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## On the Computation of Equilibrium in Discontinuous Economic Games

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 Discontinuous Economic Games**

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**Pim Heijnen**



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# On the Computation of Equilibrium in Discontinuous Economic Games

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# On the Computation of Equilibrium in Discontinuous Economic Games

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April 21, 2020

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## Abstract

In many (game-theoretic) models of price competition, mixed-strategy Nash-equilibria naturally occur. For firms, it is an equilibrium to randomly draw a price from a non-degenerate distribution whose support is an interval on the real line. Computing this distribution is a nontrivial task except in special cases. This paper proposes a procedure to numerically calculate such an equilibrium. Examples illustrate that the procedure is fast and accurate.

**JEL-codes:** C63, C71, L10

**Keywords:** mixed-strategy Nash equilibrium, numerical computation, Bertrand-Edgeworth games

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## 1 Introduction

In many price competition models, mixed-strategy Nash-equilibria naturally occur.<sup>1</sup> For firms, it is an equilibrium to randomly draw a price from a non-degenerate distribution whose support is an interval on the real line. An analytic derivation of this distribution is a nontrivial task, except in special cases. This paper proposes a procedure that allows the researcher to numerically calculate such an equilibrium in a quick and stable manner.

Before I discuss the merits of being able to numerically calculate equilibria, it is insightful to provide some detail about the kind of games that I have in mind. Consider a symmetric game in which two firms compete in prices. The aim is to find a symmetric equilibrium. Typically, the profit function of each firm is discontinuous along the line where both firms

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<sup>1</sup>While some unease arises from the concept of a mixed-strategy equilibrium, in the sense that it is not clear what it means to play a mixed strategy, for models of price competition that want to explain price dispersion for apparently homogeneous goods, mixed strategies are a godsend.

charge the same price since there exists a group of consumers who always buy from the cheapest firm but split their demand in case of equal prices. This discontinuity will destabilize any pure-strategy equilibrium since firms can substantially increase profit by setting an infinitesimally lower price. Therefore the equilibrium will be a mixed-strategy equilibrium.

Computing the mixed-strategy equilibrium is straightforward when profit only depends on the firm’s own price and whether a firm sets the lowest price or not. However, in most cases, deriving the equilibrium distribution is a cumbersome task at best and, to explore properties of the model, a simple method that does not rely on the cumbersome derivation of equilibrium conditions is valuable. Moreover, the numerical approach allows us to study more complicated games where the analytical approach is of limited use.<sup>2</sup>

While there is a substantial literature on the computation of equilibrium in games with a finite number of actions (see von Stengel (2007) for an overview) and an excellent software package to perform the actual computations in the form of Gambit (McKelvey, McLennan, & Turocy, 2014), to the best of my knowledge similar procedures for the computation of equilibrium in games with a continuum of actions are unavailable. The current paper is a first step in this direction.

The method to compute the equilibrium takes its cue from the proof of Theorem 5 in Dasgupta and Maskin (1986), which establishes the existence of an equilibrium in the type of game discussed in this paper.<sup>3</sup> The proof shows that the equilibrium of a discretized version of the game (where equilibrium existence obviously follows from Nash, 1950) converges to the equilibrium of the original game. The computational procedure presented here also approximates the equilibrium by discretizing the game.

It turns out that the choice of discretization is crucial to the performance of the algorithm. The most straightforward discretization is to restrict the action space to a finite subset of the action space, but then “ties” (where both players choose the same action) occur with positive probability. The discontinuity of the payoff function at this point leads to a poor approximation of the equilibrium. I circumvent this issue by dividing the action space into intervals. An action in the discretized game is to choose an interval and the action a player ultimately plays is a random draw from this interval, thereby smoothing out the discontinuity. Note that any strategy of the original game can be approximated with arbitrary precision by increasing the number of intervals.

The method is illustrated by three examples from the industrial organization literature. The first example is the classic “model of sales” (Varian, 1980), where there are two groups of consumers: uninformed and informed. The uninformed buy one unit from a random firm as long as the price is below their willingness to pay, whereas the informed know which

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<sup>2</sup>Since there is a tendency in economics to focus on games that are analytically solvable, examples where the analytic approach fails, are hard to find in academic journals. The only example I know of is one of my own working papers: Haan, Heijnen, and Obradovits (2019), available upon request.

