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On the benefits of saturating information in consensus networks with noise[☆]

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ABSTRACT

In a consensus network subject to non-zero mean noise, the system state may be driven away even when the disagreement exhibits a bounded response. This is unfavourable in applications since the nodes may not work properly and even be faulty outside their operating region. In this paper, we propose a new control algorithm to mitigate this issue by assigning each node a favourite interval that characterizes the nodes desired convergence region. The algorithm is implemented in a self-triggered fashion. If the nodes do not share a global clock, the network operates in a fully asynchronous mode. By this algorithm, we show that the state evolution is confined around the favourite interval and the node disagreement is bounded by a simple linear function of the noise magnitude, without requiring any priori information on the noise. We also show that if the nodes share some global information, then the algorithm can be adjusted to make the nodes evolve into the favourite interval, improve on the disagreement bound and achieve asymptotic consensus in the noiseless case.

1. Introduction

In the last two decades there has been a strong drive towards a systematic understanding of how complex large-scale systems evolve and can be efficiently monitored and controlled. Sensors and actuators are deployed over the system and exchange measurements and control signals with computing units over a communication medium, which introduces constraints on the scheduling and the duration of the transmission.

This scenario has prompted a large research activity in three directions. The first one has systematically investigated so-called event-based control methods, in which measurements are sampled and control inputs are scheduled according to a state-dependent law [1–3]. The second direction has methodically exploited the graph structure underlying large-scale systems to design distributed control laws that uses information available only locally, while achieving a global coordination task, such as consensus or synchronization [4–6]. The third research direction has investigated the combined problem, in which each local controller collects information from its neighbours and schedules new control values in an event-based fashion [7–9].

In this paper we contribute to this last line of research, according precedence to the consensus task which, in spite of its simplicity, many problems in dynamical control networks can be reduced to. In particular we focus on the *robustness of consensus to noise that is bounded but otherwise unknown*, which, in contrast to the case of consensus under noise with specific statistical properties, is a much less understood problem.

Related literature. The investigation of noisy consensus networks has been performed in the case of noise that is white [10, 11], zero-mean independent and identically distributed [12], Brownian-like [13] or martingale [14,15]. Without a statistical description of the noise, several authors have given bounds on the consensus error [16], the difference between the maximum and the minimum state over time [17], the difference [18], or the asymptotic difference [19], between the states of the agents, as a function of a suitable measure of the noise magnitude. However, these bounds come with no guarantee on the boundedness of the state, which is hard to obtain due to the presence of the zero eigenvalue in the Laplacian matrix and of noise with non-zero average. The authors in [20] developed a method that achieves approximate consensus and makes the system trajectory bounded but it requires the knowledge of the noise magnitude. State boundedness and exact consensus are achieved in [21], but the result requires the restrictive assumption on the integral of the noise absolute value to be finite. Our previous work [22] has proposed an adaptive consensus algorithm to achieve *practical* consensus, a linear dependence of the disagreement on the noise magnitude and boundedness of the state, *without* assuming any

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a priori information on the noise, except its boundedness. In the noiseless case, the node disagreement normalized by the norm of the initial state can be arbitrarily reduced. See [23] for another recent paper on the topics.

Our contribution. In this paper we propose a new method which, while still retaining the same properties of the algorithm in [22], also ensures a value of the disagreement that is independent of the norm of the initial conditions, thus guaranteeing a convergence result that is uniform over the system initial conditions.

The new algorithm assigns the nodes a "favourite" interval where the state aims at evolving and lets the nodes saturate the received information when it lies outside the interval. The favourite interval embodies the idea of a preferred operating range where the nodes are supposed to take values on. For instance, in distributed estimation, sensors are deployed to sample a physical state, like temperature or humidity, which normally varies within certain intervals. In a surveillance network, a group of agents may need to create a formation and stay within a target region, which can be characterized as an interval. More application examples include computational load sharing [24] and distributed system throttlers [25]. The favourite interval leads to a saturated control, which as shown in [26,27] is effective in contrasting perturbations from outliers and uncooperative agents. We can show that the system achieves consensus with a tolerance that can be made arbitrarily small and has a linear dependence on the noise magnitude in the noisy case. Moreover the state is confined around the given operating region for all time. The idea of using saturated version of the state to limit the system excursion is inspired by the so-called interval consensus of [24]. In this paper, we demonstrate that saturation is also useful in countering the spreading of noise over the network.

The proposed method adopts a self-triggered control scheme, which was first proposed in [28]. At each update time, each node collects the state information from the neighbours, compute its next update instant and determines the control value over the next sampling interval, resulting in a dynamical network with truly asynchronous information transmission and no global clock. Motivated by wireless communication, event/self triggered methods have been prevailing in the network control systems literature recently, since they enjoy the advantage of packet-based data exchange among the agents [29–36]. Compared with the event-triggered method, the self-triggered one does not require the agent to continuously monitor its state or listen to the communication channel, hence it can reduce the energy cost of sensing [37]. Moreover, it has been shown to be robust to various kind of malfunctions in the network, such as data loss [38] and misbehaving nodes [27].

The rest of the paper is organized as follows. In Section 2, we introduce the new self-triggered algorithm. Section 3 provides the main results showing the state convergence set and the node disagreement bound. In Section 4, we comment on the proposed algorithm and show that, by using some global information, the algorithm can be modified to achieve asymptotic consensus in the noiseless case while preserving the state boundedness in the noisy case. We then verify the main results by numerical simulations in Section 5. Section 6 provides concluding remarks. The proof of asymptotic consensus is provided in the Appendices.

2. Preliminaries

2.1. Notation

For a network with n nodes, let its topology be represented by an undirected and connected graph $G = \{I, E\}$, with $I = \{1, 2, \dots, n\}$ being the set of nodes and $E \subseteq I \times I$ being the set of edges, where $\{i, j\} \in E$, or equivalently, node i is a neighbour of node j, means that node i can receive information from node j and vice versa. We denote the set of neighbours of node i by $\mathcal{N}_i$, and let $d_i = |\mathcal{N}_i|$, $d_m = \min_{i\in I} d_i$ and $d_M = \max_{i\in I} d_i$. A path from node i to j is a sequence of nodes in I and edges in E which starts at i and ends at j.

2.2. Self-triggered interval consensus

We consider an undirected connected network with n nodes and its corresponding graph $G = \{I, E\}$. For each node i, we assume its dynamics is described by

$$\dot{x}_i = u_i \tag{1}$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input. The nodes aim at evolving towards a common value, which is not agreed upon a priori. To fulfil this task, each node needs to obtain the state information from other nodes using communication or measurements. The information transmission may be affected by noise, so we assume that node $i \in \mathcal{N}_j$ receives the following noisy state of node j

$$x^w_{ij}(t) = x_j(t) + w_{ij}(t) \tag{2}$$

where $w_{ij}(t) \in \mathbb{R}$ represents the communication noise when node j transmits information to node i at time t. Throughout the paper, the noise is assumed to be bounded in magnitude, namely $|w_{ij}(t)| \le \overline{w}$ for all $\{i, j\} \in E$ and all $t \in \mathbb{R}_{\ge 0}$. Even though $\overline{w}$ may be estimated by empirical tests in some cases, this may not be always convenient, and we assume its value is not known.

We assume that all the nodes have a so-called *favourite interval* $[p, q] \subset \mathbb{R}$ where the node would like its state to evolve and $p \le q$. Based on this favourite interval, for each neighbour $j \in \mathcal{N}_i$, node i takes the saturated version of the received state of j, namely

$$y_{ij}(t) = \text{sat}(x^w_{ij}(t)) \tag{3}$$

where

$$\text{sat}(z) = \begin{cases} p, & \text{if } z < p \\ z, & \text{if } p \le z \le q \\ q, & \text{if } z > q \end{cases} \tag{4}$$

Node i then computes the saturated noisy average

$$z^w_i(t) = \sum_{j\in\mathcal{N}_i} (y_{ij}(t) - x_i(t)) \tag{5}$$

We also denote the actual average of each node $i \in I$ by

$$\text{ave}_i(t) = \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)) \tag{6}$$

For each node $i \in I$, let $\{t^i_k\}_{k\in\mathbb{Z}_{\ge 0}}$, with $t^i_0 = 0$, be the sequence of triggering times at which node i accesses the communication network. At these times, the node collects the state information from it neighbours, updates its control action and determines the next triggering time.

