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Virtual Differential Passivity based Control for Tracking of Flexible-joints Robots

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Abstract: Based on recent advances in contraction methods in systems and control, in this paper we present the virtual differential passivity based control (v-dPBC) technique. This is a constructive design method that combines the concept of virtual systems and of differential passivity. We apply the method to the tracking control problem of flexible joints robots (FJRs) which are formulated in the port-Hamiltonian (pH) framework. Simulations on a two degrees of freedom FJR are presented to show the performance of a controller obtained with this approach.

Keywords:
Differential passivity, contraction analysis, virtual systems, port-Hamiltonian systems, flexible-joints robots.

1. INTRODUCTION

The problem of control of rigid robots has been widely studied since they are instrumental in modern manufacturing systems. However, the elasticity phenomena in the joints can not be neglected for accurate position tracking as reviewed in Nicosia and Tomei (1995). For every joint that is actuated by a motor, we need two degrees of freedom joints instead of one. Such FJRs are therefore *underactuated*. In Spong (1987) two state feedback control laws based on feedback linearization and singular perturbation are presented for a simplified model. Similarly, in de Wit et al. (2012) a dynamic feedback controller for a more detailed model is presented. In Loria and Ortega (1995) a computed-torque controller for FJRs is designed, which does not need *jerk* measurements. In Ailon and Ortega (1993) and Brogliato et al. (1995) passivity-based control (PBC) schemes are proposed. The first one is an observer-based controller which requires only motor position measurements. In the latter one a PBC controller is designed and compared with backstepping and decoupling techniques. For further details on PBC of FJRs we refer to Ortega et al. (1998) and references therein. In Astolfi and Ortega (2003), a global tracking controller based on the I&I method is introduced. From a more practical point of view, in Albu-Schäffer et al. (2007), a torque feedback is embedded into the passivity-based control approach, leading to a full state feedback controller; with this acceleration and jerk measurements are not required. In a recent work of Avila-Becerril et al. (2016), they design a dynamic controller which solves the global position tracking problem of FJRs based only on measurements of link and joint positions. The techniques mentioned above are designed for FJRs modeled as second order Euler-Lagrange (EL) systems. Most of these schemes are based on the selection of a suitable storage function that together with the dissipativity of the closed-loop system, ensures the convergence of state trajectories to the desired solution.

As an alternative to the EL formalism, the pH framework has been introduced in van der Schaft and Maschke (1995). The main characteristics of the pH framework are the existence of a Dirac structure (connects geometry with analysis), port-based network modeling and a *clear physical energy interpretation*. For the latter part, the energy function can directly be used to show the dissipativity of the systems. Some set-point controllers have been proposed for FJRs modeled as pH systems. For instance in Ortega and Borja (2014) the EL-controller for FJRs in Ortega et al. (1998) is adapted and interpreted in terms of Control by Interconnection [1] (CbI). In Zhang et al. (2014), they propose an Interconnection and Damping Assignment PBC (IDA-PBC [2]) scheme, where the controller is designed with respect to the pH representation of the EL-model in Albu-Schäffer et al. (2007). For the tracking control problem of FJRs in the pH framework, to the best of our knowledge, the only result is the one in Jardón-Kojakhmetov et al. (2016), where a singular perturbation approach is considered.

In this work we extend our previous results in Reyes-Báez et al. (2017b,a), on v-dPBC of fully-actuated mechanical systems, to solve the tracking problem of FJRs modeled as pH systems. This method relies on the *contraction* properties of the so-called virtual systems, Forni and Sepulchre (2014); Pavlov and van de Wouw (2017); Lohmiller and Slotine (1998); Sontag (2010); Wang and Slotine (2005).

[1] We refer interested readers on CbI to Ortega et al. (2008).
[2] For IDA-PBC technique see also Ortega et al. (2002).

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The paper is organized as follows: In Section 2, the preliminaries are presented. Section 3 deals with some properties of mechanical pH systems and the pH model of FJRs, together with its associated virtual mechanical system. A trajectory tracking v-dPBC scheme for FJRs is presented in Section 4. In Section 5, The performance of a v-dPBC controller is evaluated in simulation on a two-degrees of freedom FJR. Finally, in Section 6 conclusions and future research are stated.

2. CONTRACTION, DIFFERENTIAL PASSIVITY AND VIRTUAL SYSTEMS

In this paper, we adopt *the differential Lyapunov framework* for contraction analysis as in the paper Forni and Sepulchre (2014), which unifies different approaches. Some arguments will be omitted due to space limitation.