<sup>3</sup>In fact, Theorem 6 shows that, in a symmetric game (which is the focus of the present paper), there exists a symmetric mixed-strategy equilibrium.

firm charges the lowest price and buy from this firm. This is the simplest way to generate a discontinuous profit function. Since we know the mixed-strategy equilibrium for this game, it shows how well our numerical approximation performs.<sup>4</sup>

The second example is “competition on the Hotelling line”, where the distribution of consumers along the line has an atom at the point where consumers are equidistant from both firms.<sup>5</sup> The third example falls into the category of Bertrand-Edgeworth games (see Vives (2001, Chapter 5) for an overview of the literature) and is based on a symmetric version of the game in Davidson and Deneckere (1986).<sup>6</sup> In Bertrand-Edgeworth games, consumers buy from the cheapest firm (Bertrand competition), but the firms have limited capacity, which makes it attractive to set a high price and serve only those consumers who get turned down at the cheapest firm. The main purpose of these examples is to show that the method also works in more elaborate games.

The rest of the paper is structured as follows. Section 2 introduces the structure of the game. In Section 3, the details of the computational procedure are discussed. Finally, in Section 4, the numerical method is illustrated by three examples.

## 2 The game

Consider a symmetric game with two players indexed by  $i = 1, 2$ . Players choose an action  $x_i \in [\underline{x}, \bar{x}] \subset \mathbb{R}$  which results in a payoff  $\pi_i = \pi(x_i, x_j)$ , where  $j \neq i$ . Note that the payoff function is bounded and it is continuous everywhere except along the line  $x_i = x_j$ ; to be precise for all  $x \in (\underline{x}, \bar{x})$ , we have  $\lim_{y \uparrow x} \pi(y, x) > \pi(x, x) \geq \lim_{y \downarrow x} \pi(y, x)$ . Moreover there exists  $x \in (\underline{x}, \bar{x}]$  such that  $\pi(x, \underline{x}) > \pi(\underline{x}, \underline{x})$ . Note that in many economic models the payoff function is of this form, as remarked in the previous section. While this game has no symmetric equilibrium in pure strategies, there exists a symmetric equilibrium in mixed strategies (Dasgupta & Maskin, 1986, Theorem 6).

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<sup>4</sup>In this example, calculating the equilibrium is very easy, but even in minor variations of this game the calculation become very involved very quickly, cf. Obradovits (2014) who uses a couple of neat tricks to figure out the equilibrium distribution of prices. Alas, it is not easy to see how these tricks may lead to a more general procedure for solving these kind of games. The same applies to the attempts of Osborne and Pitchik (1986a, 1986b, 1987) to compute the equilibrium in Hotelling-type location games and Bertrand-Edgeworth games.

<sup>5</sup>This game is based on the price game in Anderson, Goeree, and Ramer (1997), but their focus is on specifications where there is a unique pure-strategy equilibrium. Another way to generate this kind of demand structure in Hotelling-type games is to have consumers who are located on graphs (Heijnen & Soetevent, 2018).

<sup>6</sup>Note that Davidson and Deneckere (1986) do not attempt to actually calculate equilibrium strategies.

### 3 Computational procedure

First, partition the action space into  $n$  bins:

$$\{B_1, \dots, B_n\} = \{[x_0, x_1], \dots, [x_{n-1}, x_n]\},$$

where

$$x_k = \underline{x} + \frac{\bar{x} - \underline{x}}{n} \cdot k \quad \text{for } k = 0, \dots, n.$$

Then we construct a game where action  $k = 1, \dots, n$  is to randomly pick an action from bin  $k$ . This game has a payoff matrix  $A = \{a_{k\ell}\}$ , where

$$a_{k\ell} = \left(\frac{n}{\bar{x} - \underline{x}}\right)^2 \int_{B_k} \int_{B_\ell} \pi(x, y) dy dx \quad \text{for } k, \ell = 1, \dots, n. \quad (1)$$

Note that the constant in front of the integral follows from the fact that the density function of the uniform distribution on the rectangle  $B_k \times B_\ell$  is constant and equal to the inverse of the area of  $B_k \times B_\ell$ . This game is referred to as the *discretized game*.

