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Contraction-Based Nonlinear Controller for a Laser Beam Stabilization System Using a Variable Gain

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Abstract—In this letter, we propose a contraction-based variable gain nonlinear control scheme for the laser-beam stabilizing (LBS) servo-system, which guarantees that the closed-loop system is convergent. With the variable gain acting on the velocity error, the well known waterbed effect of the low-frequency/bandwidth trade-off can be overcome. Moreover, the contraction-based framework allows us to extend the linear control performance metrics for analyzing the closed-loop nonlinear system behavior. The closed-loop system's performance is evaluated in numerical simulations under input disturbances and/or white noise measurements and its efficacy is compared to that using PID and LQG controllers.

Index Terms—Optomechanics, contraction analysis, variable gain, servo-systems, nonlinear systems.

I. INTRODUCTION

LASER beam steering systems have been used in modern engineering technologies, in which high precision and robustness are required. For instance, in laser-based manufacturing processes and printing, surgical robotics, optical communications, advanced scientific instruments in physics and astronomy, optical storage drive, bar code scanning, among others [1], [2], [3]. Control techniques for beam steering are key in the aforementioned opto-mechatronics applications. The LBS problem, roughly speaking, refers to dynamically control

the beam's direction in order to stabilize the beam's image at a target point [4]. The main difficulties for solving the LBS problem arises from the narrow beam divergence angle and vibration of the pointing system. In order to obtain precision pointing of the laser beam and high-bandwidth rejection of jitters produced by the platform vibrations, one uses active mirrors in the beam stabilizer. Then, by sampling a small percentage of the beam, the active mirrors can stabilize the beam's motion by using feedback control from position sensing detectors [1], [5]. The necessity of high accuracy in the pointing of the laser beams poses a real challenge for the successful operation due to low-frequency/bandwidth trade-off. In order to deal with these problems, many control approaches have been designed and evaluated for such systems, e.g., adaptive control [4], [6], frequency weighting method [7], PID-based controllers [5], fractional order PID control [8], H_∞ approaches [2], [3], integral resonant control [9]; to name a few. From linear control literature, the waterbed effect has been known and recognized by control practitioners where design trade-off must be made in increasing the closed-loop bandwidth and the low-frequency disturbance rejection properties at the cost of deteriorating the sensitivity to high-frequency measurement noise. On the other hand, nonlinear control schemes like nonlinear PID and sliding mode controllers [10] can take into account the low-frequency/bandwidth trade-off. Similar to the linear control counterpart, the performance in terms of noise measure attenuation is increased without necessarily deteriorating the time response of the closed-loop system. Nevertheless, the aforementioned trade-off in the nonlinear setting is less intuitive and the design procedure is not straightforward.

A different nonlinear approach, called *convergent systems or convergence* [11], has recently attracted the attention of researchers and engineers, because it naturally extends some linear control methods to the nonlinear case, and it allows us to analyze the performance of convergent nonlinear control systems by characterizing its *unique* steady-state solution; without using any linear approximation. This has successfully been applied to optical storage drives [12], [13]. The convergent systems behavior can be proved by invoking the Demidovich's sufficient condition [14]. This condition is

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generalized by the closely related notion of *contraction analysis* [15], which, under some conditions, can be shown to be equivalent to convergence as pointed out in [16].

In this letter, the design and closed-loop performance analysis of a contraction-based nonlinear controller are presented. To this end, the *differential Lyapunov framework for contraction analysis* in [17], together with a contraction-based adaptation of the standard backstepping technique, is used as a design tool, such that the closed-loop system is contractive (equivalently, convergent). In order to improve the closed-loop performance, a nonlinear variable gain in the velocity error is employed in order to handle the trade-off between the low-frequency tracking properties and the high frequency measurement noise sensitivity. The closed-loop performance is evaluated in numerical simulations in different scenarios: the nominal case, under the influence of input disturbances, with measurement noise, and under the influence of both input disturbances and measurement noise.

This letter is organized as follows. Section II contains the preliminaries of the differential Lyapunov framework, its relation to the convergent systems and the robustness property. In Section III the working principle and the model of the LBS system are described; and the controller design is proposed. Numerical simulation results are presented in Section IV, and the conclusions are given in Section V.