The control signals take values in the set $\mathcal{U} := \{-1, 0, +1\}$, and the specific quantizer of choice is $\text{sign}_\alpha : \mathbb{R} \to \mathcal{U}$, $\alpha > 0$, which is given by

$$\text{sign}_\alpha(z) := \begin{cases} \text{sign}(z), & \text{if } |z| \ge \alpha \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

The control action is given by

$$u_i(t) = \text{sign}_\varepsilon\left(z^w_i(t^i_k)\right) \tag{8}$$

for t that belongs to the left-closed and right-open time interval $[t^i_k, t^i_{k+1}[$, where $\varepsilon > 0$ is a design threshold determining the consensus accuracy.

The triggering times are given by $t_{k+1}^i = t_k^i + \Delta_k^i$, where

$$\Delta_k^i := \max\left\{\frac{|z_i^w(t_k^i)|}{4d_i}, \quad \frac{\varepsilon}{4d_i}\right\} \tag{9}$$

Notice that the algorithm is Zeno free since the inter-sampling times are always positive by construction.

Remark 1 (*On the Favourite Interval*)**.** The state favourite interval should be chosen according to the physical constraints and task requirements. In the distributed estimation problem, all the sensors may seek to agree on the value of some physical variable. In this case, the interval $[p, q]$ can be pre-set according to the operating range of the sensors. If the task is cooperative surveillance of a specific region, the multi-agent system may need to create a formation and stay within the target region. In this scenario, the interval can be set as the boundary of the region.

Remark 2 (*Heterogeneous Thresholds and Favourite Intervals*)**.** The algorithm above can be modified to assign each node a threshold ε_i. In this situation, similar results for the state convergence interval and node disagreement can be attained. We do not study the algorithm with different thresholds ε_i because it would clutter the notation even more, without adding much from a conceptual viewpoint.

For some applications it is reasonable to expect that the agents adopt different favourite intervals $[p_i, q_i]$. For this situation, provided that the intersection of all the intervals is nonempty, we can show that the state will be bounded around the intersection of all the intervals and the node disagreement will still be bounded by a value depending on the noise magnitude. However, the theoretical bounds we obtain are much conservative compared with the simulation results and improving these conservative theoretical bounds is nontrivial. Hence we decided not to present them in this paper.

3. Main result

We present the main result of the proposed self-triggered consensus method in this section. The main result as well as the preparatory statements throughout the paper hold for undirected and connected graphs G.

3.1. State convergence interval

Let $\overline{x}(t) = \max_i x_i(t)$ and $\underline{x}(t) = \min_i x_i(t)$. We start by showing that the system state is always bounded during the evolution and converges to an interval which depends on the end values of the favourite interval, the threshold ε and the noise magnitude.

Theorem 1. *Consider a network of n dynamical systems as in (1), which are interconnected over the graph G. Let each local control input be generated in accordance with (3)–(9). Then for all* $t \in \mathbb{R}_{\geq 0}$

$$\overline{x}(t) \leq \max\{\overline{x}(0),\ q + \varepsilon/d_m\}, \qquad \underline{x}(t) \geq \min\{\underline{x}(0),\ p - \varepsilon/d_m\}$$

Moreover, there exists a finite time T such that $x_i(t) \in [p - \varepsilon/d_m, q + \varepsilon/d_m]$ *for all* $t \geq T$ *and all* $i \in I$.

Proof. We only prove the upper bounds, since the conclusion for the lower bound can be derived in a similar way. Let $\overline{\gamma} = q + \varepsilon/d_m$. We first show two facts that will be used later.

Fact 1. For any node $i \in I$ and any $t' \geq 0$, if $x_i(t') \leq \overline{\gamma}$, then it will never exceed $\overline{\gamma}$ for all time $t \geq t'$.

This fact can be verified as follows. Let $t_m^i = \max\{t_k^i \in \mathbb{R}_{\geq 0}, t_k^i \leq t'\}$. If $u_i(t') \leq 0$, then $u_i(t) \leq 0$ for all $t \in [t_m^i, t_{m+1}^i[$, hence $x_i(t) \leq x_i(t') \leq \overline{\gamma}$ for all $t \in [t', t_{m+1}^i[$. While if $u_i(t') = 1$, according to (8), the noisy saturated average at t_m^i should satisfy

$$z_i^w(t_m^i) = \sum_{j\in\mathcal{N}_i}(y_{ij}(t_m^i) - x_i(t_m^i)) \geq \varepsilon \tag{10}$$

which implies

$$x_i(t_m^i) \leq \frac{1}{d_i}\Big(\sum_{j\in\mathcal{N}_i} y_{ij}(t_m^i) - \varepsilon\Big) \leq q - \varepsilon/d_i \tag{11}$$

where the last inequality comes from the definition of the saturation function (4). Consider the evolution of $x_i(t)$ for $t \in [t_m^i, t_{m+1}^i[$, we have

$$\begin{aligned} x_i(t) &= x_i(t_m^i) + u_i(t_m^i)(t - t_m^i) \leq x_i(t_m^i) + \Delta_m^i \\ &= x_i(t_m^i) + |z_i^w(t_m^i)|/(4d_i) \\ &= x_i(t_m^i) + \sum_{j\in\mathcal{N}_i}(y_{ij}(t_m^i) - x_i(t_m^i))/(4d_i) \\ &\leq x_i(t_m^i) + (q - x_i(t_m^i))/4 \\ &= 3x_i(t_m^i)/4 + q/4 \leq q - 3\varepsilon/(4d_i) < \overline{\gamma} \end{aligned}$$

where we used (10) in the third equality, (11) in the third inequality.

By the induction argument, we have $x_i(t) \leq \overline{\gamma}$ for all $t \geq t'$.

Fact 2. For any node $i \in I$ and any $t' \geq 0$, if $x_i(t') > \overline{\gamma}$, it decreases at time t', namely $u_i(t') < 0$.

This fact can be checked as follows. From Fact 1, we have $x_i(t_m^i)$ should be larger than $\overline{\gamma}$, otherwise $x_i(t') \leq \overline{\gamma}$. Consider the saturated average

$$z_i^w(t_m^i) = \sum_{j\in\mathcal{N}_i}(y_{ij}(t_m^i) - x_i(t_m^i)) < d_i(q - \overline{\gamma}) = -d_i\varepsilon/d_m \leq -\varepsilon$$

which implies that $u_i(t') = u_i(t_m^i) = -1$.

We now finalize the proof. If $\overline{x}(0) \leq \overline{\gamma}$, by Fact 1, we have that $x_i(t) \leq \overline{\gamma}$ for all $t \geq 0$. If $\overline{x}(0) > \overline{\gamma}$, for the nodes with $x_i(0) \leq \overline{\gamma}$, $x_i(t) \leq \overline{\gamma} \leq \overline{x}(0)$ holds for all $t \geq 0$ by Fact 1. While for the nodes whose initial condition is greater than $\overline{\gamma}$, according to Fact 2, their states decrease with the same constant rate until the state is no greater than $\overline{\gamma}$. This implies that $\overline{x}(t) \leq \overline{x}(0)$ for all $t > 0$, which ends the proof of the first claim. The second claim follows again from the previous analysis. ■

Remark 3 (*State Convergence Interval*)**.** There is a discrepancy between the actual state convergence interval and $[p, q]$. The actual interval to which the state converges enlarges the interval $[p, q]$ on both directions by ε/d_m. However, if the initial state is already within $[p, q]^n$, one can show that the state will never evolve outside $[p, q]^n$. To make the state converge within the favourite interval $[p, q]^n$ for any initial state $x(0)$, the agent can use $q - \varepsilon$ and $p + \varepsilon$ as the upper and lower switching points of the saturation function, respectively. This yields

$$\text{sat}(z) = \begin{cases} p + \varepsilon, & \text{if } z < p + \varepsilon \\ z, & \text{if } p + \varepsilon \leq z \leq q - \varepsilon \\ q - \varepsilon, & \text{if } z > q - \varepsilon \end{cases} \tag{12}$$

Note that (12) requires the condition $q - p > 2\varepsilon$. Based on Theorem 1, by this strategy, the state converges in finite time to the interval $[p + \varepsilon - \varepsilon/d_m, q - \varepsilon + \varepsilon/d_m] \subset [p, q]$.

