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Mitankin, Vladimir; Salgado, Cecília

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RATIONAL POINTS ON DEL PEZZO SURFACES OF DEGREE FOUR

VLADIMIR MITANKIN AND CECÍLIA SALGADO

Abstract. We study the distribution of the Brauer group and the frequency of the Brauer–Manin obstruction to the Hasse principle and weak approximation in a family of smooth del Pezzo surfaces of degree four over the rationals. We also study the geometry and arithmetic of a genus one fibration with two reducible fibres for which a Brauer element is vertical.

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1. Introduction

A del Pezzo surface of degree four X over $\mathbb{Q}$ is a smooth projective surface in $\mathbb{P}^4$ given by the complete intersection of two quadrics defined over $\mathbb{Q}$. Such surfaces have been the object of study of several papers throughout the last half century. A proeminent reason for that is the fact that they provide the simplest example of failure of the Hasse Principle for surfaces. Indeed, the simplest class of surfaces, namely those with Kodaira dimension $-\infty$, is formed by rational and ruled surfaces. The arithmetic of the latter is determined by

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that of del Pezzo and conic bundle surfaces. Del Pezzo surfaces see their level of arithmetic and geometric complexity increase inversely to its degree and those of degree at least 5 admiting a rational point are always $\mathbb{Q}$-rational. Hence quartic del Pezzo surfaces form the first class for which interesting arithmetic phenomena, as for instance failures of the Hasse Principle, can occur. They are the object of study of this paper.

A conjecture of Colliot-Thélène and Sansuc [CTS80] predicts that all such failures are explained by the Brauer–Manin obstruction. This is a cohomological obstruction developed by Manin [Man74] which exploits the fact that for a smooth, geometrically irreducible variety X over $\mathbb{Q}$ there is pairing between the set of adèles $X(\mathbf{A}_{\mathbb{Q}})$ and the Brauer group $\operatorname{Br} X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ of X. Manin showed that the set $X(\mathbb{Q})$ of rational points on X lies inside the left kernel $X(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X}$ of this paring. A *Brauer–Manin obstruction* to the Hasse principle is then present if $X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ but $X(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X} = \emptyset$. Such an obstruction may occur only if $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ is non-trivial.

Colliot-Thélène and Sansuc's conjecture is established, for a general del Pezzo surface of degree four, under Schinzel's hypothesis and the finiteness of Tate–Shafarevich groups of elliptic curves by Wittenberg [Wit07, Thm. 3.36.] when $\operatorname{Br} X = \operatorname{Br} \mathbb{Q}$ and by Várilly-Alvarado and Viray [VAV14, Thm. 1.5.] when X is of BSD type. The latter corresponds to the complete intersection in $\mathbb{P}^4$ of the following two quadrics

$$cx_3x_4 = x_2^2 - \varepsilon x_0^2,$$
$$(x_3 + x_4)(ax_3 + bx_4) = x_2^2 - \varepsilon x_1^2,$$

where $a, b, c \in \mathbb{Q}^*$ and $\varepsilon \in \mathbb{Q} \setminus \mathbb{Q}^{*2}$ with $(a - b)(a^2 + b^2 + c^2 - ab - ac - bc) \neq 0$ and $ab, \varepsilon(a^2 + b^2 + c^2 - ab - ac - bc) \notin \mathbb{Q}^{*2}$.

Quartic del Pezzo surfaces of BSD type with an adelic point always have $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$. In this setting, Jahnel and Schindler [JS17] have shown that all counter-examples to the Hasse principle form a Zariski dense set in the moduli scheme of all del Pezzo surfaces of degree four.

In this paper we consider a different family of del Pezzo surfaces of degree four given as follows. Let $\mathbf{a} = (a_0, \dots, a_4) \in \mathbb{Z}^5_{\text{prim}}$. Then define $X_{\mathbf{a}} \subset \mathbb{P}^4_{\mathbb{Q}}$ by the complete intersection

$$\begin{aligned} x_0x_1 - x_2x_3 &= 0, \\ a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 &= 0. \end{aligned} \tag{1.1}$$

We can assume that $a_0, \dots, a_4$ have no factor in common without any loss of generality, otherwise divide the second equation through that factor. Such $X_{\mathbf{a}}$ are smooth if and only if $(a_0a_1 - a_2a_3) \prod_{i=0}^4 a_i \neq 0$. Here we are interested in the family of all smooth quartic del

Pezzo surfaces given by (1.1), that is

$$\mathcal{F} = \{X_{\mathbf{a}} \text{ as in (1.1)} \ : \ \mathbf{a} \in \mathbb{Z}^5_{\mathrm{prim}} \text{ and } (a_0a_1 - a_2a_3)\prod_{i=0}^{4} a_i \neq 0\}.$$

There are numerous reasons behind our choice of this family. Firstly, different from the BSD type family, the Brauer group does witness a variation as $\mathbf{a}$ runs through $\mathbb{Z}^5$ which makes interesting the problem of studying the frequency of each possible Brauer group. Secondly, surfaces in $\mathcal{F}$ admit two distinct conic bundle structures, making their geometry and hence their arithmetic considerably more tractable. Moreover, for such surfaces the conjecture of Colliot-Thélène and Sansuc is known to hold unconditionally [CT90], [Sal86]. Finally, our surfaces can be thought of as an analogue of diagonal cubic surfaces as they also satisfy the interesting equivalence of $\mathbb{Q}$-rationality and trivial Brauer group. This is shown in Lemma 8.3 which is parallel to [CTKS87, Lem. 1.1].

We order $X_{\mathbf{a}} \in \mathcal{F}$ with respect to the naive height function $|\mathbf{a}| = \max_{0\leq i\leq 4} |a_i|$. Recall that $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) = \prod_{p\leq\infty} X_{\mathbf{a}}(\mathbb{Q}_p)$. Our first result shows that a positive proportion of $X_{\mathbf{a}} \in \mathcal{F}$ have points everywhere locally.

Theorem 1.1. *We have*

$$\lim_{B\to\infty} \frac{\#\left\{X_{\mathbf{a}} \in \mathcal{F} \ : \ |\mathbf{a}| \leq B \ \textit{and} \ X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\right\}}{B^5} = \sigma_\infty \prod_p \sigma_p > 0,$$

where σ_p, σ_∞ are local densities whose values are given in Proposition 3.2.

A first natural step towards understanding the frequency of failures of the Hasse principle for $X_{\mathbf{a}} \in \mathcal{F}$ is to understand how often $\operatorname{Br} X_{\mathbf{a}} \not\simeq \operatorname{Br}\mathbb{Q}$. It is well-understood for quartic del Pezzo surfaces that when non-trivial this quotient is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ [Man74]. For any real $B \geq 1$ let

$$N_{\#\mathcal{A}}(B) = \#\{X_{\mathbf{a}} \in \mathcal{F} \ : \ |\mathbf{a}| \leq B, X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset \text{ and } \operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq \mathcal{A}\},$$

where $\mathcal{A}$ is ether the trivial group, $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.

Our next result shows that $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is almost always of order two.

Theorem 1.2. *We have*

$$B^3 \ll N_1(B) \ll B^3(\log B)^4,$$

$$N_2(B) \sim \left(\sigma_\infty \prod_p \sigma_p\right) B^5,$$

$$N_4(B) = \frac{60}{\pi^2}B^3 + O(B^{5/2}(\log B)^2),$$

as B goes to infinity.

As we saw in Theorem 1.2 there are infinitely many $X_{\mathbf{a}}$ with $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ of order four. However, Remark 2.3 shows that such surfaces never give rise to a Brauer–Manin obstruction and thus all failures of the Hasse principle in $\mathcal{F}$ arise when $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} = \mathbb{Z}/2\mathbb{Z}$. Our next result provides an upper bound for the number of such failures and shows that they appear quite rarely in the family $\mathcal{F}$.

Theorem 1.3. *We have*

$$\#\{X_{\mathbf{a}} \in \mathcal{F} \;:\; |\mathbf{a}| \leq B, X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset \textit{ but } X_{\mathbf{a}}(\mathbb{Q}) = \emptyset\} \ll B^{9/2},$$

as B goes to infinity.

Theorem 1.3 together with Theorem 1.1 give us a better understanding of how often varieties in families have a rational point. This question has raised a significant interest lately with studies by numerous authors [Bha14], [BBL16], [BB14] [Lou18], [LS16], [Ser90], [Sof16]. An answer to it in complete generality seems out of reach with current techniques which makes results as in Theorem 1.3 especially valuable.

We say that $X_{\mathbf{a}}$ *satisfies weak approximation* if the image of $X_{\mathbf{a}}(\mathbb{Q})$ in $X_{\mathbf{a}}(\mathbf{A})$ is dense. The proof of Theorem 1.3 yields that most of the surfaces in $\mathcal{F}$ satisfy the Hasse principle but yet fail weak approximation.

Theorem 1.4. *We have*

$$\#\{X_{\mathbf{a}} \in \mathcal{F} \;:\; |\mathbf{a}| \leq B, X_{\mathbf{a}} \textit{ satisfies weak approximation}\} \ll B^{9/2},$$

as B goes to infinity.

It follows from Theorem 1.2 that the quantity in Theorem 1.4 is $\gg B^3$, since rational surfaces satisfy weak approximation. We would like to point out there are general methods for proving results as in Theorems 1.3 and 1.4 developed in [BBL16] and [Bri18]. However, these methods would yield a bound of the shape $O(B^5/(\log B))$. While our idea closely resembles the one used in [BBL16] and [Bri18], the explicit description of the Brauer group elements here allows us to get a power saving in the upper bounds obtained in Theorems 1.3 and 1.4. Thus Theorem 1.3 and 1.4 do not follow from the general tools.

Our second aim is to take advantage of the two conic bundle structures in the surfaces studied in the first part to give a throughout description of a genus one fibration with two reducible fibres for which a Brauer element is vertical. Such a fibration is known to exist thanks to [VAV14]. More precisely, we show that the two reducible fibres are of type I_4 and that the field of definition of the Mordell–Weil group of the associated elliptic surface depends on the order of the Brauer group.

This paper is organised as follows. In Section 2 we describe explicitly the two conic bundle structures on $X_{\mathbf{a}}$ and use them to compute $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$. Section 3 is dedicated to the study of the local points and the local densities σ_p and σ_∞. In Section 4 we prove Theorem 1.1. Sections 5 and 6 are dedicated to the proof of Theorem 1.2. The proofs of Theorems 1.3 and 1.4 are contained in Section 7. The final two sections are dedicated to the second aim described in the previous paragraph. Section 8 is devoted to the arithmetic of the lines on $X_{\mathbf{a}}$, giving the tools to, in Section 9 describe a genus one fibration with exactly two reducible fibres for which a Brauer element is vertical.

Notation. We fix once and for all $d = a_0a_1 - a_2a_3$. For a field k and a variety X over k we use $k(X)$ for the function field of X.

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2. Description of the Brauer group

This section is dedicated to the study of $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$. We shall give a list of explicit representatives in $\operatorname{Br} X_{\mathbf{a}}$ of the elements generating $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ which will later allow us to obtain the upper bounds in Theorems 1.3 and 1.4. This is done via a classical approach for computing the Brauer group of a conic bundle surface. We begin by first embedding

$X_{\mathbf{a}}$ in the scroll $\mathbb{F}(1,1,0)$ following [Rei97, §2]. A short summary of this is contained in [FLS18, §2]. Our method closely follows [LM20, §3].

Recall that $X_{\mathbf{a}}$ was given by (1.1). As explained in [Bro09, Ch. 2] if one of the quadrics defining $X_{\mathbf{a}}$ is of the shape $x_0x_1 - x_2x_3 = 0$, then there is a pair of morphisms $\pi_1 : X_{\mathbf{a}} \to \mathbb{P}^1$ and $\pi_2 : X_{\mathbf{a}} \to \mathbb{P}^1$ defined over $\mathbb{Q}$ each of which endows $X_{\mathbf{a}}$ with a different conic bundle structure. This can be seen in the following way. The map

$$\mathbb{F}(1,1,0) \to \mathbb{P}^4,$$
$$(s,t;x,y,z) \mapsto (sx : ty : tx : sy : z)$$

defines an isomorphism between $X_{\mathbf{a}}$ and

$$(a_0s^2 + a_2t^2)x^2 + (a_3s^2 + a_1t^2)y^2 + a_4z^2 = 0 \subset \mathbb{F}(1,1,0). \tag{2.1}$$

One can view $\mathbb{F}(1,1,0) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ as $((\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^3 \setminus 0))/\mathbb{G}_m^2$, where the action of $\mathbb{G}_m^2$ on $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^3 \setminus 0)$ is described by

$$(\lambda, \mu) \cdot (s,t;x,y,z) = (\lambda s, \lambda t; \frac{\mu}{\lambda}x, \frac{\mu}{\lambda}y, \mu z).$$

Then $\pi_1 : X_{\mathbf{a}} \to \mathbb{P}^1$ is obtained by projecting to (s,t). It is now clear that each fibre of π_1 is a conic and thus $X_{\mathbf{a}}$ is a conic bundle over the projective line.

Similarly, one obtains $\pi_2 : X_{\mathbf{a}} \to \mathbb{P}^1$ via the map

$$\mathbb{F}(1,1,0) \to \mathbb{P}^4,$$
$$(s,t;x,y,z) \mapsto (tx : sy : ty : sx : z).$$

It gives a second conic bundle structure on $X_{\mathbf{a}}$ as shown by the equation

$$X_{\mathbf{a}}: \quad (a_0t^2 + a_3s^2)x^2 + (a_1s^2 + a_2t^2)y^2 + a_4z^2 = 0 \subset \mathbb{F}(1,1,0). \tag{2.2}$$

It follows from (2.1) that the conic associated to the generic fibre of π_1 takes the shape $-a_4(a_0s^2 + a_2t^2)x^2 - a_4(a_3s^2 + a_1t^2)y^2 - z^2 = 0$. There is an associated to it quaternion algebra Q in the Brauer group of the function field of $\mathbb{P}^1$ given by

$$Q = (-a_4(a_0(s/t)^2 + a_2), -a_4(a_3(s/t)^2 + a_1)). \tag{2.3}$$

The quaternion algebra Q has a trivial residue at any closed point of $\mathbb{P}^1$ corresponding to a non-singular fibre of π_1. Its residues over the singular fibres of π_1 are described in the next lemma.

Lemma 2.1. *The following holds.*

(i) *The map $\pi_1 : X_{\mathbf{a}} \to \mathbb{P}^1$ has 4 singular geometric fibres.*

(ii) *The bad fibres lie over the zero locus of*

$$\Delta(s,t) = (a_0s^2 + a_2t^2)(a_3s^2 + a_1t^2).$$

(iii) *Assume that* $-a_0a_2, -a_1a_3 \notin \mathbb{Q}^{*2}$. *Let* T', T'' *be the closed points corresponding the zero locus of* $a_0s^2 + a_2t^2$ *and* $a_3s^2 + a_1t^2$, *respectively. They have residue fields* $\mathbb{Q}(T') = \mathbb{Q}(\sqrt{-a_0a_2})$ *and* $\mathbb{Q}(T'') = \mathbb{Q}(\sqrt{-a_1a_3})$. *The fibres over* T', T'' *have the following residues:*

$$\operatorname{Res}_{T'}(Q) = -a_0a_4d \in \mathbb{Q}(T')^*/\mathbb{Q}(T')^{*2},$$
$$\operatorname{Res}_{T''}(Q) = -a_1a_4d \in \mathbb{Q}(T'')^*/\mathbb{Q}(T'')^{*2}.$$

Proof. Follows immediately from the explicit equation (2.1) and a simple calculation. □

We continue with the structure of $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ given in the next proposition.