Let Σ be a nonlinear control system with state space $\mathcal{X}$ be the state-space of dimension N, affine in the input u,

$$\Sigma_u : \begin{cases} \dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\ y_i = h(x,t), \end{cases} \tag{1}$$

where $x \in \mathcal{X}$, $u \in \mathcal{U} \subset \mathbb{R}^n$ and $y \in \mathcal{Y}$. The vector fields $f, g_i : \mathcal{X} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ are assumed to be smooth and $h : \mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$. The input space $\mathcal{U}$ and output space $\mathcal{Y}$ are assumed to be open subsets of $\mathbb{R}^n$. System (1) in closed-loop with the uniformly smooth static feedback control law $u = \Upsilon(x,t)$ will be denoted by

$$\Sigma : \begin{cases} \dot{x} = F(x,t), \\ y = h(x,t), \end{cases} \tag{2}$$

where the vector field $F : \mathcal{X} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ is smooth.

The *variational system* a long the the trajectory $(u,x,y)(t)$ is the time-varying system $\delta\Sigma_u$, given by

$$\begin{cases} \delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} u_i \frac{\partial g_i}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} g_i \delta u_i, \\ \delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{cases} \tag{3}$$

Definition 1. (Crouch and van der Schaft (1987)). The *prolonged system* of the control system Σ_u in (1), corresponds to the system $\Sigma_u^{\delta\Sigma_u}$, that is the system described by

$$\begin{aligned} &\dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\ &y = h(x,t), \\ &\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} u_i \frac{\partial g_i}{\partial x}\delta x + \sum_{i=1}^{n} g_i(x,t)\delta u_i, \\ &\delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{aligned} \tag{4}$$

The corresponding prolonged system to (2) is

$$\Sigma^{\delta\Sigma} : \begin{cases} \dot{x} = F(x,t), \\ y = h(x,t), \\ \delta\dot{x} = \frac{\partial F}{\partial x}(x,t)\delta x, \\ \delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{cases} \tag{5}$$

with $(u,\delta u) \in T\mathcal{U}$, $(x,\delta x) \in T\mathcal{X}$, and $(y,\delta y) \in T\mathcal{Y}$.

2.1 Contraction and differential Lyapunov theory

Definition 2. (Forni and Sepulchre (2014)). A function $V : T\mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a candidate *differential or Finsler-Lyapunov function* if it satisfies, the bounds

$$c_1\mathcal{F}(x,\delta x,t)^p \leq V(x,\delta x,t) \leq c_2\mathcal{F}(x,\delta x,t)^p, \tag{6}$$

where $c_1, c_2 \in \mathbb{R}_{>0}$, p is a positive integer and $\mathcal{F}(x,\cdot,t) := \|\cdot\|_{x,t}$ is a Finsler structure, uniformly in t.

Definition 3. Consider a candidate differential Lyapunov function on $\mathcal{X}$ and the associated Finsler structure $\mathcal{F}$. For any subset $\mathcal{C} \subseteq \mathcal{X}$ and any $x_1, x_2 \in \mathcal{C}$, let $\Gamma(x_1,x_2)$ be the collection of piecewise C^1 curves $\gamma : I \to \mathcal{X}$ connecting $\gamma(0) = x_1$ and $\gamma(1) = x_2$. *The Finsler distance* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ induced by $\mathcal{F}$ is defined by

$$d(x_1,x_2) := \inf_{\Gamma(x_1,x_2)} \int_\gamma \mathcal{F}\left(\gamma(s), \frac{\partial\gamma}{\partial s}(s), t\right) ds. \tag{7}$$

The following result gives a sufficient condition for contraction in terms of differential Lyapunov functions

Theorem 1. Consider system $\Sigma^{\delta\Sigma}$ as in (5), a connected and forward invariant set $\mathcal{C} \subseteq \mathcal{X}$, and a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let V be a candidate differential Lyapunov function satisfying uniformly in t the condition

$$\dot{V}(x,\delta x,t) \leq -\alpha(V(x,\delta x,t)) \tag{8}$$

for each $(x,\delta x) \in T\mathcal{X}$. Then, on $\mathcal{C}$, system Σ in (2) is

- incrementally stable (IS) if $\alpha(s) = 0$ for each $s \geq 0$;
- asymptotically IS if α is of class $\mathcal{K}$;
- exponentially IS with rate β if $\alpha(s) = \beta s, \forall s \geq 0$.

Definition 4. (Contracting system). We say that Σ contracts V in $\mathcal{C}$ if (8) is satisfied for α of class $\mathcal{K}$. The function V is called the *contraction measure*, and $\mathcal{C}$ is *the contraction region*.

2.2 Differential passivity

Definition 5. (van der Schaft (2013)). Consider a nonlinear control system Σ_u given by (1) together with its prolonged system $\Sigma_u^{\delta\Sigma_u}$ given by (4). Then, Σ_u is called *differentially passive* if there exist a *differential storage function* function $W : T\mathcal{X} \to \mathbb{R}_{\geq 0}$ satisfying

$$\frac{dW}{dt}(x,\delta x) \leq \delta y^\top \delta u, \tag{9}$$

for all $x, \delta x, u, \delta u$. Furthermore, system (1) is called *differentially loss-less* if (9) holds with equality.