3.2. Practical consensus

We then consider the consensus property of the proposed method. As shown in the following result, the node disagreement will be bounded by a term which scales linearly with the noise magnitude. With no noise, the node disagreement is bounded by a quantity which can be made arbitrarily small by scaling the design threshold ε. In view of this feature, we refer to the result below as *practical consensus.*

Theorem 2. *Consider the same assumptions and condition as in Theorem 1, then there exists a finite time T' such that for all $i \in I$ and all $t \geq T'$,*

$$|\mathrm{ave}_i(t)| \leq 5\varepsilon/4 + \max\{d_M\varepsilon/d_m, d_M\overline{w}\}. \tag{13}$$

Remark 4 (*Pointwise Convergence*)**.** Theorem 2 shows a convergence result for general unknown but bounded noise. If $[\underline{x}(0), \overline{x}(0)] \subseteq [p, q]$ and the noise is sufficiently small such that $\overline{w} \leq \frac{\varepsilon}{2d_M}$, then by a similar proof as the one of [22, Theorem 3], we can show that $x_i(t)$ for all $i \in I$ and all $t \geq 0$ will remain in $[\underline{x}(0), \overline{x}(0)]$ and converge in a finite time to a point at which $|\mathrm{ave}_i| \leq \varepsilon + d_M\overline{w}$ for all $i \in I$. Thus, in a special yet interesting case, the state converges to a point in $[\underline{x}(0), \overline{x}(0)]$.

To prove this theorem, we need some intermediate results. We first introduce two sets for each node i as follows

$$\mathscr{S}_{i1} := \left\{t_k^i : |\mathrm{ave}_i(t_k^i)| \geq L_i\right\}, \quad \mathscr{S}_{i2} := \left\{t_k^i : |\mathrm{ave}_i(t_k^i)| < L_i\right\} \tag{14}$$

where $L_i := 5\varepsilon/4 + d_i\overline{w}'$, with $\overline{w}' = \max\{\varepsilon/d_m, \overline{w}\}$. Clearly, $t_k^i \in \mathscr{S}_{i1} \cup \mathscr{S}_{i2}$ for every $k \in \mathbb{Z}_{\geq 0}$.

The following result shows that if at certain time node i enters $\mathscr{S}_{i2}$, it will remain in this set.

Lemma 1 (*Invariant Set*)**.** *Consider the same assumptions and conditions as in Theorem 1. Consider the system evolution for $t \geq T$ with T as in Theorem 1. If $t_k^i \in \mathscr{S}_{i2}$, then $t_M^i \notin \mathscr{S}_{i1}$ for all integers $M \geq k+1$. Moreover, $|\mathrm{ave}_i(t)| < L_i$ for all $t \geq t_k^i$.*

Proof. We first show that the following inequality

$$|y_{ij}(t) - x_i(t)| \leq \overline{w}' \tag{15}$$

holds for all $t \geq T$ and all $i \in I$.

By Theorem 1, for all $t \geq T$, each node i's state satisfies $p - \varepsilon/d_m \leq x_i(t) \leq q + \varepsilon/d_m$. We consider the following three cases for $t \geq T$.

Case 1, $p \leq x_{ij}^w(t) \leq q$. By (3), $y_{ij}(t) = x_i^w(t)$. Hence,

$$|y_{ij}(t) - x_i(t)| = |x_{ij}^w(t) - x_i(t)| \leq \overline{w} \leq \overline{w}'$$

where the second inequality comes from (2).

Case 2, $x_{ij}^w(t) \geq q$. Then by (3), $y_{ij}(t) = q$. Since $x_i(t) = x_i^w(t) - w_{ij}(t)$, $q - \overline{w} \leq x_i(t) \leq q + \varepsilon/d_m$, which implies that

$$|y_{ij}(t) - x_i(t)| \leq \max\{\varepsilon/d_m, \overline{w}\} = \overline{w}'$$

Case 3, $x_{ij}^w(t) \leq p$. Then by (3) and (2), we have $y_{ij}(t) = p$ and $p - \varepsilon/d_m \leq x_i(t) \leq p + \overline{w}$, which implies

$$|y_{ij}(t) - x_i(t)| \leq \max\{\varepsilon/d_m, \overline{w}\} = \overline{w}'$$

For each node i, we then let

$$\phi_i(t) = \sum_{j\in\mathcal{N}_i}(y_{ij}(t) - x_j(t)) \tag{16}$$

By (6) and (15), we have that for all $t \geq T$,

$$z_i^w(t) = \mathrm{ave}_i(t) + \phi_i(t), \qquad |\phi_i(t)| \leq d_i\overline{w}' \tag{17}$$

Now consider the following two sub-cases,

Sub-case a. $|z_i^w(t_k^i)| \geq \varepsilon$. Without loss of generality, we assume $z_i^w(t_k^i) \geq \varepsilon$. Then $u_i(t) = 1$ for all $t \in [t_k^i, t_{k+1}^i[$ and

$$\varepsilon - \phi_i(t_k^i) \leq \mathrm{ave}_i(t_k^i) \leq L_i \tag{18}$$

For $t \in [t_k^i, t_{k+1}^i[$, the average is given by

$$\mathrm{ave}_i(t) = \mathrm{ave}_i(t_k^i) + \int_{t_k^i}^t \sum_{j\in\mathcal{N}_i}(u_j(\tau) - 1)d\tau$$

since $|u_j(t)| \leq 1$, we have $\mathrm{ave}_i(t) \leq \mathrm{ave}_i(t_k^i) \leq L_i$ and

$$\begin{aligned}\mathrm{ave}_i(t) &\geq \mathrm{ave}_i(t_k^i) - 2d_i(t - t_k^i) \geq \mathrm{ave}_i(t_k^i) - 2d_i\Delta_k \\ &= \mathrm{ave}_i(t_k^i) - \frac{1}{2}|z_i^w(t_k^i)| = \mathrm{ave}_i(t_k^i) - \frac{1}{2}(\mathrm{ave}_i(t_k^i) + \phi_i(t_k^i)) \\ &= \frac{1}{2}(\mathrm{ave}_i(t_k^i) - \phi_i(t_k^i)) \geq \frac{1}{2}(\varepsilon - 2\phi_i(t_k^i)) \\ &\geq \varepsilon/2 - d_i\overline{w}' > -L_i\end{aligned} \tag{19}$$

where the third inequality comes from (18), the fourth from (17) and the last from the definition of L_i.

Sub-case b. $|z_i^w(t_k^i)| < \varepsilon$. Then $\Delta_k^i = \varepsilon/(4d_i)$, $u_i(t) = 0$ for all $t \in [t_k^i, t_{k+1}^i[$ and

$$\begin{aligned}|\mathrm{ave}_i(t_k^i)| &= |z_i^w(t_k^i) - \phi_i(t_k^i)| \leq |z_i^w(t_k^i)| + |\phi_i(t_k^i)| \\ &< \varepsilon + d_i\overline{w}' = L_i - \varepsilon/4\end{aligned}$$

Hence for $t \in [t_k^i, t_{k+1}^i[$, the average satisfies

$$\begin{aligned}|\mathrm{ave}_i(t)| &= |\mathrm{ave}_i(t_k^i) + \int_{t_k^i}^t \sum_{j\in\mathcal{N}_i} u_j(\tau)d\tau| \\ &\leq |\mathrm{ave}_i(t_k^i)| + |\int_{t_k^i}^t \sum_{j\in\mathcal{N}_i} u_j(\tau)d\tau| \\ &\leq |\mathrm{ave}_i(t_k^i)| + |d_i\Delta_k^i| \\ &< L_i - \varepsilon/4 + \varepsilon/4 = L_i\end{aligned}$$

By induction as above for all the integers $M \geq k+1$ and all the time intervals $[t_{M-1}^i, t_M^i[$, we prove the result. ∎

The next result shows that the average preserves the sign as long as its absolute value remains large enough.

