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Abstract Motivated by some string and plaster models dating back from the late 19th and early 20th century, this note recalls some of the early history of the classification of plane cubic curves over the real numbers. Examples of different classifications are provided, showing their connection with some of the models in the Schilling collection.

The building of concrete mathematical models and dynamical instruments for higher education received quite some impulse in the nineteenth century at many universities in Europe as well as in the United States. In Europe collections of models were constructed at the polytechnic schools in Germany [6, 8] during the second half of the nineteenth century. This reached and became popular at universities in the United States a little later (see for instance [7]).

Fig. 1 shows two of these models.

The one on the left is made of strings which all (at least in theory) pass through a common point, indicating that the model represents a cone over a certain curve in space. The plaster model on the right shows a sphere containing colored curves on its surface. In fact both models illustrate the same geometric phenomena: particular species occurring in a classification of plane cubic curves. It is exactly the beginnings (up to the early 20th century) of this subject and its models that we intend to describe in the text below.

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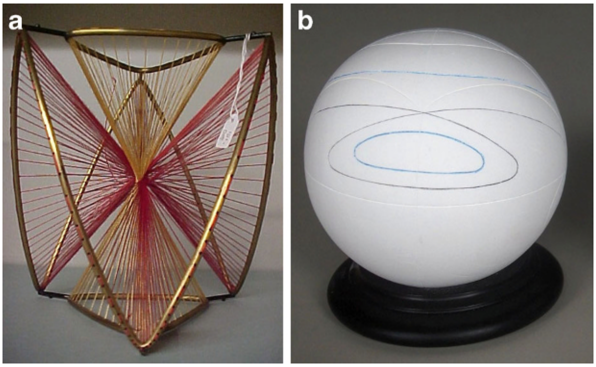


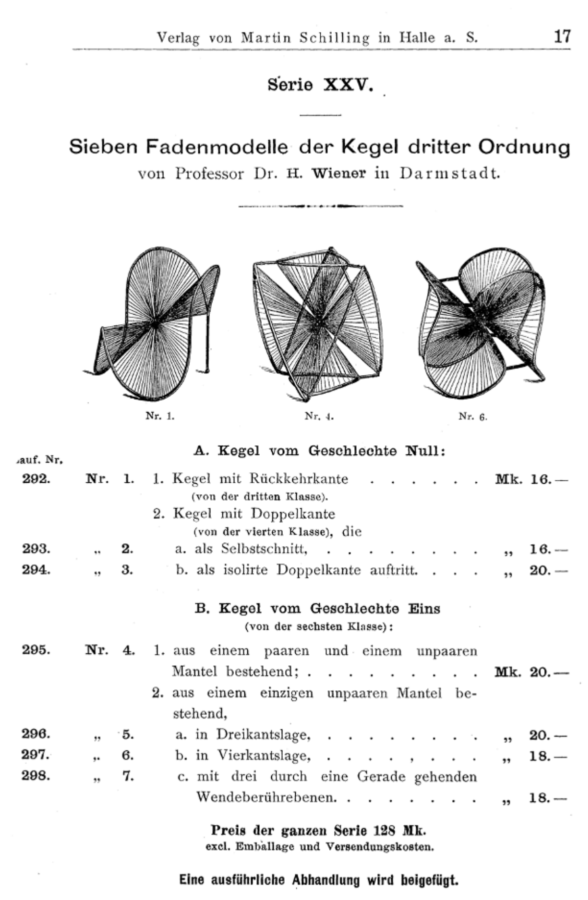
Fig. 1 Two models, **a** from Special Collections of the University of Amsterdam (reproduced with permission); **b** from Bernoulli Institute, University of Groningen (photo authors)

1 The history behind the models

In Germany, Ludwig Brill, brother of the mathematician Alexander von Brill, began to reproduce and sell copies of quite a number of mathematical models, and in 1880 he founded a firm for the production of them. This firm was taken over in 1899 by Martin Schilling who renamed it. Schilling's 1911 catalog [18] describes forty series consisting of almost four hundred models and devices and contains the name of the models and a short mathematical explanation. In some cases this is accompanied by a drawing. Several texts describe the origin and development of this model making, for example (with a focus on the Göttingen collection) [16], and also [17] (mostly discussing H. Wiener's collection).