Proposition 2.2. *Let* $(*)$ *denote the condition that* $-a_0a_4d \notin \mathbb{Q}(\sqrt{-a_0a_2})^{*2}$, $-a_1a_4d \notin \mathbb{Q}(\sqrt{-a_1a_3})^{*2}$ *and that one of* $-a_0a_2$, $-a_1a_3$ *or* a_0a_1 *is not in* $\mathbb{Q}^{*2}$. *Then we have*

$$\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } a_0a_1, a_2a_3, -a_0a_2 \in \mathbb{Q}^{*2} \text{ and } -a_0a_4d \notin \mathbb{Q}^{*2}, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (*), \\ \{\mathrm{id}\} & \text{if } -a_0a_4d \in \mathbb{Q}(\sqrt{-a_0a_2})^{*2} \text{ or } -a_1a_4d \in \mathbb{Q}(\sqrt{-a_1a_3})^{*2}. \end{cases}$$

Proof. Let $T = s/t$ be the variable on the base $\mathbb{P}^1$ of the conic bundle $\pi_1 : X_{\mathbf{a}} \to \mathbb{P}^1$. Let T', T'' be the closed points of $\mathbb{P}^1$ corresponding to the zero loci of $a_0s^2 + a_2t^2$ and $a_3s^2 + a_1t^2$, respectively. Recall that $Q \in \operatorname{Br}\mathbb{Q}(T)$ given in (2.3) is the quaternion algebra corresponding to the generic fibre of π_1. Let $\alpha \in \operatorname{Br} X_{\mathbf{a}}$. Then the image of α in the Brauer group $\operatorname{Br}\mathbb{Q}(X_{\mathbf{a}})$ of the function field of $X_{\mathbf{a}}$ is the pull-back π_1^*A of some $A \in \operatorname{Br}\mathbb{Q}(T)$ by [CTSD94, Thm. 2.2.1]. This is illustrated in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Br}\mathbb{Q} \arrow[r] \arrow[d, "\pi_1^*"] & \operatorname{Br}\mathbb{Q}(T) \arrow[r] \arrow[d, "\pi_1^*"] & \bigoplus \mathrm{H}^1(\mathbb{Q}[T]/(P(T)), \mathbb{Q}/\mathbb{Z}) \\ 0 \arrow[r] & \operatorname{Br} X_{\mathbf{a}} \arrow[r] & \operatorname{Br}\mathbb{Q}(X_{\mathbf{a}}) \arrow[r] & \bigoplus \mathrm{H}^1(\mathbb{Q}(Y), \mathbb{Q}/\mathbb{Z}), \end{tikzcd} \tag{2.4}$$

where the top sum is taken over all irreducible polynomials $P(T) \in \mathbb{Q}[T]$ and the bottom sum is taken over all integral subvarieties Y of $X_{\mathbf{a}}$ of codimension 1.

Assume first that $-a_0a_4d \in \mathbb{Q}(\sqrt{-a_0a_2})^{*2}$. By Lemma 2.1 Q has a trivial residue at T'. Moreover, [CTSD94, Thm. 2.2.1] and Remark 2.2.3 after it imply that the residue $\operatorname{Res}_{T'}(A)$ of A at T' is trivial and that $\operatorname{Res}_{T''}(A)$ is either trivial or equal to $\operatorname{Res}_{T''}(Q)$. In both cases

[CTSD94, Thm. 2.2.1] implies that $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is trivial. Similarly, $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is trivial if $-a_1a_4d \in \mathbb{Q}(\sqrt{-a_1a_3})^{*2}$.

Assume now that $-a_0a_4d \notin \mathbb{Q}(\sqrt{-a_0a_2})^{*2}$ and that $-a_1a_4d \notin \mathbb{Q}(\sqrt{-a_1a_3})^{*2}$. If $-a_0a_2$, $-a_1a_3$ are not squares of $\mathbb{Q}^*$, then T' and T'' are both degree two points. By [CTSD94, Thm. 2.2.1] there are only two possibilities for α. Let $\alpha_1 = \pi_1^*A_1$ and $\alpha_2 = \pi_1^*A_2$ realise them, that is

- $\operatorname{Res}_{T'}(A_1) = -a_0a_4d$ and $\operatorname{Res}_{T''}(A_1) = 1$,
- $\operatorname{Res}_{T'}(A_2) = 1$ and $\operatorname{Res}_{T''}(A_2) = -a_1a_4d$.

By Faddeev's reciprocity law which is the top line of (2.4) we have $A_1 + A_2 = Q$ modulo elements of $\operatorname{Br}\mathbb{Q}$. Thus by [CTSD94, Thm. 2.2.1] we have $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$ generated by α_1. A similar analysis shows the claim in the remaining cases covered by $(*)$.

Assume now that $a_0a_1, a_2a_3, -a_0a_2 \in \mathbb{Q}^{*2}$ and $-a_0a_4d \notin \mathbb{Q}^{*2}$. Then $T' = \{T_1', T_2'\}$ and $T'' = \{T_1'', T_2''\}$ consist of two degree one points and Q has a non-trivial residue equal to $-a_0a_4d$ at each of these degree one points. By [CTSD94, Thm. 2.2.1] there are six possibilities $\alpha_i = \pi_1^*A_i$ for α which we list below.

- $\operatorname{Res}_{T_1'}(A_1) = \operatorname{Res}_{T_2'}(A_1) = -a_0a_4d$ and $\operatorname{Res}_{T_1''}(A_1) = \operatorname{Res}_{T_2''}(A_1) = 1$,
- $\operatorname{Res}_{T_1''}(A_2) = \operatorname{Res}_{T_2''}(A_2) = -a_0a_4d$ and $\operatorname{Res}_{T_1'}(A_2) = \operatorname{Res}_{T_2'}(A_2) = 1$,
- $\operatorname{Res}_{T_1'}(A_3) = \operatorname{Res}_{T_1''}(A_3) = -a_0a_4d$ and $\operatorname{Res}_{T_2'}(A_3) = \operatorname{Res}_{T_2''}(A_3) = 1$,
- $\operatorname{Res}_{T_2'}(A_4) = \operatorname{Res}_{T_2''}(A_4) = -a_0a_4d$ and $\operatorname{Res}_{T_1'}(A_4) = \operatorname{Res}_{T_1''}(A_4) = 1$,
- $\operatorname{Res}_{T_1'}(A_5) = \operatorname{Res}_{T_2''}(A_5) = -a_0a_4d$ and $\operatorname{Res}_{T_1''}(A_5) = \operatorname{Res}_{T_2'}(A_5) = 1$,
- $\operatorname{Res}_{T_2'}(A_6) = \operatorname{Res}_{T_1''}(A_6) = -a_0a_4d$ and $\operatorname{Res}_{T_1'}(A_6) = \operatorname{Res}_{T_2''}(A_6) = 1$.

Again on the level of residues and by Faddeev's reciprocity law we conclude that neither of the α_i is trivial in $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ and that we have the relations

$$A_1 + A_2 = A_3 + A_4 = A_5 + A_6 = Q \bmod \operatorname{Br}\mathbb{Q},$$
$$A_1 + A_3 = A_6 \bmod \operatorname{Br}\mathbb{Q}.$$

Thus $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is generated by α_1, α_3. This completes the proof. □

Remark 2.3. Note that in the case when $-a_0a_2 \in \mathbb{Q}^{*2}$ there is an obvious rational point $(s,t;x,y,z) = (1 : \sqrt{-a_0/a_2}; 1 : 0 : 0)$ on $X_{\mathbf{a}}$. Thus there is no Brauer–Manin obstruction to the existence of rational points on $X_{\mathbf{a}}$ if $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is of order four.

2.1. **Generators.** In order to study the Brauer–Manin obstruction to the Hasse principle or to weak approximation we can assume that $X_{\mathbf{a}}$ has non-trivial Brauer group, that is

$-a_1a_4d \notin \mathbb{Q}(\sqrt{-a_1a_3})^{*2}, -a_0a_4d \notin \mathbb{Q}(\sqrt{-a_0a_2})^{*2}$. Moreover, by Theorem 1.2 the number of surfaces with Brauer group of order four is negligible compared to the bounds in Theorems 1.3 and 1.4 and thus we can further assume that $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$.

We continue with an explicit description of a generator of $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$. Let $A \in \operatorname{Br}\mathbb{Q}(\mathbb{P}^1)$ be given by

$$A = (a_0(s/t)^2 + a_2, -a_0a_4d).$$

It is clear that the image $\alpha = \pi_1^* A \in \operatorname{Br}\mathbb{Q}(X_{\mathbf{a}})$ of A is unramified along each irreducible divisor of $X_{\mathbf{a}}$ except possibly on $D = \{a_0(s/t)^2 + a_2 = 0\} \subset X_{\mathbf{a}}$. Along D one has

$$-a_0a_4d = (a_0a_4z/ty)^2.$$

Thus $\operatorname{Res}_{\mathbb{Q}(D)} \alpha = -a_0a_4d \in \mathbb{Q}(D)^*/\mathbb{Q}(D)^{*2}$ is trivial which implies that α is unramified on D. Alternatively, one can check that $\mathbb{Q}(D) = \mathbb{Q}(\sqrt{-a_0a_2}, \sqrt{-a_0a_4d})(T)$ where $-a_0a_4d$ is clearly a square. Since $X_{\mathbf{a}}$ is smooth we can apply Grothendieck's purity theorem (the bottom line of (2.4)) to conclude that α lies inside the image of $\operatorname{Br} X_{\mathbf{a}} \to \operatorname{Br}\mathbb{Q}(X_{\mathbf{a}})$. Moreover, it follows by Proposition 2.2 that $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$ and that α is a non-trivial element there.

3. Local points

3.1. **Local solubility.** In this subsection we are concerned with the existence of local points on $X_{\mathbf{a}}$ given in (1.1). To do so we first define an equivalence relation on $\mathbb{Z}_{\geq 0}^5$ in the spirit of [BBL16, §2]. We say that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \sim (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$$

if and only if at least one of the following holds.

- $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_1, \beta_0, \beta_2, \beta_3, \beta_4)$,
- $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_2, \beta_3, \beta_0, \beta_1, \beta_4)$,
- $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\beta_0, \beta_1, \beta_2, \beta_3,)$ and $\alpha_4 \equiv \beta_4 \bmod 2$.
- There are $k, \ell, m, n \in \mathbb{Z}$ satisfying $k + \ell = m + n$ such that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_0 + 2k, \beta_1 + 2\ell, \beta_2 + 2m, \beta_3 + 2n, \beta_4).$$

- There is some $k \in \mathbb{Z}$ such that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_0 + k, \beta_1 + k, \beta_2 + k, \beta_3 + k, \beta_4 + k).$$

The above equivalence relation has the property that for each field K containing $\mathbb{Q}$ and each $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}_{\geq 0}^5$ with

$$(\mathrm{v}_p(a_0), \mathrm{v}_p(a_1), \mathrm{v}_p(a_2), \mathrm{v}_p(a_3), \mathrm{v}_p(a_4)) \sim (\mathrm{v}_p(a_0'), \mathrm{v}_p(a_1'), \mathrm{v}_p(a_2'), \mathrm{v}_p(a_3'), \mathrm{v}_p(a_4'))$$

we have $X_{\mathbf{a}}(K) \neq \emptyset$ if and only if $X_{\mathbf{a}'}(K) \neq \emptyset$. We will make great use of this fact. Unlike in [BBL16, §2] in our setting when we quotient $\mathbb{Z}_{\geq 0}^5$ by the equivalence relation $\sim$ we do not get a finite list of representatives. Thus we require a more involved approach in order to understand the local solubility of $X_{\mathbf{a}}$. For convenience, if p is an odd prime let

$$\left[\frac{a}{p}\right] = \left(\frac{a/p^{\mathrm{v}_p(a)}}{p}\right),$$

where the second entry is the Legendre symbol. We shall give necessary and sufficient conditions for the existence of local points on $X_{\mathbf{a}}$ in the next proposition.

Proposition 3.1. *Let $p \neq 2$ be a place of $\mathbb{Q}$. Then we have $X_{\mathbf{a}}(\mathbb{Q}_p) = \emptyset$ if and only if one of the following holds.*

(i) *$p = \infty$ and all a_i have the same sign,*

(ii) *$p = 3$, and the following hold*
- $\mathrm{val}_3(a_0) \equiv \mathrm{val}_3(a_1) \equiv \mathrm{val}_3(a_2) \equiv \mathrm{val}_3(a_3) \not\equiv \mathrm{val}_3(a_4) \bmod 2$,
- $\mathrm{val}_3(a_0a_1) = \mathrm{val}_3(a_2a_3)$,
- $\left[\frac{-a_0a_2}{3}\right] = \left[\frac{-a_0a_3}{3}\right] = \left[\frac{-a_1a_2}{3}\right] = \left[\frac{-a_1a_3}{3}\right] = -1.$

(iii) *p is an odd prime, (i,j) is either $(0,1)$ or $(2,3)$, we have $\{\ell, m\} = \{0,1,2,3\} \setminus \{i,j\}$ and the following hold*
- $\mathrm{v}_p(a_i) \equiv \mathrm{v}_p(a_j) \equiv \mathrm{v}_p(a_4) \bmod 2$,
- $\mathrm{v}_p(a_\ell) \equiv \mathrm{v}_p(a_m) \not\equiv \mathrm{v}_p(a_i) \bmod 2$,
- $\mathrm{v}_p(a_ia_j) = \mathrm{v}_p(a_\ell a_m)$ *and* $\left[\frac{-a_ia_4}{p}\right] = \left[\frac{-a_ja_4}{p}\right] = -1$ *or*
  $\mathrm{v}_p(a_ia_j) > \mathrm{v}_p(a_\ell a_m)$ *and* $\left[\frac{-a_ia_4}{p}\right] = \left[\frac{-a_ja_4}{p}\right] = \left[\frac{-a_\ell a_m}{p}\right] = -1.$

(iv) *p is an odd prime, (i,j,k) is one of $(0,1,2)$, $(0,1,3)$, $(2,3,0)$ or $(2,3,1)$ as an ordered triple, $\{\ell\} = \{0,1,2,3\} \setminus \{i,j,k\}$ and the following hold*
- $\mathrm{v}_p(a_i) \equiv \mathrm{v}_p(a_j) \equiv \mathrm{v}_p(a_k) \bmod 2$,
- $\mathrm{v}_p(a_\ell) \equiv \mathrm{v}_p(a_4) \not\equiv \mathrm{v}_p(a_i) \bmod 2$,
- $\mathrm{v}_p(a_ia_j) > \mathrm{v}_p(a_ka_\ell)$,
- $\left[\frac{-a_ia_k}{p}\right] = \left[\frac{-a_ja_k}{p}\right] = \left[\frac{-a_\ell a_4}{p}\right] = -1.$

Proof. One easily verifies that if $p = \infty$, then $X(\mathbb{R}) \neq \emptyset$ if and only if two of the a_i have different signs since this is equivalent to the bottom quadratic from in (1.1) being indefinite.

