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Clustering-Based Model Reduction of Laplacian Dynamics With Weakly Connected Topology

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***Abstract*—This article studies the structure-preserving model reduction of Laplacian dynamics, which represent weakly connected directed networks with diffusive couplings. The notion of clusterability is introduced to guarantee a bounded reduction error, and a clustering algorithm is then proposed to partition the nodes into clusters, such that the nodes in each cluster form a connected subgraph of the original network. Then, a reduced-order model, which is established using the generalized balanced form of the original network, preserves the weakly connection structure and consensus property. Finally, the effectiveness of the proposed approach is illustrated by a numerical example.**

***Index Terms*—Graph clustering, graph topology, model order reduction, network systems.**

I. Introduction

A network system captures the collective behavior of a number of interacting dynamical subsystems. One of important properties of network systems is consensus, which occurs when certain agreements are reached via exchanging the information among the nodes [1]. In applications, e.g., formation control of mobile vehicles, coordination of distributed sensors, or balancing in chemical kinetics, consensus networks are modeled as Laplacian dynamics [2]–[4], which are commonly used to describe diffusion processes, e.g., information or energy spreading in networks. The model reduction of Laplacian dynamics is motivated by challenges caused by large scale and high complexity of networks that hinder both theoretical analysis and experimental studies. In this article, we address a structure-preserving model reduction problem for directed networks exhibiting consensus properties, and we aim for a lower-dimensional model that approximates the behavior of the original network with an acceptable accuracy and without being too expensive to evaluate.

The mainstream approach for reducing the complexity of networks is *graph clustering*, which is relevant to the problem of community or cluster detection for static graphs (see [5]). By partitioning the nodes into disjoint clusters, the nodes in each cluster can be assimilated into a single node such that the original topology is simplified, and its essential structure is retained. For dynamical networks, the clustering process has to take into account the evolution of node states driven by external input signals. The methods in [6]–[13] formulate the model reduction problem of *undirected networks* via the Petrov–Galerkin framework, where the projections are generated from selected clusters. A pioneering extension of the clustering approach to the reduction of directed networks is presented in [14], where clusters are formed based on the notion of reducibility, characterized by the uncontrollability of local states. However, the projection of this method heavily relies on the assumption that the network matrix has only one simple zero eigenvalue, whose corresponding eigenvectors (i.e., the *Frobenius eigenvectors*) has all strictly positive entries. In [15], the proposed clustering approach is restricted to directed networks with strongly connected topology. Moreover, in [6], [10], [12], and [14], the connection topology in each cluster is not considered, i.e., some nodes may be separated from the others in the same cluster, which actually complicates the interpretation of the reduced-order network in terms of the original network.

In contrast to the existing results, this article investigates a model reduction scheme for *weakly connected* directed networks with *diffusive couplings*. The studied systems describe a more general class of networks than undirected networks and strongly connected networks. In practice, weakly connected structures are found in various network applications, e.g., vehicle formations, self-synchronizing sensor networks, pagerank algorithms, and social networks (see [2], [3], [16], and [17]). One of the major difficulties comes from the semistability of the network system, in which the network matrix, different from [14], [15], may contain multiple semisimple zero eigenvalues, and the corresponding eigenvectors may have rows with only zeros. Furthermore, weakly connected networks may only achieve local consensus rather than a global one. In our model reduction method, the local consensus phenomenon of weakly connected networks is also taken into account, such that the obtained reduced networks can preserve the consensus property. However, this consideration is not specified in [14].

To tackle the above difficulties, this article introduces the concept of *generalized balanced* digraphs, by which the projection matrices are generated. Moreover, node *clusterability* is defined for weakly connected networks, and it is shown that merging clusterable nodes guarantees an bounded approximation error. The concept of nodal dissimilarity is used for quantifying the difference between a pair of clusterable nodes, which then leads to an iterative clustering algorithm to yield a set of *proper clusters*, achieving a small model approximation error between the full-order and reduced-order networks. This method preserves a connection topology of the nodes in each cluster such that each cluster forms a connected subgraph of the original network.

The rest of this article is organized as follows: In Section II, we introduce the model of directed networks and the clustering-based projection for reducing network models. In Section III, the clusterability is defined, and a greedy algorithm is proposed to generate proper clusters. Then, the scheme is illustrated through an example in Section IV. Finally, concluding remarks are made in Section V.

Notation: Denote $\mathbb{R}$ as the set of real numbers. Let $\mathbb{W}$ be a subspace of $\mathbb{R}^n$, then $\mathbb{W}^\perp$ denotes the orthogonal complement of $\mathbb{W}$ in $\mathbb{R}^n$. The cardinality of a set $\mathcal{V}$ is denoted by $|\mathcal{V}|$, and The identity matrix of size

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n is given as I_n, and $\mathbb{1}_n$ denotes a n-entries vector of all ones. The subscript n is omitted when no confusion arises. $\mathbf{e}_i$ is the ith column vector of I_n, and $\mathbf{e}_{ij} = \mathbf{e}_i - \mathbf{e}_j$. The trace, rank, image, and nullspace of A are denoted by $\mathtt{tr}(A)$, $\mathtt{rank}(A)$, $\mathtt{im}(A)$, and $\mathtt{ker}(A)$, respectively.

II. Preliminaries and Problem Settings

In this section, we provide necessary preliminaries on graph theory, based on which the model of a directed network is given in the form of Laplacian dynamics. The clustering-based projection framework is then introduced.

A. Directed Network Systems

We briefly recap some definitions in graph theory, (see [18] and [19] for more details). A digraph graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is composed of a node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and a directed edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each elements in $\mathcal{E}$ is an ordered pair of $\mathcal{V}$, and if $(i, j) \in \mathcal{E}$, we say that the edge is directed from vertex i to vertex j. A *directed path* connecting nodes i_0 and i_n is a sequence of edges of the form (i_{k-1}, i_k), $k = 1, \ldots, n$. A node i is reachable from j, if there is a directed path from j to i. A *strongly connected component* (SCC) is a subgraph of $\mathcal{G}$ in which any pair of nodes are reachable from each other. It is obvious that any digraph $\mathcal{G}$ can be partitioned into a series of SCCs that do not share any common nodes. If an SCC has only outflows, i.e., there is no edges from our other components to this SCC, it is then called a *root strongly connected component* (RSCC). For a weighted weakly connected graph $\mathcal{G}$, we denote $\mathcal{W} \in \mathbb{R}^{n\times n}$ as its *weighted adjacency matrix*, whose (i, j) entry, denoted by $\mathcal{W}_{ij}$, is positive if the directed edge $(j, i) \in \mathcal{E}$, and $\mathcal{W}_{ij} = 0$ otherwise. Then, $\mathcal{G}$ can be represented by a *Laplacian matrix*, which is defined by

$$\mathcal{L} = \mathtt{diag}(\mathcal{W}\mathbb{1}) - \mathcal{W}. \tag{1}$$

The behavior of a directed consensus network evolving over $\mathcal{G}$ is presented by the following Laplacian dynamics:

$$\mathbf{\Sigma} : \begin{cases} \dot{x}(t) = -\mathcal{L}x(t) + \mathcal{F}w(t) \\ y(t) = Hx(t) \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^p$, and $y(t) \in \mathbb{R}^q$ are the vectors of node states, external inputs, and measurement signals, respectively. The input and output matrix $\mathcal{F}$ and H represent the distribution of the external inputs and measurements, respectively. The Laplacian matrix $\mathcal{L}$ represents a weighted digraph with diffusive couplings among the nodes.