Observe that as  $n \rightarrow \infty$ , any strategy of the original game can be approximated by our discretization. Moreover this discretization is very close to the discretization that Dasgupta and Maskin (1986) use to proof existence of a symmetric equilibrium in mixed strategies. In particular, Dasgupta and Maskin choose  $n$  points from that action space and make sure that the distance between any point and its closest neighbor is sufficiently small. For any rectangle  $B_k \times B_\ell$  where the payoff function is continuous, the integral in (1) can of course be replaced by a single action in that rectangle where the payoff is equal to  $a_{k\ell}$ . The only difference between the approach taken here and in Dasgupta and Maskin is for rectangles where the payoff function is not continuous. The numerical illustration in Section 4.2 will show that the choice to smooth out the payoff function in rectangles where discontinuities appear increases the accuracy of the calculations.

To calculate the Nash-equilibrium of the discretized game, let  $f_i$  be a vector on the  $n$ -dimensional simplex which represents the strategy of player  $i$ , i.e. the  $k$ -th entry of the vector is the probability that player  $i$  selects bin  $k$ . The expected payoff of player  $i$  is  $f_i^\top A f_j$ , where  $j \neq i$ . It follows from Mangasarian and Stone (1964) that a symmetric Nash equilibrium is the solution of a quadratic program, i.e. let

$$(f^*, \gamma^*) = \arg \min_{f, \gamma} \gamma - f^\top A f \quad (2)$$

such that

$$A f \leq \gamma \iota \quad (3)$$

$$\iota^\top f = 1 \quad (4)$$

$$f \geq 0 \quad (5)$$



where  $\iota = (1, \dots, 1)$ . Then  $f^*$  is a symmetric Nash equilibrium of the discretized game and  $\gamma^* = f^{*\top} A f^*$  is the equilibrium payoff. In the computations below, I find Nash-equilibria by solving the quadratic program above.<sup>7</sup>

The Nash equilibrium of the discretized game is an  $\varepsilon$ -equilibrium of the original game, i.e. by deviating from the equilibrium the increase in the payoff is at most  $\varepsilon$ . Since the payoff function is bounded, there exists an  $\varepsilon$  for which this is true, however, to assess the accuracy of the approximation, it can be useful to calculate the minimal value of  $\varepsilon$ . Define

$$\pi^*(x) = \frac{n}{\bar{x} - \underline{x}} \sum_{\ell=1}^n f_{\ell}^* \int_{B_{\ell}} \pi(x, y) dy$$

as the payoff of deviating to action  $x$  if the other player sticks to the Nash-equilibrium. Observe that if  $f_k^* > 0$ , then a player will receive an expected payoff of  $\gamma^*$  if he randomly picks an action in bin  $k$ . Therefore there exists an action in bin  $k$  whose payoff is at least  $\gamma^*$ . Hence  $\max_x \pi^*(x) \geq \gamma^*$ . Let  $\varepsilon^* \equiv \max_x \pi^*(x) - \gamma^* \geq 0$  denote the maximum payoff of deviating. The symmetric Nash equilibrium of the discretized game is an  $\varepsilon$ -equilibrium of the original game for all  $\varepsilon \geq \varepsilon^*$ . We refer to  $\varepsilon^*$  as the *accuracy* of the equilibrium. We conjecture that  $\varepsilon^* \rightarrow 0$  as  $n \rightarrow \infty$ . Remark that:

- I focus on a symmetric equilibrium of a symmetric two-player game. Three questions naturally arise: (1) Is there a way to calculate all symmetric equilibria? (2) What if the game is not symmetric? and (3) What if the game has more than two players? Ad (1): In most symmetric economic games, the symmetric equilibrium appears to be unique. Therefore, in practice, this is not of great importance. Ad (2) and (3): Asymmetric two-player games and symmetric  $n$ -player games can easily be handled by modifying the program in (2–5) in an appropriate fashion. Since Mangasarian and Stone (1964) actually consider the general asymmetric two-player case, this only requires a bit of tweaking for the symmetric  $n$ -player case.
- The computations that I report below were executed with Matlab on a run-of-the-mill desktop. Hence, the time to calculate the equilibrium can be drastically improved by people with better programming skills and access to better hardware than me. Moreover,  $n = 1024$  was the largest-scale problem that ran decently on my setup: again there is scope for improvement in that direction.
- In solving the quadratic program, I used the Matlab-routine `fmincon` from the `optimization`-package instead of the `quadprog`-routine, i.e. I used a general minimization routine (using the sequential quadratic programming algorithm) instead of an algorithm specifically designed for quadratic programming. The problem is