II. PRELIMINARIES

A. Differential Lyapunov Theory and Contraction Analysis

Let¹ Σ_u be a nonlinear control system with N -dimensional stat manifold \mathcal{X} and affine in the input u , described by

$$\Sigma_u : \begin{cases} \dot{x} = f(x, t) + \sum_{i=1}^n g_i(x, t)u_i, \\ y = h(x, t), \end{cases} \quad (1)$$

where $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. The vector fields $f, g_i : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow T\mathcal{X}$ and $h : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathcal{Y}$ are assumed to be smooth. The input space \mathcal{U} and the output space \mathcal{Y} are open subsets of \mathbb{R}^n ; and $T\mathcal{X} = \bigcup T_x\mathcal{X}$ is the tangent bundle, with $T_x\mathcal{X}$ the tangent space of \mathcal{X} at x . System (1) in closed-loop with the control law $u = \gamma(x, t)$ will be denoted by

$$\Sigma : \begin{cases} \dot{x} = F(x, t), \\ y = h(x, t). \end{cases} \quad (2)$$

Definition 1 [19]: The *prolonged system* Σ_u^δ associated to Σ_u in (1), comprises the original system Σ_u , *together* with its variational dynamics, that is the total system

$$\begin{aligned} \dot{x} &= f(x, t) + \sum_{i=1}^n g_i(x, t)u_i, \\ y &= h(x, t), \\ \delta\dot{x} &= \frac{\partial f}{\partial x}(x, t)\delta x + \sum_{i=1}^n u_i \frac{\partial g_i}{\partial x}\delta x + \sum_{i=1}^n g_i\delta u_i, \\ \delta y &= \frac{\partial h}{\partial x}(x, t)\delta x. \end{aligned} \quad (3)$$

¹We refer interested reader to [18] and references therein, for a self-contained treatment of the contraction approach to incremental stability.

Respectively, the prolonged system Σ^δ of Σ in (2) is

$$\begin{cases} \dot{x} = F(x, t), & y = h(x, t), \\ \delta\dot{x} = \frac{\partial F}{\partial x}(x, t)\delta x, & \delta y = \frac{\partial h}{\partial x}(x, t)\delta x. \end{cases} \quad (4)$$

Definition 2 [17]: A function $V : T\mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a candidate *differential Lyapunov function* (dLF) if it satisfies

$$c_1 \|\delta x\|_x^p \leq V(x, \delta x, t) \leq c_2 \|\delta x\|_x^p, \quad (5)$$

where $c_1, c_2 \in \mathbb{R}_{>0}$, p is a positive integer and $\|\cdot\|_x^p$ is a Finsler metric (structure), uniformly in t .

For any subset $\mathcal{C} \subseteq \mathcal{X}$ and any $x_1, x_2 \in \mathcal{C}$, let $\Gamma(x_1, x_2)$ be the collection of piecewise C^1 curves $\gamma : I \rightarrow \mathcal{X}$ connecting $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The *Finsler distance* $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ induced by the dLF V is

$$d(x_1, x_2) := \inf_{\Gamma(x_1, x_2)} \int_{\gamma} V\left(\gamma(s), \frac{\partial \gamma}{\partial s}(s), t\right) ds. \quad (6)$$

Theorem 1 [17]: Consider system (4), a connected and forward² invariant set $\mathcal{C} \subseteq \mathcal{X}$, and a function $\kappa : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. Let V be a candidate dLF satisfying the relation

$$\dot{V}(x, \delta x, t) \leq -\kappa(V(x, \delta x, t)) \quad (7)$$

for all $(x, \delta x) \in T\mathcal{X}$ and all $t > t_0$. Then, system (2) is

- incrementally stable (IS) if $\kappa(s) = 0$ for each $s \geq 0$;
- asymptotically IS if κ is of class³ \mathcal{K} ;
- exponentially IS with rate β if $\kappa(s) = \beta s, \forall s \geq 0$.

Definition 3 (*Contractive System*): We say that Σ is contractive if for V on \mathcal{C} , it holds that (7) is satisfied for α of class \mathcal{K} . The subset \mathcal{C} is the *contraction region*.