Let p be an odd prime now. It is clear that at least three of the a_i have the same parity of their p-adic valuations. Let $\alpha = (\mathrm{v}_p(a_0), \mathrm{v}_p(a_1), \mathrm{v}_p(a_2), \mathrm{v}_p(a_3), \mathrm{v}_p(a_4))$. We distinguish between the following cases.

(a) If $\mathrm{v}_p(a_0) \equiv \mathrm{v}_p(a_2) \equiv \mathrm{v}_p(a_4) \bmod 2$ we set $x_1 = x_3 = 0$. It thus suffices to show that the diagonal projective conic $a_0x_0^2 + a_2x_2^2 + a_4x_4^2 = 0$ has a p-adic point. In view of $\sim$ since $\mathrm{v}_p(a_0), \mathrm{v}_p(a_2), \mathrm{v}_p(a_4)$ have the same parity we can assume that $p \nmid a_0a_2a_4$. Such conics are known to have a smooth $\mathbb{F}_p$-point which is easily verified for example by fixing $x_4 \in \mathbb{F}_p^*$ and then counting the number of possible values that $a_0x_0^2$ and $a_2x_2^2 + a_4x_4^2$ can take. Moreover, a smooth $\mathbb{F}_p$-point on the conic lifts to a $\mathbb{Z}_p$-point on $X_{\mathbf{a}}$ with $x_1 = x_3 = 0$ by Hensel's lemma. Indeed, the Jacobian matrix of $X_{\mathbf{a}}$ is

$$J(\mathbf{x}) = \begin{pmatrix} x_1 & x_0 & -x_3 & -x_2 & 0 \\ 2a_0x_0 & 2a_1x_1 & 2a_2x_2 & 2a_3x_3 & 2a_4x_4 \end{pmatrix}.$$

The above argument shows that either the minor $J_{1,2}$ or the minor $J_{3,4}$ has a determinant a unit in $\mathbb{F}_p$ when evaluated at $\mathbf{x}$. The same analysis applies if a_0, a_3, a_4 or a_1, a_2, a_4 or a_1, a_3, a_4 have the same parity of their p-adic valuations.

(b) If $\mathrm{v}_p(a_0) \equiv \mathrm{v}_p(a_1) \equiv \mathrm{v}_p(a_4) \bmod 2$ we can assume that $\mathrm{v}_p(a_2) \equiv \mathrm{v}_p(a_3) \not\equiv \mathrm{v}_p(a_0) \bmod 2$, otherwise we fall into (a). If $p \nmid (a_0, a_1, a_4)$ we can apply a similar argument to the one in (a) by solving $a_0x_0^2 + a_1x_1^2 + a_4x_4^2 \equiv 0 \bmod p$ first and then taking x_2, x_3 such that $x_2x_3 \equiv x_0x_1 \bmod p$. Thus $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$. Moreover, if $\mathrm{v}_p(a_0a_1) < \mathrm{v}_p(a_2a_3)$, then $\alpha \sim (0, 0, 2k+1, 1, 0)$ for some $k \geq 0$ and hence $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$.

Assume that $\mathrm{v}_p(a_0a_1) \geq \mathrm{v}_p(a_2a_3)$. Recall (2.1). Our investigation continues with an analysis of the following system

$$\begin{aligned} a_0(sx)^2 + a_1(ty)^2 + a_4z^2 &= 0, \\ a_2(tx)^2 + a_3(sy)^2 &= 0. \end{aligned}$$

We have $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$ if $\left[\frac{-a_0a_4}{p}\right] = 1$ by setting $t = y = 0$. Similarly, $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$ if $\left[\frac{-a_1a_4}{p}\right] = 1$.

If $\mathrm{v}_p(a_0a_1) = \mathrm{v}_p(a_2a_3)$, then $\alpha \sim (2, 0, 1, 1, 0)$ and hence one can conclude that $X_{\mathbf{a}}(\mathbb{Q}_p) = \emptyset$ if $\left[\frac{-a_0a_4}{p}\right] = \left[\frac{-a_1a_4}{p}\right] = -1$ by looking at the p-adic valuation of possible solutions to the system above.

On the other hand, if $\mathrm{v}_p(a_0a_1) > \mathrm{v}_p(a_2a_3)$, then $\alpha \sim (2m, 2, 1, 1, 2)$ for some $m > 0$ and hence $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$ if $\left[\frac{-a_2a_3}{p}\right] = 1$ since this condition implies the existence of a smooth $\mathbb{F}_p$-point on $X_{\mathbf{a}}$ mod p.

We assume from now on that $\mathrm{v}_p(a_0a_1) > \mathrm{v}_p(a_2a_3)$ and

$$\left[\frac{-a_0a_4}{p}\right] = \left[\frac{-a_1a_4}{p}\right] = \left[\frac{-a_2a_3}{p}\right] = -1.$$

We claim that $X_{\mathbf{a}}(\mathbb{Q}_p) = \emptyset$. If there was a point in $X_{\mathbf{a}}(\mathbb{Q}_p)$, then by looking at the possible p-adic valuations of its coordinates we see that at least one of $\left[\frac{-a_0a_4}{p}\right]$, $\left[\frac{-a_1a_4}{p}\right]$ or $\left[\frac{-a_2a_3}{p}\right]$ has to be trivial, a contradiction! We treat the case $\mathrm{v}_p(a_2) \equiv \mathrm{v}_p(a_3) \equiv \mathrm{v}_p(a_4) \bmod 2$ in a similar way.

(c) If $\mathrm{v}_p(a_0) \equiv \mathrm{v}_p(a_1) \equiv \mathrm{v}_p(a_2) \not\equiv \mathrm{v}_p(a_3) \equiv \mathrm{v}_p(a_4) \bmod 2$, then the analysis is very similar to the one in (b) modulo the fact that there is no symmetry in a_3 and a_4. Here (2.1) transforms into

$$\begin{aligned} a_0(sx)^2 + a_1(ty)^2 + a_2(tx)^2 &= 0,\\ a_3(sy)^2 + a_4z^2 &= 0. \end{aligned}$$

Once again we begin by observing that $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$ unless

$$\left[\frac{-a_0a_2}{p}\right] = \left[\frac{-a_1a_2}{p}\right] = \left[\frac{-a_3a_4}{p}\right] = -1,$$

which we assume from now on. It is clear that $\alpha \sim (2m, 0, 0, 2\ell+1, 1)$ for some $m, \ell \geq 0$. If $2m < 2\ell + 1$, then we reduce once more to finding a p-adic point on the conic $a_0u^2 + a_1v^2 + a_2w^2 = 0$ and thus $X_{\mathbf{a}}(\mathbb{Q}_p) \neq \emptyset$. On the other hand, if $2m > 2\ell+1$, then as in (b) we see that $X_{\mathbf{a}}(\mathbb{Q}_p) = \emptyset$. An identical analysis applies in the remaining cases of (v).

(d) If $\mathrm{v}_p(a_0) \equiv \mathrm{v}_p(a_1) \equiv \mathrm{v}_p(a_2) \equiv \mathrm{v}_p(a_3) \not\equiv \mathrm{v}_p(a_4) \bmod 2$, then via the above relation we can assume that $\alpha = (2m, 0, 0, 0, 1)$ for some $m > 0$ if $\mathrm{v}_p(a_0a_1) > \mathrm{v}_p(a_2a_3)$. Once again by (2.1) we need to analyse

$$a_0(sx)^2 + a_1(ty)^2 + a_2(tx)^2 + a_3(sy)^2 = 0. \tag{3.1}$$

It thus suffices to show that $a_1u^2 + a_2v^2 + a_3w^2 = 0$ has a p-adic point. As explained in (a) this is always the case. A similar argument applies when $\mathrm{v}_p(a_0a_1) < \mathrm{v}_p(a_2a_3)$.

Assume now that $\mathrm{v}_p(a_0a_1) = \mathrm{v}_p(a_2a_3)$ and thus we have $\alpha \sim (0,0,0,0,1)$. In view of (3.1) there is a smooth $\mathbb{F}_p$-point on $X_{\mathbf{a}}$ if $-a_ia_j \bmod p \in \mathbb{F}_p^{*2}$ for some $(i,j) \in \{(0,2),(0,3),(1,2),(1,3)\}$. On the other hand, if the reduction of all of the above $-a_ia_j \bmod p$ is a non-square in $\mathbb{F}_p$, then any $\mathbb{F}_p$-point must satisfy $tx \neq 0$. The change of variables $s/t = X$, $y/x = Y$ reduces the problem to showing the existence of an $\mathbb{F}_p$-point on $Y^2 = -(a_0X^2 + a_2)/(a_3X^2 + a_1)$. In fact, our assumptions imply that any $\mathbb{F}_p$-point on it must be smooth if it exists. Clearly, the existence of such an $\mathbb{F}_p$-point is equivalent to the existence of an $\mathbb{F}_p$-point on the genus one curve

$$C: \quad Y^2 = -(a_0X^2 + a_2)(a_3X^2 + a_1).$$

The Hasse–Weil bound implies that $\#C(\mathbb{F}_p) \geq p+1-2\sqrt{p}$. This quantity is clearly positive if $p \geq 5$. On the other hand, if $p = 3$, then the condition $-a_ia_j \bmod p$ a non-square in $\mathbb{F}_p$ for each $(i,j) \in \{(0,2),(0,3),(1,2),(1,3)\}$ says that $a_0 \equiv a_1 \equiv a_2 \equiv a_3 \bmod 3$. One easily checks that in this case $X_{\mathbf{a}}(\mathbb{Q}_3) = \emptyset$. This completes the proof.

□

3.2. **Local densities.** We continue with the study of the proportion of everywhere locally soluble $X_{\mathbf{a}} \in \mathcal{F}$. For each place p of $\mathbb{Q}$ let $X_{\mathbf{a},p} = X_{\mathbf{a}} \times_{\mathbb{Q}} \mathbb{Q}_p$ and define

$$\Omega_p = \left\{\mathbf{a} \in \mathbb{Z}_p^5 \ : \ p \nmid \mathbf{a}, X_{\mathbf{a},p} \text{ smooth and } X_{\mathbf{a},p}(\mathbb{Q}_p) \neq \emptyset\right\},$$
$$\Omega_\infty = \left\{\mathbf{a} \in [-1,1]^5 \ : \ X_{\mathbf{a},\infty} \text{ smooth and } X_{\mathbf{a},\infty}(\mathbb{R}) \neq \emptyset\right\}.$$

Let μ_p be the normalised Haar measure on $\mathbb{Z}_p^5$ such that $\mu_p\left(\mathbb{Z}_p^5\right) = 1$ and let μ_∞ be the Lebesgue measure on $\mathbb{R}^5$. The local densities σ_p corresponding to $X_{\mathbf{a}}$ are then defined by

$$\sigma_p = \mu_p\left(\Omega_p\right), \quad \sigma_\infty = \frac{\mu_\infty(\Omega_\infty)}{\mu_\infty(\{\mathbf{a} \in [-1,1]^5 \ : \ X_{\mathbf{a},\infty} \text{ smooth}\})}.$$

We shall estimate σ_p in the next proposition. For technical reasons we avoid the calculation of σ_2 here. However, it is easy to see that $\sigma_2 > 0$, for example by calculating the proportion of $\mathbf{a} \in \mathbb{Z}_2^5$ such that a_i are all units in $\mathbb{Z}_2$ and $a_1 \equiv -a_4 \bmod 8$.

Proposition 3.2. *We have $\sigma_\infty = 15/16$ and for each odd prime p the following holds.*

$$\sigma_p = \begin{cases} \frac{63693071}{66355200} & \text{if } p = 3, \\ 1 - \frac{1}{2p^2} + \frac{9}{4p^3} + O\left(\frac{1}{p^4}\right) & \text{if } p > 3. \end{cases}$$

Proof. One clearly has

$$\frac{\mu_\infty(\{\mathbf{a} \in [-1,1]^5 \ : \ (a_0a_1 - a_2a_3)\prod_{i=0}^4 a_i = 0\})}{\mu_\infty([-1,1]^5)} = 0,$$

since each of the conditions $a_i = 0$ or $a_0a_1 - a_2a_3 = 0$ defines a proper subspace of $\mathbb{R}^5$. Thus

$$\sigma_\infty = \frac{\mu_\infty(\Omega_\infty)}{\mu_\infty([-1,1]^5)}.$$

On the other hand, by Lemma 3.1(i) we have $X_{\mathbf{a},\infty}(\mathbb{R}) = \emptyset$ if and only if all a_i have the same sign. Thus

$$\sigma_\infty = 1 - \frac{\mu_\infty(\{\mathbf{a} \in [-1,1]^5 \ : \ \text{all } a_i \text{ have the same sign}\})}{\mu_\infty([-1,1]^5)} = 1 - \frac{1}{2^4} = \frac{15}{16},$$

as claimed.

Assume now that p is an odd prime. Once again we have

$$\mu_p(\{\mathbf{a} \in \mathbb{Z}_p^5 \ : \ (a_0a_1 - a_2a_3)\prod_{i=0}^{4} a_i = 0\}) = 0,$$

which allows us to safely ignore this condition from now on. The conditions on local solubility for $X_{\mathbf{a},p}$ in this case are given in Lemma 3.1(iv), (v) and in (iii) if $p = 3$. We continue with calculating the proportion of surfaces $X_{\mathbf{a},p}$ with $\mathbf{a} \in \mathbb{Z}_p^5$ satisfying each of these conditions.