From the large collection of models, as already mentioned, we here focus on those describing a classification of plane cubic curves. Several texts classifying plane cubic curves exist, starting with the famous appendix *Enumeratio Linearum Tertii Ordinis* written by Isaac Newton in 1704 as an appendix in his book *Opticks* (see [13]). For an English translation including an abundance of notes, see [21pp. 588–645]. Newton's work was extended in 1746 by the English mathematician Patrick Murdoch [12]. Möbius in 1852 did this again [11], soon in 1864 followed by Arthur Cayley [3]. The PhD thesis [2] at the University of Amsterdam by Hermann Gottfried Breijer in 1893 also treats the subject. This work was presented in the Dutch Royal Academy of Sciences by Breijer's thesis supervisor Diederik Korteweg [10]. Apparently unaware of the results by Breijer and Korteweg, in 1901 Hermann Wiener [22] published a very similar classification. The Italian mathematician Giulio Giraud in 1910 presented an expository text [9] on the theory of plane cubic curves, including (§5 loc. sit.) a classification which can be regarded as an explicit version of Wiener's (and, although no reference is given, the one by Breijer and Korteweg). Lastly, we mention the elaborate text [7] published around 1920 by the American mathematician Arnold Emch.

Some models in Schilling's collection illustrate these classifications of plane cubic curves. Around 1900, Schilling describes the seven string models from Series XXV as follows (Fig. 2).



Verlag von Martin Schilling in Halle a. S. 17

Serie XXV.

Sieben Fadenmodelle der Kegel dritter Ordnung

von Professor Dr. H. **Wiener** in Darmstadt.

Nr. 1. Nr. 4. Nr. 6.

A. Kegel vom Geschlechte Null:

Lauf. Nr.				
292.	Nr. 1.	1. Kegel mit Rückkehrkante . . . . (von der dritten Klasse).	Mk.	16.—
		2. Kegel mit Doppelkante (von der vierten Klasse), die		
293.	„ 2.	a. als Selbstschnitt, ...	„	16.—
294.	„ 3.	b. als isolirte Doppelkante auftritt. . . .	„	20.—

B. Kegel vom Geschlechte Eins
(von der sechsten Klasse):

295.	Nr. 4.	1. aus einem paaren und einem unpaaren Mantel bestehend;	Mk.	20.—
		2. aus einem einzigen unpaaren Mantel bestehend,		
296.	„ 5.	a. in Dreikantslage, ...	„	20.—
297.	„ 6.	b. in Vierkantslage, ...	„	18.—
298.	„ 7.	c. mit drei durch eine Gerade gehenden Wendeberührebenen.	„	18.—

Preis der ganzen Serie 128 Mk.
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Eine ausführliche Abhandlung wird beigefügt.

Fig. 2 Description of the models in Series XXV in Schilling's catalog

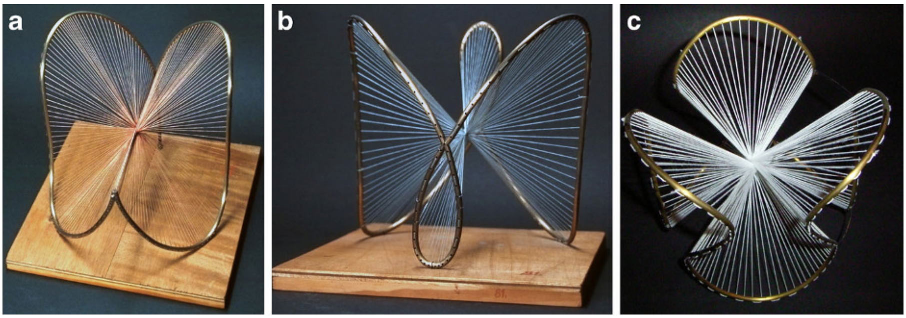

Fig. 3 Three string models, Series XXV. **a**, **b** © 2012 Collection of Mathematical Models, University of Göttingen (Reproduced by permission); **c** Bernoulli Institute, University of Groningen (photo authors)