If additionally, the differential storage function is required to be a differential Lyapunov function, then differential passivity implies contraction when the input is $u = 0$.

Lemma 1. Consider system Σ_u in (1). Suppose there exists a differential transformation $\delta\tilde{x} = \Theta(x,t)\delta x$ such that the variational dynamics $\delta\Sigma_u$ in (3) takes the form

$$\begin{aligned} \delta\dot{\tilde{x}} &= [\Xi(\tilde{x},t) - \Upsilon(\tilde{x},t)]\,\Pi(\tilde{x},t)\delta\tilde{x} + \Psi(\tilde{x},t)\delta u, \\ \delta\tilde{y} &= \Psi(\tilde{x},t)^\top\Pi(\tilde{x},t)\delta\tilde{x}, \end{aligned} \tag{10}$$

where $\Pi(\tilde{x},t)$ is a Riemannian metric, $\Xi(\tilde{x},t) = -\Xi^\top(\tilde{x},t)$, $\Upsilon(\tilde{x},t) = \Upsilon^\top(\tilde{x},t)$ and δy the variational output. If the following inequality holds

$$\delta\tilde{x}^\top\left[\dot{\Pi}(\tilde{x},t) - 2\Pi(\tilde{x},t)\Upsilon(\tilde{x},t)\Pi(\tilde{x},t)\right]\delta\tilde{x} \leq -\alpha(W) \tag{11}$$

with α of class $\mathcal{K}$. Then, Σ_u is differentially passive from δu to $\delta\tilde{y}$ with respect to the differential storage function

$$W(\tilde{x},\delta\tilde{x}) = \frac{1}{2}\delta\tilde{x}^\top\Pi(\tilde{x},t)\delta\tilde{x}. \tag{12}$$

2.3 Contraction and differential passivity of virtual systems

Definition 6. (Virtual system). Consider systems Σ and Σ_u, given by (2) and (1), respectively. Suppose that $\mathcal{C}_v \subseteq \mathcal{X}$ and $\mathcal{C}_x \subseteq \mathcal{X}$ are connected and forward invariant sets. A *virtual system* associated to Σ is defined as the system

$$\Sigma_v : \begin{cases} \dot{x}_v = \Phi_v(x_v, x, t), \\ y_v = h_v(x_v, x, t), \end{cases} \tag{13}$$

in the state $x_v \in \mathcal{C}_v$ and parametrized by $x \in \mathcal{C}_x$, where $\Phi : \mathcal{C}_v \times \mathcal{C}_x \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ and $h_v : \mathcal{C}_v \times \mathcal{C}_x \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ satisfy the condition

$$\Phi_v(x, x, t) = F(x, t) \quad \text{and} \quad h_v(x, x, t) = h(x, t), \tag{14}$$

for all $t \geq t_0$. Similarly, a *virtual control system* for Σ_u is defined as the control system

$$\Sigma_{uv} : \begin{cases} \dot{x}_v = \Gamma(x_v, x, u, t), \\ y_v = h_v(x_v, x, t), \qquad \forall t \geq t_0, \end{cases} \tag{15}$$

in the state $x_v \in \mathcal{X}$ and parametrized by $x \in \mathcal{X}$, the output $y_v \in \mathcal{Y}$, where $h_v : \mathcal{C}_v \times \mathcal{C}_x \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$ and $\Gamma : \mathcal{C}_v \times \mathcal{C}_x \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ satisfy

$$\begin{aligned} \Gamma(x, x, u, t) &= f(x, t) + G(x, t)u, \\ h_v(x, x, t) &= h(x, t), \qquad \forall u, \forall t \geq t_0. \end{aligned} \tag{16}$$

Theorem 2. (Virtual contraction). Consider Σ and Σ_v in (2) and (13), respectively. Let $\mathcal{C}_v \subseteq \mathcal{X}$ and $\mathcal{C}_x \subseteq \mathcal{X}$ be two connected and forward invariant sets. Suppose that Σ_v is uniformly contracting with respect to x_v. Then, for any $x_0 \in \mathcal{C}_x$ and $x_{v0} \in \mathcal{C}_v$, each solution to Σ_v asymptotically converges to the solution of Σ.

If Theorem 2 holds, then the actual system Σ is said to be *virtually contracting*. In case of the virtual control system Σ_{uv} is differentially passive, then the actual control system Σ_u is said to be *virtually differentially passive.*

2.4 Virtual differential passivity based control

The design procedure[3] of *virtual differential passivity based control (v-dPBC)* is divided in three main steps:

(1) Design the virtual system (15) for system (1).
(2) Design the feedback $u = \eta(x_v, x, t) + \omega$ for (15) such that the closed-loop virtual system is differentially passive for the input-output pair $(\delta y_v, \delta\omega)$ and has a desired trajectory $x_d(t)$ as steady-state solution.
(3) Define the controller for system (1) as $u = \eta(x, x, t)$.