We begin with (iii). Firstly, we partition the set $S^{(iii)} = S_1^{(iii)} \cup S_1^{(iii)}$ of all $\mathbf{a} \in \mathbb{Z}_p$ satisfying Lemma 3.1(iii) into two disjoint subsets. Here $S_1^{(iii)}$ is the subset of $S^{(iii)}$ consisting of those $\mathbf{a}$ with $\mathrm{v}_p(a_4)$ odd and $S_2^{(iii)}$ is its complement in $S^{(iii)}$. To compute the measure of $S_1^{(iii)}$ let

$$\begin{aligned} &\mathrm{v}_p(a_0) = 2k, \quad \mathrm{v}_p(a_1) = 2\ell, \quad \mathrm{v}_p(a_2) = 2m, \\ &\mathrm{v}_p(a_3) = 2n, \quad \mathrm{v}_p(a_4) = 2r+1. \end{aligned}$$

It is clear that under the above parametrisation the condition $p \nmid \mathbf{a}$ is equivalent to the minimum of k, ℓ, m, n being 0. We have $\left[\frac{-a_0a_2}{3}\right] = \left[\frac{-a_0a_3}{3}\right] = \left[\frac{-a_1a_2}{3}\right] = \left[\frac{-a_1a_3}{3}\right] = -1$ with probability 1/8. The proportion of $a \in \mathbb{Z}_p$ with $\mathrm{v}_p(a) = t$ is $p^{-t}(1 - 1/p)$. Thus we have

$$\begin{aligned} \mu_p(S_1^{(iii)}) &= \frac{1}{8}\left(1 - \frac{1}{p}\right)^5 \sum_{r \geq 0} \frac{1}{p^{2r+1}} \left(\sum_{\substack{k,\ell,m,n \geq 0 \\ k+\ell=m+n}} \frac{1}{p^{2k+2\ell+2m+2n}} - \sum_{\substack{k,\ell,m,n > 0 \\ k+\ell=m+n}} \frac{1}{p^{2k+2\ell+2m+2n}} \right) \\ &= \frac{(p-1)^2(p^4+1)^2}{8p^4(p+1)^3(p^2+1)^2}. \end{aligned}$$

To study $S_2^{(iii)}$ let

$$\begin{aligned} &\mathrm{v}_p(a_0) = 2k+1, \quad \mathrm{v}_p(a_1) = 2\ell+1, \quad \mathrm{v}_p(a_2) = 2m+1, \\ &\mathrm{v}_p(a_3) = 2n+1, \quad \mathrm{v}_p(a_4) = 2r. \end{aligned}$$

Note that in this case the condition $p \nmid \mathbf{a}$ is equivalent to $r = 0$. Then

$$\mu_p(S_2^{(iii)}) = \frac{1}{8}\left(1-\frac{1}{p}\right)^5 \sum_{\substack{k,\ell,m,n\geq 0\\ k+\ell=m+n}} \frac{1}{p^{2k+2\ell+2m+2n+4}} = \frac{(p-1)^2(p^4+1)}{8p(p+1)^3(p^2+1)^3}.$$

Thus we find that

$$\mu_p(S^{(iii)}) = \mu_p(S_1^{(iii)}) + \mu_p(S_2^{(iii)}) = \frac{(p-1)^2(p^2-p+1)(p^4+1)(p^4+p^3+p^2+p+1)}{8p^4(p+1)^3(p^2+1)^3}.$$

We continue with the contribution from Lemma 3.1(iv). Let $S^{(iv)}$ denote the corresponding set. Recall that in Lemma 3.1(iv) we can have (i,j) either $(0,1)$ or $(2,3)$. We first partition $S^{(iv)} = S_{(0,1)}^{(iv)} \cup_{(2,3)}^{(iv)}$, where in $S_{(0,1)}^{(iv)}$ we have taken $(i,j) = (0,1)$ and in $S_{(2,3)}^{(iv)}$ we have taken $(i,j) = (2,3)$. The sets $S_{(0,1)}^{(iv)}$ and $S_{(2,3)}^{(iv)}$ are clearly disjoint. One easily verifies that $\mu_p(S_{(0,1)}^{(iv)}) = \mu_p(S_{(2,3)}^{(iv)})$.

We begin by further partitioning $S_{(0,1)}^{(iv)} = S_1^{(iv)} \cup S_2^{(iv)}$ into two disjoint sets where $S_1^{(iv)}$ consists of those $\mathbf{a} \in \mathbb{Z}_p^5$ with $\mathrm{v}_p(a_4)$ odd. To find the measure of $S_1^{(iv)}$ we interpret the condition $p \nmid \mathbf{a}$ as $\min\{\mathrm{v}_p(a_2), \mathrm{v}_p(a_3)\} = 0$. A similar analysis as before now yields that $\mu_p(S_1^{(iv)})$ equals

$$\left(1-\frac{1}{p}\right)^5 \left(\frac{1}{2} \sum_{\substack{k,\ell,m,r\geq 0\\ k+\ell+1=m}} \frac{1}{p^{2k+2\ell+2m+2r+3}} + \frac{1}{4} \sum_{\substack{k,\ell,r\geq 0\\ k+\ell\geq m>0}} \frac{1}{p^{2k+2\ell+2m+2r+3}} + \frac{1}{8} \sum_{k,\ell,r\geq 0} \frac{1}{p^{2k+2\ell+2r+3}} \right).$$

We conclude that

$$\mu_p(S_1^{(iv)}) = \frac{p^3(p^6+5p^4-p^2+1)}{8(p-1)^4(p+1)^4(p^2+1)^2}.$$

Similarly, we have

$$\begin{aligned}
\mu_p(S_2^{(iv)}) = {} & \frac{1}{4}\left(1-\frac{1}{p}\right)^5 \sum_{m,n\geq 0} \frac{1}{p^{2m+2n+2}} \left(\sum_{\substack{k,\ell,r\geq 0\\ k+\ell=m+n+1}} \frac{1}{p^{2k+2\ell+2r}} - \sum_{\substack{k,\ell,r>0\\ k+\ell=m+n+1}} \frac{1}{p^{2k+2\ell+2r}} \right) \\
& + \frac{1}{8}\left(1-\frac{1}{p}\right)^5 \sum_{m,n\geq 0} \frac{1}{p^{2m+2n+2}} \left(\sum_{\substack{k,\ell,r\geq 0\\ k+\ell>m+n+1}} \frac{1}{p^{2k+2\ell+2r}} - \sum_{\substack{k,\ell,r>0\\ k+\ell>m+n+1}} \frac{1}{p^{2k+2\ell+2r}} \right) \\
& = \frac{(p-1)^2(4p^4+p^2-2)}{8p(p+1)^3(p^2+1)^3}.
\end{aligned}$$

Thus we obtain that the contribution from Lemma 3.1(iv) is

$$\mu_p(S^{(iv)}) = 2(\mu_p(S_1^{(iv)})+\mu_p(S_2^{(iv)})) = \frac{(p+1)(4p^4+p^2-2)(p-1)^6+p^4(p^8+6p^6+4p^4+1)}{4p(p^2-1)^4(p^2+1)^3}.$$

Lastly, we deal with Lemma 3.1(v). With the convention as above we have $S^{(v)} = S^{(v)}_{(0,1,2)} \cup S^{(v)}_{(0,1,3)} \cup S^{(v)}_{(2,3,0)} \cup S^{(v)}_{(2,3,1)}$. Each of this four disjoint sets has the same measure. We proceed with $S^{(v)}_{(0,1,2)}$. As before we have $S^{(v)}_{(0,1,2)} = S_1^{(v)} \cup S_2^{(v)}$, where $S_1^{(iv)}$ consists of those $\mathbf{a} \in \mathbb{Z}_p^5$ with $\mathrm{v}_p(a_4)$ odd. Then

$$\begin{aligned}
\mu_p(S_1^{(v)}) &= \frac{1}{8}\left(1-\frac{1}{p}\right)^5 \sum_{n,r\geq 0} \frac{1}{p^{2n+2r+2}} \left(\sum_{\substack{k,\ell,m\geq 0\\ k+\ell>m+n}} \frac{1}{p^{2k+2\ell+2m}} - \sum_{\substack{k,\ell,m>0\\ k+\ell>m+n}} \frac{1}{p^{2k+2\ell+2m}} \right),\\
&= \frac{(p-1)(2p^4+p^2+2)}{8p(p+1)^4(p^2+1)^2},\\
\mu_p(S_2^{(v)}) &= \frac{1}{8}\left(1-\frac{1}{p}\right)^5 \sum_{k,\ell,m\geq 0} \frac{1}{p^{2k+2\ell+2m+3}} \left(\sum_{\substack{n,r\geq 0\\ k+\ell\geq m+n}} \frac{1}{p^{2n+2r}} - \sum_{\substack{n,r>0\\ k+\ell\geq m+n}} \frac{1}{p^{2n+2r}} \right)\\
&= \frac{(p-1)(p^8+2p^6+4p^4+2p^2+1)}{8p(p+1)^4(p^2+1)^3}.
\end{aligned}$$

Hence we get

$$\mu_p(S^{(v)}) = 4(\mu_p(S_1^{(v)}) + \mu_p(S_2^{(v)})) = \frac{(p-1)(p^8+4p^6+7p^4+5p^2+3)}{2p(p+1)^4(p^2+1)^3}.$$

What is left is to take into account that

$$\sigma_p = \begin{cases} 1-\mu_p(S^{(iii)})-\mu_p(S^{(iv)})-\mu_p(S^{(v)}), & \text{if } p=3,\\ 1-\mu_p(S^{(iv)})-\mu_p(S^{(v)}, & \text{if } p>3. \end{cases}$$

Thus $\sigma_3 = 63693071/66355200$ and if $p>3$ we get

$$\begin{aligned}
\sigma_p &= -\frac{-4p^{15}+6p^{13}-9p^{12}+44p^{11}-74p^{10}+129p^9-173p^8}{4(p-1)^4p(p+1)^4(p^2+1)^3}\\
&\quad + \frac{165p^7-157p^6+145p^5-120p^4+119p^3-87p^2+36p-8}{4(p-1)^4p(p+1)^4(p^2+1)^3}\\
&= 1-\frac{1}{2p^2}+\frac{9}{4p^3}+O\left(\frac{1}{p^4}\right).
\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.1

We will now prove Theorem 1.1. It suffices to apply [BBL16, Thm. 1.3] to the morphism $f : \mathcal{F} \to \mathbb{P}^4$ which projects each $X_{\mathbf{a}}$ to its coordinate vector $\mathbf{a}$. In order to do so we need that $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ and that the fibre of f above each codimension one point of $\mathbb{P}^4$ is split, i.e., it contains a geometrically integral open subscheme. The former is assured by our arrangements, while for the latter it is enough to check that the singular fibres of f are split. The singular locus is determined by $a_0 \cdots a_4(a_0a_1 - a_2a_3) = 0$. We do a case by case analysis and deal with the following separately.

(i) $a_0 = 0$,
(ii) $a_4 = 0$,
(iii) $a_0a_1 - a_2a_3 = 0$.

Notice that $a_i = 0$, for $i = 1, 2, 3$, is analogous to case (i) thanks to the symmetry in the equations definig $X_{\mathbf{a}}$.

For case (i) the fibre is defined via

$$x_0x_1 - x_2x_3 = 0,$$
$$a_1x_1^2 + \cdots a_4x_4^2 = 0.$$

Consider the chart $x_1 = 1$. In it, x_0 is determined by $x_0 = x_2x_3$ and thus the fibre is birational to a smooth quadric surface in $\mathbb{P}^3$ which is clearly split. For cases (ii) and (iii), we use the conic bundle representation. The fibres have equations given by:

$$\text{(ii):} \quad (a_0s^2 + a_2t^2)x^2 + (a_3s^2 + a_1t^2)y^2 = 0,$$
$$\text{(iii):} \quad (a_0s^2 + a_2t^2)(x^2 + y^2) + a_4z^2 = 0.$$

Notice that both are irreducible. Indeed, otherwise the generic fibres of each conic bundle would be reducible and hence singular. This is clearly not the case as both admit smooth fibres. Thus all conditions of [BBL16, Thm. 1.3] are fulfilled and Theorem 1.1 follows from it. We thank Dan Loughran for pointing out the above argument to us. □

5. Surfaces with Brauer group of order 4

This section is dedicated to the proof of the asymptotic formula for $N_4(B)$ appearing in Theorem 1.2.

Proposition 5.1. *We have*

$$N_4(B) = \frac{60}{\pi^2}B^3 + O(B^{5/2}(\log B)^2).$$

Proof. Proposition 2.2 implies that the quotient $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is of order 4 if and only if a_0a_1, $-a_0a_2$, a_2a_3 are all rational squares and $-a_0a_4d \notin \mathbb{Q}^{*2}$. Moreover, if these conditions are met, then $(1:0:\sqrt{a_0/a_2}:0:0) \in X_{\mathbf{a}}(\mathbb{Q})$ and thus we need not worry about local solubility. One easily verifies that a_0, a_1 need to have the same sign, and similarly for a_2, a_3 while a_0, a_2 need to have different signs. Thus we can distinguish between four cases depending on the sign of each a_i, these are

$$\text{(i) } a_0, a_1, a_4 > 0,\ a_2, a_3 < 0, \quad \text{(iii) } a_0, a_1 > 0,\ a_2, a_3, a_4 < 0,$$
$$\text{(ii) } a_0, a_1, a_4 < 0,\ a_2, a_3 > 0, \quad \text{(iv) } a_0, a_1 < 0, a_2,\ a_3, a_4 > 0.$$

One checks that (i) and (ii) contribute to the same amount in $N_4(B)$ via the map $\mathbf{a} \mapsto -\mathbf{a}$ and so do (iii) and (iv). On the other hand, (i) and (iv) have equal contribution in $N_4(B)$ as seen by the map $(a_0, a_1, a_2, a_3, a_4) \mapsto (a_2, a_3, a_0, a_1, a_4)$. Thus if $N_4^{(i)}(B)$ is the contribution in $N_4(B)$ coming from (i), we then clearly have

$$N_4(B) = 4N_4^{(i)}(B). \tag{5.1}$$

We shall now treat $N_4^{(i)}(B)$. Firstly, we need to understand how often $d = 0$ under the assumptions a_0a_1, $-a_0a_2$, $a_2a_3 \in \mathbb{Q}^{*2}$. This is done in the next lemma.

Lemma 5.2. *We have*

$$\#\left\{(a_0, \dots, a_3) \in \mathbb{Z}^4 \cap [-B, B]^4 \ :\ a_0a_1, -a_0a_2, a_2a_3 \in \mathbb{Q}^{*2} \text{ and } a_0a_1 = a_2a_3\right\} \ll B(\log B)^3.$$

Proof. For a quadruple $(a_0, \dots, a_3) \in \mathbb{Z}^4$ the conditions $a_0a_1, -a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}$ imply that $a_0, \dots, a_3$ must obey the following factorisation

$$a_0 = mk^2b_0^2, \quad a_1 = mk^2b_1^2, \quad a_2 = -m\ell^2b_2^2, \quad a_3 = -m\ell^2b_3^2,$$

with $(k, \ell) = (b_0, b_1) = (b_2, b_3) = 1$. It suffices to consider the case where $k, \ell, m, b_0, b_1, b_2, b_3$ are all positive integers.