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<sup>7</sup>It is more common to use the Lemke-Howson algorithm (Lemke & Howson, 1964), but for large  $n$  this algorithm is very slow compared solving a quadratic program. For instance, for  $n = 1024$ , the game in Section 4.1 takes 43 seconds to solve when using the quadratic program approach, but the Lemke-Howson algorithm was running for 11 hours (and 500,000 pivots) and had not found a solution yet. It appears that Lemke-Howson does not scale up well.

that the objective function is not necessarily convex (which the `quadprog`-routine requires), nonetheless in all cases I achieved convergence to a global minimum (which is zero by construction).

- Constructing the payoff matrix is the most computationally-intensive part of the procedure since it requires the evaluation of  $n^2$  double integrals. I used simple Monte Carlo integration, which seemed to perform well. Note that one can choose to only smooth the payoff function for bins where a discontinuity is present in order to increase the speed of computation.

## 4 Illustrations

To conclude the paper, the computational method is illustrated with three examples. The first example is based on Varian (1980), where the equilibrium can be derived analytically. This allows us to see how accurate the method is. The second example is a model of horizontal product differentiation akin to the price game in Anderson et al. (1997), which shows that the equilibrium distribution may have unexpected features. The third example is based on Davidson and Deneckere (1986) and shows that the discretization process will lead to a decent approximation of the cumulative distribution function, but the accuracy of approximation of the probability distribution function may be lower.

### 4.1 A model of sales

There is a unit mass of consumers who demand one unit of a good as long as the price does not exceed 1. A fraction  $\lambda \in (0, 1)$  of the consumers are informed, the rest are uninformed. Two firms, who face no cost production, simultaneously set prices. The informed consumers buy from the cheapest firm, the uninformed split equally between the two firms. When both firms set equal prices, the informed also split equally between the two firms.

In terms of the notation of the previous section,  $\underline{x} = 0$ ,  $\bar{x} = 1$  and

$$\pi(x, y) = \begin{cases} \frac{1}{2}(1 - \lambda)x & \text{if } x > y \\ \frac{1}{2}x & \text{if } x = y \\ \frac{1}{2}(1 + \lambda)x & \text{if } x < y \end{cases}$$

where  $x$  and  $y$  denote respectively own price and price of the competitor.

This game has a mixed-strategy equilibrium with a distribution

$$F(x) = 1 - \frac{(1 - \lambda)(1 - x)}{2\lambda x}$$

$n$	Accuracy	Payoff	Time
16	2	0.4075	1
64	2	0.3982	1
256	3	0.3999	2
1024	6	0.4000	43

Table 1: Accuracy indicates the largest integer  $m$  such that  $\varepsilon^* \leq 10^{-m}$ , payoff is the equilibrium payoff of the discretized game (which is 0.4 in the actual equilibrium of the game), time is time needed to calculate the equilibrium, measured in seconds.

and support  $[(1 - \lambda)/(1 + \lambda), 1]$ . Equilibrium profit is  $(1 - \lambda)/2$  (cf. Varian, 1980). We take  $\lambda = 1/5$ . Table 1 shows the performance of our approach for several values of  $n$ , the number of bins. Note that for relatively small values of  $n$ , we get a decent approximation of the equilibrium payoff, starting from  $n = 64$  the payoff is correct up to two decimals. However, to get a decent approximation of the equilibrium price distribution,  $n$  should be quite large: at  $n = 1024$ , the maximum distance between  $F$  and the equilibrium of the discretized game drops to approx.  $3 \times 10^{-4}$ .