Throughout the article, we assume that the underlying digraph $\mathcal{G}$ of the network system $\mathbf{\Sigma}$ is *weakly connected*, namely, replacing all the directed edges of $\mathcal{G}$ with undirected ones and yields a connected (undirected) graph. Note that a weakly connected digraph is the most general topology characterization of a dynamic network in (2), and it allows for multiple RSCCs. Without loss of generality, we suppose $\mathcal{G}$ can be decomposed to $\bar{m}$ SCCs, namely $\mathcal{S}_1, \ldots, \mathcal{S}_{\bar{m}}$, and the set $\mathcal{S}_1, \ldots, \mathcal{S}_m$ are all the RSCCs. Denote $\mathcal{S}_R := \mathcal{S}_1 \cup \mathcal{S}_2 \cdots \cup \mathcal{S}_m$. Then, the Laplacian matrix is written as

$$\mathcal{L}(\mathcal{G}) = \left[\begin{array}{ccc:c} \mathcal{L}_{\alpha 1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mathcal{L}_{\alpha m} & 0 \\ \hdashline \mathcal{L}_{\gamma 1} & \cdots & \mathcal{L}_{\gamma m} & \mathcal{L}_\beta \end{array}\right] \tag{3}$$

where the first m diagonal blocks, $\mathcal{L}_{\alpha i} \in \mathbb{R}^{|\mathcal{S}_i|\times|\mathcal{S}_i|}$ $(i = 1, \ldots, m)$, are the Laplacian matrices associated with the m RSCCs, while $\mathcal{L}_\beta$ corresponds to the remaining nodes in $\mathcal{G}$, i.e., $\mathcal{V} \setminus \mathcal{S}_R$ and $\mathcal{L}_{\gamma i} \in \mathbb{R}^{|\mathcal{V}\setminus\mathcal{S}_R|\times|\mathcal{S}_i|}$ indicates the incoming edges from $\mathcal{S}_i$ to $\mathcal{V} \setminus \mathcal{S}_R$.

Note that $\mathcal{L}$ has semisimple zero eigenvalues with the multiplicity m, and the real part of each nonzero eigenvalue of $\mathcal{L}$ is strictly positive. Thus, the network system $\mathbf{\Sigma}$ is *semistable*. i.e., $\lim_{t\to\infty} e^{-\mathcal{L}t}$ exists [20].

Remark 1: If $m = 1$, $\mathcal{G}$ is a digraph containing one spanning tree, and $\mathcal{L}$ has one simple zero eigenvalue. Particularly, if all the nodes in $\mathcal{G}$ are reachable to each other, then $\mathcal{G}$ is strongly connected, and the Frobenius eigenvector of $\mathcal{L}$ (i.e., the left eigenvector corresponding to the zero eigenvalue) has all positive entries [14], [15].

B. Clustering-Based Projection

We construct the reduced-order network model using projection matrices that are generated by graph clustering. Consider a connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. *Graph clustering* is to find nonempty subsets $\mathcal{C}_i$ (called *clusters*), $i = 1, 2, \ldots, r$ such that $\mathcal{V} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r$, and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$, $\forall\, i, j = 1, 2, \ldots, r$.

Definition 1: The characteristic matrix of a clustering $\mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_r$ is denoted by a binary matrix $\Pi \in \mathbb{R}^{n\times r}$, whose (i, j)-entry is defined by

$$\Pi_{ij} := \begin{cases} 1, & \text{node } i \in \mathcal{C}_j \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

As each node is included in a unique cluster, each row of Π has a single 1 entry while all the others are 0. Thus, the property $\Pi\mathbb{1}_n = \mathbb{1}_r$ holds. For a given clustering of $\mathcal{G}$, with $|\mathcal{V}| = n$, the clustering-based projection is applied to construct a reduced network system as

$$\hat{\mathbf{\Sigma}} : \begin{cases} \dot{z}(t) = -\hat{\mathcal{L}}z(t) + \hat{\mathcal{F}}w(t) \\ \hat{x}(t) = \hat{H}x(t) \end{cases} \tag{5}$$

with $z(t) \in \mathbb{R}^r$, $\hat{\mathcal{L}} := \Pi^\dagger\mathcal{L}\Pi$, $\hat{\mathcal{F}} = \Pi^\dagger\mathcal{F}$, and $\hat{H} = H\Pi$, where $\Pi^\dagger \in \mathbb{R}^{r\times n}$ is full row rank such that $\Pi^\dagger\Pi = I_r$.

Denote the transfer matrices of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ by

$$\eta(s) = H\left(sI_n + \mathcal{L}\right)^{-1}F, \quad \text{and} \quad \hat{\eta}(s) = \hat{H}(sI_r + \hat{\mathcal{L}})^{-1}\hat{F} \tag{6}$$

respectively. For the rest of this article, we address the following model reduction problem.

Problem 1: Given a directed network system $\mathbf{\Sigma}$ in (2), find matrices Π and $\Pi^\dagger$ such that the obtained reduced-order model $\hat{\mathbf{\Sigma}}$ in (5) preserves a network structure and approximates the behavior of $\mathbf{\Sigma}$, i.e., $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is bounded and small.

III. Main Results

The strategy of constructing a reduced-order network system is discussed. First, we construct the projection matrices to guarantee the boundedness of the error $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$, and then, we develop an effective algorithm to find appropriate clusters that make reduction error small.

A. Clusterability

The network matrices in [14] and [15] have a single zero eigenvalue whose corresponding left eigenvector has all positive value. In that case, any clustering will result in a bounded reduction error. However, this property does no longer hold in the weakly connected case, where the network matrix $\mathcal{L}$ may contain multiple zero eigenvalues. To characterize the clusters that guarantee the boundedness of reduction error, the notion of clusterability is introduced.