Fig. 4 Description of models nr 2 in Series XVII in Schilling's catalog

2. **Die sieben Haupttypen der ebenen Curven 3. Ordnung,** nach Möbius auf einer Kugel dargestellt. Unter Leitung von Professor Dr. *Brill* modelliert von cand math. *Dollinger* in Tübingen. Zwei Modelle in Gips. (Durchmesser 10 cm.) Preis zusammen Mark 12.—.

Fig. 5 Plaster models, Series XVII nr. 2a and 2b, University of Groningen (photo authors)

The Series XXV string models were designed by Hermann Wiener. His 1901 text [22] describes the models and the mathematics behind them. Fig. 3 shows, from left to right, using the description as given in the displayed page: Nr. 1 (292), Nr. 2 (293), and Nr. 6 (297). Fig. 3a shows Nr. 4 (295). Note that Dyck's 1892 catalog, under nr. 181, mentions five earlier string models by Wiener which do not belong to Brill's collection. The Addendum to Dyck's catalog, which appeared in 1893, mentions under nr. 183a Wiener's seven string models. So one may conclude that Wiener actually designed his series of seven in 1892/93.

Two plaster models from Series XVII illustrate the same classification of plane cubic curves, and seem to have been designed around the same time. An advertisement in the American Journal of Mathematics from 1890 mentions only 16 series, as does the 1888 catalogue by Brill. Dyck's 1892 catalog mentions the new models, somewhat later introduced as Series XVII 2a and 2b of Brill's catalog; see [4] for a slightly more recent reference.

They are announced in Fig. 4 and shown in Fig. 5, and consist of two plaster balls with coloured curves drawn on them. According to catalogs [6] and [18] they were designed by Tübingen University mathematics student H. Dollinger directed by Alexander von Brill (who came to Tübingen in 1884).

The page from Schilling's catalogue displayed in Fig. 2 contains a clue how Series XVII 2ab and Series XXV are related. Indeed, the models in Series XXV are described as "Kegel" (cones), so sets of lines passing through some common point in space. Intersecting these cones with a ball centered around this common point yields the colored curves in XVII 2ab.

2 The geometry

As one reads in the description in Schilling's catalog, all models from both series XVII and XXV 2ab illustrate a classification of cubic curves due to Möbius. From a modern point of view, the curves in question are given as embedded in the real projective plane $P^2(R)$.

Geometrically, $P^2(R)$ can be regarded as the space of all lines in R^3 that pass through the origin. In this setup, curves in the projective plane are given as families of lines. When making a model using this kind of representation, curves can be given as a collection of threads (each representing a line) passing through a common point. This is what one finds in Series XXV.

Instead of taking lines through the origin O, one can also take a sphere S^2 with center O and regard $P^2(R)$ as the set of pairs of antipodal points on S^2 (so, pairs obtained by intersecting S^2 with a line through O). In modern language, one starts with an irreducible cubic polynomial $f \in R[x,y]$. The (real) plane cubic curve associated to f consists of the set of zero's of f in R^2. This corresponds to the lines through the origin in R^3 whose points satisfy the homogeneous equation $F(X,Y,Z)=0$, with $F(X,Y,Z)=Z^3 f(X/Z, Y/Z)$. Apart from lines passing through a point $(a,b,1)$ with $f(a,b)=0$, the step from f to F adds "asymptotic directions": solutions $(a,b,0)$ to $F=0$ with a, b not both zero correspond to lines through the origin in R^3 not intersecting the plane given by $Z=0$, and a, b such that the "direction" (a,b) in the plane is asymptotic to the plane curve defined by $f=0$. If the sphere is taken with equation $X^2+Y^2+Z^2=1$, the curve on the sphere is given as