Lemma 2. *Consider the same assumptions and conditions as in Theorem 1, Consider the system evolution for $t \geq T$, with T as in Theorem 1. For any $i \in I$ and any positive integer M, if $|\mathrm{ave}_i(t_{k+m}^i)| \geq L_i$ for $m = 0, 1, \ldots, M$, then $\mathrm{sign}(\mathrm{ave}_i(t_{k+m}^i)) = \mathrm{sign}(\mathrm{ave}_i(t_k^i))$, for $m = 1, 2, \ldots, M+1$*

Proof. Notice that, since $t > T$, the inequality (17) always holds in the following analysis. Suppose w.l.o.g that $\mathrm{ave}_i(t_k^i) \geq L_i > 0$, we know

$$z_i^w(t_k^i) = \mathrm{ave}_i(t_k^i) + \phi_i(t_k^i) \geq L_i - d_i\overline{w}' = \varepsilon + \varepsilon/4 > \varepsilon$$

This implies that $u_i(t) = 1$ for all $t \in [t_k^i, t_{k+1}^i[$. Hence same as (19), we have

$$\mathrm{ave}_i(t_{k+1}^i) \geq \frac{1}{2}(\mathrm{ave}_i(t_k^i) - \phi_i(t_k^i)) \geq \frac{1}{2}(L_i - d_i\overline{w}') \geq \frac{5}{8}\varepsilon$$

which has the same sign as $\mathrm{ave}_i(t_k^i)$. ∎

Proof of Theorem 2. Notice that

$$\begin{aligned}L_i &= 5\varepsilon/4 + d_i\overline{w}' = 5\varepsilon/4 + d_i\max\{\varepsilon/d_m, \overline{w}\} \\ &\leq 5\varepsilon/4 + \max\{d_M\varepsilon/d_m, d_M\overline{w}\}\end{aligned}$$

Accordingly, it is sufficient to show that there exists a time $t' \geq T$ with T given as in Theorem 1, such that $|\mathrm{ave}_i(t)| < L_i$ for all $t \geq t'$.

We claim that there exist a finite sampling time $t_s^i \geq T$ such that $|\mathrm{ave}_i(t_s^i)| < L_i$. Suppose this is not true, assume that $|\mathrm{ave}_i(t_k^i)| \geq L_i$ holds for every $t_k^i \geq T$. By Lemma 2, the average preserves its sign. Assume without loss of generality that $\mathrm{ave}_i(t_k^i) \geq L_i$ for all $t_k^i \geq T$. Let $t_r^i = \min\{t_k^i : t_k^i \geq T\}$, then by Lemma 2, $u_i(t) = 1$ for all $t \geq t_r^i$ and $\mathrm{sign}(\mathrm{ave}_i(t_k^i)) = \mathrm{sign}(\mathrm{ave}_i(t_r^i)) > 0$ for all $t_k^i \geq t_r^i$. However, this implies that $x_i(t)$ will increase to infinity, which contradicts Theorem 1 and proves the existence of finite t_s^i.

Now consider the system evolution for the time $t \geq t_s^i$. Since $|\mathrm{ave}_i(t_s^i)| < L_i$, by the invariant property shown in Lemma 1, we have $|\mathrm{ave}_i(t)| < L_i$ for all $t \geq t_s^i$. This ends the proof. ■

4. Asymptotic consensus

In the noiseless case, the algorithm in Section 2 only achieves practical consensus. In this section, we show that the algorithm can be modified to achieve *asymptotic* consensus, i.e., the node disagreements converge to zero asymptotically, with positive lower bounded inter-sampling times by requiring some global information. As shown later, with this modification, all nodes employ a common time-varying threshold $\varepsilon(t)$ and control magnitude $\alpha(t)$. Compared with the algorithm in Section 2, this in fact implies that the nodes have access to a common clock. Nonetheless, we emphasize that the nodes still need not to synchronously communicate states.

In detail, the system dynamics (1) becomes

$$\dot{x}_i(t) = \alpha(t) u_i(t) \tag{20}$$

where the control input between two successive sampling times t_k^i and t_{k+1}^i is given by

$$u_i(t) = \mathrm{sign}_{\varepsilon(t_k^i)}(z_i^w(t_k^i)) \tag{21}$$

and the inter-sampling time satisfies

$$\Delta_k^i := \max\left\{ \frac{|z_i^w(t_k^i)|}{4\alpha(t_k^i) d_i}, \quad \frac{\varepsilon(t)}{4\alpha(t_k^i) d_i} \right\} \tag{22}$$

The signals $\varepsilon(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\alpha(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are positive monotone decreasing functions and satisfy the conditions

$$\lim_{t\to\infty} \varepsilon(t) = \lim_{t\to\infty} \alpha(t) = 0, \tag{23}$$

$$\int_t^\infty \alpha(\tau) d\tau = +\infty, \quad \frac{\varepsilon(t)}{\alpha(t)} \geq c \tag{24}$$

for all $t \geq 0$, with c being a positive value. The first two equalities guarantee that the system will achieve consensus and the last inequality is used to rule out the Zeno behaviour.

We can show the following result on the state boundedness.

Theorem 3. *Consider a network of n dynamical systems as in (20), which are interconnected over the graph G. Let each local control input be generated in accordance with (21) and (22). If $\varepsilon(t)$ and $\alpha(t)$ satisfy (23), then for all $t \in \mathbb{R}_{\geq 0}$,*

$$\overline{x}(t) \leq \max\{\overline{x}(0), q + \varepsilon(0)/d_m\},$$
$$\underline{x}(t) \geq \min\{\underline{x}(0), p - \varepsilon(0)/d_m\}$$

Moreover, $\limsup_{t\to+\infty} \overline{x}(t) = q$ *and* $\liminf_{t\to+\infty} \underline{x}(t) = p$.

The consensus property can be described as follows.

Theorem 4. *Consider the same assumptions and conditions as in Theorem 3. If $\varepsilon(t)$ and $\alpha(t)$ satisfy (23), then for all $i \in I$,* $\limsup_{t\to\infty} |\mathrm{ave}_i(t)| \leq d_M \overline{w}$.

The proofs of Theorems 3 and 4 are provided in Appendices A and B. Here we stress that by the time-varying threshold and control magnitude in (23), when $w \equiv 0$, the algorithm achieves exact consensus while making the state converge to the desired favourite interval. Moreover, in the noisy case, the system state is still confined within the desired favourite interval. Finally, if the noise converges to zero, one can verify that the consensus error will approach zero.

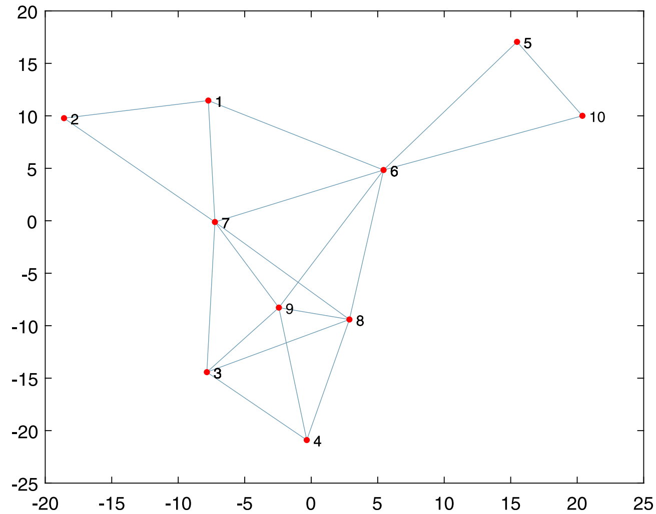


Fig. 1. Graph of the consensus network considered in Section 5.