Before proceeding, we first extend the definition of balanced graphs from strongly connected graphs to weakly connected ones. A strongly connected graph is *balanced* if the indegree and outdegree of each node in the graph are equal [19].

Definition 2: A weakly connected digraph $\mathcal{G}$ is generalized balanced if each RSCC is balanced, i.e., $\sum_{j\in\mathcal{S}_k} w_{ij} = \sum_{j\in\mathcal{S}_k} w_{ji}, \forall\, i \in \mathcal{S}_k, \forall\, k \in \{1, 2, \ldots, m\}$.

We show that any directed network system in (2) can be converted to a generalized balanced form.

Lemma 1: Consider a directed network system $\mathbf{\Sigma}$ in (2). There always exists an equivalent representation

$$\begin{cases} M\dot{x}(t) = -Lx(t) + Fw(t) \\ y(t) = Hx(t) \end{cases} \tag{7}$$

where L is the Laplacian of a generalized balanced digraph, and $M \in \mathbb{R}^{n\times n}$ is diagonal and positive definite such that $F = M\mathcal{F}$ and $L = M\mathcal{L}$.

Proof: Note that each $\mathcal{L}_{ri}$ in (3) represents a strongly connected digraph. From [19], there exists a vector of all positive entries ν_i such that $\mathcal{L}_{\alpha i}^\top \nu_i = 0$ for all $i = \{1, \ldots, m\}$. Thus, $\mathtt{diag}(\nu_i)\mathcal{L}_{\alpha i}$ represents a balanced directed subgraph, i.e.,

$$\mathbb{1}^\top \mathtt{diag}(\nu_i)\mathcal{L}_{\alpha i} = 0 \text{ and } \mathtt{diag}(\nu_i)\mathcal{L}_{\alpha i}\mathbb{1} = 0. \tag{8}$$

Let

$$M := \mathtt{diag}\left(\left[\nu_1^\top, \ldots, \nu_m^\top, \nu_r^\top\right]\right) \tag{9}$$

with ν_r an arbitrary positive vector. Then, $L := M\mathcal{L}$ represents a generalized balanced digraph by Definition 2. ■

Then, the clusterability of nodes is defined using the generalized balanced representation of $\mathbf{\Sigma}$ in (7).

Definition 3: Consider a directed network system $\mathbf{\Sigma}$ in (2). The nodes i and j are clusterable if $\mathbf{e}_{ij} \in \mathtt{ker}(L)^\perp$ and $\mathbf{e}_{ij} \in \mathtt{ker}(L^\top)^\perp$ simultaneously, where L is defined in (7). Furthermore, a clustering of $\mathcal{G}$ is proper, if all the nodes in each cluster are clusterable.

Furthermore, the physical meaning of the clusterability is explained in the following lemma.

Lemma 2: The nodes i and j are clusterable if and only if the following conditions hold.

1) The nodes i and j reach consensus, i.e., when $w = 0$

$$\lim_{t\to\infty} [x_i(t) - x_j(t)] = 0$$

for all initial conditions.

2) The nodes i and j are either contained in the same RSCC or $i, j \in \mathcal{V}\backslash\mathcal{S}_R$.

Proof: Consider the decomposition

$$\mathcal{L} = \begin{bmatrix} U & \bar{U} \end{bmatrix} \begin{bmatrix} 0 & \\ 0 & \bar{D} \end{bmatrix} \begin{bmatrix} V \\ \bar{V} \end{bmatrix} \tag{10}$$

where $\bar{D} \in \mathbb{R}^{(n-m)\times(n-m)}$ is Hurwitz with m the algebraic multiplicity of the zero eigenvalue of $\mathcal{L}$, and $U, V \in \mathbb{R}^{n\times m}$ satisfy

$$\mathcal{L}U = 0,\ V^\top\mathcal{L} = 0,\ \text{ and }\ V^\top U = I_m \tag{11}$$

It then follows from $L = M\mathcal{L}$ that $\mathtt{im}(U) = \mathtt{ker}(L)$, $\mathtt{im}(M^{-1}V) = \mathtt{ker}(L^\top)$, with M given in (9). By Definition 3, the clusterability of the nodes i and j is equivalent to

$$\mathbf{e}_{ij}^\top U = 0,\ \text{ and }\ \mathbf{e}_{ij}^\top M^{-1} V = 0. \tag{12}$$

Note that for any initial condition $x_0 \in \mathbb{R}^n$, the zero input response of $\mathbf{\Sigma}$ satisfies $\lim_{t\to\infty} e^{-\mathcal{L}t}x_0 = UV^\top x_0$. Thus, the nodes i and j reach consensus for all initial conditions x_0, if and only if the ith and jth rows of U coincide, i.e., $\mathbf{e}_{ij}^\top U = 0$. Furthermore, from Lemma 1, $\mathbf{e}_{ij}^\top M^{-1}V = 0$ holds if and only if the nodes i and j belongs to the same RSCC, or $i, j \in \mathcal{V}\backslash\mathcal{S}_R$. In the latter case, $\mathbf{e}_i^\top V = 0$, for all $i \in \mathcal{V}\backslash\mathcal{S}_R$. ■

Remark 2: The clusterability of two nodes does not simply mean that the two nodes are in the same SCC, or they reach consensus at the same final value. Particularly, the clusterability of different types of directed networks are discussed. If $\mathcal{G}$ is strongly connected, we have $U = \frac{1}{\sqrt{n}}\mathbb{1}_n$, and $V \in \mathbb{R}^n$ has all positive entries such that $\mathbb{1}^\top L = \mathbb{1}^\top M\mathcal{L} = 0$. Thus, all the nodes are clusterable. If $\mathcal{G}$ is a weakly connected digraph containing m RSCCs, local consensus is achieved in each RSCC, and thus the nodes in the same RSCC are clusterable. For a pair of nodes in $\mathcal{V}\backslash\mathcal{S}_R$ (may be in different SCCs), they are clusterable if and only if their states reach consensus.

The clusterability determines the feasibility of a graph clustering. More specifically, $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is bounded if and only if clusterable nodes are aggregated.

Theorem 1: Consider the directed network system $\mathbf{\Sigma}$ in (2) and its reduced-order model $\hat{\mathbf{\Sigma}}$ in (5) with

$$\hat{\mathcal{L}} = \left(\Pi^\top M\Pi\right)^{-1}\Pi^\top L\Pi, \text{ and } \hat{\mathcal{F}} = \left(\Pi^\top M\Pi\right)^{-1}\Pi^\top F \tag{13}$$

where M, L, and F are defined in (7). For all input matrix F, the error $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$ if and only if Π characterizes a proper clustering of $\mathcal{G}$.