$$\left\{\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)\right\}$$

where (a, b, c) runs over the nontrivial solutions of $F(X, Y, Z)=0$. The $9=7+2$ models that have been mentioned all represent real curves of degree three, either represented as sets of lines (threads, Series XXV, 7 models) through the origin in R^3 or as points on S^2 (the surface of a plaster ball, Series XVII 2ab; 2 balls with 3 curves drawn on one of them and 4 on the other). The models illustrate the seven different types of real cubic curves as classified by Möbius (see [11] and the text below).

3 Why seven curves?

In order to understand the mathematics in Möbius' classification, we recall some well-known definitions and results on cubic curves.

3.1 Classification of plane cubic curves

Given a plane cubic curve C, which means $C \subset P^2(R)$ is defined as the zeros of an irreducible, homogeneous $F \in R[X, Y, Z]$ of total degree 3. An *inflection point* or a *flex point* p of C is a non-singular point of C such that the intersection of the tangent line at p with the curve C has multiplicity 3. The tangent line to C at a flex point p is called a *flex line* of C.

It is a well-known result (compare [20], Thm. 6.4 and [15]) that every irreducible cubic curve over R has a real flex point and can be given after a linear change of variables defined over R, by $X^3 + aX^2Z + bXZ^2 + cZ^3 - Y^2Z = 0$ with real constants a, b, c. This form is called the *Weierstrass form* of the curve C.

Using the above standard form, Newton [13, 21] classified the irreducible cubic curves over R into 5 types (and for each type he identified various curves in $\mathbb{R}^2$, depending on a choice of embedding $\mathbb{R}^2$ into $\mathbb{P}^2(\mathbb{R})$). Newton in fact starts from the given standard form, but he did not provide a proof of the fact that any homogeneous cubic allows a (linear) change of variables bringing it in the desired form. Plücker [14] refined Newton's classification and gave a detailed and complete proof (see [1]).

4 Newton's classification

For curves given by an equation

$$Y^2Z = X^3 + aX^2Z + bXZ^2 + cZ^3$$

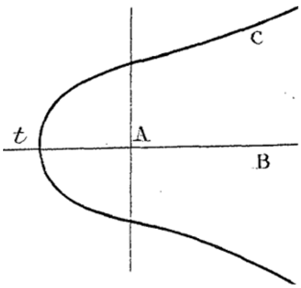

Fig. 6 *Parabola pura*: $g(x)$ has precisely one real zero and this zero is simple. In this case the curve is non-singular and its real locus has one component

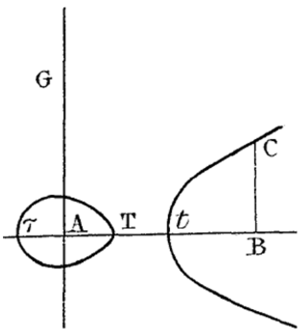

Fig. 7 *Parabola campaniformis cum ovali*: $g(x)$ has three distinct real roots. In this case the curve is non-singular and its real locus has two components

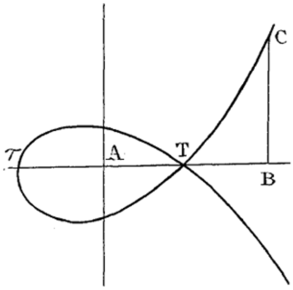

Fig. 8 *Parabola nodata:* $g(x)$ has a real double root α and a real simple root β with $\alpha > \beta$. Then the curve has an ordinary double point with real tangents

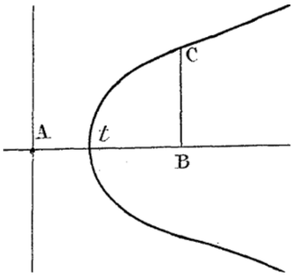

Fig. 9 *Parabola punctata*: $g(x)$ has a real double root α and a real simple root β with $\alpha < \beta$. Then the curve has an isolated double point with complex conjugate tangents