Trajectory tracking control via v-dPBC The above method can be directly applied to solve the trajectory tracking problem, which for system (1) is stated as follows:

Tracking problem: Given a desired trajectory $x_d(t)$, design a control law $u(x, t)$ for system (1) such that $x(t) \to x_d(t)$ as $t \to \infty$, uniformly.

Proposed solution: in above step (2), split the control as

$$\eta(x_v, x, t) := u_{ff}(x_v, x, t) + u_{fb}(x_v, x, t) \tag{17}$$

such that

- The *feedforward-like* term u_{ff} ensures that the closed-loop virtual system has the desired trajectory $x_d(t)$ as particular solution.
- The *feedback* action u_{fb} commands the closed-loop system to be differentially passive in a connected and forward complete set $\mathcal{C} \subseteq \mathcal{X}$.

[3] The use of virtual systems for control design was already considered in Jouffroy and Fossen (2010) and Manchester et al. (2015).

3. MECHANICAL PORT-HAMILTONIAN SYSTEMS

Ideas in the previous section will be applied to mechanical pH systems framework,van der Schaft and Maschke (1995).

Definition 7. A port-Hamiltonian system with N dimensional state space manifold $\mathcal{X}$, input and output spaces $\mathcal{U} = \mathcal{Y} \subset \mathbb{R}^m$, and Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$, is given by

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^\top(x) \frac{\partial H}{\partial x}(x), \end{aligned} \tag{18}$$

where $g(x)$ is a $N \times m$ input matrix, and $J(x)$, $R(x)$ are the interconnection and dissipation $N \times N$ matrices which satisfy $J(x) = -J^\top(x)$ and $R(x) = R^\top(x) \geq 0$.

In the specific case of a standard mechanical system with generalized coordinates q on the configuration space $\mathcal{Q}$ of dimension n and velocity $\dot{q} \in T_q\mathcal{Q}$, the Hamiltonian function is given by the total energy

$$H(x) = \frac{1}{2} p^\top M^{-1}(q) p + P(q), \tag{19}$$

where $x = (q, p) \in T^*\mathcal{Q} := \mathcal{X}$ is the state, $P(q)$ is the potential energy, $p := M(q)\dot{q}$ is the momentum and the inertia matrix $M(q)$ is symmetric and positive definitive. Then, the pH system (18) takes the form

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -I & -D(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u, \\ y &= B^\top(q) \frac{\partial H}{\partial p}(q, p), \end{aligned} \tag{20}$$

with matrices

$$J(x) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}; R(x) = \begin{bmatrix} 0 & 0 \\ 0 & D(q) \end{bmatrix}; g(x) = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}, \tag{21}$$

where $D(q) = D^\top(q) \geq 0$ being the damping matrix and I and 0 are the $n \times n$ identity, respectively, zero matrices. The input force matrix $B(q)$ has rank $m \leq n$.

3.1 Alternative representation of mechanical pH systems

In this part, we propose an alternative representation[4] for mechanical pH systems by exploiting their structural properties. This motivated by the work of Arimoto and Miyazaki (1984) on EL systems. Consider the relation

$$-\frac{\partial}{\partial q} \left(\frac{1}{2} \dot{q}^\top M(q) \dot{q} \right) = \left[S_L(q, \dot{q}) - \frac{1}{2} \dot{M}(q) \right] \dot{q}. \tag{22}$$

where $S_L(q, \dot{q})$ is a skew-symmetric matrix whose (k, j)-th element of matrix $S_L(q, \dot{q})$ is given by

$$S_{Lkj}(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{n} \left\{ \frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q) \right\} \dot{q}_i. \tag{23}$$

In order to express (22) on $T^*\mathcal{Q}$, the Legendre transformation of the kinetic (co-)energy in the brackets of the left hand side of (22) and the definition of momentum implies

[4] Similar ideas are addressed in Stadlmayr and Schlacher (2008), Sarras et al. (2012) and Zada and Belda (2016).

$$\frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}(q)p\right) = \left[S_H(q,p) - \frac{1}{2}\dot{M}(q)\right] M^{-1}(q)p. \tag{24}$$

where $S_H(q,p) := S_L(q, M^{-1}(q)p)$ is skew-symmetric. With above, (20) takes the following structure

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & -(E(q,p)+D(q))\end{bmatrix}\begin{bmatrix}\frac{\partial P}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix}u,$$
$$y_E = \begin{bmatrix}0 & B^\top\end{bmatrix}\begin{bmatrix}\frac{\partial P}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix}, \tag{25}$$

with $E(x) := S_H(q,p) - \frac{1}{2}\dot{M}(q)$. System (25) keeps the (cyclo-)passivity with respect to the storage function (19).