Proof: The $\mathcal{H}_2$-norm of the approximation error is given by

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \|\xi(t) - \hat{\xi}(t)\|_2^2 dt \tag{14}$$

where $\xi(t) := e^{-\mathcal{L}t}F$ and $\hat{\xi}(t) := \Pi e^{-\hat{\mathcal{L}}t}\Pi^\dagger F$ are the impulse responses of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, respectively. Since both $\xi(t)$ and $\hat{\xi}(t)$ are smooth functions over $t \geq 0$, the integral in (14) is finite if and only if the error $\xi(t) - \hat{\xi}(t)$ exponentially converges to zero. Hence, for any H and F matrices with proper dimensions, $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$ is equivalent to

$$\mathcal{J} = \Pi\hat{\mathcal{J}}\Pi^\dagger. \tag{15}$$

where $\mathcal{J} := \lim_{t\to\infty} e^{-\mathcal{L}t}$ and $\hat{\mathcal{J}} := \lim_{t\to\infty} e^{-\hat{\mathcal{L}}t}$.

To prove the "if" part, we assume $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r\}$ to be a proper clustering of $\mathcal{G}$. We first verify that

$$U = \Pi\Pi^\dagger U, \text{ and } V^\top = V^\top\Pi\Pi^\dagger. \tag{16}$$

Without loss of generality, assume that

$$\Pi = \mathtt{blkdiag}\left(\mathbb{1}_{|\mathcal{C}_1|}, \mathbb{1}_{|\mathcal{C}_2|}, \ldots, \mathbb{1}_{|\mathcal{C}_r|}\right). \tag{17}$$

Accordingly, the matrices U and V in (11) are partitioned as $U^\top = [U_1^\top, \ldots, U_r^\top]$ and $V^\top = [V_1^\top, \ldots, V_r^\top]$. Meanwhile, $\Gamma = \Pi\Pi^\dagger$ becomes block diagonal with the ith diagonal entry

$$\Gamma_i = \mathbb{1}_{|\mathcal{C}_i|}\left(\mathbb{1}_{|\mathcal{C}_i|}^\top M_i \mathbb{1}_{|\mathcal{C}_i|}\right)^{-1}\mathbb{1}_{|\mathcal{C}_i|}^\top M_i \tag{18}$$

where M_i is the corresponding principal submatrix in N. Then, the equations in (16) hold if and only if

$$U_i = \Gamma_i U_i,\ \text{ and }\ V_i^\top = V_i^\top\Gamma_i. \tag{19}$$

It follows from Lemma 2 that $U_i = \mathbb{1}_{|\mathcal{C}_i|}$, which leads to the first equation in (19). Moreover, as the nodes in $\mathcal{C}_i$ are clusterable, Lemma 2 implies that these nodes are either contained in the same RSCC or in the set $\mathcal{V}\backslash\mathcal{S}_R$, implying that $V_i = M_i\mathbb{1}_{|\mathcal{C}_i|}$ or $V_i = 0$. Thus, the second equation in (19) also holds. Let $\hat{U} := \Pi^\dagger U$ and $\hat{V}^\top := V^\top\Pi$, which yield $\hat{\mathcal{L}}\hat{U} = \Pi^\dagger\mathcal{L}\Pi\Pi^\dagger U = \Pi^\dagger\mathcal{L}U = 0$, and $\hat{V}^\top\hat{\mathcal{L}} = V^\top\Pi\Pi^\dagger\mathcal{L}\Pi = V^\top\mathcal{L}\Pi = 0$. Due to $\hat{V}^\top\hat{U} = V^\top\Pi\Pi\Pi^\dagger U = V^\top U = I_m$, we obtain

$$\hat{\mathcal{J}} := \hat{U}\hat{V}^\top = \Pi^\dagger UV^\top\Pi \tag{20}$$

and thus, we obtain $\Pi\hat{\mathcal{J}}\Pi^\dagger = \Pi\Pi^\dagger UV^\top\Pi\Pi\Pi^\dagger = UV^\top = \mathcal{J}$, which implies $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$.

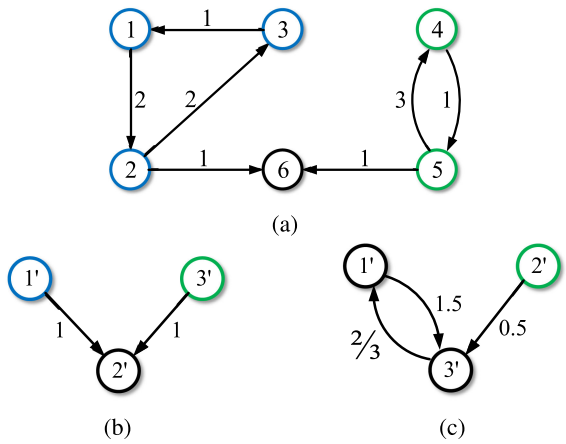

Fig. 1. Illustration of clusterability. (a) Directed graph, with a proper clustering $\mathcal{C}_1 = \{1,2,3\}$, $\mathcal{C}_2 = \{4\}$, and $\mathcal{C}_3 = \{5,6\}$. (b) Reduced network, obtained by the proper clustering, gives a bounded approximation error. (c) Reduced network obtained by an improper clustering, resulting in an unbounded approximation error.

For the "only if" part, $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is assumed to be bounded for all F, equivalently, (15) holds. Similarly, we consider a block diagonal structure of Π as in (17) such that (15) is presented as

$$\begin{bmatrix} U_1 \\ \vdots \\ U_r \end{bmatrix} \left[V_1^\top, \cdots, V_r^\top\right] = \begin{bmatrix} \mathbb{1}_{|\mathcal{C}_1|}\tilde{U}_1 \\ \vdots \\ \mathbb{1}_{|\mathcal{C}_r|}\tilde{U}_r \end{bmatrix} \left[\tilde{V}_1^\top \Pi_1^\dagger, \cdots, \tilde{V}_r^\top \Pi_r^\dagger\right] \tag{21}$$

where $\Pi_i^\dagger := (\mathbb{1}_{|\mathcal{C}_i|}^\top M_i \mathbb{1}_{|\mathcal{C}_i|})^{-1}\mathbb{1}_{|\mathcal{C}_i|}^\top M_i$, $\tilde{U}_i := \mathbf{e}_i^\top \tilde{U}$ and $\tilde{V}_i := \mathbf{e}_i^\top \tilde{V}$, with $\tilde{U}$ and $\tilde{V}$ fulfilling

$$\mathtt{im}(\tilde{U}) = \mathtt{ker}(\hat{\mathcal{L}}),\ \mathtt{im}(\tilde{V}) = \mathtt{ker}(\hat{\mathcal{L}}^\top),\ \text{ and } \tilde{V}^\top \tilde{U} = I. \tag{22}$$