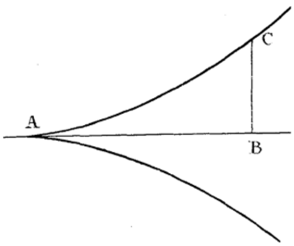

Fig. 10 *Parabola cuspidata*: $g(x)$ has a triple root. In this case the curve has a cusp

Newton identified 5 types depending on $g(x) := x^3 + ax^2 + bx + c$, as shown in Figs. 6, 7, 8, 9 and 10 (the drawings sketch the zeros of $f(x, y) := y^2 - g(x)$ in R^2; they are taken from the English translation [19] of Newton's original text).

5 Möbius' classification

Möbius studied the five types in Newton's classification in terms of their real flex points. It is known that irreducible cubic curves have at most 9 flex points over C and they can either have either 1 or 3 flex points over R. The *Parabola nodata* and *cuspidata* have one real flex point. In this case, Möbius' and Newton's classification coincide; the two types are shown in Fig. 11.

The remaining three "Parabolas" in Newton's classification have three real flex points, and these points are collinear. Möbius now considers the three flex lines l_1, l_2 and l_3 at the three flex points and also the line l containing the three real flex

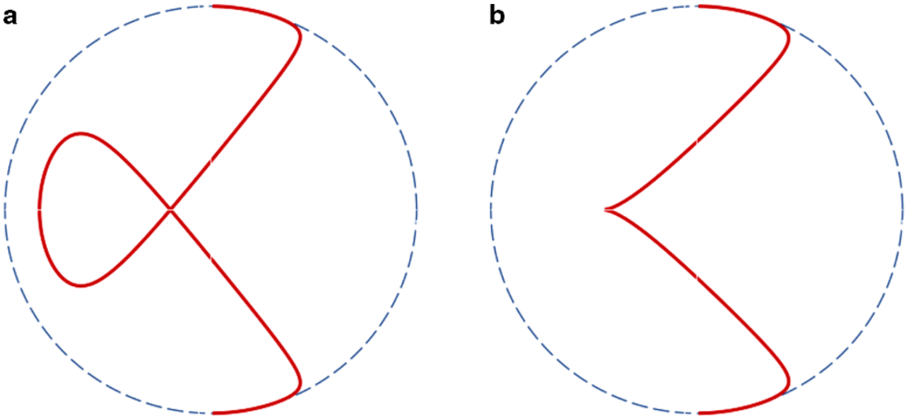

Fig. 11 *Parabalo nodata* and *Parabola cuspidata* on a sphere

points. Via the antipodal map $\Phi\colon S^2 \to P^2(R)$ these four lines correspond to four large circles $\Phi^{-1}(l_j)$, $\Phi^{-1}(l) \subset S^2$ which give a tiling of the sphere S^2. It can happen that this tiling contains only triangles; if not then it contains both triangles and quadrangles. Taking this into account, and by considering whether the curve lies in the area of the quadrangles or in the area of the triangles, Möbius arrives at a slightly finer classification compared to Newton, by distinguishing the cases where indeed 3 real flex points exist:

0-curves The tiling of the sphere contains only triangles. This can only happen for the *Parabola Pura*.

3-curves The tiling of the sphere contains both triangles and quadrangles, and the curve lies in the area of the triangles. This happens for every *Parabola Campaniformis cum ovali* as well as for every *Parabola punctata*. Moreover it can happen for the *Parabola pura*. An example of a 3-curve is shown in Fig. 12.

4-curves The tiling of the sphere contains both triangles and quadrangles, and the curve lies in the area of the quadrangles. As for the 0-curves, this can only happen for the *Parabola Pura*. An example of this type of curve is shown in Fig. 13.

So considering how the curve is located with respect to the flex lines and the line passing through the flex points, Möbius subdivides the "*parabola pura*" of Newton into 3 distinct types. Model XVII 2b in the Schilling collection, shown as the rightmost plaster ball in Fig. 5, shows exactly this subdivision.

