2\Delta^2 c^2 g^4+\Delta^3 cg^4+\Delta^7 g-\Delta c^2 g^5}{(\Delta^3+\Delta^2 g+g^2 c)^2}, \end{aligned}$$

where it is straightforward to note that the trace is negative. For the determinant to be positive, we verify that $2\Delta c^2 g^3 + \Delta^2 cg^3 + \Delta^7 > c^2 g^4$ holds true for strictly positive parameters, and therefore we have an asymptotically stable point. A similar calculation can be done to prove that the equilibrium point in Corollary 1 is still asymptotically stable. By linearising about the reduced equilibrium point and multiplying all the factors by $2g+c$, the calculation simplifies as $\mathrm{Tr}(\bar{J}_3) = -g^2 - 3cg < 0$ and $\det(\bar{J}_3) = cg^3 + c^2 g^2 > 0$. Therefore, the equilibrium in *Case 3* is asymptotically stable. Finally, when we assume that the first condition in Corollary 2 holds true, the above trace and determinant become: $\mathrm{Tr}(\bar{J}_3) = -g^2 < 0$ and $\det(\bar{J}_3) = 0$, which implies that the corresponding fixed point is asymptotically stable. This concludes our proof. ■

Proof of Theorem 3. We calculate the trace and the determinant of matrix $A_k(\theta)$. The trace is calculated as $\mathrm{Tr}(A_k(\theta)) = -\eta_k \Delta u(\tilde{\tau})(\theta_1 + \theta_2) - g\theta_1 - c\theta_2$, which is negative definite. The determinant is calculated as in the following: $\det(A_k(\theta)) = \eta_k^2 \Delta u(\tilde{\tau})\theta_1\theta_2 + \eta_k \Delta u(\tilde{\tau}) c\theta_2^2 + cg\theta_1\theta_2$, which is positive definite. Therefore, since the trace is positive and the determinant is negative, system (7) is locally asymptotically stable. We now compute $\Delta = \mathrm{Tr}(A_k(\theta))^2 - 4\det(A_k(\theta)) = (-\eta_k \Delta u(\tilde{\tau})(\theta_1 + \theta_2) - g\theta_1 - c\theta_2)^2 - 4(\eta_k^2 \Delta u(\tilde{\tau})\theta_1\theta_2 + \eta_k \Delta u(\tilde{\tau}) c\theta_2^2 + cg\theta_1\theta_2) = \eta_k^2 \Delta u(\tilde{\tau})^2(\theta_1^2 + \theta_2^2 - 3\theta_1\theta_2) + \eta_k \Delta u(\tilde{\tau})(g\theta_1\theta_2 + g\theta_1^2 + c\theta_1\theta_2 - 3\theta_2^2) - 3cg\theta_1\theta_2$. We can calculate the eigenvalues as $\lambda_{1,2} = \frac{\mathrm{Tr}\pm\sqrt{\Delta}}{2}$; in the case of $\eta_k = 0$, the eigenvalues are:

$$\lambda_{1,2} = \left(\frac{-g\theta_1 - c\theta_2 + i\sqrt{3cg\theta_1\theta_2}}{2}, \frac{-g\theta_1 - c\theta_2 - i\sqrt{3cg\theta_1\theta_2}}{2}\right),$$

while in the case $\eta_k = 1$, we have:

$$\lambda_{1,2} = \frac{-\Delta u(\tilde{\tau})(\theta_1+\theta_2) - g\theta_1 - c\theta_2 \pm \sqrt{\Delta_{\eta_k=1}}}{2},$$

where $\Delta_{\eta_k=1} = \Delta u(\tilde{\tau})^2(\theta_1^2 + \theta_2^2 - 3\theta_1\theta_2) + \Delta u(\tilde{\tau}(g\theta_1\theta_2 + g\theta_1^2 + c\theta_1\theta_2 - 3\theta_2^2) - 3cg\theta_1\theta_2$. This concludes our proof. ■

Proof of Theorem 4. To prove that $[x_{1,k}^*, x_{2,k}^*]^T = -A_k^{-1}(\theta)c_k(\theta)$ are the fixed points of system (7), we need to prove that matrix $A_k(\theta)$ is invertible, i.e. $\det(A_k(\theta)) \neq 0$, and that $(-A_k^{-1}(\theta)c_k(\theta))$ is element-wise positive. From Theorem 3, we know that $\det(A_k(\theta)) > 0$, and therefore the matrix is invertible. To prove the second part of the theorem, we calculate the equilibria as in the following:

$$\begin{aligned} \begin{bmatrix} x_{1,k}^* \\ x_{2,k}^* \end{bmatrix} &= -A_k^{-1}(\theta)c_k(\theta) \\ &= -\frac{\mathrm{adj}(A_k(\theta))}{\eta_k(\eta_k \Delta u(\tilde{\tau})^2\theta_1\theta_2 + \Delta u(\tilde{\tau})c\theta_2^2) + cg\theta_1\theta_2} c_k(\theta) \\ &= \frac{1}{\det(A_k(\theta))} \begin{bmatrix} cg\theta_1\theta_2 + \eta_k \Delta u(\tilde{\tau}) g\theta_1^2 - \eta_k^2 \Delta u(\tilde{\tau})^2 \theta_1^2 \\ -g^2\theta_1^2 + \eta_k^2 \Delta u(\tilde{\tau})^2 \theta_1\theta_2 + \eta_k \Delta u(\tilde{\tau}) g\theta_1^2 \end{bmatrix}. \end{aligned}$$

This concludes our proof. ■

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