3.2 Virtual mechanical systems

A virtual system associated to (20) is given by

$$\dot{x}_v = \begin{bmatrix}0_n & I\\ -I & -(E(x)+D(q))\end{bmatrix}\begin{bmatrix}\frac{\partial P}{\partial q_v}(q_v)\\ \frac{\partial H}{\partial p_v}(x_v)\end{bmatrix} + \begin{bmatrix}0\\ B(q)\end{bmatrix}u$$
$$y_v = \begin{bmatrix}0 & B^\top(q)\end{bmatrix}\begin{bmatrix}\frac{\partial P}{\partial q_v}(q_v)\\ \frac{\partial H}{\partial p_v}(x_v)\end{bmatrix}, \tag{26}$$

in the state $x_v = (q_v, p_v) \in T^*\mathcal{Q}$ and parametrized by the state variable x, which has as particular solution $x_v = x$. Remarkably, for every state trajectory $x(t)$ of (25), the virtual system (26) is also (cyclo-)passive with respect to storage function given by

$$H_v(x_v,t) = \frac{1}{2}p_v^\top M^{-1}(q)p_v + P(q_v). \tag{27}$$

Notice that for the storage function (27), $\frac{\partial H_v}{\partial q_v} = \frac{\partial P}{\partial q_v}$ holds. Then, the virtual system (26) can be rewritten as

$$\dot{x}_v = [J_v(x) - R_v(x)]\frac{\partial H_v}{\partial x_v}(x_v,t) + g(x)u$$
$$y_v = g^\top(x)\frac{\partial H_v}{\partial x_v}(x_v,t), \tag{28}$$

with $g(x)$ as in (21) and matrices

$$J_v(x) = \begin{bmatrix}0_n & I\\ -I & -S_H\end{bmatrix},\ R_v(x) := \begin{bmatrix}0 & 0\\ 0 & (D - \frac{1}{2}\dot{M})\end{bmatrix}, \tag{29}$$

where $J_v = -J_v^\top$ qualifies as interconnection matrix and $R_v = R_v^\top$. However, $R_v \geq 0$ is not required; we say that system (28) is a mechanical pH-like system.

4. TRAJECTORY TRACKING CONTROLLER OF FLEXIBLE-JOINT ROBOTS

4.1 Flexible-joints Robots as port-Hamiltonian systems

Flexible rotational joints robots are a particular class of mechanical systems (21), where the generalized position is split as $q = [q_\ell^\top, q_m^\top]^\top \in \mathcal{Q} = \mathcal{Q}_{n_\ell} \times \mathcal{Q}_{n_m}$, where q_ℓ and q_m are the $n_\ell-$ links and the n_m-motors generalized positions, respectively; with $\dim\mathcal{Q} = n_\ell + n_m$. The inertia and damping matrices are partitioned into $M(q) = \mathrm{diag}\{M_\ell(q_\ell), M_m(q_m)\}$ and $D(q) = \mathrm{diag}\{D_\ell(q_\ell), D_m(q_m)\}$, where $M_\ell(q_\ell)$ and $M_m(q_m)$ are the link and motors inertias; similarly $D_\ell(q_\ell)$ and $D_m(q_m)$ are the link and motor damping matrices, respectively. The potential energy is

$$P(q) = P_\ell(q_\ell) + P_m(q), \tag{30}$$

which is the sum of the links potential energy $P_\ell(q_\ell)$ and the joints potential energy $P_m(q) = \frac{1}{2}\zeta^\top K\zeta$, with $\zeta := q_m - q_\ell$ and $K \in \mathbb{R}^{n\times n}$ a symmetric, positive definitive matrix of stiffness coefficients. The input acts only in the motor state, that is $\mathrm{rank}(B(q)) = n_m$. We follow the standard structural assumptions in Spong (1987)

- The relative displacement ζ (deflection) at each joint is small, such that the spring's dynamics is linear.
- The $i-$th motor, which drives the $i-th$ link, is mounted at the $(i-1)$-th link.
- Each motor center of mass is on the rotation axes.
- Motors angular velocity is due to their own spinning.

Thus, a flexible-joints robot can be modeled as an underactuated pH system of the form (21), given by

$$\begin{bmatrix}\dot{q}_\ell\\ \dot{q}_m\\ \dot{p}_\ell\\ \dot{p}_m\end{bmatrix} = \begin{bmatrix}0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m}\\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m}\\ -I_{n_\ell} & 0_{n_m} & -D_\ell & 0_{n_m}\\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -D_m\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}0_{n_\ell}\\ 0_{n_m}\\ 0_{n_\ell}\\ B_m(q_m)\end{bmatrix}u,$$
$$y = B_m(q_m)^\top\frac{\partial H}{\partial p_m}(x), \tag{31}$$