B. Contractive Systems and Convergent Dynamics

Definition 4 (*Convergent System* [14]): System Σ in (2) is said to be convergent if

- 1) all solutions $x(t)$ are well-defined for all $t \in [t_0, \infty)$ and all initial conditions $t_0 \in \mathbb{R}, x(t_0) \in \mathcal{X}$.
- 2) there exists a unique solution $\bar{x}(t)$ in \mathcal{X} , called a *steady state solution*, defined and bounded for all $t \in \mathbb{R}$.
- 3) the solution $\bar{x}(t)$ is globally asymptotically stable.

A sufficient condition for convergent behavior is given in the following theorem, the so-called Demidovich condition.

Theorem 2: Consider system Σ in (2). Suppose that there exists matrices $P = P^\top > 0$ and $Q = Q^\top > 0$ satisfying

$$P \frac{\partial F}{\partial x}(x, t) + \frac{\partial F^\top}{\partial x}(x, t)P \leq -Q, \quad \forall x \in \mathcal{C}, \forall t \in \mathbb{R}. \quad (8)$$

Then, system Σ is exponentially convergent in \mathcal{C} .

Following Definition 3, in a contractive system any pair of neighboring solutions x_1 and x_2 convergence towards each other by condition (7), due to the distance $d(x_1, x_2)$ shrinks exponentially, see (6). However, nothing is said about the stability properties of the solutions x_1 and x_2 . On the other hand, by Definition 4, in a convergent system all of its solutions converge to a *unique* globally attractive steady-state solution \bar{x} .

²System (2) is said to be *forward complete* if for every initial condition x_0 , the corresponding solution $x(t)$ is defined for all $t \geq 0$. The set $\mathcal{C} \subseteq \mathcal{X}$ is *forward invariant* for (2) if the system is forward complete and $\forall x_0 \in \mathcal{C}$ the solution $x(t) \in \mathcal{C}, \forall t \geq 0$, [20].

³The class of continuous and strictly increasing functions $\kappa : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ for which $\kappa(0) = 0$ is denoted by \mathcal{K} .

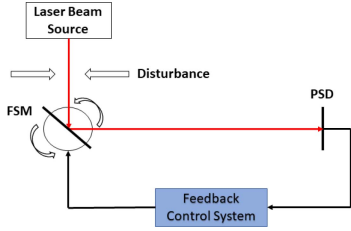


Fig. 1. Working principle of the LBS system.

It can be shown that under compactness assumption on \mathcal{C} in Theorem 1, both notions are equivalent [16]. Moreover, Theorem 2 can be seen as a particular case of Theorem 1 by taking as dLF to

$$V(x, \delta x, t) = \frac{1}{2} \delta x^\top P \delta x. \quad (9)$$

C. Robustness of Contractive Systems

Contractive systems exhibit inherent robustness to bounded perturbations and uncertainties. The robustness properties indicated in [21] are described in terms of dLFs and Finsler distances in the following lemma.

Lemma 1: Consider the perturbed system

$$\dot{x}_p = F(x_p, t) + p(x_p, t), \quad (10)$$

with state $x_p \in \mathcal{X}$, where the perturbation term $p(x_p, t)$ is uniformly bounded in t a constant $p \in \mathbb{R}_{>0}$. Suppose the unperturbed system Σ in (2) is contractive for $\kappa(s) = \beta s$ with respect to Theorem 1. Then, the trajectories of the perturbed system (10) verify the following bounds

$$d(x, x_p) \leq \xi e^{-\beta t} d(x(0), x_p(0)) + \frac{\xi p}{\beta}, \quad (11)$$

where ξ is the condition number of a quadratic dLF.

In the case of the dLF in (9) suppose that P can be rewritten as $P = \Theta^\top \Theta$, with $\Theta^\top = \Theta > 0$. Then, ξ in (11) is the condition number of Θ .

III. CONTRACTION-BASED CONTROL OF THE LBS

A. LBS System Working Principle and Model

The LBS system consists of a low-power stationary laser beam source pointing at a fast steering mirror (FSM) that rotates around a pivot, see Figure 1. The reflected beam is picked up by a high-resolution position sensing detector (PSD). The PSD measures the relative displacement of the beam from the nominal position and the mirror mechanism is actuated using a high-bandwidth voice coil [1].