The condition $a_0a_1 = a_2a_3$ becomes now $k^2b_0b_1 = \ell^2b_2b_3$ and since $(k, \ell) = 1$ we must have $k^2 \mid b_2b_3$ and $\ell^2 \mid b_0b_1$. Write $k = k_2k_3$ and $\ell = \ell_0\ell_1$ so that $b_0 = \ell_0^2c_0$, $b_1 = \ell_1^2c_1$, $b_2 = k_2^2c_2$ and $b_3 = k_3^2c_3$. Thus $c_0c_1 = c_2c_3$. Writing $c_0 = r_0s_0$ and $c_1 = r_1s_1$ now gives

$$a_0 = mk_2^2k_3^2\ell_0^4r_0^2s_0^2, \quad a_1 = mk_2^2k_3^2\ell_1^4r_1^2s_1^2, \quad a_2 = m\ell_0^2\ell_1^2k_2^4r_0^2r_1^2, \quad a_3 = m\ell_0^2\ell_1^2k_3^4s_0^2s_1^2,$$

with $(k_2k_3, \ell_0\ell_1) = (\ell_0r_0d_0, \ell_1r_1d_1) = (k_2r_0r_1, k_3d_0d_1) = 1$. Forgetting the coprimality conditions and summing over r_0, r_1 first now proves the claim. This completes the proof. $\square$

We continue with the study of $N_4^{(i)}(B)$. By definition it equals the following quantity

$$\#\{\mathbf{a} \in \mathbb{Z}^5 : 0 < a_0, a_1, -a_2, -a_3, a_4 \leq B, a_0a_1, -a_0a_2, a_2a_3 \in \mathbb{Q}^{*2} \text{ and } -a_0a_4d \notin \mathbb{Q}^{*2}\}.$$

Let us first make the following change $(a_2, a_3) \mapsto (-a_2, -a_3)$ so that all of the coordinates of $\mathbf{a}$ are positive integers. Thus

$$\begin{aligned} N_4^{(i)}(B) &= \sum_{\substack{a_0,\dots,a_4 \leq B \\ a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2} \text{ and } -a_0a_4d \notin \mathbb{Q}^{*2}}} 1 \\ &= \sum_{\substack{a_0,\dots,a_4 \leq B \\ a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}}} 1 - \sum_{\substack{a_0,\dots,a_4 \leq B \\ -a_0a_4d, a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}}} 1 - \sum_{\substack{a_0,\dots,a_4 \leq B \\ d=0, a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}}} 1. \end{aligned}$$

Lemma 5.2 implies that the last sum is $O(B^2(\log B)^3)$, the extra exponent of B coming from the additional sum over a_4. We shall soon see that the second sum is also relatively small and thus the main contribution comes from the first sum. To do so we first write

$$N_4^{(i)}(B) = M_4(B) \left(\sum_{a_4 \leq B} 1 - \sum_{\substack{a_4 \leq B \\ -a_0a_4d \in \mathbb{Q}^{*2}}} 1 \right) + O(B^2(\log B)^3), \tag{5.2}$$

where

$$M_4(B) = \sum_{\substack{a_0,\dots,a_3 \leq B \\ a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}}} 1.$$

We continue with the two sums over a_4 appearing above. Clearly,

$$\sum_{a_4 \leq B} 1 = B + O(1). \tag{5.3}$$

On the other hand, writing $a_4 = tu_4^2$ and $-a_0d = wu_5^2$ in the second sum with t, w square-free shows that the the condition $-a_0a_4d \in \mathbb{Q}^{*2}$ is fulfilled only if $t = w$. Hence one obtains

$$\sum_{\substack{a_4 \leq B \\ -a_0a_4d \in \mathbb{Q}^{*2}}} 1 \ll \sum_{u_4 \leq \sqrt{B}} \sum_{\substack{t \leq B/u_4^2 \\ t=w}} 1 \ll B^{1/2}. \tag{5.4}$$

We now proceed with $M_4(B)$. As in Lemma 5.2 the conditions $a_0a_1, a_0a_2, a_2a_3 \in \mathbb{Q}^{*2}$ are detected by the factorisation

$$a_0 = mk^2b_0^2, \quad a_1 = mk^2b_1^2, \quad a_2 = m\ell^2b_2^2, \quad a_3 = m\ell^2b_3^2,$$

with $(k, \ell) = (b_0, b_1) = (b_2, b_3) = 1$. Thus

$$M_4(B) = \sum_{m \le B} \sum_{\substack{k,\ell \le \sqrt{B/m} \\ (k,\ell)=1}} \sum_{\substack{b_0,b_1 \le \sqrt{B/mk^2} \\ (b_0,b_1)=1}} \sum_{\substack{b_2,b_3 \le \sqrt{B/m\ell^2} \\ (b_2,b_3)=1}} 1. \tag{5.5}$$

To continue we need an asymptotic formulae for the inner sums over b_0, b_1 and b_2, b_2. A standard computation in analytic number theory using the fact that the condition $(a, b) = 1$ is detected by the indicator function $\sum_{t|(a,b)} \mu(t)$ shows that for $X \ge 1$ a real number we have

$$\sum_{\substack{a,b \le X \\ (a,b)=1}} 1 = \frac{6}{\pi^2} X^2 + O(X \log X).$$

We apply this twice in (5.5), once for the sum over b_2, b_3 and once for the sum over b_0, b_1. This gives

$$M_4(B) = \frac{36}{\pi^4} B^2 \sum_{m \le B} \frac{1}{m^2} \sum_{\substack{k,\ell \le \sqrt{B/m} \\ (k,\ell)=1}} \frac{1}{k^2 \ell^2} + O(B^{3/2} (\log B)^2).$$

All of the three sums appearing above are absolutely convergent. We first complete the sum over ℓ, it equals $\zeta(2) \prod_{p|k} (1 - 1/p^2)$. The error in $M_4(B)$ coming from this completion is $O(B^{3/2})$. We do the same for the sum over k. The function inside the sum is multiplicative and hence this sum has an Euler product, it is $\prod_p (1 + 1/p^2)$. The error here is once again negligible compared to $B^{3/2} (\log B)^2$. Lastly, we complete the sum over m to get

$$M_4(B) = \prod_p \left(1 + \frac{1}{p^2}\right) B^2 + O(B^{3/2} (\log B)^2). \tag{5.6}$$

What is left is to combine (5.1), (5.2), (5.3), (5.4), (5.6) and to observe that the infinite product above equals $\zeta(2)/\zeta(4) = 15/\pi^2$. This gives

$$N_4(B) = \frac{60}{\pi^2} B^3 + O(B^{5/2} (\log B)^2),$$

which completes the proof of Proposition 5.1. □

6. Proof of Theorem 1.2

In order to prove Theorem 1.2 we first need to show that there are only a few surfaces in $\mathcal{F}$ with trivial $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q}$. This is done in the next proposition.

Proposition 6.1. *We have*

$$N_1(B) \ll B^3 (\log B)^4.$$

Proof. Recall that by Proposition 2.2 we have $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ trivial if and only if one of $-a_0a_4d$, $-a_1a_4d$, a_2a_4d, a_3a_4d is a non-zero rational square. Let $N_1^{(0)}(B)$ be the number of those $X_{\mathbf{a}} \in \mathcal{F}$ with $-a_0a_4d \in \mathbb{Q}^{*2}$ and $|\mathbf{a}| \le B$. Define $N_1^{(1)}(B), N_1^{(2)}(B), N_1^{(3)}(B)$ in a similar fashion according the the conditions $-a_1a_4d$, a_2a_4d, $a_3a_4d \in \mathbb{Q}^{*2}$. Forgetting the assumption on the existence of local points everywhere we clearly have

$$N_1(B) \ll N_1^{(0)}(B) + N_1^{(1)}(B) + N_1^{(2)}(B) + N_1^{(3)}(B).$$

We shall now explain how to treat $N_1^{(0)}(B)$, the analysis of the other quantities being similar. Letting

$$a_0 = w_0v_0^2, \quad a_4 = w_4v_4^2, \quad -a_0a_1 + a_2a_3 = w_5v_5^2,$$

with w_0, w_4, w_5 square-free allows us to see that $-a_0a_4d \in \mathbb{Q}^{*2}$ is equivalent to $w_0w_4w_5$ being a non-zero rational square. Thus we must have $w_0 = st, w_4 = sw, w_5 = tw$ with stw square-free. On the other hand, $-stv_0^2a_1 + a_2a_3 = twv_5^2$ implies that $t \mid a_2a_3$. Write $a_2 = t_2b_2$ and $a_3 = t_3b_3$, where $t = t_2t_3$ with $\mu^2(t_2t_3) = 1$. Thus

$$b_2b_3 - a_1v_0^2s - v_5^2w = 0.$$

Let $S, T_2, T_3, W, A_1, B_2, B_3, V_0, V_4, V_5 \gg 1$ run through the powers of 2. We shall also require then to satisfy the following conditions

$$ST_2T_3V_0^2 \ll B, \quad A_1 \ll B, \quad T_2B_2 \ll B, \quad T_3B_3 \ll B, \quad SWV_4^2 \ll B$$

and finally $WV_5^2 \ll B_2B_3 + A_1SV_0^2$. We break the the quantity $N_1^{(0)}(B)$ into sums $\Sigma = \Sigma(S, T_2, T_3, W, B_1, B_2, B_3, V_0, V_4, V_5)$ over the dyadic ranges $s \in (S/2, S]$, $t_2 \in (T_2/2, T_2]$ and so on.

What follows is an application of an upper bound of Heath-Brown [HB84, Lem. 3] which states that if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a integer vector with coprime coordinates, then

$$\#\{\mathbf{x} \in \mathbb{Z}^3_{\text{prim}} : |x_i| \le X_i, i = 1, 2, 3 \text{ and } \boldsymbol{\alpha}\cdot\mathbf{x} = 0\} \ll 1 + \frac{X_1X_2X_3}{\max|\alpha_iX_i|}.$$

We choose $\mathbf{x} = (a_1, b_2, w)$ and let $h = \gcd(a_1, b_2, w)$. Thus $X_1 = A_1$, $X_2 = B_2$, $X_3 = W$, $\alpha_1 = -sv_0^2$, $\alpha_2 = b_3$ and $\alpha_3 = v_5^2$. Let $\mathbf{x}' = \mathbf{x}/h$ and $\boldsymbol{\alpha}' = \boldsymbol{\alpha}/\gcd(sv_0^2, b_2, v_5^2)$. Applying Heath-Brown's bound for the number of $\mathbf{x}'$ with coordinates at most X_i/h and satisfying $\mathbf{x}'\cdot\boldsymbol{\alpha}' = 0$ now gives

$$\#\{\mathbf{x} \in \mathbb{Z}^3 : |x_i| \le X_i \text{ and } \boldsymbol{\alpha}\cdot\mathbf{x} = 0\} \ll \sum_{h \le \min\{A_1, B_2, W\}} \left(1 + \frac{X_1X_2X_3\gcd(sv_0^2, b_3, v_5^2)}{h^2\max|\alpha_iX_i|}\right).$$

It is clear that when we open the brackets the first sum is $O(W)$. On the other hand, the second sum over h is convergent and the error coming from its tail is negligible. Thus summing over the remaining variables in the first sum and replacing $\max|\alpha_i X_i|$ by $(\alpha_1 X_1 \alpha_2 X_2 \alpha_3 X_3)^{1/3}$ in the second sum gives

$$\Sigma \ll ST_2T_3WB_3V_0V_4V_5 + T_2T_3(WA_1B_2)^{2/3}V_4 \sum_{s,b_3,v_0,v_4} \frac{\gcd(sv_0^2, b_3, v_5^2)}{(sb_3v_0^2v_5^2)^{1/3}}.$$

Running the same argument with $\mathbf{x} = (a_1, b_3, w)$ and then using the elementary fact that $\min\{B_2, B_3\} \ll (B_2B_3)^{1/2}$ now gives

$$\begin{aligned}\Sigma \ll & ST_2T_3W(B_2B_3)^{1/2}V_0V_4V_5 + T_2T_3(WA_1B_3)^{2/3}V_4 \sum_{s,b_2,v_0,v_4} \frac{\gcd(sv_0^2, b_2, v_5^2)}{(sb_2v_0v_5)^{1/3}} \\ & + T_2T_3(WA_1B_2)^{2/3}V_4 \sum_{s,b_3,v_0,v_4} \frac{\gcd(sv_0^2, b_3, v_5^2)}{(sb_3v_0v_5)^{1/3}}.\end{aligned}$$

Our analysis so far showed that

$$\begin{aligned}N_1^{(0)}(B) \ll & \sum_{S,T_2,T_3,W,A_1,B_2,B_3,V_0,V_4,V_5} ST_2T_3W(B_2B_3)^{1/2}V_0V_4V_5 \\ & + \sum_{S,T_2,T_3,W,A_1,B_2,B_3,V_0,V_4,V_5} T_2T_3(WA_1B_3)^{2/3}V_4 \sum_{s,b_2,v_0,v_4} \frac{\gcd(sv_0^2, b_2, v_5^2)}{(sb_2v_0v_5)^{1/3}} \\ & + \sum_{S,T_2,T_3,W,A_1,B_2,B_3,V_0,V_4,V_5} T_2T_3(WA_1B_2)^{2/3}V_4 \sum_{s,b_3,v_0,v_4} \frac{\gcd(sv_0^2, b_3, v_5^2)}{(sb_3v_0v_5)^{1/3}}.\end{aligned}$$

Let S_1, S_2 and S_3 denote the first, the second and the third sum above, respectively.

We claim that

$$S_1 \ll B^3(\log B)^3.$$

Indeed, recall the simple bound

$$\sum_{A=2^i \leq X} A^\theta \ll \begin{cases} X^\theta & \text{if } \theta > 0, \\ \log X & \text{if } \theta = 0, \\ 1 & \text{if } \theta < 0, \end{cases} \tag{6.1}$$

which follows from the fact that we are summing the terms of a geometric progression. With the arrangements made above we have guaranteed that

$$U_0 \ll \left(\frac{B}{ST_2T_3}\right)^{1/2}, \quad U_4 \ll \left(\frac{B}{SW}\right)^{1/2}, \quad U_5 \ll \frac{B}{(T_2T_3W)^{1/2}}.$$

Thus applying (6.1) to the sums over B_2, B_3, U_0, U_4, U_5 gives

$$S_1 \ll B^3 \sum_{S,T_2,T_3,W,A_1} \frac{1}{(T_2T_3)^{1/2}}.$$

We then apply (6.1) to the sums over the remaining variables to get

$$S_1 \ll B^3(\log B)^3,$$

which proves the claim.

Finally, we claim that

$$S_2 \ll B^3(\log B)^4, \quad S_3 \ll B^3(\log B)^4.$$

We will prove the above bound for S_2, the analysis for S_3 being similar. We begin by studying of the sum over s, b_2, v_0, v_4 appearing in S_2. Write

$$s = ks', \quad b_2 = kmn^2c_2, \quad v_0 = mnu_0, \quad v_5 = kmnu_5.$$

Then $\gcd(sv_0^2, b_2, v_5^2) = kmn^2$ and thus

$$\sum_{s,b_2,v_0,v_4} \frac{\gcd(sv_0^2, b_2, v_5^2)}{(sb_2v_0v_5)^{1/3}} \ll \sum_{k,m,n,s',c_2,u_0,u_5} \frac{1}{(km^2s'c_2u_0^2u_5^2)^{1/3}},$$

where the sum is over $ks' \ll S$, $kmn^2c_2 \ll B_2$, $mnu_0 \ll V_0$ and $kmnu_5 \ll V_5$. Summing over s', u_0, c_2 and u_5 then gives

$$\sum_{s,b_2,v_0,v_4} \frac{\gcd(sv_0^2, b_2, v_5^2)}{(sb_2v_0v_5)^{1/3}} \ll (SB_2)^{2/3}(V_0V_5)^{1/3} \sum_{k,m,n} \frac{1}{(kmn)^2}.$$

The sums over k, m, n are convergent and their tails after completion have only negligible contribution. Thus

$$S_2 \ll \sum_{S,T_2,T_3,W,A_1,B_2,B_3,V_0,V_4,V_5} S^{2/3}T_2T_3W^{2/3}(A_1B_2B_3)^{2/3}(V_0V_5)^{1/3}V_4.$$

Proceeding in a similar fashion as in the study of S_1 now gives the claim which completes the proof. □

We continue with a lower bound for the quantity analysed in the previous proposition.