where p_ℓ and p_m are the links and motors momenta, $p = [p_\ell^\top, p_m^\top]^\top$ and $B_m(q_m)$ is the input matrix associated to the motors. System (31) can be rewritten as (25), with

$$E(x) = \begin{bmatrix}S_\ell(q_\ell,p_\ell) - \frac{1}{2}\dot{M}_\ell & 0_{n_m}\\ 0_{n_\ell} & S_m(q_m,p_m) - \frac{1}{2}\dot{M}_m\end{bmatrix}, \tag{32}$$

with $S_\ell^\top = -S_\ell$ and $S_m^\top = -S_m$. With this specification, the virtual system (26) corresponding to (31) is

$$\dot{x}_v = \begin{bmatrix}0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m}\\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m}\\ -I_{n_\ell} & 0_{n_m} & -(E_\ell + D_\ell) & 0_{n_m}\\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -(E_m + D_m)\end{bmatrix}\frac{\partial H_v}{\partial x_v} + gu$$
$$y_v = g^\top(x)\frac{\partial H_v}{\partial x_v}(x_v,t). \tag{33}$$

4.2 Tracking controller design for FJRs

Notice that the link dynamics in (31) is fully-actuated by the force $K\zeta$. This is rewritten as $K\zeta = K(q_{mv} - q_{md}) + u_\ell$, where $q_{md} = q_\ell + K^{-1}u_\ell$ and u_ℓ is given in Lemma 2

Lemma 2. (Reyes-Báez et al. (2017b)). Suppose $q_{mv} = q_{md}$. Consider a smooth trajectory $x_{\ell d} = (q_{\ell d}, p_{\ell d}) \in T^*\mathcal{Q}_\ell$, with $n_\ell = \dim\mathcal{Q}_\ell$. Introduce the change coordinates

$$\tilde{x}_{\ell v} := \begin{bmatrix}\tilde{q}_{\ell v}\\ \sigma_{\ell z}\end{bmatrix} = \begin{bmatrix}q_{\ell v} - q_{\ell d}\\ p_{\ell v} - p_{\ell r}\end{bmatrix}, \tag{34}$$

and define the auxiliary momentum reference as

$$p_{\ell r} := M_\ell(q_\ell)(\dot{q}_{\ell d} - \phi_\ell(\tilde{q}_{\ell v})), \tag{35}$$

with $\phi_\ell : \mathcal{Q} \to T\mathcal{Q}_\ell$ and a positive definite Riemannian metric $\Pi_\ell : \mathcal{Q}_\ell \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_\ell \times n_\ell}$ satisfying the inequality

$$\dot{\Pi}_\ell(\tilde{q}_{\ell v},t) - \Pi_\ell(\tilde{q}_{\ell v},t)\frac{\partial \phi_\ell}{\partial \tilde{q}_{\ell v}}(\tilde{q}_{\ell v}) - \frac{\partial \phi_\ell^\top}{\partial \tilde{q}_{\ell v}}(\tilde{q}_{\ell v}) \times \Pi_\ell(\tilde{q}_{\ell v},t) \leq -2\beta_\ell(\tilde{q}_{\ell v},t)\Pi_\ell(\tilde{q}_{\ell v},t), \tag{36}$$

with $\beta_\ell > 0$, uniformly in t. Suppose that the i-th row of Π_ℓ is a conservative vector field. Consider the link dynamics in (31), its corresponding link virtual system in (33) and the composite control $u_\ell(x_{\ell v}, x_\ell, t) := u_{\ell ff} + u_{\ell fb}$ with

$$\begin{aligned} u_{\ell ff} &= \dot{p}_{\ell r} + \frac{\partial P_\ell}{\partial q_\ell} + \left[E_\ell + D_\ell\right] M_\ell^{-1}(q(t)) p_{\ell r}, \\ u_{\ell fb} &= -\int_{0_{n_\ell}}^{\tilde{q_{\ell v}}} \Pi_\ell(\xi_{\ell v}, t) \mathrm{d}\xi_{\ell v} - K_{\ell d} M_\ell^{-1} \sigma_{\ell v} + \omega_\ell, \end{aligned} \tag{37}$$

where $K_{\ell d} > 0$ and ω_ℓ is an external input. Then, the link virtual system of (33) in closed-loop with (37) is differentially passive for the input-output pair $(\delta\omega_\ell, \delta y_{\sigma_{\ell v}})$, with $\delta y_{\sigma_{\ell v}} = B_\ell^\top M_\ell^{-1} \delta\sigma_{\ell v}$ and differential storage function

$$V_\ell(\tilde{x}_{\ell v}, \delta\tilde{x}_{\ell v}, t) = \frac{1}{2} \delta\tilde{x}_{\ell v}^\top \begin{bmatrix} \Pi_\ell(\tilde{q}_{\ell v}, t) & 0_{n_\ell} \\ 0_{n_\ell} & M_\ell^{-1} \end{bmatrix} \delta\tilde{x}_{\ell v}. \tag{38}$$