The transfer function of the actuator is given by [1]:

$$P(s) = \frac{Y(s)}{V_c(s)} = \frac{K}{\tau s^2 + s}, \quad (12)$$

where $K = 2200$ and $\tau = 0.005$ sec are the open-loop gain and time constant, respectively; $Y(s)$ is the FSM position and $V_c(s)$ the voltage control input. By introducing the state variables given by the position $x_1 = y(t)$ and velocity $x_2 = \dot{y}(t)$, a realization for the transfer function (12) is given by:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -ax_2 + bV_c \end{bmatrix}, \quad (13)$$

with $x = (x_1, x_2) \in \mathcal{X} = \mathbb{R}^2$, $a := 1/\tau$ and $b := K/\tau$.

B. Contraction-Based Variable Gain Controllers Design

For designing the control scheme, a contraction-based adaptation of the standard backstepping technique is used. While the goal of standard backstepping is to recursively construct Lyapunov functions that ensure asymptotic stability of the overall closed-loop system [20], in the contraction-based counterpart, the goal is to recursively construct dLF that ensure the *contractive behavior* of the overall closed-loop system; as first presented in [18].

Proposition 1: Consider a reference trajectory $x_d(t) = (x_{1d}(t), x_{2d}(t)) \in \mathcal{X}$ for the LBS system in (13). Let the error coordinates be $\tilde{x} = x_d(t) - x$. Then, system (13) in closed-loop with the feedback control law given by

$$V_c = \frac{a}{b} \dot{x}_{1d} + \frac{1}{b} \ddot{x}_{1d} + \frac{1}{b} \operatorname{sech}^2(\tilde{x}_1) \dot{x}_{1d} + \frac{a}{b} \tanh(\tilde{x}_1) - \frac{1}{b} \operatorname{sech}^2(\tilde{x}_1) x_2 + \frac{k_p}{Tb} \tilde{x}_1 + \frac{k_1 x_2^2}{x_2^2 + k_2} \left(\frac{1}{Tb} \dot{x}_{1d} - \frac{1}{Tb} x_2 + \frac{1}{Tb} \tanh(\tilde{x}_1) \right), \quad (14)$$

is contractive, with $k_p, k_1, k_2 \in \mathbb{R}_{>0}$, with dLF given by

$$V_2(\tilde{x}_1, \delta \tilde{x}_1, \tilde{x}_2, \delta \tilde{x}_2) = \frac{1}{2} k_p \delta \tilde{x}_1^2 + \frac{1}{2} T \delta \tilde{x}_2^2. \quad (15)$$

Proof: As a first design step, consider the position error $\tilde{x}_1 = x_{1d} - x_1$, whose dynamics is given by

$$\dot{\tilde{x}}_1 = \dot{x}_{1d} - x_2. \quad (16)$$

Assume that x_2 is artificial controller for (16) given by

$$x_2 = \tilde{x}_2 - \alpha(\tilde{x}_1, t), \quad (17)$$

where \tilde{x}_2 is a new state and $-\alpha(x_1)$ is an action that makes the dynamics of \tilde{x}_1 error in (16) contractive. Substitution of (17) in (16) results in the ‘‘closed-loop’’ dynamics

$$\dot{\tilde{x}}_1 = \dot{x}_{1d} - \tilde{x}_2 + \alpha(\tilde{x}_1, t). \quad (18)$$

In order to ensure that $\tilde{x}_1 = 0$ is a solution of (18) when $\tilde{x}_2 = 0$, the following expression is chosen for $\alpha(x_1)$:

$$\alpha(\tilde{x}_1, t) := -\dot{x}_{1d} - \phi(\tilde{x}_1), \quad (19)$$

with $\phi(0) = 0$. The resulting closed-loop system reads as

$$\dot{\tilde{x}}_1 = -\tilde{x}_2 - \phi(\tilde{x}_1), \quad (20)$$

whose associated prolonged system is given by

$$\Sigma_1^\delta : \begin{cases} \dot{\tilde{x}}_1 = -\tilde{x}_2 - \phi(\tilde{x}_1) \\ \delta \dot{\tilde{x}}_1 = -\delta \tilde{x}_2 - \frac{\partial \phi(\tilde{x}_1)}{\partial \tilde{x}_1} \delta \tilde{x}_1 \end{cases}. \quad (21)$$