## 4.2 Horizontal product differentiation

Consider a market where two firms, indexed by  $i = 1, 2$  and who face no cost of production, are active. Firm  $i$  sets a price  $x_i$ . There is a unit mass of consumers. A fraction  $\lambda$  consumers consider the firms' products to be perfect substitutes. The remaining consumers attach a value  $v_i$  to the consumption of firm  $i$ 's product. Suppose that  $\delta \equiv v_2 - v_1$  is uniformly distributed on  $[-1/2, 1/2]$ , i.e. there is horizontal product differentiation. The consumer buys from firm 1 if and only if  $v_1 - x_1 \geq v_2 - x_2 \implies \delta \leq x_2 - x_1$ . Demand for firm 1 from this group of consumers is  $(1 - \lambda)$  times the probability that  $\delta \leq x_2 - x_1$ . The payoff function for firm  $i$  is then given by

$$\pi_i = \begin{cases} x_i & \text{if } x_i - x_j \leq -\frac{1}{2} \\ [\lambda + (1 - \lambda)(\frac{1}{2} - (x_i - x_j))]x_i & \text{if } x_i - x_j \in (-\frac{1}{2}, 0) \\ \frac{1}{2}x_i & \text{if } x_i - x_j = 0 \\ (1 - \lambda)[\frac{1}{2} - (x_i - x_j)]x_i & \text{if } x_i - x_j \in (0, \frac{1}{2}) \\ 0 & \text{if } x_i - x_j \geq \frac{1}{2} \end{cases}$$

where  $j \neq i$ . Observe that consumers always buy from either firm 1 or firm 2, hence there is no natural upper bound on prices. This means that we need to establish some upper bound on price, which requires either experimentation or a bit of preliminary analysis. For this particular game, there is a pure-strategy equilibrium when  $\lambda = 0$  at  $x_1 = x_2 = 1/2$ . Given that competition intensifies if more consumers think that the goods are perfect sub-

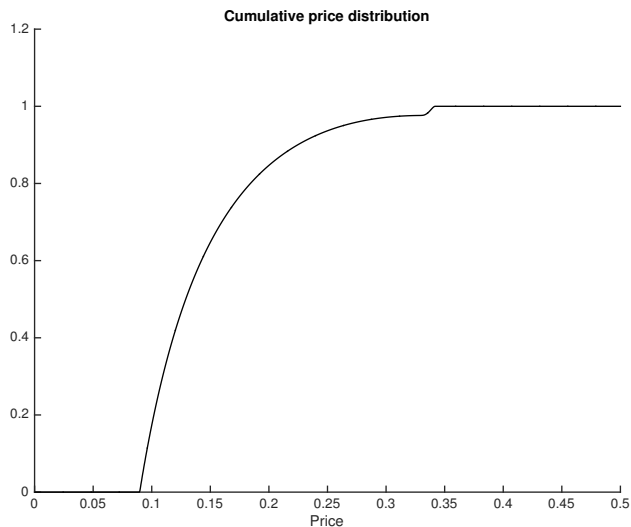


Figure 1: Equilibrium price distribution for  $\lambda = 2/5$  (cumulative).

stitutes, it seems that we can restrict the support to  $[0, 1/2]$ .<sup>8</sup> Figure 1 shows the cumulative equilibrium price distribution for  $\lambda = 2/5$ . It shows that prices range from roughly 0.09 to 0.35. The equilibrium shows a surprising feature. While most of the probability mass is clustered around the lower end of the price distribution with a small chance of drawing a high price, at the top end of the distribution we see another “lump” of probability mass: it happens relatively often that a firm sets a very high price.

For this particular game, a discretization of the game along the lines of Dasgupta and Maskin (1986), where the player’s actions are restricted to  $n$  points evenly distributed on the interval  $[0, 1/2]$ , does not work very well. For  $n = 1024$ , the accuracy is only  $\approx 10^{-4}$  compared to  $\approx 10^{-6}$  for the preferred method and Matlab’s minimization routine struggles to find the global minimum. More worryingly, when the number of gridpoints is increased to 2048 (which takes forever to calculate), the accuracy does not increase. This is a clear example where the smoothing of the payoff function produces better results.