Proposition 1. Consider the virtual system of FJRs in (33). Suppose that the hypotheses in Lemma 2 hold for the link dynamics with the controller u_ℓ given by (37). Let the motor reference state be given by $x_{md} = (q_{md}, p_{md}) \in T^*\mathcal{Q}_m$, with $q_{md} = q_\ell + K^{-1} u_\ell$ and $n_m = \dim\mathcal{Q}_m$. Consider the following change of coordinates

$$\tilde{x}_{mv} := \begin{bmatrix} \tilde{q}_{mv} \\ \sigma_{mv} \end{bmatrix} = \begin{bmatrix} q_{mv} - q_{md} \\ p_{mv} - p_{mr} \end{bmatrix}, \tag{39}$$

and define the auxiliary motor momentum reference as

$$p_{mr} := M_m \left[\dot{q}_{md} - \phi_m(\tilde{q}_{mv}) - \Pi_m^{-1} K^\top M_\ell^\top \sigma_{\ell v}\right], \tag{40}$$

where $\phi_m : \mathcal{Q}_m \to T_{\tilde{q}_v}\mathcal{Q}_m$ and a positive definite Riemannian metric $\Pi_m : \mathcal{Q}_m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_m \times n_m}$ satisfying

$$\begin{aligned} \dot{\Pi}_m(\tilde{q}_{mv}, t) - \Pi_m(\tilde{q}_{mv}, t) \frac{\partial\phi_m}{\partial\tilde{q}_{mv}}(\tilde{q}_{mv}) - \frac{\partial\phi_m^\top}{\partial\tilde{q}_{mv}}(\tilde{q}_{mv}) \\ \times \Pi_m(\tilde{q}_{mv}, t) \leq -2\beta_m(\tilde{q}_{mv}, t)\Pi_m, \end{aligned} \tag{41}$$

with $\beta_m(\tilde{q}_{mv}, t) > 0$. Suppose that the i-th row of Π_m is a conservative vector field. Consider also system (31), its corresponding virtual system (33) and the control law given by $u(x_z, x, t) := u_{mff} + u_{mfb}$ with

$$\begin{aligned} u_{mff} &= \dot{p}_{mr} + k\zeta + \left[E_m + D_m\right] M_m^{-1}(q_m) p_{mr}, \\ u_{mfb} &= -\int_{0_{n_m}}^{\tilde{q_{mv}}} \Pi_m \mathrm{d}\xi_{mv} - K_{md} M_m^{-1} \sigma_{mv} + \omega, \end{aligned} \tag{42}$$

where $K_{md} > 0$ and ω and external input. Then, the closed-loop virtual system (33) is differentially passive with respect to input-output pair $(\delta\omega, \delta y_{\sigma_{mv}})$, with $\delta y_{\sigma_{mv}} = B_m^\top M_m^{-1} \delta\sigma_{mv}$ and differential storage function

$$V(\tilde{x}_v, \delta\tilde{x}_v, t) = \frac{1}{2} \delta\tilde{x}_v^\top \begin{bmatrix} \Pi(\tilde{q}_v, t) & 0_n \\ 0_n & M^{-1}(q) \end{bmatrix} \delta\tilde{x}_v, \tag{43}$$

with $\overline{\Pi} = \mathrm{diag}\{\Pi_\ell(\tilde{q}_{\ell v}), \Pi_m(\tilde{q}_{mv})\}$. Furthermore, the closed-loop variational dynamics of (33) preserves the structure of (10) in coordinates $\tilde{x}_v$, with

$$\begin{aligned} \Pi(\tilde{x}_v, t) &= \mathrm{diag}\{\overline{\Pi}, M^{-1}(q)\}, \\ \Xi(\tilde{x}_v, t) &= \mathrm{diag}\left\{\frac{\partial\phi}{\partial\tilde{q}_v}\overline{\Pi}^{-1}, [E(x) + D(q) + K_d]\right\}, \\ \Upsilon(\tilde{x}_v, t) &= \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & -\Pi_m^{-1}K^\top & I_{n_m} \\ -I_{n_\ell} & K\Pi_m^{-1} & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & 0_{n_m} \end{bmatrix}, \end{aligned} \tag{44}$$

where $\phi(\tilde{q}_v) = [\phi_\ell^\top, \phi_m^\top]^\top$ and $K_d = \mathrm{diag}\{K_\ell, K_m\}$.

Remark 1. From (44), it follows that the closed-loop virtual variational dynamics of the differentially passive system (33) can be written as the feedback interconnection between the variational link and motor dynamics; as stated in van der Schaft (2013).