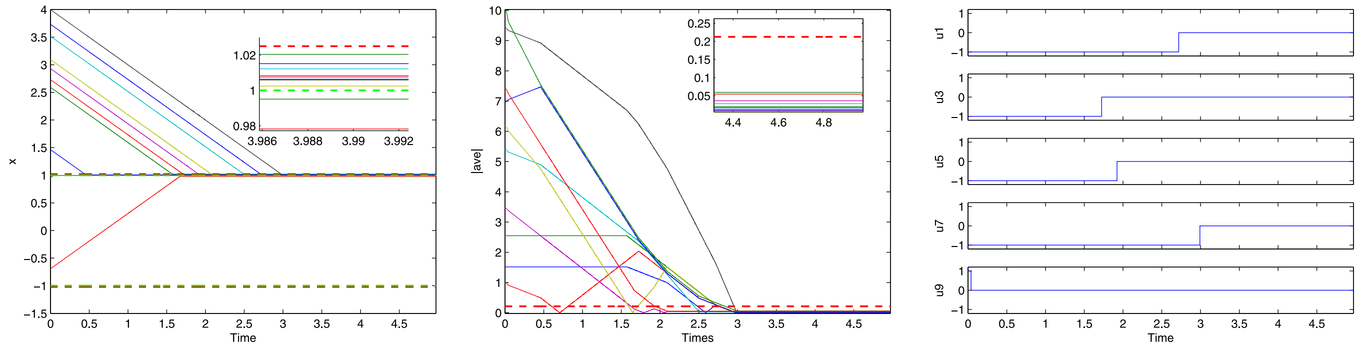
Remark 5 (*Benefit of the Proposed Method*)**.** To the best of our knowledge, most of the works addressing the case of noise assume prior knowledge of the noise magnitude, see, *e.g.*, [20]. The case of noise with unknown magnitude is considered in [22]. It employs an adaptive threshold which scales dynamically with the state absolute value. By this adaptive threshold, we show that the state bound depends on a term proportional to $\overline{w}$ and inversely proportional to the parameter ε. Furthermore, the bound on the local disagreement $|\mathrm{ave}_i(t)|$ is a linear function of $\overline{w}$ plus a bilinear function of the parameter ε and $\max_{i\in I} |x_i(0)|$. Thus the node disagreement may be large when the norm of the initial state is large. Moreover, as $\varepsilon \to 0$, the bounds on the state diverges. Hence we cannot simultaneously achieve asymptotic consensus in the noiseless case and make the system state bounded in the presence of noise. In this paper, we avoid these drawbacks by keeping the threshold unchanged and using the notion of favourite interval. By Theorems 1 and 2, the state evolution is bounded even for $\varepsilon \to 0$ and the node disagreements converge in finite time to a set whose size is independent of the magnitude of the initial states. Hence, with the new method, we can force the node disagreements to converge to a smaller set by choosing smaller ε. Moreover, by Theorems 3 and 4, the algorithm can be adjusted to steer $\mathrm{ave}_i(t)$ asymptotically to zero when $w = 0$ without compromising the state boundedness in the noisy case.

5. Numerical examples

In this section, we perform simulations to verify the results. We consider a 10-node network with the communication graph illustrated in Fig. 1. From the figure, we know that $d_m = 2$ and $d_M = 6$. For all the simulations we assume $[p, q] = [-1, 1]$.

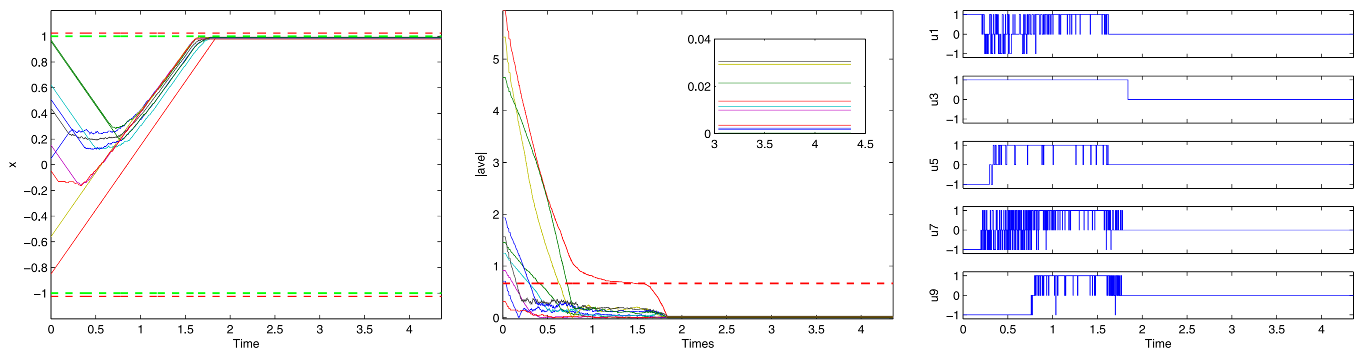
5.1. Constant threshold

In this subsection we assume the consensus threshold $\varepsilon = 0.05$. We first consider the noiseless situation and let the initial state of each node be generated as a random value within the interval $[-2, 4]$. The simulation result is given in Fig. 2. Fig. 2(a)



(a) State (b) Absolute value of the average (c) Control inputs at the odd nodes

Fig. 2. Network behaviour for constant threshold and $\overline{w} = 0$.



(a) State (b) Absolute value of the average (c) Control inputs at the odd nodes

Fig. 3. Network behaviour for constant threshold and positive random noise with $\overline{w} = 0.1$.

shows the state trajectory, where the green dashed lines represent the boundary of $[p, q]$ and the red dashed lines represent the real state bounds ($\overline{\gamma} = 1.025$ and $\underline{\gamma} = -1.025$) from Theorem 3. Fig. 2(b) shows the absolute values of the averages, where the red dashed line represents the bound on the average from Theorem 2 (0.2125). Fig. 2(c) shows the control inputs for the nodes with odd indices. From the figure, we can find that the nodes which are outside $[\underline{\gamma}, \overline{\gamma}]$ move toward it with constant velocity until they converge inside the interval. All the nodes enter the set $[\underline{\gamma}, \overline{\gamma}]$ within 3 s. There are some nodes whose states are larger than q, in conformity with Theorem 1. From Fig. 2(b), all the absolute averages are less than the bound given in Theorem 2 within 3 s and the absolute averages are much less than the theoretical bound. Moreover, from these simulations it appears that all the control inputs eventually become zero even though we do not provide yet a proof of this.

In the second example, we consider the noisy case where $w_{ij}(t), \{i, j\} \in E$ is a positive random noise with $\overline{w} = 0.1$. The initial state of each node is generated randomly within $[-1, 1]$. The results are presented in Fig. 3. From these figures, we can see that the states are driven by the noise to increase, however, they never exceed the upper bound of the favourite interval. The absolute values of all the averages become smaller than the theoretical bound (0.6625) within 2 s. The bound on the absolute averages value is 0.035, much less than the theoretical bound. The control inputs become zero after 2 s. This is reasonable since when all the states are within $[q - \varepsilon/d_M, q]$, with positive noise, all $y_{ij}(t)$ are also within $[q - \varepsilon/d_M, q]$ and the control inputs for all nodes will be zero by (8).

In the third example, we consider the noisy case where $w_{ij}(t), \{i, j\} \in E$ is a random noise with $\overline{w} = 0.1$, with initial state of each node generated randomly within $[-1, 1]$. The results are presented in Fig. 4. From these figures, we can see that the bound on the absolute averages value is 0.17, less than the theoretical bound (0.6625). Moreover, the control inputs are non-zeros for most of the time.