The matrices $U_i, V_i \in \mathbb{R}^{|\mathcal{C}_i| \times m}$ are the corresponding submatrices of U and V, respectively. Then, (21) yields $U_i V_j^\top = \mathbb{1}_{|\mathcal{C}_i|}\tilde{U}_i \tilde{V}_j^\top \Pi_j^\dagger = \alpha_{ij} \cdot \mathbb{1}_{|\mathcal{C}_i|}\mathbb{1}_{|\mathcal{C}_j|}^\top M_j$, $\forall\, i,j = 1,2,\ldots,r$, with a scalar $\alpha_{ij} := \tilde{U}_i \tilde{V}_j^\top \cdot (\mathbb{1}_{|\mathcal{C}_j|}^\top M_j \mathbb{1}_{|\mathcal{C}_j|})^{-1}$. It follows that $\mathbf{e}_{ij}^\top U_k = 0$ and $V_k^\top N_k^{-1}\mathbf{e}_{ij} = 0$, $\forall i,j \in \mathcal{C}_k$. Note that $LU = 0$ and $V^\top M^{-1} L = 0$ with L defined in (7). Thus, we obtain for all $i,j \in \mathcal{C}_k$, $\mathbf{e}_{ij} \in \mathtt{ker}(L)^\perp$ and $\mathbf{e}_{ij} \in \mathtt{ker}(L^\top)^\perp$. As the result holds for all clusters $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$, the graph clustering is proper by Definition 3. ■

Note that $\hat{\mathcal{L}} \in \mathbb{R}^{r\times r}$ in (13) is also a lower-dimensional Laplacian matrix that represents a digraph with fewer nodes. Thus, the reduced-order model $\hat{\boldsymbol{\Sigma}}$ preserves the structure of a directed network and the local consensus property.

Remark 3: When $\mathcal{L}$ has only one simple zero eigenvalue, all the nodes in the network $\boldsymbol{\Sigma}$ are clusterable, i.e., any clustering of $\mathcal{G}$ is proper, and thus $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$ is always guaranteed. In this case, the proposed projection in (13) is essentially identical to that in [14] and can be reduced to it.

Remark 4: For a weakly connected network, we can find a set of *maximal proper clusters*, $\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2, \ldots, \bar{\mathcal{C}}_\kappa$, where all the nodes in the each cluster are clusterable, while every two nodes from distinct clusters are not. From Remark 2, it is clear that $\kappa \geq m+1$. Furthermore, to guarantee the boundedness of the reduction error, it is necessary for the order of the reduced-order network system to be larger or equal to κ.

Example 1: A weakly connected graph is illustrated in Fig. 1(a), which contains three SCCs, i.e., $\mathcal{S}_1 = \{1,2,3\}, \mathcal{S}_2 = \{4,5\}$, and $\mathcal{S}_3 = \{6\}$. $\mathcal{S}_1$ and $\mathcal{S}_2$ are the RSCCs. The weighted Laplacian matrix of the network is written as

$$\mathcal{L} = \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ \hdashline 0 & -1 & 0 & 0 & -1 & 2 \end{array}\right] \tag{23}$$

which has two zero eigenvalues, and the corresponding right and left eigenvectors are be characterized by

$$U^\top = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0 & 0 & 0.125 \\ 0 & 0 & 0 & 0.25 & 0.25 & 0.125 \end{bmatrix}$$

$$V^\top = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

as defined in (10). Let $\mathcal{F} = H = I_6$, and we can choose $M = \mathtt{diag}(2,1,1,1,1,3)$ to obtain the system in (7) with a Laplacian matrix L associating with a generalized balanced digraph. Consider two different graph clustering

$$\tilde{\mathcal{C}}_1 = \{1,2,3\},\ \tilde{\mathcal{C}}_2 = \{4\},\ \tilde{\mathcal{C}}_3 = \{5,6\}$$

$$\mathcal{C}_1 = \{1,2\},\ \mathcal{C}_2 = \{3,4\},\ \mathcal{C}_3 = \{5,6\}$$

which leads to reduced-order network systems in form of (5) with

$$\hat{\mathcal{L}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{bmatrix},\quad \hat{\mathcal{L}}' = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 2 \end{bmatrix} \tag{24}$$

respectively. Both two matrices represent digraphs, as shown in Fig. 1(b) and (c), respectively. But only the digraph in Fig. 1(b) is obtained by a proper clustering, thus it gives an approximation error: $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2} = 0.7852$, while the latter reduced network yields an unbounded reduction error.

B. Dissimilarity and Graph Clustering

We present a greedy algorithm to choose a proper clustering of $\mathcal{G}$ such that $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is small. To this end, we introduce the notion of *dissimilarity*, which is defined only for a pair of clusterable nodes, to characterize their difference in terms of their state responses with respect to external inputs.

Definition 4: Consider the system $\boldsymbol{\Sigma}$ in (2). Let $\mathcal{I}$ be a set collecting all the clusterable pairs, i.e., $(i,j) \in \mathcal{I}$, for all clusterable nodes i and j. Then, the **dissimilarity** is defined for each element in $\mathcal{I}$ as

$$\mathcal{D}_{ij} = \mathcal{D}_{ji} := \|\mathbf{e}_{ij}^\top (sI_n + \mathcal{L})^{-1}\mathcal{F}\|_{\mathcal{H}_2},\ \forall\, (i,j) \in \mathcal{I}. \tag{25}$$

The similar definition for undirected networks can be found in [10] and [12]. A direct computation of the $\mathcal{H}_2$-norm for each pair of clusterable nodes can be rather expensive for a large-scale network. Therefore, we utilize a novel controllability Gramian to facilitate the dissimilarity computation.

Theorem 2: Consider the system $\boldsymbol{\Sigma}$ in (2). The dissimilarity between two clusterable nodes i and j is computed as

$$\mathcal{D}_{ij} = \sqrt{\mathbf{e}_{ij}^\top \mathcal{P} \mathbf{e}_{ij}} \tag{26}$$

where $\mathcal{P}$ is the unique solution of the following equations:

$$\begin{cases} \mathcal{L}\mathcal{P} + \mathcal{P}\mathcal{L}^\top = (I_n - \mathcal{J})\mathcal{F}\mathcal{F}^\top (I_n - \mathcal{J}^\top) \\ \mathcal{J}\mathcal{P}\mathcal{J}^\top = 0 \end{cases} \tag{27}$$

with $\mathcal{J} := \lim_{t\to\infty} e^{-\mathcal{L}t}$ a constant matrix.