Using the same criteria, none of the other 4 types given by Newton results in a finer subdivision. Hence in total, the classification by Möbius consists of 7 types. In his paper [11], Möbius indeed shows that all three refined cases for the *Parabola pura* occur; see Fig. 14. He refers to the in total 7 types obtained like this by rather uninformative names; later Cayley [3] uses names for them which are more suggestive. These are listed in Table 1.

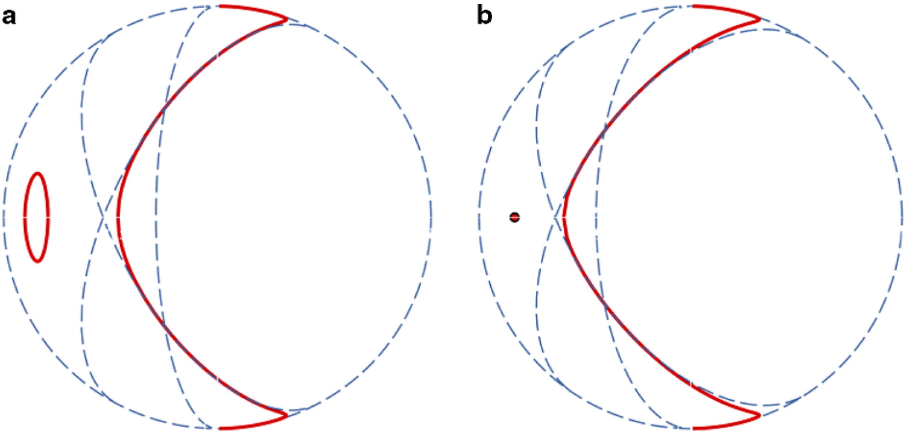

Fig. 12 Every *Parabola campaniformis cum ovali* and every *Parabola punctata* turns out to be a 3-curve, as is illustrated here on a sphere

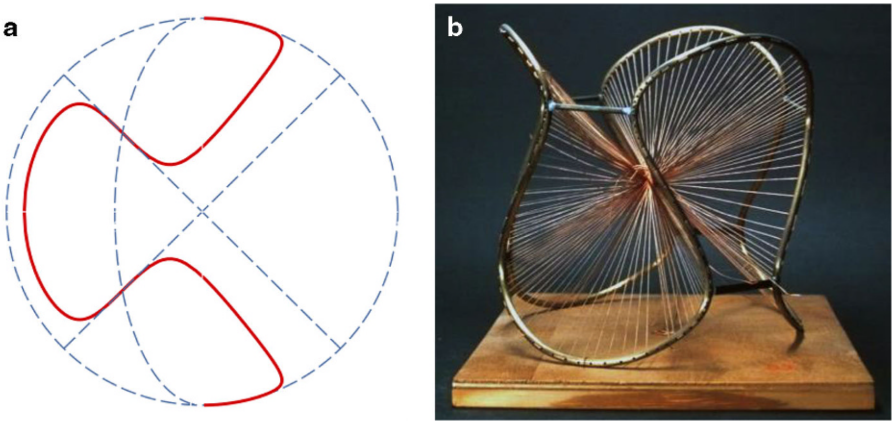

Fig. 13 Example of a 4-curve (*Parabola Pura*) on a sphere (**a**) and on a string model (**b**) from Series XXV.