Proposition 6.2. *We have*

$$N_1(B) \gg B^3.$$

Proof. To show this we shall count the number of $\mathbf{a} \in \mathbb{Z}^5_{\text{prim}}$ coming from the following subfamily

$$\mathcal{F}' = \{X_{\mathbf{a}} \in \mathcal{F} \ : \ a_0a_1 - a_2a_3 \in \mathbb{Q}^{*2} \text{ and } a_4 = -a_0\}.$$

Clearly, $(1:0:0:0:1) \in X_{\mathbf{a}}(\mathbb{Q})$ for each $X_{\mathbf{a}} \in \mathcal{F}'$ and thus we need not worry about the existence of local points. Moreover, Proposition 2.2 implies that all varieties in $\mathcal{F}'$ have a trivial Brauer group and thus

$$N_1(B) \gg \#\{X_{\mathbf{a}} \in \mathcal{F}' \ : \ |\mathbf{a}| \leq B\}.$$

Let $N_1'(B)$ denote the cardinality of the set on the right hand side above. We then have

$$N_1'(B) = \sum_{k \leq B\sqrt{2}} \sum_{\substack{|a_0|,|a_1|,|a_2|,|a_3| \leq B \\ a_0a_1 - a_2a_3 = k^2}} 1 \gg \sum_{k \leq B/2} \sum_{\substack{a_0^2+a_1^2+a_2^2+a_3^2 \leq B^2 \\ a_0a_1 - a_2a_3 = k^2}} 1.$$

The inner sum in the far right hand side corresponds to the number of matrices in $M_2(\mathbb{Z})$ with non-zero entries of height at most B with respect to the standard Euclidean norm whose discriminant is equal to k^2. We shall apply [DRS93, Ex. 1.6] to this sum. Since in [DRS93, Ex. 1.6] quadruples with $a_i = 0$ are allowed we must first guarantee that the contribution from these $\mathbf{a}$ is negligible. Indeed, let $a_0 = 0$. Then a_1 can be chosen almost arbitrarily in the region $a_1^2 + a_2^2 + a_3^2 \leq B^2$ and the choice of a_2 uniquely determined a_3 for each fixed k. Thus

$$\sum_{k \leq B/2} \sum_{\substack{a_1^2+a_2^2+a_3^2 \leq B^2 \\ -a_2a_3 = k^2}} 1 \ll B \sum_{k \leq B/2} \tau(k^2) \ll B^{2+\varepsilon},$$

where ε is arbitrarily small positive number and $\tau(k^2)$ denotes the number of divisors of k^2. Applying [DRS93, Ex. 1.6] and taking the above into account now gives

$$N_1'(B) \gg B^2 \sum_{k \leq B/2} \sum_{d | k^2} \frac{1}{d}.$$

Changing the order of summation and applying a standard asymptotic formula for the sum over k above shows that $N_1'(B) \gg B^3$ which completes the proof. □

We are now in position to prove Theorem 1.2. Recall that there are only three possibilities for $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q}$. With the notation set up earlier in the introduction we then have

$$N_2(B) = \#\left\{X_{\mathbf{a}} \in \mathcal{F} \ : \ |\mathbf{a}| \leq B \text{ and } X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset\right\} - N_1(B) - N_4(B).$$

On one hand, $N_4(B) = O(B^3)$ by Proposition 5.1 and $B^3 \ll N_1(B) \ll B^3(\log B)^4$ by Propositions 6.1 and 6.2. On the other hand, the remaining quantity above was studied in Theorem 1.1. This proves Theorem 1.2. □

7. Proof of Theorems 1.3 and 1.4

We begin with the following simple lemma.

Lemma 7.1. *Let $p > 7$ be a prime and let $a, b, c \in \mathbb{F}_p^*$. Then there exist $u_1, u_2 \in \mathbb{F}_p$ such that $u_1^2 + b \in \mathbb{F}_p^{*2}$ and $u_2^2 + b \in \mathbb{F}_p^* \setminus \mathbb{F}_p^{*2}$. Moreover, $u_1^2 + c$ and $u_2^2 + c$ are both units in $\mathbb{F}_p$ and $a(u_1^2 + b)(u_1^2 + c), a(u_2^2 + b)(u_2^2 + c) \in \mathbb{F}_p^{*2}$.*

Proof. Consider the projective curve C defined over $\mathbb{F}_p$ by $v^2 = u^2 + bt^2$. This is a smooth quadric with an $\mathbb{F}_p$-rational point and thus it is isomorphic to $\mathbb{P}^1$. Hence $\#C(\mathbb{F}_p) = p + 1$. If $-b \notin \mathbb{F}_p^{*2}$, then all $\mathbb{F}_p$-points on C satisfy $v \neq 0$. Alternatively, if $-b \in \mathbb{F}_p^{*2}$ there are two points in $C(\mathbb{F}_p)$ with $v = 0$. Finally, the divisor given by $t = 0$ consists of two $\mathbb{F}_p$-points. We conclude that

$$\#\{(u, v) \in \mathbb{F}_p^2 \ : \ v^2 = u^2 + b \neq 0\} = p - 2 - \left(\frac{-b}{p}\right).$$

In particular, there exists $u_1 \in \mathbb{F}_p$ such that $u_1^2 + b \in \mathbb{F}_p^{*2}$ when the above quantity is positive.

To see the existence of u_2 as in the statement we apply a similar argument to show that

$$\#\{(u, v) \in \mathbb{F}_p^2 \ : \ u^2 + b \neq 0\} = p^2 - \left(1 + \left(\frac{-b}{p}\right)\right) p.$$

Thus we have

$$\#\left(\{(u, v) \in \mathbb{F}_p^2 \ : \ u^2 + b \neq 0\} \setminus \{(u, v) \in \mathbb{F}_p^2 \ : \ v^2 = u^2 + b \neq 0\}\right) > p^2 - 3p + 1.$$

It is clear that if $p > 3$, then both $p - 2 - \left(\frac{-b}{p}\right)$ and $p^2 - 3p + 1$ are positive. Moreover, if $p > 7$, then both quantities are at least $(p + 1)/2$ and thus we can choose u_1 and u_2 so that the assumptions of the statement are satisfied. This completes the proof of Lemma 7.1. □

Recall that the Brauer–Manin obstruction is known to be the only obstruction to the Hasse principle and weak approximation for $X_{\mathbf{a}} \in \mathcal{F}$ since such surfaces are conic bundles with four degenerate geometric fibres [CT90], [Sal86]. A Brauer–Manin obstruction to the existence of rational points on $X_{\mathbf{a}}$ is present only if $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$ by Remark 2.3. Moreover, we can assume that $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$ since by Theorem 1.2 the number of surfaces with $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \not\simeq \mathbb{Z}/2\mathbb{Z}$ is negligible compared to the bounds we need to show in order to prove Theorems 1.3 and 1.4. In this case, $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is generated by

$$(a_0(x_0/x_2)^2 + a_2, -a_0a_4d) = ((x_0/x_2)^2 + a_2/a_0, -a_0a_4d) \in \operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}.$$

Let $\alpha = ((x_0/x_2)^2 + a_2/a_0, -a_0a_4d)$ as an element of $\operatorname{Br} X_{\mathbf{a}}$.

For a detailed background on the Brauer–Manin obstruction we refer the reader to [CTS19, Ch. 12] or [Poo17, Ch. 8]. What is important for us is that the Brauer–Manin set $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X_{\mathbf{a}}}$ is determined by the adelic points on $X_{\mathbf{a}}$ for which the sum of local invariant maps inv_p of α vanish. In particular, $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X_{\mathbf{a}}}$ is defined as the kernel of the map $X(\mathbf{A}_{\mathbb{Q}}) \to \mathbb{Q}/\mathbb{Z}$ in the following diagram

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook] \arrow[d, "\mathrm{ev}_\alpha"'] & X(\mathbf{A}_{\mathbb{Q}}) \arrow[d, "\mathrm{ev}_\alpha"] \arrow[dr] & \\
\operatorname{Br}\mathbb{Q} \arrow[r] & \bigoplus_{p\le\infty} \operatorname{Br}\mathbb{Q}_p \arrow[r, "\sum_p \operatorname{inv}_p"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}.$$

For a point $\mathbf{x}_p \in X_{\mathbf{a}}(\mathbb{Q}_p)$ the value of $\operatorname{inv}_p(\mathrm{ev}_\alpha(\mathbf{x}_p))$ is either 0 or 1/2. Thus if there is a prime p for which this map surjects on $\{0, 1/2\}$ we can modify the adèle $(\mathbf{x}_p)$ at p to get a point inside $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X_{\mathbf{a}}}$. This shows that there is no Brauer–Manin obstruction to the Hasse principle in this case. It implies further that $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}})^{\operatorname{Br} X_{\mathbf{a}}}$ is a strict subset of $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}})$ and thus weak approximation fails.

Assume now that there is a prime $p > 7$ dividing a_4 to an odd power and such that p does not divide $a_0a_1a_2a_3d$. On one hand, the invariant map $\operatorname{inv}_p(\mathrm{ev}_\alpha(\mathbf{x}_p))$ depends only on the Hilbert symbol $((x_0/x_2)^2 + a_2/a_0, -a_0a_4d)_p$ as seen by the following formula

$$\operatorname{inv}_p(\mathrm{ev}_\alpha(\mathbf{x}_p)) = \frac{1 - ((x_0/x_2)^2 + a_2/a_0, -a_0a_4d)_p}{4}.$$

On the other hand, in view of (2.1) Lemma 7.1 applied with $u = x_0/x_2$, $a = a_0a_3$, $b = a_2/a_0$ and $c = a_1/a_3$ together with Hensel's lemma shows the existence of $\mathbf{x}_p', \mathbf{x}_p'' \in X_{\mathbf{a}}(\mathbb{Q}_p)$ for which the Hilbert symbol takes both values $-1, 1$. Therefore, in order to obtain the upper bounds in Theorems 1.3 and 1.4 we can count all $\mathbf{a} \in \mathbb{Z}^5_{\mathrm{prim}}$ of height at most B failing the above assumption.

It is clear that we only need to deal with divisibility conditions on the coordinates of $\mathbf{a}$. Since the sign of each a_i is irrelevant to such conditions and we aim to prove only upper bounds we can freely assume that $a_i > 0$ for $i = 0, \ldots, 4$. Let $k = \gcd(a_4, a_0)$,

$\ell = \gcd(a_4/k, a_1)$, $m = \gcd(a_4/k\ell, a_2)$ and $n = \gcd(a_4/k\ell m, a_3)$. Then write

$$a_0 = kb_0, \quad a_1 = \ell b_1, \quad a_2 = mb_2, \quad a_3 = nb_3, \quad a_4 = 2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mnsb_4^2, \tag{7.1}$$

with $v_p = \mathrm{v}_p(a_4)$ for $p = 2, 3, 5, 7$ and s being the square-free part of $a_4/2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mn$. Our analysis above shows that if there is a Brauer–Manin obstruction to the Hasse principle, then for every prime $p \mid s$ we must have $b_0b_1 - b_2b_3 \equiv 0 \bmod p$. Since s is square-free this condition is equivalent to $b_0b_1 - b_2b_3 \equiv 0 \bmod s$. We therefore obtain the upper bound

$$N_{\mathrm{Br}}(B) \ll \#\{\mathbf{a} \in \mathbb{Z}^5_{\mathrm{prim}} : |a_i| \leq B, \mathbf{a} \text{ satisfies (7.1) and } b_0b_1 - b_2b_3 \equiv 0 \bmod s\}.$$

Let $N(B)$ denote the quantity on the right hand side above. Forgetting the coprimality conditions except those between s and b_i now gives

$$N(B) \ll \sum_{\substack{k,\ell,m,n,s,v_2,v_3,v_5,v_7,b_0,b_1.b_2 \\ (s,b_0b_1b_2)=1}} \mu^2(s) \sum_{\substack{b_3 \leq B/n \\ b_3 \equiv b_0b_1b_2^{-1} \bmod s}} 1, \tag{7.2}$$

where the summation is taken over $kb_0, \ell b_1, mb_2, nb_3, 2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mnsb_4^2 \leq B$. We treat the inner sum in a standard way by detecting the condition $b_3 \equiv b_0b_1b_2^{-1} \bmod s$ via the orthogonality of the characters modulo s. This gives

$$\sum_{\substack{b_3 \leq B/n \\ b_3 \equiv b_0b_1b_2^{-1} \bmod s}} 1 = \frac{B}{n\varphi(s)} + O(B^{1/2}\log B),$$

where $\varphi(s)$ is the Euler totient function. We then apply the above asymptotic formula in (7.2). Forgetting the remaining coprimality conditions and the squarefreeness of s and then summing over b_4 shows that

$$N(B) \ll \sum_{k,\ell,m,n,s,v_i,b_0,b_1,b_2} \left(\left(\frac{B}{2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mns} \right)^{1/2} + O(1) \right) \left(\frac{B}{n\varphi(s)} + O(B^{1/2}\log B) \right),$$

where the summation is once again taken over $kb_0, \ell b_1, mb_2, 2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mns \leq B$. Hence

$$N(B) \ll B^{3/2} \sum_{k,\ell,m,n,s,v_i,b_0,b_1,b_2} \frac{1}{(2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mn^3)^{1/2}s\varphi(s)} + O(B^4(\log B)^2).$$

Since $s \sim \varphi(s)$ on average, the above sum over s is convergent. Thus summing over b_0, b_1, b_2 gives

$$N(B) \ll B^{3/2} \sum_{k,\ell,m,n,v_i} \frac{1}{(2^{v_2}3^{v_3}5^{v_5}7^{v_7}k\ell mn^3)^{1/2}} \left(\frac{B}{k} + O(1)\right)\left(\frac{B}{\ell} + O(1)\right)\left(\frac{B}{m} + O(1)\right) + O(B^4(\log B)^2.$$

It is now clear that $N(B) \ll B^{9/2}$. Since the quantities appearing in Theorems 1.3 and 1.4 are $O(N(B))$ this proves Theorems 1.3 and 1.4. □

8. Arithmetic of the lines on X

In what follows we will analyse how the conditions of Proposition 2.2 on the coefficients $\mathbf{a} = (a_0 : \cdots : a_4)$ reflect on the arithmetic of the lines on the del Pezzo surface $X_{\mathbf{a}}$. The results in this section hold over an arbitray number field, but for the sake of consistency we work over $\mathbb{Q}$.