Proof: Following [15], we define

$$\mathcal{P}_n = \int_0^\infty (e^{-\mathcal{L}t} - \mathcal{J})\mathcal{F}\mathcal{F}^\top (e^{-\mathcal{L}^\top t} - \mathcal{J}^\top)dt \tag{28}$$

as the pseudo controllability Gramian of $\mathbf{\Sigma}$. It is shown that the $\mathcal{P}_n = \mathcal{P}$, which is the unique solution of (27) (see [10] for a similar proof).

Since i and j are clusterable, we obtain from Lemma 2 that $\mathbf{e}_{ij}^\top \mathcal{J}\mathcal{F} = 0$. Let $g(t) := \mathbf{e}_{ij}^\top e^{-\mathcal{L}t}\mathcal{F}$. By the definition of the $\mathcal{H}_2$ norm [21], we then have

$$\begin{aligned}\mathcal{D}_{ij}^2 &= \int_0^\infty \mathbf{e}_{ij}^\top e^{-\mathcal{L}t}\mathcal{F}\mathcal{F}^\top e^{-\mathcal{L}^\top t}\mathbf{e}_{ij}dt \\ &= \int_0^\infty \mathbf{e}_{ij}^\top \left(e^{-\mathcal{L}t} - \mathcal{J}\right)\mathcal{F}\mathcal{F}^\top \left(e^{-\mathcal{L}^\top t} - \mathcal{J}^\top\right)\mathbf{e}_{ij}dt \end{aligned} \tag{29}$$

which gives (26). ■

Remark 5: We can apply a blockwise computation of the matrix $\mathcal{P}$. Following the structure of $\mathcal{L}$ in (3), we can partition the input matrix $\mathcal{F}$ as $\mathcal{F}^\top = [\mathcal{F}_1^\top \;\; \mathcal{F}_2^\top \;\; \dots, \mathcal{F}_m^\top \;\; \mathcal{F}_\beta^\top]$ and $\mathcal{J} = \lim_{t\to\infty} e^{-\mathcal{L}t} = \texttt{blkdiag}(\nu_1\mathbb{1}_{|\mathcal{S}_1|\nu_k^\top}, \dots, \nu_m\mathbb{1}_{|\mathcal{S}_m|\nu_k^\top}, 0)$, with ν_k the left Frobenius eigenvector of $\mathcal{L}_{\alpha k}$, see the proof of Lemma 1. Consequently, we obtain

$$\mathcal{P} = \left[\begin{array}{ccc:c} \mathcal{P}_{\alpha 1} & \cdots & 0 & \mathcal{P}_{\gamma 1}^\top \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mathcal{P}_{\alpha m} & \mathcal{P}_{\gamma m}^\top \\ \hdashline \mathcal{P}_{\gamma 1} & \cdots & \mathcal{P}_{\gamma m} & \mathcal{P}_\beta \end{array}\right] \tag{30}$$

where $\mathcal{P}_{\alpha k}$ (pseudocontrollability Gramians), $\mathcal{P}_{\gamma k}$, for $k = 1, \dots, m$, and $\mathcal{P}_\beta$ are solved by the following matrix equations:

$$\begin{cases} \mathcal{L}_{\alpha k}\mathcal{P}_{\alpha k} + \mathcal{P}_{\alpha k}\mathcal{L}_{\alpha k}^\top = (I - \nu_k\mathbb{1}^\top)\mathcal{F}_k\mathcal{F}_k^\top(I - \mathbb{1}\nu_k^\top) \\ \qquad \nu_k\mathbb{1}^\top\mathcal{P}_{\alpha k}\mathbb{1}\nu_k^\top = 0 \end{cases} \tag{31}$$

$$\mathcal{L}_\beta\mathcal{P}_{\gamma k} + \mathcal{P}_{\gamma k}\mathcal{L}_{\alpha k}^\top = \mathcal{F}_\beta\mathcal{F}_k^\top(I - \mathbb{1}\nu_k^\top) - \mathcal{L}_{\gamma k}\mathcal{P}_{\gamma k} \tag{32}$$

$$\mathcal{L}_\beta\mathcal{P}_\beta + \mathcal{P}_\beta\mathcal{L}_\beta^\top = \mathcal{F}_\beta\mathcal{F}_\beta^\top - \sum_{k=1}^m \left(\mathcal{L}_{\gamma k}\mathcal{P}_{\gamma k}^\top + \mathcal{P}_{\gamma k}\mathcal{L}_{\gamma k}^\top\right). \tag{33}$$

Note that $-\mathcal{L}_\beta$ is Hurwitz, implying that both (32) and (33) have unique solutions. Now, instead of directly solving large-scale equations in (27), we tackle multiple problems in smaller dimensions, in which all the blocks in (30) are computed by a cascading approach. Specifically, for all $i = 1, 2, \dots, m$, we solve $\mathcal{P}_{\alpha k}$ in (31) which is then substituted to (32) yielding $\mathcal{P}_{\gamma k}$. As a result, $\mathcal{P}_{\gamma k}$ is computed in (33) using all $\mathcal{P}_{\gamma k}$.

Roughly speaking, a smaller value of $\mathcal{D}_{ij}$ indicates a smaller amount of input energy required to steer the nodes i and j to reach consensus. Intuitively, clustering a pair of nodes with smaller dissimilarity results in a smaller approximation error. Thus, we present a greedy algorithmic procedure to generate clusters of a given network based on its node dissimilarities. Different from the methods in [10], [12], [14], our algorithm takes into account the interconnection topology of the original network. More precisely, it preserves the connection topology in each cluster such that the nodes in each cluster form a connected subgraph of the original network.

First, a direct simplification of the original network can be applied. If there exists an RSCC $\mathcal{S}_k$, $k \in \{1, 2, \dots, m\}$ such that $\mathcal{F}_k = 0$, we can remove all the nodes in $\mathcal{S}_k$. If there exists a pair of nodes i, j such that $\mathcal{D}_{ij} = 0$, we then merge i, j as a single node. Note that both operations do not yield any approximation error, since they essentially remove uncontrollable component of the system.

Next, we consider a network system $\mathbf{\Sigma}$ of n nodes, which satisfies $\mathcal{F}_k \neq 0$ for all $k \in \{1, 2, \dots, m\}$, and $\mathcal{D}_{ij} \neq 0$, for all $i, j \in \mathcal{V}$. Define

Algorithm 1: Connection-Preserving Clustering Algorithm.