5.2. Time-varying threshold

In this subsection, we consider the threshold and control magnitude function from [28], given in the following form

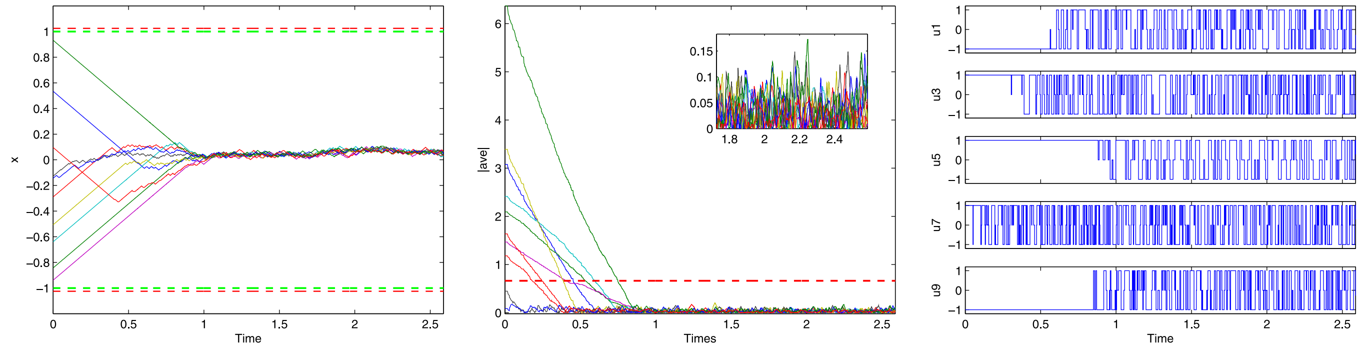
$$\alpha(t) = 1/(1+t), \qquad \varepsilon(t) = c\alpha(t)$$

where $c = 0.05$ is the constant mentioned in Section 4.

We only present the simulation for the noiseless case since the numerical result for the noisy case in this subsection is similar to that in the last subsection. In the simulation, the initial state is the same as that of the first example in the constant threshold case. The simulation results are given in Fig. 5. In Fig. 5(b), the red dashed line represents the bound calculated in (13) with $\varepsilon = 0.05$. From these figures, we can see that the absolute value of the average decreases asymptotically and all the states approach $[p, q]$.

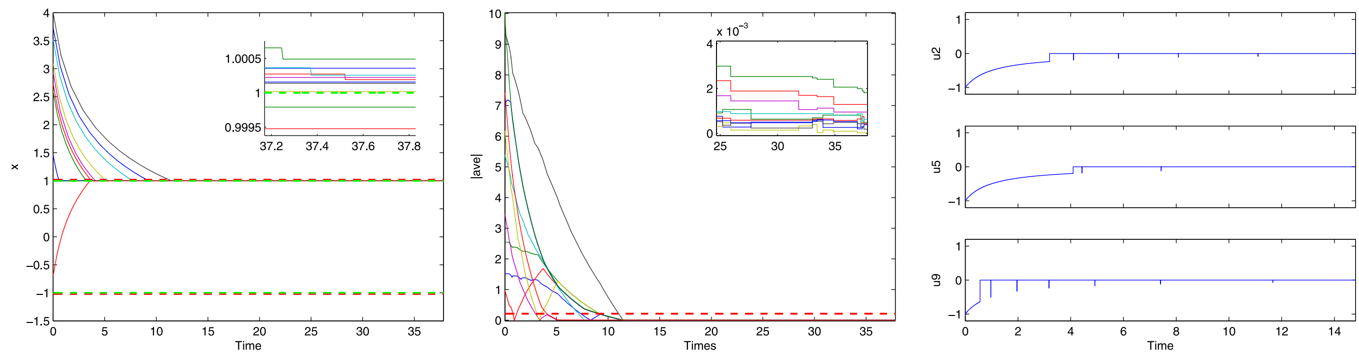
6. Conclusion

In this paper, we presented a self-triggered coordination method in the presence of communication noise. The nodes adopt a common favourite interval of evolution and saturate the received states from the neighbours. For constant thresholds, the method can achieve approximate consensus and make the system state converge within a set around the favourite interval in finite time. Compared with our previous result [22], the node disagreement in this paper is independent of the initial condition



(a) State (b) Absolute value of the average (c) Control inputs at the odd nodes

Fig. 4. Network behaviour for constant threshold and random noise with $\overline{w} = 0.1$.



(a) state (b) Absolute value of the average (c) Control inputs of part of the nodes

Fig. 5. Network behaviour with time-varying threshold $\varepsilon(t) = \frac{1}{20(t+1)}$ and $\overline{w} = 0$.

of the system. Moreover, with time-varying threshold and control magnitude, the algorithm can achieve asymptotic consensus in the noiseless case, while preserving state boundedness in the noisy case.

A challenging direction is to extend the result in this paper to the situation that each node adopts a private favourite interval which may be different from those of the others. Other important topics are the study of the convergence rate and the extension to systems with more complex dynamics. However, the presence of measurement noise makes the investigation challenging and at the moment we only have partial answers to these envisioned extensions. Nevertheless, we think that the results in this paper are a valuable starting point for further exploration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Mingming Shi: Conceptualization, Methodology, Software, Investigation, Writing - original draft, Visualization. **Claudio De Persis:** Conceptualization, Methodology, Supervision, Writing - review & editing, Project administration. **Pietro Tesi:** Conceptualization, Methodology, Writing - review & editing, Supervision.

Appendix A. Proof of Theorem 3

Notice that under the algorithm (20)–(22), the two facts in Theorem 1 also hold for $\overline{\gamma}$ defined with $\varepsilon = \varepsilon(0)$, hence the first claim can be derived by the same analysis in Theorem 1.

For the second claim, we need some intermediate supporting results. For any $\eta \in (0, \varepsilon(0)]$, let $\overline{\gamma}(\eta) = q + \eta/d_m$ and $\underline{\gamma}(\eta) = p - \eta/d_m$. We then prove the following two lemmas for the upper bound. Similar results can be established for the lower bound, we omit the proofs due to space limitations.

Lemma 3. *For any node $i \in I$ and any $t' \geq 0$, if the state $x_i(t') \leq \overline{\gamma}(\eta)$, then it will never exceed $\overline{\gamma}(\eta)$ for all time $t \geq t'$.*

Proof. Let $t^i_m = \max\{t^i_k \in \mathbb{R}_{\geq 0}, t^i_k \leq t'\}$. If $u_i(t') \leq 0$, then $u_i(t) \leq 0$ for all $t \in [t^i_m, t^i_{m+1}[$, hence $x_i(t) \leq x_i(t') \leq \overline{\gamma}$ for all $t \in [t', t^i_{m+1}[$. If instead, $u_i(t') = 1$, $z^w_i(t^i_m) \geq \varepsilon(t^i_m)$ according to (21), which implies

$$x_i(t^i_m) \leq \frac{1}{d_i}\sum_{j\in\mathcal{N}_i}(\mathrm{sat}(x^w_{ij}(t^i_m)) - \varepsilon(t^i_m)) \leq q - \varepsilon(t^i_m)/d_i \tag{A.1}$$

where the last inequality comes from the definition of the saturation function (4). Consider the evolution of $x_i(t)$ for $t \in [t^i_m, t^i_{m+1}[$, we have

$$\begin{aligned} x_i(t) &= x_i(t^i_m) + \int_{t^i_m}^{t} \alpha(\tau)u_i(\tau)d\tau \leq x_i(t^i_m) + \alpha(t^i_m)(t^i_{m+1} - t^i_m) \\ &= x_i(t^i_m) + \frac{1}{4d_i}|z^w_i(t^i_m)| = x_i(t^i_m) + \frac{1}{4d_i}\sum_{j\in\mathcal{N}_i}(y_{ij}(t^i_m) - x_i(t^i_m)) \\ &\leq x_i(t^i_m) + (q - x_i(t^i_m))/4 = 3x_i(t^i_m)/4 + q/4 \\ &\leq q - 3\varepsilon(t^i_m)/(4d_i) < \overline{\gamma}(\eta) \end{aligned}$$

where the first inequality follows from that $\alpha(t)$ is a time decreasing function, the second equality from (22) and the third inequality from (A.1).