Corollary 1. (Tracking controller for FJRs). Consider the controller (42). Then, all solutions of (31) in closed-loop with the controller $u(x, x, t)$ converges exponentially to the desired trajectory $x_d(t)$ with rate

$$\beta = \min\{\min\{\beta_\ell, \beta_m\}, \lambda_{\min}\{D + K_d\}\lambda_{\min}\{M^{-1}\}\}. \tag{45}$$

5. EXAMPLE: A FLEXIBLE-JOINTS RR ROBOT

We consider a FJR with $n_\ell = n_m = 2$ in (31). The parameters of the system are shown in Table 1,

Parameter	Value	Parameter	Value
$m_{\ell 1}$	$1.510kg$	$r_{\ell 2}$	$0.00055m$
$m_{\ell 2}$	$0.873kg$	$\ell_{\ell 1}$	$0.343m$
$I_{\ell 1}$	$0.0392kg \cdot m^2$	$\ell_{\ell 2}$	$0.267m$
$I_{\ell 2}$	$0.00808kg \cdot m^2$	$D_{\ell 1}$	$0.8\ N \cdot s/m$
$r_{\ell 1}$	$0.00159m$	$D_{\ell 2}$	$0.55N \cdot s/m$
m_{m1}	$0.23kg$	D_{m1}	$0.2\ N \cdot s/m$
m_{m2}	$0.01kg$	D_{m2}	$0.2N \cdot s/m$
K_1	$90N/m$	K_2	$90N/m$
ϕ_ℓ	$\Lambda_\ell \tilde{q}_{\ell v}$	ϕ_m	$\Lambda_m \tilde{q}_{mv}$
Λ_ℓ	$\mathrm{diag}\{10, 5\}$	Λ_m	$\mathrm{diag}\{15, 7\}$
Π_ℓ	Λ_ℓ	Π_m	Λ_m
$K_{\ell d}$	$\mathrm{diag}\{10, 5\}$	K_{md}	$\mathrm{diag}\{8, 3\}$

Table 1. Simulation parameters.

The link dynamics inertia matrix is given by

$$M_\ell(q_\ell) = \begin{bmatrix} a_1 + a_2 + 2b\cos(q_{\ell 2}) & a_2 + b\cos(q_{\ell 2}) \\ a_2 + b\cos(q_{\ell 2}) & a_2 \end{bmatrix} \tag{46}$$

with constants $a_1 = m_{\ell 1}r_1^2 + m_{\ell 2}\ell_1^2 + I_{\ell 1}$; $a_2 = m_{\ell 2}r_{\ell 2}^2 + I_{\ell 2}$; $b = m_{\ell 2}\ell_{\ell 1}r_{\ell 2}$, and motor inertia is $M_m(q_m) = \mathrm{diag}\{m_{m1}, m_{m2}\}$. The performance of the closed-loop system is shown in Figure 1; both, the position and momentum error converge to zero and the control is bounded.

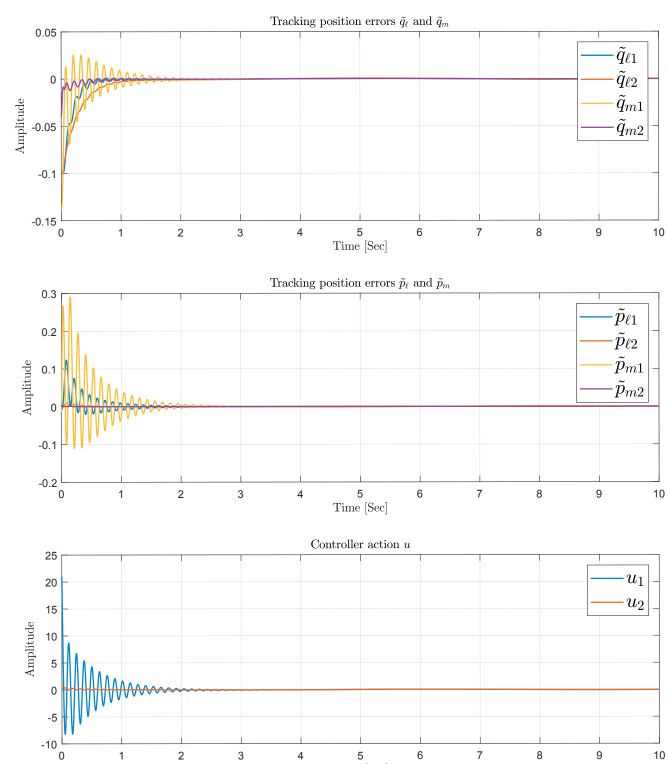

Fig. 1. Closed-loop trajectories and control signal.

6. CONCLUSION

The v-dPBC method solved the tracking control in FJRs. The closed-loop virtual system preserves the variational dynamics form (10), which can be seen as the feedback interconnection of two differentially passive subsystems. The controller $u(x, x, t)$ solves the tracking problem in FJRs. Simulations confirm the theoretical results. A major implementation drawback of our controller presented here is that we require acceleration and jerk measurements.

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