Input: $\mathcal{L}$, $\mathcal{F}$, and desired order r
Output: $\hat{\mathcal{L}}$ and $\hat{\mathcal{F}}$

```
Compute the dissimilarity D_ij by Theorem 2.
Remove all the nodes in S_k, if F_k = 0, k = 1, 2, ..., m, and
    merge the nodes with dissimilarity 0;
Construct the distance matrix X in (34);
c ← rank(diag(X1) − X)
repeat
    Find (u, v) = arg_{(i,j)∈E∩I} max X_ij;
    X_uv ← 0, and X_vu ← 0;
    c ← rank(diag(X1) − X);
until c ≤ r
Generate Π, and compute L̂ and F̂ according to (13).
```

a *distance matrix* of $\mathbf{\Sigma}$, denoted by $\mathcal{X}$, whose (i, j) entry is

$$\mathcal{X}_{ij} = \mathcal{X}_{ji} = \begin{cases} \mathcal{D}_{ij}, & \text{if } (i,j) \in \mathcal{E} \cap \mathcal{I} \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

where $\mathcal{E}$ is the edge set of the original network, and $\mathcal{E} \cap \mathcal{I}$ collects all the adjacent and clusterable pairs of nodes. Note that $\mathcal{X}$ can be seen as a weighted adjacency matrix of an undirected graph $\mathcal{G}_D$, which contains multiple disconnected subgraphs, and each subgraph is composed of clusterable nodes in $\mathcal{G}$. Then, a clustering procedure is taken as follows.

We find the edge of $\mathcal{G}_D$ with the largest weight, namely, the largest dissimilarity among all pairs in $\mathcal{E} \cap \mathcal{I}$. Then, we remove this edge in $\mathcal{G}_D$ such that a new distance matrix is obtained. We then repeat the above step to remove more edges until r connected components are left in $\mathcal{G}_D$. In every step, to check the number of connected components in $\mathcal{G}_D$, we compute the Laplacian matrix of $\mathcal{G}_D$

$$L_D = L_D^T = \mathsf{diag}(\mathcal{X}\mathbb{1}) - \mathcal{X}. \tag{35}$$

From [18], the rank of L_D is equal to the number of connected components. The detailed process is shown in Algorithm 1.

Remark 6: As $\mathcal{X}$ is defined in (34), the unclusterable nodes are already separated into different connected components of the initial $\mathcal{G}_D$. The subsequent operation is just to split the connected components into more smaller ones. Thus, the resulting graph clustering is proper. Moreover, each edge in $\mathcal{G}_D$ also corresponds an edge in the original graph $\mathcal{G}$. Thus, the obtained r clusters form node-disjoint subgraphs of $\mathcal{G}$.

With a proper graph clustering of the original network, the reduced-order network system $\hat{\mathbf{\Sigma}}$ in (5) achieves a bounded approximation error, which can be evaluated by using the pseudo controllability Gramian in (28).

Theorem 3: Consider the network system $\mathbf{\Sigma}$ in (2) and the reduced-order network model $\hat{\mathbf{\Sigma}}$ obtained from a proper clustering. Then, the reduction error is computed as

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}^2 = \texttt{tr}\left[H\left(\mathcal{P}_n + \Pi\mathcal{P}_r\Pi^\top - 2\Pi\mathcal{P}_x\right)H^\top\right] \tag{36}$$

where $\mathcal{P}_n$ and $\mathcal{P}_r$ are the pseudocontrollability Gramians of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, respectively. $\mathcal{P}_x := \tilde{\mathcal{P}}_x - \Pi^\dagger\mathcal{J}\Pi\tilde{\mathcal{P}}_x\mathcal{J}^\top \in \mathbb{R}^{r\times n}$ with $\tilde{\mathcal{P}}_x$ an arbitrary solution of the Sylvester equation

$$\hat{\mathcal{L}}^\top\tilde{\mathcal{P}}_x + \tilde{\mathcal{P}}_x\mathcal{L}^\top = \Pi^\dagger(I - \mathcal{J})FF^\top(I - \mathcal{J}^\top). \tag{37}$$

Proof: The approximation error is characterized by

$$\eta(s) - \hat{\eta}(s) := \eta_e(s) = \mathcal{C}(sI_{n+r} - \mathcal{A})^{-1}\mathcal{B} \tag{38}$$

with

$$\mathcal{A} = -\begin{bmatrix} \mathcal{L} & 0 \\ 0 & \hat{\mathcal{L}} \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} \mathcal{F} \\ \hat{\mathcal{F}} \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} H & -H\Pi \end{bmatrix}.$$

Note that $\eta_e(s) \in \mathcal{H}_2$ due to Theorem 1. Denote $\mathcal{J}_e := \lim_{t\to\infty} e^{\mathcal{A}t} = \mathtt{blkdiag}(\mathcal{J}, \hat{\mathcal{J}})$. Then, from (15), we obtain $\mathcal{C}\mathcal{J}_e\mathcal{B} = H\mathcal{J}\mathcal{F} - H\Pi\hat{\mathcal{J}}\Pi^\dagger\mathcal{F} = 0$. Therefore

$$\begin{aligned} \|\eta_e(s)\|_{\mathcal{H}_2}^2 &= \mathtt{tr}\left(\int_0^\infty \mathcal{C}e^{\mathcal{A}t}\mathcal{B}\mathcal{B}^\top e^{\mathcal{A}^\top t}\mathcal{C}^\top dt\right) \\ &= \mathtt{tr}\left[\int_0^\infty \mathcal{C}(e^{\mathcal{A}t} - \mathcal{J}_e)\mathcal{B}\mathcal{B}^\top(e^{\mathcal{A}^\top t} - \mathcal{J}_e^\top)\mathcal{C}^\top dt\right] \\ &:= \mathtt{tr}(\mathcal{C}\mathcal{P}_e\mathcal{C}^\top) \end{aligned} \tag{39}$$

where $\mathcal{P}_e$ is defined as the pseudocontrollability Gramian of the error system $\eta_e(s)$ (see the definition in [15]). Moreover, similar to Lemma 2, $\mathcal{P}_e$ is solved as the unique solution of

$$\mathcal{A}\mathcal{P}_e + \mathcal{P}_e\mathcal{A}^\top + (I - \mathcal{J}_e)\mathcal{F}\mathcal{F}^\top(I - \mathcal{J}_e^\top) = 0 \tag{40}$$

$$\mathcal{J}_e\mathcal{P}_e\mathcal{J}_e^\top = 0. \tag{41}$$

Consider an arbitrary solution of (40), which is partitioned as

$$\tilde{\mathcal{P}}_e = \begin{bmatrix} \tilde{\mathcal{P}}_n & \tilde{\mathcal{P}}_x^\top \\ \tilde{\mathcal{P}}_x & \tilde{\mathcal{P}}_r \end{bmatrix}, \text{ with } \tilde{\mathcal{P}}_n \in \mathbb{R}^{n\times n}, \tilde{\mathcal{P}}_r \in \mathbb{R}^{k\times k}. \tag{42}$$