Considering both cases, we obtain that $x_i(t) \le \overline{\gamma}(\eta)$ for all $t \ge t'$ by induction. ∎

Lemma 4. *For any $\eta \in (0, \varepsilon(0)]$, let $t'(\eta)$ be a time satisfying $\varepsilon(t'(\eta)) \le \eta$. If $x_i(t'(\eta)) > \overline{\gamma}(\eta)$, then there exists a time $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$.*

Proof. First, since $\varepsilon(t)$ is a positive time decreasing function with limit value being zero, $t'(\eta)$ always exists. Let $t^i_m = \max\{t^i_k \in \mathbb{R}_{\ge 0}, t^i_k \le t'(\eta)\}$. From Lemma 3, we have $x_i(t^i_m)$ must be larger than $\overline{\gamma}(\eta)$, otherwise $x_i(t'(\eta)) \le \overline{\gamma}(\eta)$. We consider the saturated average at t^i_m,

$$z^w_i(t^i_m) = \sum_{j\in\mathcal{N}_i}(\mathrm{sat}(x^w_{ij}(t^i_m)) - x_i(t^i_m)) < d_i(q - \overline{\gamma}(\eta)) = -d_i\eta/d_m \le 0$$

This implies that $u_i(t) \in \{0, -1\}$ for all $t \in [t^i_m, t^i_{m+1}[$.

Assume $u_i(t) = -1$ for all $t \in [t^i_m, t^i_{m+1}[$. Suppose there exists no $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$, then for all $t^i_k \ge t^i_{m+1}$, $x_i(t^i_k) > \overline{\gamma}(\eta)$ and the saturated average should satisfy

$$z^w_i(t^i_k) < d_i(q - \overline{\gamma}(\eta)) = -\frac{d_i\eta}{d_m} \le -\eta \le -\varepsilon(t'(\eta)) < -\varepsilon(t^i_k) \quad \text{(A.2)}$$

where the second inequality comes from $d_m \le d_i$ for all $i \in I$, the third from the assumption that $\varepsilon(t'(\eta)) \le \eta$ and the last from the property that $\varepsilon(t)$ is time-decreasing and $t^i_k \ge t^i_{m+1} > t'(\eta)$. This, along with the property that $u_i(t) = -1$ for $t \in [t^i_m, t^i_{m+1}[$, shows that $u_i(t) = -1$ for all $t \ge t^i_m$. However, by $\int_t^{+\infty}\alpha(s)ds = +\infty$, this would imply that $x(t) = x_i(t^i_m) + \int_{t^i_m}^t \alpha(s)u(s)ds = x_i(t^i_m) - \int_{t^i_m}^t \alpha(s)ds$ diverges to $-\infty$ as $t \to +\infty$. This contradicts the assumption that there is no $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$. Hence there must exist a finite time $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$.

Next assume $u_i(t) = 0$ for all $t \in [t^i_m, t^i_{m+1}[$. We have $x_i(t^i_{m+1}) = x_i(t^i_m) > \overline{\gamma}(\eta)$. Again suppose the instant $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$ does not exist, then for all the sampling time $t^i_k \ge t_{m+1}$, $z^w_i(t^i_k)$ should also satisfy (A.2). This implies that $u_i(t) = -1$ for all $t \ge t^i_{m+1}$. By the same argument as in the case $u_i(t) = -1$ for $t \in [t^i_m, t^i_{m+1}[$, we have $\lim_{t\to+\infty} x(t) = -\infty$. This leads to a contradiction and proves the existence of t''. ∎

As a final step for the proof of Theorem 3, we show that $\limsup_{t\to+\infty} x_i(t) = q$ and $\liminf_{t\to+\infty} x_i(t) = p$ for all $i \in I$. For each $\eta \in (0, \varepsilon(0)]$, let $t'(\eta) = \min\{t \ge 0, \varepsilon(t) \le \eta\}$. For each node $i \in I$, if $x_i(t'(\eta)) \le \overline{\gamma}(\eta)$, then $x_i(t) \le \overline{\gamma}(\eta)$ for all $t \ge t'(\eta)$ by Lemma 3. If instead $x_i(t'(\eta)) > \overline{\gamma}(\eta)$, by Lemma 4, there should exist a finite time $t'' > t'(\eta)$ such that $x_i(t'') \le \overline{\gamma}(\eta)$. By Lemma 3, we further have $x_i(t) \le \overline{\gamma}(\eta)$ for all $t \ge t''$. This shows that for any $\eta \in (0, \varepsilon(0)]$, there exists a time $T_1(\eta) \ge t'(\eta)$ such that $x_i(t) \le \overline{\gamma}(\eta)$ for all $i \in I$ and all $t \ge T_1(\eta)$. For the lower bound, by the same analysis, we have that there exists a finite time $T_2(\eta) \ge t'(\eta)$ such that $x_i(t) \ge \underline{\gamma}(\eta)$ for all $i \in I$ and all $t \ge T_2(\eta)$. Let $T(\eta) = \max\{T_1(\eta), T_2(\eta)\}$, then $x_i(t) \in [\underline{\gamma}(\eta), \overline{\gamma}(\eta)]$ for all $i \in I$ and all $t \ge T(\eta)$. As $t \to +\infty$, $\overline{\gamma}(\eta) \to \overline{\gamma}(0)$ and $\underline{\gamma}(\eta) \to \underline{\gamma}(0)$, all the states will be within $[p, q]$.

Appendix B. Proof of Theorem 4

To prove this theorem, we need some intermediate results. For each $\eta \in (0, \varepsilon(0)]$, we introduce two sets for node i as

$$\mathscr{S}_{i1}(\eta) := \left\{t^i_k : |\mathrm{ave}_i(t^i_k)| \ge L_i(\eta)\right\} \quad \text{(B.1)}$$

$$\mathscr{S}_{i2}(\eta) := \left\{t^i_k : |\mathrm{ave}_i(t^i_k)| < L_i(\eta)\right\} \quad \text{(B.2)}$$

where $L_i(\eta) := 5\eta/4 + d_i\overline{w}'(\eta)$, with $\overline{w}'(\eta) = \max\{\eta/d_m, \overline{w}\}$. Clearly, $t^i_k \in \mathscr{S}_{i1}(\eta) \cup \mathscr{S}_{i2}(\eta)$ for every $k \in \mathbb{Z}_{\ge 0}$.

The following result shows that if at certain time node i enters $\mathscr{S}_{i2}(\eta)$, it will indefinitely remain in this set.

Lemma 5 (*Invariant Set*)**.** *For the network of n dynamical systems as in (20), which are interconnected over the graph G, let each local control input be generated in accordance with (21) and (22) with $\varepsilon(t)$ and $\alpha(t)$ satisfying (23). For each $\eta \in (0, \varepsilon(0)]$, define $T(\eta)$ as in the proof of Theorem 3. If $t^i_k \ge T(\eta)$ and belongs to $\mathscr{S}_{i2}(\eta)$, then $t^i_M \notin \mathscr{S}_{i1}(\eta)$ for all integers $M \ge k+1$. Moreover, $|\mathrm{ave}_i(t)| < L_i(\eta)$ for all $t \ge t^i_k$.*

Proof. By the proof of Theorem 3, for each $\eta \in (0, \varepsilon(0)]$, $T(\eta)$ exists and satisfies $\varepsilon(T(\eta)) \le \eta$. Moreover, for all $i \in I$ and all $t \ge T(\eta)$, $p - \eta/d_m = \underline{\gamma}(\eta) \le x_i(t) \le \overline{\gamma}(\eta) = q + \eta/d_m$. Then by the same proof as for inequality (15), for all $t \ge T(\eta)$, we have $|y_{ij}(t) - x_i(t)| \le \overline{w}'(\eta)$. Hence for all $t \ge T(\eta)$, it holds $|\phi_i(t)| \le d_i\overline{w}'(\eta)$, with $\phi_i(t)$ given in (16).