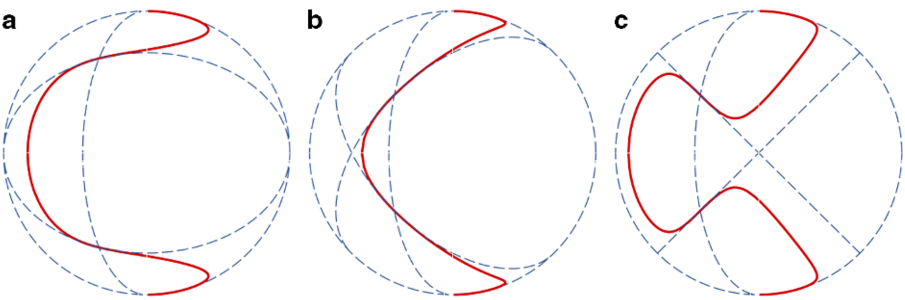

Fig. 14 **a** to **c** The three types of *Parabolas Pura*. **a** to **c** 0-curve, 3-curve, and 4-curve

Table 1 Names of the seven types according to Newton, Möbius, and Cayley, respectively

Parabola	Möbius	Cayley
Pura (0-curve)	Gattung 5	Neutral simplex cone
Pura (3-curve)	Gattung 1	Trilateral simplex cone
Pura (4-curve)	Gattung 4	Quadrilateral simplex cone
Camp. cum ovali	Gattung 3	Complex cone
Nodata	Gattung 6	Crunodal cone
Punctata	Gattung 2	Acnodal cone
Cuspidata	Gattung 7	Cuspidal cone

6 H.G. Breijer and H. Wiener

To conclude this paper, we mention how Breijer [2] and H. Wiener [22] arrive at essentially the same classification as Möbius.

Breijer starts from a real (cubic) curve C in the projective plane $\mathbb{P}^2$, and any point in $\mathbb{P}^2(\mathbb{R})$. Now he asks how many (real) lines through this point are tangent to C. In this way $\mathbb{P}^2(\mathbb{R})\setminus C$ is subdivided into regions bounded by C and by the flex lines. This approach is in fact motivated by work of Plücker, who introduced what is now called the "class curve" or "dual curve" of C.

Breijer (and Korteweg who somewhat simplified his results) show that in case the cubic curve contains more than one real flex point (as is the case whenever the irreducible cubic contains no singular points), one can choose coordinates such that one flex point is the point $(0:1:0)\in\mathbb{P}^2$ with corresponding flex line $Z=0$, and the other two flex points are $(0:\pm1:1)\in\mathbb{P}^2$. The cubic is now given by an equation

$$Y^2Z = X^3 + (bX + Z)^2Z$$

and the two remaining flex lines have equation $Y=\pm(bX+Z)$. The different types are described in terms of how the curve together with the three flex lines partition $\mathbb{P}^2(\mathbb{R})$ (or the sphere S^2):

- if $4b^3>27$, one has a *campaniformis cum ovali*, and in this case there are regions with 0, with 2, with 4, and with 6 tangent lines passing through any point in the interior of that region;
- if $b = \frac{3}{2}\sqrt[3]{2}$ we have a *parabola punctata* (and there are regions with 0, 2, and 4 tangent lines passing through any given interior point);
- for $0 < b < \frac{3}{2}\sqrt[3]{2}$ one obtains a *parabola pura* which is a 3-curve. For the tangent lines count, one finds the numbers 0, 2, and 4.
- the case $b=0$ yields a different kind of *parabola pura*, namely a 0-curve. Here the tangent line count only yields 2 and 4;
- finally, if $b<0$ one obtains a *parabola pura* which is a 4-curve. Here the outcomes of the tangent line count are 2, 4, and 6.