Following Swinnerton-Dyer [SD99] we detect the double fours that give rise to Brauer classes. Firstly we show that a del Pezzo surface of degree 4 given by (1.1) has a trivial Brauer group if and only if it is rational over the ground field (see Lemma 8.3). In particular, no $\mathbb{Q}$-minimal del Pezzo surface of degree 4 given by (1.1) has a trivial Brauer group. We notice that the largest orbit of lines has size four. This together with [BBFL07, Prop. 13] tells us that surfaces given by (1.1) with non-trivial Brauer group have orbits of lines of sizes $(2,2,2,2,2,2,2,2)$, $(2,2,2,2,4,4)$ or $(4,4,4,4)$. We note that for a del Pezzo surface of degree 4 with a conic bundle structure the sizes of the orbits of lines are determined by the order of the Brauer group (but, of course, not vice-versa as a surface with eight pairs of conjugate lines can have both trivial or non-trivial Brauer group for example). On the other hand, if one assumes that the Brauer group is non-trivial then the size of the orbits does determine that of the Brauer group (see Lemma 8.9). Moreover, given a Brauer element, we describe in detail a genus one fibration with exactly two reducible fibres as in [VAV14] for which a non-trivial element is vertical. We obtain a rational elliptic surface by blowing up four points, namely two singular points of fibres of the conic bundle 2.1 together with two singular points of fibres of the conic bundle 2.2. The field of definition of the Mordell–Weil group of the elliptic fibration is determined by the size of the Brauer group of $X_{\mathbf{a}}$. In general, it is fully defined over a biquadratic extension. We also show that the reducible fibres are both fo type I_4.

8.1. **Conic bundles and lines.** Let $X_{\mathbf{a}}$ be given by (1.1). Then it admits two conic bundle structures given by (2.1) and (2.2). Each conic bundle structure has two pairs of conjugate singular fibres with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ acting on the 4 lines that form each of the two pairs. The intersection behaviour of the lines on $X_{\mathbf{a}}$ is described in Figure 8.1. Together, these 8 pairs of lines give the 16 lines on $X_{\mathbf{a}}$.

We now assign a notation to work with the lines. Given $i \in \{1, \cdots, 4\}$, the union of two lines L_i^+ and L_i^- will denote the components of a singular fibre of the conic bundle (2.1). Similarly, the union of two lines M_i^+ and M_i will denote the singular fibres of the conic bundle (2.2). More precisely, using the variables $(x_0 : x_1 : x_2 : x_3 : x_4)$ to describe the conic bundles, we have the following

$$x_0x_1 = x_2x_3 = -\sqrt{-\frac{a_2}{a_0}}, x_4 = \pm\sqrt{\frac{d}{-a_0a_4}}x_1, \tag{$L_1^{\pm}$}$$

$$x_0x_1 = x_2x_3 = \sqrt{-\frac{a_2}{a_0}}, \ x_4 = \pm\sqrt{\frac{d}{-a_0a_4}}x_1, \tag{$L_2^{\pm}$}$$

$$x_0x_1 = x_2x_3 = -\sqrt{-\frac{a_1}{a_3}}, \ x_4 = \pm\sqrt{\frac{d}{a_3a_4}}x_2, \tag{$L_3^{\pm}$}$$

$$x_0x_1 = x_2x_3 = \sqrt{-\frac{a_1}{a_3}}, \ x_4 = \pm\sqrt{\frac{d}{a_3a_4}}x_2, \tag{$L_4^{\pm}$}$$

$$x_0x_1 = x_2x_3 = -\sqrt{-\frac{a_0}{a_3}}, \ x_4 = \pm\sqrt{\frac{d}{a_3a_4}}x_2, \tag{$M_1^{\pm}$}$$

$$x_0x_1 = x_2x_3 = \sqrt{-\frac{a_0}{a_3}}, \ x_4 = \pm\sqrt{\frac{d}{a_3a_4}}x_2, \tag{$M_2^{\pm}$}$$

$$x_0x_1 = x_2x_3 = -\sqrt{-\frac{a_2}{a_1}}, \ x_4 = \pm\sqrt{\frac{d}{-a_0a_4}}x_1, \tag{$M_3^{\pm}$}$$

$$x_0x_1 = x_2x_3 = \sqrt{-\frac{a_2}{a_1}}, \ x_4 = \pm\sqrt{\frac{d}{-a_0a_4}}x_1. \tag{$M_4^{\pm}$}$$

One can readily determine the intersection behavior of these lines, which we describe in Lemma 8.1. We also take the opportunity to identify fours and double fours defined over small field extensions. Recall that a *four* in a del Pezzo surface of degree 4 is a set of four skew lines that do not all intersect a fifth one. A *double four* is four together with the four lines that meet three lines from the original four ([BBFL07, Def. 8]).

Lemma 8.1. *Let $i, j, k, l \in \{1, \cdots, 4\}$ with $j \neq i$. Consider L_i^+, L_i^-, M_i^+ and M_i^- as above. Then*

(a) *L_i^+ intersects L_i^-, M_i^- and M_j^+, while L_i^- intersects L_i^+, M_i^+, and M_j^-.*
(b) *M_i^+ intersects M_i^-, L_i^- and L_j^+, while M_i^- intersects M_i^+, L_i^+ and L_j^-.*
(c) *The lines $L_i^+, L_j^+, M_k^-, M_l^-$ and the lines $L_i^-, L_j^-, M_k^+, M_l^+$, with $i+j \equiv k+l \equiv 3 \bmod 4$, form two fours defined over the same field extension $L/\mathbb{Q}$ of degree at most 2. Together they form a double four defined over $\mathbb{Q}$.*

Proof. Statements (a) and (b) are obtained by direct calculations. For the line L_1, for instance, one sees readily that it intersects $L_1^-, M_1^-, M_2^+, M_3^+$ and M_4^+ respectively at the points $(-\sqrt{\frac{-a_2}{a_0}} : 0 : 1 : 0 : 0), (-\sqrt{\frac{-a_2}{a_0}} : -\sqrt{\frac{-a_0}{a_3}} : 1 : -\sqrt{\frac{a_2}{a_3}} : -\sqrt{\frac{d}{a_4 a_3}}), (-\sqrt{\frac{-a_2}{a_0}} : \sqrt{\frac{-a_0}{a_3}} : 1 : \sqrt{\frac{a_2}{a_3}} : \sqrt{\frac{d}{a_4 a_3}}), (-\sqrt{\frac{-a_2}{a_0}} : -\sqrt{\frac{-a_2}{a_1}} : 1 : \frac{a_2}{\sqrt{a_0 a_1}} : -\sqrt{\frac{d a_2}{a_4 a_0 a_1}})$ and $(-\sqrt{\frac{-a_2}{a_0}} : \sqrt{\frac{-a_2}{a_1}} : 1 : -\frac{a_2}{\sqrt{a_0 a_1}} : \sqrt{\frac{d a_2}{a_4 a_0 a_1}})$. Part (c) follows from (a) and (b). To see that one of such fours is defined over an extension of degree at most 2, note that each subset $\{L_i^+, L_j^+\}$ and $\{M_k^-, M_l^-\}$ is defined over the same extension of degree 2. For instance, taking $i = 1, j = 2, k = 3$ and $l = 4$, we see that the four is defined over $\mathbb{Q}(\sqrt{-a_0 a_4 d})$. The double four is defined over $\mathbb{Q}$ since both $\{L_i^+, L_j^+, L_i^-, L_j^-\}$ and $\{M_k^+, M_l^+, M_k^-, M_l^-\}$ are Galois invariant sets. $\square$

Among the 40 distinct fours on a del Pezzo surface of degree 4, the ones that appear in the previous lemma are special. More precisely, given a four as in Lemma 8.1 such that its field of definition has degree $d \in \{1, 2\}$, the smallest degree possible among such fours, then any other four is defined over an extension of degree at least d.

Definition 8.2. Given a four as in Lemma 8.1 part (c), we call it a *minimal four* if the field of definition of its lines has the smallest degree among such fours.

Borrowing the definition of complexity as in [FLS18, Def. 1.4], that is the sum of the degrees of the fields of definition of the non-split fibres, we see that the conic bundles in $X_{\mathbf{a}}$ have complexity at most four. This allows us to obtain in our setting the following.

Lemma 8.3. *Let $X_{\mathbf{a}}$ be given by* (1.1) *such that $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$. Then $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q}$ is trivial if and only if $X_{\mathbf{a}}$ is rational.*

Proof. The *if* implication holds for any rational variety since $\operatorname{Br} X_{\mathbf{a}}$ is a birational invariant. To prove the non-trivial direction, we make use of [KM17] which shows that conic bundles of complexity at most 3 with a rational point are rational. Firstly, note that if $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q}$ is trivial, then either $-a_0 a_4 d \in \mathbb{Q}(\sqrt{-a_0 a_2})^{*2}$ or $-a_1 a_4 d \in \mathbb{Q}(\sqrt{-a_1 a_3})^{*2}$ by Proposition 2.2

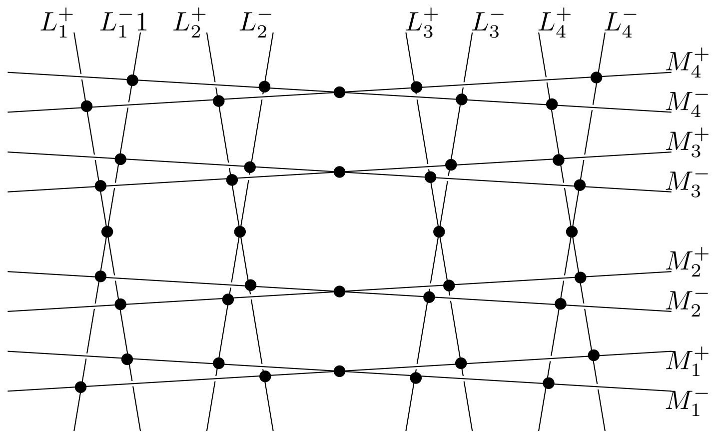

FIGURE 1. The lines on $X_{\mathbf{a}}$ and their intersection behavior. The intersection points of pairs of lines are marked with $\bullet$.

and thus the complexity of the conic bundle π_1 is at most 2. It remains to show that $X_{\mathbf{a}}$ admits a rational point. This follows from the independent work in [CT90] and [Sal86] that show that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for conic bundles with 4 degenerate geometric fibres. Hence $X_{\mathbf{a}}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ and the triviality of $\operatorname{Br} X_{\mathbf{a}} / \operatorname{Br} \mathbb{Q}$ imply the existence of a rational point on $X_{\mathbf{a}}$. □

Remark 8.4. Lemma 8.3 is parallel to [CTKS87, Lem. 1] which deals with diagonal cubic surfaces whose Brauer group is trivial. Moreover, a simple exercise shows that in our case, if the Brauer gorup is trivial, then the surface is a blow up of a Galois invariant set of four points in the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$, while the diagonal cubic satisfying the hypothesis of [CTKS87, Lem. 1] is a blow up of an invariant set of six points in the projective plane. The Picard group over the ground field of the former is of rank four while that of the latter has rank three.

8.2. **Brauer elements and double fours.** The following two results of Swinnerton-Dyer allow one to describe Brauer elements via the lines in a double four, and determine the order of the Brauer group.

The first result is presented exactly as stated in [BBFL07, Thm. 10].

Theorem 8.5. *Let X be a del Pezzo surface of degree 4 and α a non-trivial element of $\operatorname{Br} X$. Then α can be represented by an Azumaya algebra in the following way: there is a double-four defined over $\mathbb{Q}$ whose constituent fours are not rational but defined over $\mathbb{Q}(\sqrt{b})$, for some non-square $b \in \mathbb{Q}$. Further, let V be a divisor defined over $\mathbb{Q}(\sqrt{b})$ whose class is* *the sum of the classes of one line in the double-four and the classes of the three lines in the double-four that meet it, and let V' be the Galois conjugate of V . Let h be a hyperplane section of S. Then the $\mathbb{Q}$-rational divisor $D = V + V' - 2h$ is principal, and if f is a function whose divisor is D then α is represented by the quaternion algebra (f, b).*

The following can be found at [SD93, Lem. 11].

Lemma 8.6. $\mathrm{Br}\, X_{\mathbf{a}}$ *cannot contain more than three elements of order 2, and it contains as many as three if and only if the lines in S can be partitioned into four disjoint cohyperplanar sets T_i, $i = 1, .., 4$, with the following properties:*

(1) *the union of any two of the sets T_i is a double-four;*
(2) *each of the T_i is fixed under the absolute Galois group;*
(3) *if γ is half the sum of a line λ in some T_i, the two lines in the same T_i that meet λ, and one other line that meets λ, then no such γ is in* $\mathrm{Pic}\, X_{\mathbf{a}} \otimes \mathbb{Q} + \mathrm{Pic}\, \bar{X}_{\mathbf{a}}$.

We proceed to analyze how the conic bundle structures in $X_{\mathbf{a}}$ and the two results above can be used to describe the Brauer group of $X_{\mathbf{a}}$.

8.3. **The general case.** We first describe the general case, i.e., on which the combinations of coefficients that appear in Proposition 2.2 are not squares.

Proposition 8.7. *Let $X_{\mathbf{a}}$ be such that its defining equations satisfy $(*)$ of Proposition 2.2. Then there are exactly two distinct double fours on $X_{\mathbf{a}}$ defined over $\mathbb{Q}$ with constituent fours defined over a quadratic extension. In other words, there are exactly 4 minimal fours which pair up in a unique way to form two double fours defined over $\mathbb{Q}$.*

Proof. Part (c) of Lemma 8.1 tells us that the minimal fours are given by the double four formed by the fours $\{L_1^+, L_2^+, M_3^-, M_4^-\}, \{L_1^-, L_2^-, M_3^+, M_4^+\}$ and that formed by $\{L_3^+, L_4^+, M_1^-, M_2^-\}$ and $\{L_3^-, L_4^-, M_1^+, M_2^+\}$. By the hypothesis, each four is defined over a quadratic extension and the two double fours are defined over $\mathbb{Q}$. The hypothesis on the coefficients of the equations defining $X_{\mathbf{a}}$ also imply that any other double four is defined over a non-trivial extension of $\mathbb{Q}$. For instance, consider a distinct four containing L_1^+. For a double four containing this four to be defined over $\mathbb{Q}$, we need that the second four contains L_1^- and that one of the fours contains L_2^+ and the other L_2^-. The hypothesis that each four is defined over a degree two extension gives moreover that L_2^+ is in the same four as L_1 and hence, due to their intersecting one of the lines, M_1^+ and M_2^+ cannot be in the same four. We are left with $L_3^+, L_4^+, M_3^+, M_4^+$ and their conjugates. But if L_3^+ is in one of the fours then L_3^- would be in the other four. This is impossible as neither L_3^+ nor L_3^-

intersect L_1^- or L_2^-, and each line on a double four intersects three lines of the four that does not contain it. □

Corollary 8.8. *Let $X_{\mathbf{a}}$ be as above. Then $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q}$ is of order 2.*

Proof. This is a direct consequence of Proposition 8.7 together with Theorem 8.5. □

We shall now allow further assumptions on the coefficients of $X_{\mathbf{a}}$ to study how they influence the field of definition of double fours and hence the Brauer group.

8.4. **Trivial Brauer group.** Suppose that one of $-a_0a_4d, -a_1a_4d, a_2a_4d, a_3a_4d$ is in $\mathbb{Q}^{*2}$. Assume, to exemplify, that $-a_0a_4d$ is a square. Consider the conic bundle structure given by (2.1). Then the lines L_1^+ and L_2^+ are conjugate and, clearly, do not intersect. Indeed, they are components of distinct fibres of (2.1). Contracting them we obtain a del Pezzo surface of degree 6. Since, by assumption, $X_{\mathbf{a}}$ has points everywhere locally, the same holds for the del Pezzo surface of degree 6 by Lang–Nishimura [Lan54], [Nis55]. As the latter satisfies the Hasse principle, it has a $\mathbb{Q}$-point. In particular, $X_{\mathbf{a}}$ is rational, which gives us an alternative proof of Lemma 8.3.