Now, consider the following function as a candidate dLF for (21)

$$V_1(\tilde{x}_1, \delta \tilde{x}_1) = \frac{1}{2} k_p \delta \tilde{x}_1^2, \quad (22)$$

where $k_p \in \mathbb{R}_{>0}$. Then, the time derivative along the prolonged system (21) satisfies

$$\dot{V}_1(\tilde{x}_1, \delta \tilde{x}_1) = -k_p \delta \tilde{x}_1 \delta \tilde{x}_2 - k_p \frac{\partial \phi(\tilde{x}_1)}{\partial \tilde{x}_1} \delta \tilde{x}_1^2. \quad (23)$$

In order to ensure that (23) satisfies the contraction inequality in (7) when $\delta\tilde{x}_2 = 0$, the function $\phi(\tilde{x}_1)$ should satisfy

$$-k_p \frac{\partial \phi(\tilde{x}_1)}{\partial \tilde{x}_1} \leq -\beta_1, \quad (24)$$

for a $\beta_1 > 0$. A solution⁴ to this inequality is given by $\phi(\tilde{x}_1) := \tanh(\tilde{x}_1)$. Indeed, since $0 < \text{sech}(\tilde{x}_1) \leq 1$, then

$$-k_p \frac{\partial \phi}{\partial \tilde{x}_1} = -k_p \text{sech}^2(\tilde{x}_1) \Rightarrow \beta_1 = k_p \inf(\text{sech}^2(\tilde{x}_1)) > 0. \quad (25)$$

It follows that $\dot{V}_1(\tilde{x}_1, \delta\tilde{x}_1) < -\beta_1 \delta\tilde{x}_1^2$, and therefore \tilde{x}_1 converges to 0 exponentially, whenever $\tilde{x}_2 = 0$.

In the second step of the design, the goal is to ensure that $(\tilde{x}_2, \delta\tilde{x}_2) = (0, 0)$ holds. To this end, consider the dynamics of \tilde{x}_2 from (17) and (13) in error coordinate, that is

$$\dot{\tilde{x}}_2 = -a(\tilde{x}_2 - \alpha(\tilde{x}_1, t)) + bV_c + \dot{\alpha}(x_{1d} - \tilde{x}_1). \quad (26)$$

Thus, the complete closed-loop dynamics, taking into account (20) and (26), can be written as:

$$\Sigma_{V_c} : \begin{cases} \dot{\tilde{x}}_1 = -\phi(\tilde{x}_1) - \tilde{x}_2 \\ \dot{\tilde{x}}_2 = -a(\tilde{x}_2 - \alpha(\tilde{x}_1, t)) + bV_c + \dot{\alpha}(\tilde{x}_1, t), \end{cases} \quad (27)$$

whose variational system is expressed as:

$$\begin{cases} \delta\dot{\tilde{x}}_1 = -\frac{\partial \phi(\tilde{x}_1)}{\partial \tilde{x}_1} \delta\tilde{x}_1 - \delta\tilde{x}_2 \\ \delta\dot{\tilde{x}}_2 = -a\delta\tilde{x}_2 + a \frac{\partial \alpha}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1 + b\delta V_c \\ \quad + \frac{\partial \dot{\alpha}}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1. \end{cases} \quad (28)$$

The corresponding prolonged system associated to Σ_{V_c} in (27), is the system composed of (27) and (28). In order to prove the contractivity of the overall system, we consider the following candidate dLF

$$V_2(\tilde{x}_1, \delta\tilde{x}_1, \tilde{x}_2, \delta\tilde{x}_2) = V_1(\tilde{x}_1, \delta\tilde{x}_1) + \frac{1}{2} T \delta\tilde{x}_2^2 > 0. \quad (29)$$

Direct computation shows that

$$\dot{V}_2(\tilde{x}, \delta\tilde{x}) = \dot{V}_1(\tilde{x}_1, \delta\tilde{x}_1) + T \delta\tilde{x}_2 \delta\dot{\tilde{x}}_2. \quad (30)$$

By substituting (23) to (30) we obtain

$$\begin{aligned} \dot{V}_2(\tilde{x}, \delta\tilde{x}) &= -k_p \delta\tilde{x}_1 \delta\tilde{x}_2 - k_p \frac{\partial \phi(\tilde{x}_1)}{\partial \tilde{x}_1} \delta\tilde{x}_1^2 - aT \delta\tilde{x}_2^2 \\ &\quad + Ta \frac{\partial \alpha}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1 \delta\tilde{x}_2 + Tb \delta V_c \delta\tilde{x}_2 \\ &\quad + T \frac{\partial \dot{\alpha}}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1 \delta\tilde{x}_2. \end{aligned} \quad (31)$$