We consider the following two cases,

Case 1. $|z^w_i(t^i_k)| \ge \varepsilon(t^i_k)$. Without loss of generality, we assume $z^w_i(t^i_k) \ge \varepsilon(t^i_k)$, then $u_i(t) = 1$ for all $t \in [t^i_k, t^i_{k+1}[$ and

$$\varepsilon(t^i_k) - \phi_i(t^i_k) \le \mathrm{ave}_i(t^i_k) < L_i(\eta) \quad \text{(B.3)}$$

For $t \in [t^i_k, t^i_{k+1}[$, the average satisfies

$$\mathrm{ave}_i(t) = \mathrm{ave}_i(t^i_k) + \int_{t^i_k}^t \alpha(\tau)\sum_{j\in\mathcal{N}_i}(u_j(\tau) - 1)d\tau \quad \text{(B.4)}$$

Since $|u_j(t)| \le 1$, we have $\mathrm{ave}_i(t) \le \mathrm{ave}_i(t^i_k) < L_i(\eta)$ by (B.3) and

$$\begin{aligned}\mathrm{ave}_i(t) &\ge \mathrm{ave}_i(t^i_k) - 2d_i\int_{t^i_k}^t \alpha(\tau)d\tau \\ &> \mathrm{ave}_i(t^i_k) - 2d_i\alpha(t^i_k)(t^i_{k+1} - t^i_k) \\ &= \mathrm{ave}_i(t^i_k) - \frac{1}{2}|z^w_i(t^i_k)| = \mathrm{ave}_i(t^i_k) - \frac{1}{2}(\mathrm{ave}_i(t^i_k) + \phi_i(t^i_k)) \\ &= \frac{1}{2}(\mathrm{ave}_i(t^i_k) - \phi_i(t^i_k)) \ge \frac{1}{2}(\varepsilon(t^i_k) - 2\phi_i(t^i_k)) \\ &\ge \varepsilon(t^i_k)/2 - d_i\overline{w}'(\eta) > -L_i(\eta)\end{aligned} \quad \text{(B.5)}$$

where the second inequality comes from the assumption that $\alpha(t)$ is a time decreasing function, the first equality from (22), the third inequality from (B.3), the fourth inequality from the bound on $|\phi_i(t)|$ and the last from the definition of $L_i(\eta)$.

Case 2. $|z^w_i(t^i_k)| < \varepsilon(t^i_k)$. We have $\Delta^i_k = \varepsilon(t^i_k)/(4d_i)$ and $u_i(t) = 0$ for all $t \in [t^i_k, t^i_{k+1}[$. Since $\varepsilon(t)$ is time decreasing and $t^i_k \ge T(\eta)$ with $\varepsilon(T(\eta)) \le \eta$, we have $\varepsilon(t^i_k) \le \eta$ and

$$\begin{aligned}|\mathrm{ave}_i(t^i_k)| &= |z^w_i(t^i_k) - \phi_i(t^i_k)| \le |z^w_i(t^i_k)| + |\phi_i(t^i_k)| < \varepsilon(t^i_k) + d_i\overline{w}'(\eta) \\ &= 5\varepsilon(t^i_k)/4 + d_i\overline{w}'(\eta) - \varepsilon(t^i_k)/4 \le L_i(\eta) - \varepsilon(t^i_k)/4\end{aligned} \quad \text{(B.6)}$$

where the first equality follows from (17). Hence for $t \in [t^i_k, t^i_{k+1}[$, the average satisfies

$$\begin{aligned}|\mathrm{ave}_i(t)| &\le |\mathrm{ave}_i(t^i_k)| + \left|\int_{t^i_k}^t \alpha(\tau)\sum_{j\in\mathcal{N}_i}u_j(\tau)d\tau\right| \\ &< |\mathrm{ave}_i(t^i_k)| + d_i\alpha(t^i_k)\Delta^i_k \\ &< L_i(\eta) - \varepsilon(t^i_k)/4 + \varepsilon(t^i_k)/4 = L_i(\eta),\end{aligned}$$

where the last inequality comes from (B.6). As before, by induction for all the integers $M \ge k+1$ and all the corresponding time intervals $[t^i_{M-1}, t^i_M[$, we prove the result. ∎

The next result shows that the average preserves the sign as long as its absolute value remains large enough compared with $L_i(\eta)$.

Lemma 6. *For the network of n dynamical systems as in (20), which are interconnected over the graph G, let each local control input be generated in accordance with (21) and (22) with $\varepsilon(t)$ and $\alpha(t)$ satisfying (23). For each $\eta \in (0, \varepsilon(0)]$, define $T(\eta)$ as in the proof of Theorem 3. For any $i \in I$ and any positive integer M, if $t_k^i \geq T(\eta)$ and $|\text{ave}_i(t_{k+m}^i)| \geq L_i(\eta)$ for $m = 0, 1, \ldots, M$, then $\text{sign}(\text{ave}_i(t_{k+m}^i)) = \text{sign}(\text{ave}_i(t_k^i))$ for $m = 1, 2, \ldots, M+1$.*

Proof. Since $t_k^i > T(\eta)$, $|\phi_i(t)| \leq d_i\overline{w}'(\eta)$ always holds for $t \geq t_k^i$. Suppose w.l.o.g that $\text{ave}_i(t_k^i) \geq L_i(\eta) > 0$, we know

$$z_i^w(t_k^i) = \text{ave}_i(t_k^i) + \phi_i(t_k^i) \geq L_i(\eta) - d_i\overline{w}'(\eta) = 5\eta/4 > \varepsilon(t_k^i)$$

where the last inequality descends from $\varepsilon(t)$ being a time decreasing signal and $t_k^i \geq T(\eta)$, with $\varepsilon(T(\eta)) \leq \eta$. This implies that $u_i(t) = 1$ for all $t \in [t_k^i, t_{k+1}^i[$. Hence

$$\begin{aligned}\text{ave}_i(t_{k+1}^i) &> \text{ave}_i(t_k^i) - 2d_i\alpha(t_k^i)\Delta_k^i = \frac{1}{2}(\text{ave}_i(t_k^i) - \phi_i(t_k^i)) \\ &\geq \frac{1}{2}(L_i(\eta) - d_i\overline{w}'(\eta)) \geq \frac{5}{8}\eta\end{aligned}$$

where the first equality comes from (B.5). This shows that $\text{ave}_i(t_{k+1}^i)$ has the same sign as $\text{ave}_i(t_k^i)$. ■

We then finalize the proof of Theorem 4. First notice that

$$L_i(\eta) = 5\eta/4 + d_i\overline{w}'(\eta) \leq 5\eta/4 + \max\{d_M\eta/d_m, d_M\overline{w}\}$$

and, $L_i(\eta)$ approaches $d_M\overline{w}$ as η approaches zero. Then we show that for any $\eta \in (0, \varepsilon(0)]$, there exists a finite time $t_i'(\eta) \geq T(\eta)$ with $T(\eta)$ given in the Proof of Theorem 3, such that $|\text{ave}_i(t)| < L_i(\eta)$ for all $t \geq t_i'(\eta)$. To see this, we first claim that there exists a finite sampling time $t_s^i(\eta) \geq T(\eta)$, such that $|\text{ave}_i(t_s^i(\eta))| < L_i(\eta)$. Suppose this is not true W.l.o.g. we assume $\text{ave}_i(t_k^i) \geq L_i(\eta)$ for all $t_k^i \geq T(\eta)$. Let $t_r^i = \min\{t_k^i : t_k^i \geq T(\eta)\}$, then by Lemma 6 and the definition of $L_i(\eta)$, $\text{sign}(\text{ave}_i(t_k^i)) = \text{sign}(\text{ave}_i(t_r^i)) > 0$ for all $t_k^i \geq T(\eta)$ and $u_i(t) = 1$ for all $t \geq t_r^i$. However, this implies that $x_i(t) = x_i(t_r^i) + \int_{t_r^i}^{\infty}\alpha(\tau)d\tau$ will increase to infinity, which contradicts Theorem 3 and proves the existence of a finite $t_s^i(\eta) \geq T(\eta)$ with the property $|\text{ave}_i(t_s^i(\eta))| < L_i(\eta)$. Since $|\text{ave}_i(t_s^i(\eta))| < L_i(\eta)$, $t_s^i(\eta) \in \mathscr{L}_{i2}(\eta)$. This along with $t_s^i(\eta) \geq T(\eta)$ shows that $|\text{ave}_i(t)| < L_i(\eta)$ for all $t \geq t_s^i(\eta) \geq T(\eta)$ by Lemma 5. The existence of $t_i'(\eta)$ follows by letting $t_i'(\eta) = t_s^i(\eta)$.

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