8.5. **Brauer group of order four.** For the last case, assume that $a_0a_1, a_2a_3, -a_0a_2 \in \mathbb{Q}^{*2}, -a_0a_4d, -a_1a_4d, a_2a_4d, a_3a_4d \notin \mathbb{Q}^{*2}$. We produce two double fours that give distinct Brauer classes. Firstly note that all the singular fibres of the two conic bundles are defined over $\mathbb{Q}$. In particular, their singularities are $\mathbb{Q}$-rational points and thus there is no Brauer–Manin obstruction to the Hasse principle. Moreover, every line is defined over a quadratic extension, but no pair of lines can be contracted since each line intersects its conjugate. Secondly, note that since $-a_0a_2$ is a square, thus $\mathbb{Q}(\sqrt{-a_0a_4d}) = \mathbb{Q}(\sqrt{a_2a_4d})$. We have the double four as above, given by $L_1^+, L_2^+, M_3^-, M_4^-$ and the correspondent intersecting components, and a new double four given by $\{L_1^+, L_3^+, M_2^-, M_4^-\}, \{L_2^+, L_4^+, M_1^-, M_4^-\}$, which under this hypothesis is formed by two *minimal* fours.

The Picard group of $X_{\mathbf{a}}$ is generated by $L_1^+, L_2^+, L_3^+, L_4^+$, a smooth conic and a section, say M_1^+ of the conic fibration (2.1). We can apply Lemma 8.6 with $T_i = \{L_i^+, L_i^-, M_i^+, M_i^-\}$ to check that in this case the Brauer group has indeed size four.

Lemma 8.9. *Let $X_{\mathbf{a}}$ be as above. Assume that $X_{\mathbf{a}}$ does not contain a pair of skew conjugate lines, equivalently $X_{\mathbf{a}}$ is not $\mathbb{Q}$-rational. Then the following hold:*

(i) $\# \operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} = 4$ *if an only if the set of lines on $X_{\mathbf{a}}$ has orbits of size* $(2, 2, 2, 2, 2, 2, 2, 2)$.

(ii) $\#\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} = 2$ *if an only if the set of lines on* $X_{\mathbf{a}}$ *has orbits of size* $(2,2,2,2,4,4)$ *or* $(4,4,4,4)$.

Proof. This is an application of [SD93, Lem. 11] or a reinterpretation of Proposition 2.2 together with the description of the lines given in this section and the construction of Brauer elements via fours given by Swinnerton-Dyer (see for instance [SD93, Lem. 10] and [BBFL07, Thm 10.] for the construction of the Brauer elements via fours). □

9. A genus 1 fibration and vertical Brauer elements

In what follows we will give a description of the genus 1 fibration $X_{\mathbf{a}} \dashrightarrow \mathbb{P}^1$ from [VAV14] for which a given Brauer element is vertical. First we recall some basic facts about elliptic surfaces. We then obtain the Brauer element and the genus 1 fibration as in [VAV14] to afterwards reinterpret it in our special setting of surfaces admitting two non-equivalent conic bundles over the ground field. We study how the order of the Brauer group influences the arithmetic of this genus 1 fibration. More precisely, after blowing up the base points of the genus one pencil, we show that the field of definition of its Mordell–Weil group depends on the size of the Brauer group.

9.1. **Background on elliptic surfaces.** Let k be a number field.

Definition 9.1. An *elliptic surface* over k is a smooth projective surface X together with a morphism $\mathcal{E} : X \to B$ to some curve B whose generic fibre is a smooth curve of genus 1. If it admits a section we call the fibration *jacobian*. In that case, we fix a choice of section to act as the identity element for each smooth fibre. The set of sections is in one-to-one correspondence with the $k(B)$-points of the generic fibre, hence it has a group structure and its called the *Mordell–Weil group* of the fibration, or of the surface if there is no doubt on the fibration considered.

Remark 9.2. If X is a rational surface and an elliptic surface, we call it a *rational elliptic surface*. If the fibration is assumed to be minimal, i.e., no fibre contains (-1)-curves as components, then by the adjunction formula the components of reducible fibres are (-2)-curves. In that case, the sections are precisely the (-1)-curves and the fibration is jacobian over a field of definition of any of the (-1)-curves.

Given a smooth, projective, algebraic surface X its Picard group has a lattice structure with bilinear form given by the intersection pairing. If X is an elliptic surface then, thanks to the work of Shioda, we know that its Mordell–Weil group also has a lattice structure, with a different bilinear pairing [Shi90]. Shioda also described the Néron–Tate height pairing via intersections with the zero section and the fibre components. This allows us to determine, for instance, if a given section is of infinite order and the rank of the subgroup generated by a subset of sections. We give a brief description of the height pairing below.

Definition 9.3. Let $\mathcal{E} : X \to B$ be an elliptic surface with Euler characteristic χ. Let O denote the zero section and P, Q two sections of $\mathcal{E}$. The Néron–Tate height pairing is given by

$$\langle P, Q\rangle = \chi + P \cdot O + Q \cdot O - P \cdot Q - \sum_{F \in \text{ reducible fibres}} \operatorname{contr}_F(P, Q),$$

where $\operatorname{contr}_F(P, Q)$ denotes the contribution of the reducible fibre F to the pairing and depends on the type of fibre (see [Shi90, §8] for a list of all possible contributions). Upon specializing at $P = Q$ we obtain a formula for the height of a section (point in the generic fibre):

$$h(P) = \langle P, P\rangle = 2\chi + 2P \cdot O - \sum_{F \in \text{ reducible fibres}} \operatorname{contr}_F(P).$$

Remark 9.4. The contribution of a reducible fibre depends on the components that P and Q intersect. In this article we deal only with fibres of type I_4, thus for the sake of completion and brevity we give only its contribution. Denote by Θ_0 the component that is met by the zero section, Θ_1 and Θ_3 the two components that intersect Θ_0, and let Θ_2 be the component opposite to Θ_0. If P and Q intersect Θ_i and Θ_j respectively, with $i \leq j$ then $\operatorname{contr}_{I_4}(P, Q) = \frac{i(4-j)}{4}$.

9.2. **Vertical elements.**

Definition 9.5. Given an arbitrary[1] fibration $\pi : X \to \mathbb{P}^1$, the vertical Picard group, denoted by Pic_{vert} is the subgroup of the Picard group generated by the irreducible components of the fibres of π. The vertical Brauer group Br_{vert} is given by the algebras in $\mathrm{Br}\,\mathbb{Q}(\mathbb{P}^1)$ that give Azumaya algebras when lifted to X (see [Bri06, Def. 3]).

Lemma 9.6. *Assume that the Brauer group of $X_{\mathbf{a}}$ is non-trivial. Let $F = L_1^+ + L_2^+ + M_3^+ + M_4^+$ and $F' = L_1^- + L_2^- + M_3^- + M_4^-$. The pencil of hyperplanes spanned by F and F' gives a genus one fibration with exactly two reducible fibres on $X_{\mathbf{a}}$ which are of type I_4, for which a non-trivial element of its Brauer group is vertical. The other Brauer elements are horizontal divisors.*

[1]that is, not necessarily elliptic

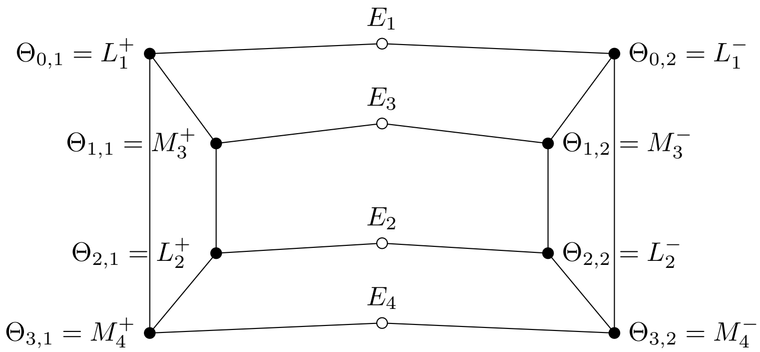

FIGURE 2. The reducible fibres of the genus one fibration $\mathcal{E}$. The eight $\bullet$ denote fibre components and the four $\circ$ denote sections given by the blow up of the 4 base points.

Proof. The linear system spanned by F and F' is a subsystem of $|-K_{X_{\mathbf{a}}}|$. Hence it gives a genus one pencil on $X_{\mathbf{a}}$. Its base points are precisely the four singular points of the following fibres of the conic bundle fibrations: $L_1^+ \cup L_1^-, L_2^+ \cup L_2^-, M_3^+ \cup M_3^-, M_4^+ \cup M_4^-$. The blow up of these four base points produces a geometrically rational elliptic surface[2] with two reducible fibres given by the strict transforms of F and F'. Since each of the latter is given by four lines in a square configuration and the singular points of this configuration are not blown up, these are of type I_4. There are no other reducible fibres as the only (-2)-curves are the ones contained in the strict transforms of F and F'. Let $\mathcal{E}$ denote the fibration map.

The Azumaya algebra (f, b) with f and b as in Theorem 8.5 taking as double four the components of F and F', gives a Brauer element which is vertical for the genus one fibration $\mathcal{E}$. Indeed, the lines that give such a double four are clearly in Pic_{vert} and hence the algebra (f, b) lies on $H^1(\mathbb{Q}, \mathrm{Pic}_{vert})$. By [Bri06, Prop. 4] it gives an element of the form $\mathcal{E}^*\mathcal{A}$, where $\mathcal{A}$ is in $\mathrm{Br}\,\mathbb{Q}(\mathbb{P}^1)$.

The other Brauer elements on $X_{\mathbf{a}}$ are described by double fours, i.e., pairs of sets of four (-1)-curves on $X_{\mathbf{a}}$, subject to intersection conditions. Since each component intersects each reducible fibre in exactly one point, after passing to its field of definition, these give sections of the genus one fibration. That is, such Brauer elements are horizontal with respect to this genus one fibration.

□

[2] not necessarily with a section over the ground field

9.3. **Mordell–Weil meets Brauer.** In what follows we will keep the letter $\mathcal{E}$ for the genus one fibration on the blow up surface just described.

Lemma 9.7. *Let $X_{\mathbf{a}}$ and $\mathcal{E}$ be as above. Then the following hold:*

(i) *If $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq \mathbb{Z}/2\mathbb{Z}$, then the genus 1 fibration $\mathcal{E}$ is an elliptic fibration over q quadratic extension, i.e., admits a section. Moreover, it admits a section of infinite order over a further quadratic extension. The Mordell–Weil group of $\mathcal{E}$ is fully defined over at most a third quadratic extension.*

(ii) *If $\operatorname{Br} X_{\mathbf{a}}/\operatorname{Br}\mathbb{Q} \simeq (\mathbb{Z}/2\mathbb{Z})^2$, then $\mathcal{E}$ is an elliptic fibration with a 2-torsion section and a section of infinite order over $\mathbb{Q}$. Moreover, the full Mordell–Weil group of $\mathcal{E}$ is defined over a quadratic extension.*

Proof. To prove (i) notice that the hypothesis of Proposition 2.2 imply that the four blown up points form two distinct orbits of Galois conjugate points. To exemplify, we work with the genus one fibration given by F and F' as in Lemma 9.6. Let P_i be the intersection point of L_i^+ and L_i^- for $i = 1, 2$ and that of M_i^+ and M_i^-, for $i = 3, 4$. Denote by E_i the exceptional curve after the blow up of P_i. Then $\{E_1, E_2\}$ and $\{E_3, E_4\}$ give two pairs of conjugate sections of $\mathcal{E}$. Moreover, the sections on a pair intersect opposite, i.e., disjoint, components of the fibres given by F and F'. Fixing one as the zero section of $\mathcal{E}$, say E_1, then a height computation gives that E_2 is the 2-torsion section of $\mathcal{E}$. Indeed, as we have fixed E_1 as the zero section, the strict transform of L_1^+ and L_1^- are the zero components of the fibres F and F', respectively. We denote them by $\Theta_{0,j}$ with $j = 1, 2$ respectively. Keeping the standard numbering of the fibre components, the strict transforms of L_2 and L_2' are denoted by $\Theta_{2,j}$, with $j = 1, 2$, respectively. Finally, in this notation, M_3^+ and M_3^- correspond to $\Theta_{1,j}$ while M_4^+ and M_4^- correspond to $\Theta_{3,j}$, for $j = 1, 2$ respectively.

To compute the height of the section E_2 we need the contribution of each I_4 to the pairing which in this case is 1 (see [Shi10, §11] for details on the height pairing on elliptic surfaces and the contribution of each singular fibre to it). We have thus

$$\langle E_2, E_2 \rangle = 2 - 0 - 1 - 1 = 0.$$

In particular, E_2 is a torsion section. Since E_2 is distinct from the zero section E_1 and such fibrations admit torsion of order at most 2 (see [Per90] for the list of fibre configurations and torsion on rational elliptic surfaces), we conclude that E_2 is a 2-torsion section. The two other conjugate exceptional divisors E_3 and E_4 give sections of infinite order as one can see, for example, after another height pairing computation.

To show (ii) it is enough to notice that the hypothesis of Proposition 2.2 imply that the four base points of the linear system spanned by F and F' are defined over $\mathbb{Q}$. From the discussion above we have that the zero section, the 2-torsion and also a section of infinite order, say E_3, are defined over $\mathbb{Q}$ since each of them is an exceptional curve above a $\mathbb{Q}$-point. The height matrix of the sections E_3 and E_4 has determinant zero, hence the section E_4 is linearly dependent on E_3. Moreover, it follows from the Shioda–Tate formula for $\mathrm{Pic}(X)^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}$ that any section defined over $\mathbb{Q}$ of infinite order is linearly dependent on E_3. Indeed, the rank of the Picard group of the rational elliptic surface is 6 since that of $X_{\mathbf{a}}$ has rank 2 and we blow up 4 rational points. The non-trivial components of the two fibres of type I_4 give a contribution of 3 to the rank. The other 3 come from the zero section, a smooth fibre and a section of infinite order, say E_3. For a second section of infinite order which is independent in the Mordell–Weil group of E_3 one can consider the pull-back of a line in $X_{\mathbf{a}}$. The hypothesis on the Brauer group implies that $X_{\mathbf{a}}$ has no line defined over $\mathbb{Q}$ but each is defined over a quadratic extension. $\square$

Remark 9.8. The arguments above rely only on the presence of two conic bundle structures on $X_{\mathbf{a}}$ and the Galois action on its singular fibres. Thus they can be generalized to any del Pezzo surface of degree four with two conic bundle structures and a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on its singular fibres.

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Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: vmitankin@mpim-bonn.mpg.de

Instituto de Matemática, UFRJ, Centro de Tecnologia - Bloco C, Cidade Universitária, Athos da Silveira Ramos 149, Ilha do Fundão, Rio de Janeiro, RJ, 21941–909, Brazil and Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: odaglas.ailicec@gmail.com