Pick the variational control action δV_c in (31) as follows

$$\delta V_c = -\frac{a}{b} \frac{\partial \alpha}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1 - \frac{1}{b} \frac{\partial \dot{\alpha}}{\partial \tilde{x}_1}(\tilde{x}_1, t) \delta\tilde{x}_1 + \frac{1}{Tb} \delta u, \quad (32)$$

where δu is the variational of an additional control input. Substituting (32) to (31) yields

$$\dot{V}_2(\tilde{x}, \delta\tilde{x}) = -k_p \delta\tilde{x}_1 \delta\tilde{x}_2 - k_p \frac{\partial \phi}{\partial \tilde{x}_1} \delta\tilde{x}_1^2 - aT \delta\tilde{x}_2^2 + \delta u \delta\tilde{x}_2. \quad (33)$$

Since one aims to design a control strategy allowing to neutralize the effect of external disturbances while maximizes the

⁴Notice that the solution to (24) is not unique. For instance $\phi(\tilde{x}_1) = \arctan(\tilde{x}_1)$ is another solution. The selection of $\phi(\tilde{x}_1) = \tanh(\tilde{x}_1)$ here is mainly motivated by its monotonicity and boundedness properties.

error decay rate, we can propose the following controller's term which guarantees that $\dot{V}_2(\tilde{x}_1, \delta\tilde{x}_1, \tilde{x}_2, \delta\tilde{x}_2) < 0$,

$$\delta u = k_p \delta\tilde{x}_1 - K_v^{var}(x_2) \delta\tilde{x}_2, \quad (34)$$

with the variable derivative gain function be chosen as

$$K_v^{var}(x_2) := \frac{k_1 x_2^2}{x_2^2 + k_2}. \quad (35)$$

Straightforward substitution yields

$$\dot{V}_2 = -k_p \text{sech}^2(\tilde{x}_1) \delta\tilde{x}_1^2 - aT \delta\tilde{x}_2^2 - \frac{k_1 x_2^2}{x_2^2 + k_2} \delta\tilde{x}_2^2 < 0, \quad (36)$$

which fulfills Theorem 1 for contraction. Finally, the controller (14) is obtained by path integration on the variational states from (32) and (34). This completes the proof. ■

Remark 1: The idea behind the construction of the control law (14) is similar to this one of the 2-DOF linear controllers. Note that the control (14) can be decomposed in a feedforward, a feedback and a combination of feedback and feedforward components pondered with a variable gain which depends on the velocity. The feedback part guarantees that the closed-loop is uniformly convergent. The feedforward part shapes the steady-state response to be $x_d(t)$. Variable gain control is useful to overcome the trade-off between low-frequency tracking properties and high-frequency measurement noise.

IV. NUMERICAL SIMULATIONS

The LBS problem is a constant set-point regulation problem, which can be solved by the developed controller (14), by considering as reference to $x_d = (\bar{x}_1, 0)$, with \bar{x}_1 constant. Notice that the variable derivative gain term $K_v^{var}(x_2)$ in (35) can be seen as a filter-like depending on the square of the velocity signal, and acting on the velocity error. This helps for reaching a smart trade-off between low-frequency disturbance rejection (mechanical vibrations for the LBS) and high-frequency measurement noise in the PSD.

For the case of study, the desired position is $x_{1d} = 1\text{mm}$ and the following gains values are taken for the simulation $k_p = 900$, $k_1 = 1$, $k_2 = 0.1$. Five different scenarios are considered for evaluating the performance of the closed-loop LBS servo system. The first scenario is considering nominal values of the system parameters. For the second scenario, uncertainty in the system parameters is present as follows $K_i = K + \Delta K$ as uncertainty for the nominal open-loop gain K and $\tau_i = \tau + \Delta\tau$ for the nominal open loop time constant τ , with $\Delta K = 0.8K$ and $\Delta\tau = 0.6\tau$, respectively. In the third scenario, an input disturbance given by a sine function of amplitude 1 and 12 rad/s of frequency is introduced. A position measurement white noise signal with 0.0001 noise power is added in the fourth scenario. Finally, the fifth scenario consist of adding the external disturbance in the input and white noise in the position measurement simultaneously. The position error and control signal time response are presented in Figure 2 and 3, respectively.