### 4.3 Capacity constraints with a proportional sharing rule

Consider a market where two firms, indexed by  $i = 1, 2$  and who face no cost of production, are active. Consumers have downward-sloping demand  $D(x)$ . Firm  $i$  sets a price  $x_i$ , but no firm can serve more than  $K$  consumers (capacity constraint). The game is as follows: when  $x_i < x_j$ , firm  $i$  gets the first pick of the consumers and demand is  $D(x_i)$  if capacity

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<sup>8</sup>In general, by restricting the action space to the smallest possible interval, we can speed up the calculation of the Nash-equilibrium. Of course one needs to be careful not to exclude actions that are part of the Nash-equilibrium.

is sufficient. In case  $K < D(x_i)$ , there are  $1 - K/D(x_i)$  unsatisfied consumers, who go to firm  $j$  and demand  $D(x_j)$  units. The payoff for firm  $i$  is

$$\pi_i = \begin{cases} x_i \times \min\{D(x_i), K\} & \text{if } x_i < x_j \\ x_i \times \min\{D(x_i)/2, K\} & \text{if } x_i = x_j \\ x_i \times \min\left\{\max\left\{0, D(x_i)\left(1 - \frac{K}{D(x_j)}\right)\right\}, K\right\} & \text{if } x_i > x_j \end{cases}$$

Take  $D(x) = 1 - x$ .<sup>9</sup> Davidson and Deneckere (1986) show that for  $1/4 \leq K \leq 1$ , there is only a symmetric equilibrium in mixed strategies. Note that  $x_i \in [0, 1]$ .<sup>10</sup>

Figure 2 shows the cumulative equilibrium distribution for  $K = 2/3$ , which appears unremarkable: the steep slope at the lower end of the support reveals that price tend to be low and occasionally the firms set a very high price. The scatter plot in Figure 3 shows the probabilities per bin. Note the probabilities per bin fluctuate between a low value and a high value. This has little impact on the smoothness of the cumulative distribution function, but it does show that the discretization of the game can lead to equilibrium behavior, that is unlikely to be observed in a game with a continuous action space: one should always examine the outcome critically.

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<sup>9</sup>As per usual, this is shorthand for  $D(x) = \min\{\max\{1 - x, 0\}, 1\}$ .

<sup>10</sup>This is a proportional sharing rule, where the lowest-price firm does not give priority to consumers with the highest willingness-to-pay. Under a surplus-maximizing rule (cf. Kreps and Scheinkman, 1983), the game will have a drastically different equilibrium.

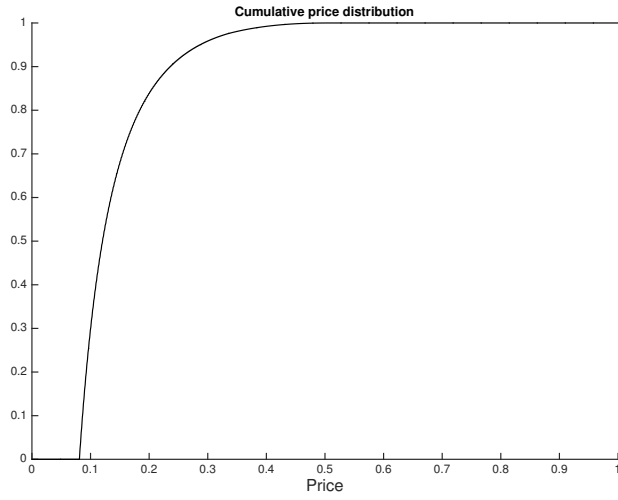


Figure 2: Equilibrium price distribution for  $K = 2/3$  (cumulative).

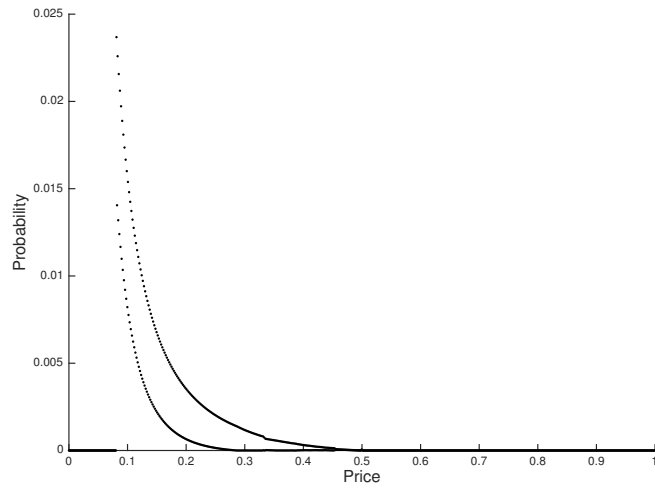


Figure 3: Equilibrium price distribution for  $K = 2/3$ .

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