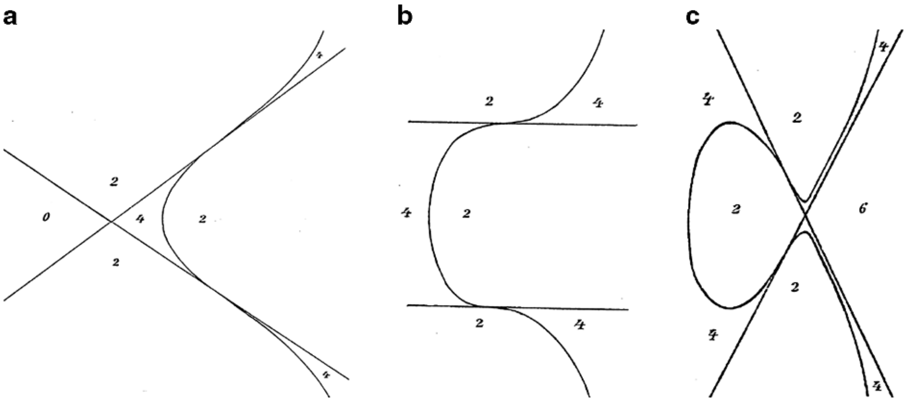

Fig. 15 Three drawings from Breijer's thesis showing, respectively, a pura 3-curve; a pura 0-curve; a pura 4-curve

The last three possibilities mentioned above are illustrated in Fig. 15. The remaining two cases from Newton's classification (*nodata* and *cuspidata*) are described from this point of view as well; in total Breijer, although using different criteria, finds precisely the same classification as Möbius did.

Eight years after Breijer's thesis H. Wiener presents a very similar classification, in which he distinguishes 13 instead of 7 types of curves. Much of this appears to be inspired by the book [5] on cubic curves by Heinrich Durège. G. Giraud [9] explains Wiener's classification by first reducing to the case of homogeneous equations

$$X^3 + Y^3 + Z^3 + 6\lambda XYZ = 0.$$

In case $\lambda = -1/2$ the given polynomial has a factor $X + Y + Z$, so it is reducible. For all other real values of λ one obtains cubic curves without singularities. The points $(0:1:-1)$, $(1:0:-1)$, and $(1:-1:0)$ are flex points with flex lines respectively $Y + Z = 2\lambda X$, $X + Z = 2\lambda Y$, and $X + Y = 2\lambda Z$. This readily leads to a classification:

- For $\lambda < -1/2$ the curve has two real components (so, using Newton's terminology it is a *campaniformis cum ovali*). Still, Wiener distinguishes three cases here:

II if $\lambda = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$, the curve is a so-called harmonic cubic (in modern language: a curve with an automorphism of order 4 fixing one of the flex points);

I, III Wiener also separates the cases $-\frac{1}{2} - \frac{1}{2}\sqrt{3} < \lambda < -\frac{1}{2}$ and $\lambda < -\frac{1}{2} - \frac{1}{2}\sqrt{3}$. However the reasoning behind this appears somewhat obscure.

- For $\lambda > -1/2$ the curve has one real component (in fact, it is Newton's *parabola pura*). Wiener considers three very special cases and then divides the remaining possibilities into four more kinds:

IX if $\lambda = 0$, the curve is a so-called *equianharmonic* one (in modern language: it has an automorphism of order 6 fixing a flex point). Moreover in this case the Hessian (determinant of the matrix $(\frac{\partial^2 F}{\partial X_i \partial X_j})$ where $F=0$ is the equation of the curve) factors into three linear forms over R;

VII if $\lambda = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$, one has a harmonic curve (now with only one real component);

V if $\lambda = 1$, the curve is equianharmonic and the Hessian factors into one real and two complex conjugate linear forms; the curve is a 0-curve.

IV For $\lambda > 1$ one obtains 4-curves;

VI, VIII, X in the three remaining intervals between –1/2 and 1, one obtains 3-curves. Again it is not clear why Wiener separates these cases.

In addition to these 10 types of irreducible cubics not containing any singular point, Wiener mentions the usual singular ones (*nodata*: XI and *cuspidata*: XII and *punctata*: XIII). Although his classification consists of more types than the earlier ones, presumably the observation that several types have exactly the same appearance kept him from designing models for his refined cases. Wiener restricted himself to making models illustrating the classification given by Möbius.

7 Conclusion

This text discusses an example of how classical models may illustrate a geometric classification. In particular, we note how models from Series XXV and XVII 2ab in Schilling's collection show different representations of the types of plane cubic curves as described by Möbius [11]. Some history and mathematics behind the models together with pictures of most of them is presented. Hopefully this helps to understand and appreciate the beauty of these geometric objects.

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