It then leads to three equations as follows:

$$\begin{cases} \mathcal{L}\tilde{\mathcal{P}}_n + \tilde{\mathcal{P}}_n\mathcal{L}^\top = (I_n - \mathcal{J})\mathcal{F}\mathcal{F}^\top(I_n - \mathcal{J}^\top) & \text{(43a)} \\ \hat{\mathcal{L}}\tilde{\mathcal{P}}_r + \tilde{\mathcal{P}}_r\hat{\mathcal{L}}^\top = (I_r - \hat{\mathcal{J}})\hat{\mathcal{F}}\hat{\mathcal{F}}^\top(I_r - \hat{\mathcal{J}}^\top) & \text{(43b)} \\ \hat{\mathcal{L}}\tilde{\mathcal{P}}_x + \tilde{\mathcal{P}}_x\mathcal{L}^\top = (I_r - \hat{\mathcal{J}})\hat{\mathcal{F}}\mathcal{F}^\top(I_n - \mathcal{J}^\top) & \text{(43c)} \end{cases}$$

where (43c) is equivalent to (37) due to $\hat{\mathcal{J}}\Pi^\dagger = \Pi^\dagger\mathcal{J}\Pi\Pi\Pi^\dagger = \Pi^\dagger\mathcal{J}$. Since both $\mathcal{P}_e$ and $\tilde{\mathcal{P}}_e$ are solutions of (40), it follows from (41) that $\tilde{\mathcal{P}}_e - \mathcal{P}_e = \mathcal{J}(\tilde{\mathcal{P}}_e - \mathcal{P}_e)\mathcal{J}^T = \mathcal{J}\tilde{\mathcal{P}}_e\mathcal{J}^\top$. Following the definition in (28), we define $\mathcal{P}_o$ and $\mathcal{P}_r$ as the pseudocontrollability Gramians of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, respectively. Then, we obtain

$$\begin{bmatrix} \tilde{\mathcal{P}}_n - \mathcal{J}\tilde{\mathcal{P}}_n\mathcal{J}^\top & \tilde{\mathcal{P}}_x^\top - \mathcal{J}\tilde{\mathcal{P}}_x^\top\hat{\mathcal{J}}^\top \\ \tilde{\mathcal{P}}_x - \hat{\mathcal{J}}\tilde{\mathcal{P}}_x\mathcal{J}^\top & \tilde{\mathcal{P}}_r - \hat{\mathcal{J}}\tilde{\mathcal{P}}_r\hat{\mathcal{J}}^\top \end{bmatrix} := \begin{bmatrix} \mathcal{P}_n & \mathcal{P}_x^\top \\ \mathcal{P}_x & \mathcal{P}_r \end{bmatrix}.$$

Thus, the reduction error in (36) is obtained from (39). ■

IV. Numerical Examples

The effectiveness of the proposed approach is shown by a large-scale network example in Fig. 2, which contains three strongly connected digraphs. The data is available from the Harwell–Boeing Sparse Matrix Collection (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing).

Note that the methods in [14] and [15] are not applicable to this case. In the simulation, we select 1, 152, 728 as the input nodes and 246, 615, 733 as the measured nodes. The original network is reduced using Algorithm 1 and a random clustering scheme, respectively. Their comparison is shown in Fig. 3, where the reduction errors in terms of the $\mathcal{H}_2$-norm are evaluated under different reduced order from 10 to 700. Clearly, the proposed method has a much better performance, as the reduction error shows a rapidly decay. In Fig. 4, the topologies of reduced-order networks with different dimensions $r = 60$ and $r = 120$ are plotted. The reduction error in the two cases are given by 0.0028 and 0.0012, respectively, which are much smaller than the maximal dissimilarity 0.8839. Hence, the reduced-order network with 80 nodes provides a rather accurate approximation of the original 735-node network. In conclusion, this example shows that the proposed clustering method is feasible and effective in reducing the scale of a complex directed network.

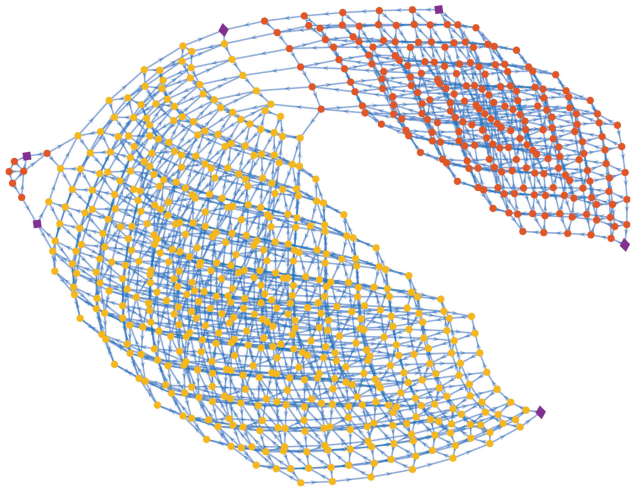

Fig. 2. Weakly connected directed network consisting of 735 nodes, where the controlled and measured nodes are labeled by diamonds and squares, respectively.

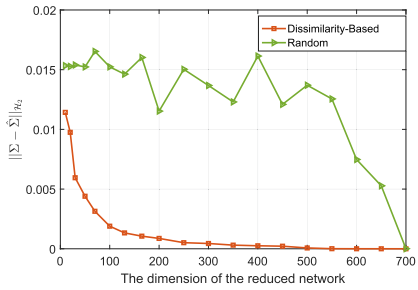

Fig. 3. Approximation error comparisons between the proposed algorithm and the random clustering algorithm.

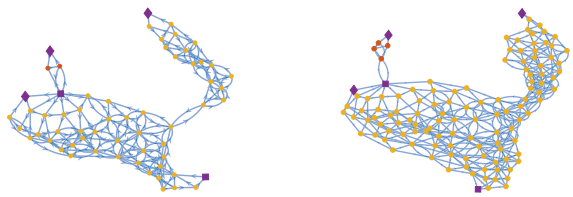

Fig. 4. Reduced directed networks of different dimensions. (a) $r = 120$. (b) $r = 60$.

V. Conclusion

This article has proposed a structure-preserving model reduction scheme for Laplacian dynamics of weakly connected networks that are semistable and have local consensus properties. The notion of clusterability is given to classify nodes that are mergeable, and pairwise dissimilarity of nodes quantify the difference among clusterable nodes. A graph clustering algorithm is applied to cluster clusterable and adjacent nodes that behave similarly. Consequently, it is guaranteed that the $\mathcal{H}_2$ reduction error is bounded, and the nodes in each cluster form a connected subgraph of the original network.

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