From Figure 2, it is clear that in the nominal case the position error converges to zero in an exponential manner. Similarly, in the case under parameters uncertainty, the position error goes to zero, but with a different convergence rate according to the bounds in (11). Similarly, in the case of the

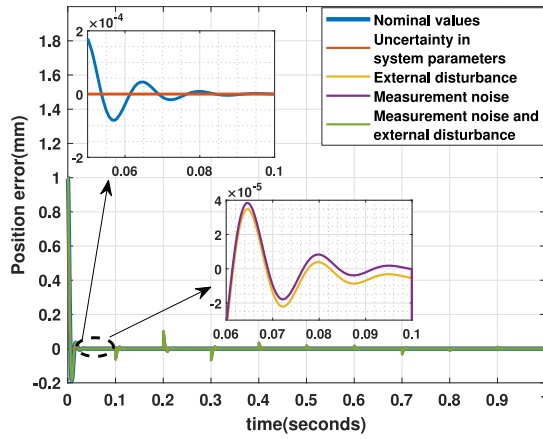


Fig. 2. Closed-loop position error time response with the controller (14).

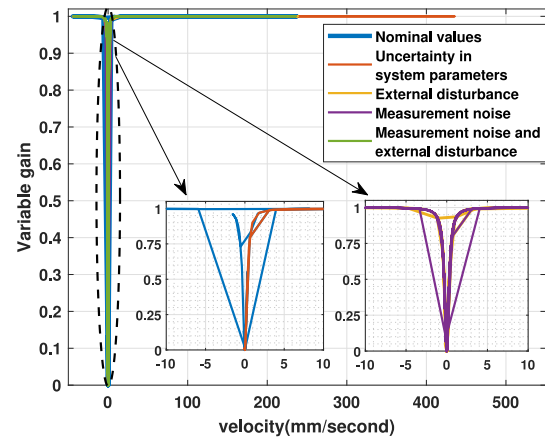


Fig. 4. Variable gain vs velocity.

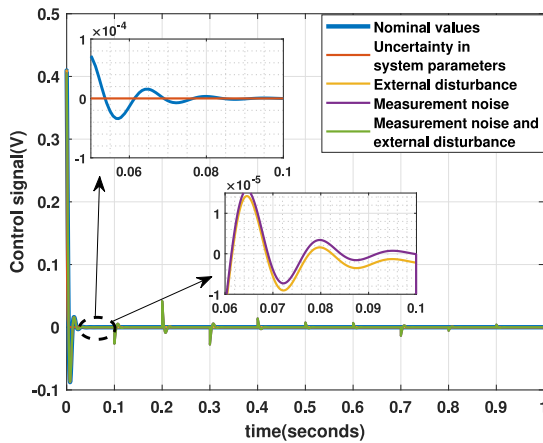


Fig. 3. Time response of the contraction-based controller (14).

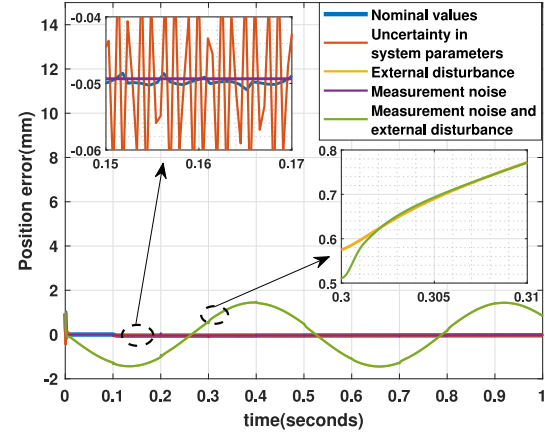


Fig. 5. Position error time response under the PID controller.

time-varying input external disturbance, the convergence is guaranteed by (11), and the steady state position error remains in a neighborhood of zero, modulated by the amplitude of the disturbance. Nevertheless, when white noise is considered, despite of the closed-loop system trajectories remain bounded, the steady-state error presents small peaks which decrease as the simulation time elapses. Notice that in the fifth case where both input disturbance and noise measurement are considered, the disturbance is rejected as in the third scenario, but the effect of the noise is dominating the steady-state time response of the position error.

The control signal under the five scenarios exhibits a similar performance for the steady state error. As expected, the control effort in the cases under noise measurements is not smooth as in the other ones. The behavior of the variable gain (35) is shown in Figure 4, for the simulation values of k_1 and k_2 .

As mentioned in the introduction, a number of strategies have been used to control the LBS system. Among those, the PID scheme presented in [22] is used for comparison purposes, in which the derivative term is implemented with a low-pass filter. The time response of the position error in closed-loop with the PID controller is shown in Figure 5.

In the nominal case, the system maintains a small state error, which is accentuated and oscillates randomly in the case of uncertainty in the system parameters. Then, when the external

sine disturbance is added to the input, the error enters in a stable oscillatory regime, but never converges to zero. When white noise is added to the state measurements, the position error time response converges to a value close to zero almost like in the nominal case; this can be attributed to the low-pass filter added in the derivative part. Finally, both input disturbance and white noise are added to the simulation; it is clear from Figure 4 that despite the noise is somehow filtered, the oscillating disturbance drives the closed-loop steady state response, and not the reference.

A. Qualitative and Quantitative Performance Comparison

In order to qualitatively (steady-state performance) and quantitatively (transient performance) compare the closed-loop LSB servo system performance under the proposed contraction-based scheme, the PID, Linear Quadratic Gaussian (LQG) [23], [24], Contraction-Based With Variable Gain (CBWVG) and Contraction-Based Without Variable Gain (CBWoVG) controllers, the root mean square (RMS) index and the \mathcal{L}_2 norm performance of the position error are considered, respectively. That is, the quantities

$$\text{RMS} = \sqrt{\frac{1}{T} \int_0^T \|\tilde{x}_1(t)\|^2 dt}, \quad \mathcal{L}_2 = \sqrt{\int_0^T \|\tilde{x}_1(t)\|^2 dt} \quad (37)$$

TABLE I
PERFORMANCE ANALYSIS WITH RMS INDEX

	PID	LQG	CBWVG	CBWoBG
Nominal system values	0.01759	0.0178	0.01807	0.01854
Uncertainty in parameters	0.03013	0.02295	0.01659	0.01704
External disturbance	1.08506	1.30359	0.02611	0.02896
Measurement noise	0.03133	0.02672	0.02672	0.02818
Measurement noise and external disturbance	1.08689	1.30513	0.02818	0.03098

TABLE II
PERFORMANCE ANALYSIS WITH \mathcal{L}_2 NORM

	PID	LQG	CBWVG	CBWoBG
Nominal system values	0.03932	0.03979	0.04042	0.04145
Uncertainty in parameters	0.0676	0.05156	0.03708	0.03811
External disturbance	2.4262	2.91485	0.05838	0.06476
Measurement noise	0.07006	0.05975	0.06301	0.06929
Measurement noise and external disturbance	2.43052	2.91856	0.06301	0.06928

where $T = 5\text{sec}$ represents the simulation or experimentation time. The interpretation of the \mathcal{L}_2 norm is the following: the highest \mathcal{L}_2 value means the poorest performance (transient performance) [25]. The same principle applies to the RMS index (steady-state performance) [24]. The simulation results for the different scenarios are summarized in Tables I and II.

In general, it can be said that the overall performance of the proposed controller exhibits superior robust performance than the PID, LQG and CBWoBG schemes, since it can deal very well with external disturbances and keeps the steady state error in a neighborhood of zero of small ratio when noise measurements are considered. However, the PID and LQG schemes exhibit a little better performance in the nominal and only measurement noise scenarios.

V. CONCLUSION

In this letter, a contraction-based nonlinear scheme was proposed to control the LBS servo system. By means of contractivity/convergence, the closed-loop system was shown to have robust properties against input disturbances and parameter uncertainty; and the existence of a unique steady state solution given by the reference trajectory was ensured. Moreover, the use of a variable gain acting on the velocity error was shown to be useful to overcoming the waterbed effect with the trade-off between low-frequency tracking properties and high-frequency measurement noise which affects the design of PID and LQG controllers. We have shown that by utilizing a contraction-based nonlinear controller for stabilizing a linear plant we can attain simultaneous performance benefits that cannot be attained by any linear control methods. The simulation results confirm the theoretical developments.

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