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Linear differential equations with finite differential Galois group

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ABSTRACT

For a finite irreducible subgroup $H \subset \mathrm{PSL}(C^n)$ and an irreducible, H-invariant curve $Z \subset \mathbb{P}(C^n)$ such that $C(Z)^H = C(t)$, a standard differential operator $L_{st} \in C(t)[\frac{d}{dt}]$ is constructed. For $n = 2$ this is essentially Klein's work. For $n > 2$ an actual calculation of L_{st} is done by computing an evaluation of invariants $C[X_1, \ldots, X_n]^H \to C(t)$ and applying a scalar form of a theorem of E. Compoint in a "Procedure". Also in some cases where Z is unknown evaluations are produced.
This new method is tested for $n = 2$ and for three irreducible subgroups of SL_3. This supplements [18]. The theory developed here relates to and continues classical work of H.A. Schwarz, G. Fano, F. Klein and A. Hurwitz.

1. Introduction and summary

Let C denote an algebraically closed field of characteristic zero. Let k be $C(z)$ and let $\overline{k}$ denote the algebraic closure of k. Both fields are provided with the C-linear derivation

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$f \mapsto f'$ with $z' = 1$. The positive but not explicit or constructive answer to the inverse problem of Galois theory is:

For any finite group G there is a Galois extension $\ell \supset k$ with group G.

Indeed, a proof for the complex case uses analytic tools, in particular the "Riemann Existence Theorem". The proof for any field C as above is deduced from the complex case. There is an extensive literature on solving the inverse problem *explicitly* for certain finite groups.

A finite Galois extension $\ell \supset k$ can be given as the splitting field of a polynomial P in $k[T]$. Sometimes, a more efficient way is to describe $\ell \supset k$ as the Picard–Vessiot field of a linear differential operator L in $k[\partial]$ with $\partial = \frac{d}{dz}$. From a polynomial P for $\ell \supset k$ one can easily compute a differential operator L for $\ell \supset k$, see [18, §1] and [8, §2]. The other direction is far more complicated (see (ii) below).

Let π denote the profinite Galois group of $\overline{k}/k$. There is a well known bijection between the monic differential operators $L \in k[\partial]$ of order n, such that all solutions are algebraic over k, and the C-vector spaces $V \subset \overline{k}$ of dimension n which are stable under π.

Indeed, one associates to L the π-stable space $\{f \in \overline{k} \mid L(f) = 0\}$ (i.e., the contravariant solution space). On the other hand, let the π-stable $V \subset \overline{k}$ have basis $b_1, \dots, b_n$ over C. There is a unique operator $L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0$ with all $a_i \in \overline{k}$ such that all $L(b_j) = 0$. The uniqueness and the π-stability of V imply that all $a_i \in k$.

A more abstract way to compare differential equations and Galois extensions $\ell \supset k$ is the following. The category $\mathrm{Diff}_{\overline{k}/k}$ that we study here, has as objects the finite dimensional differential modules M over k which become trivial over the field $\overline{k}$. This condition on M is equivalent to M having a finite differential Galois group. The morphisms in this category are the k-linear maps that commute with differentiation.

Let $Repr_\pi$ denote the category of the (continuous) representations of π on finite dimensional C-vector spaces. *The functor* $\mathrm{Diff}_{\overline{k}/k} \to Repr_\pi$, which associates to a differential module M its (covariant) solution space $\ker(\partial, \overline{k} \otimes_k M)$, is known to be *an equivalence of (Tannakian) categories.*

The aim of this paper is to make this equivalence of categories explicit for special cases. There are two directions to consider:

(i) Compute a differential operator connected to a given representation of a given finite group and some additional data.

(ii) Describe or construct the Picard–Vessiot field for a given module $M \in \mathrm{Diff}_{\overline{k}/k}$, when M is represented by a differential operator L.

We recall some earlier results on (i) and (ii).

Regarding (i): The Schwarz' list (see [18] for a modern version) and Klein's theorem (e.g., see [1] and [2]) are classical results for the special case of order $n = 2$. We recall the statement of Klein's theorem:

for each of the irreducible subgroups $G \subset \mathrm{PSL}(C^2)$ (so $G \in \{D_n, A_4, S_4, A_5\}$), there is a standard order two differential operator L_{st} having G as projective differential Galois group. It has the universal property that any order two differential operator with projective group G is a "weak pullback" (see Definition 3.6) of L_{st}.

In the case $n = 3$ Hurwitz' paper [12] produces examples. This method was refined in [18]. Klein's theorem is generalized in, e.g., [2,20–22]. Not much seems to have been done for $n > 3$. Here (Section 3) we treat the general case.
Regarding (ii): This was initiated by J. Kovacic in his paper [14] dealing with $n = 2$. There are many subsequent papers [23,10,11] considering small n. For general n there is work of E. Compoint and M.F. Singer [6,7]. The paper [4] discusses the particular case of hypergeometric differential equations.

We now describe the present paper, which is mainly concerned with (i) but also contributes to (ii) by exploiting invariant theory for finite groups and Compoint's work [6].

Section 2 associates to a differential operator $L \in k[\partial]$ with all solutions in $\overline{k}$, geometric objects: a Picard–Vessiot curve, a Fano curve, Schwarz maps, projective differential Galois groups and an evaluation of invariants.

In Section 3 Klein's theorem for order two is generalized, resulting in a *subtle* construction of a *standard differential operator* L_{st} (Theorem 3.1). The data for this construction are a finite irreducible subgroup $H \subset \mathrm{PSL}(C^n)$, an H-invariant irreducible curve $Z \subset \mathbb{P}(C^n)$ such that the normalization of Z/H has genus zero and a variable z with $C(Z/H) = C(z)$. In the construction of L_{st} the group H is replaced by a subgroup $\tilde{H} \subset \mathrm{SL}(C^n)$ which is *minimal* such that $\tilde{H} \to H$ is surjective.

The "universal property" of L_{st} is the following:

any differential operator L with projective differential Galois group isomorphic to H and Fano curve isomorphic to Z is a weak pullback of L_{st} (see 3.1 and 3.7). This clarifies and extends the work of [2,20–22].

Section 4. For order $n = 2$ the Fano curve is by definition $\mathbb{P}(C^2)$ and the computation of the standard operators L_{st} is easy and produces the classical operators. For $n > 2$ however, the construction of L_{st} as described in Section 3 does not in an obvious way result in a computation of this operator. A *new method* for the computation of L_{st} is introduced. We derive a "scalar version" of Compoint's theorem (see 4.2) which is roughly the following. Let the homogeneous polynomials $f_1, \dots, f_N$ be generators for the ring of invariants $C[X_1, \dots, X_n]^{\tilde{H}}$. An evaluation of the invariants is a suitable homomorphism $ev : C[X_1, \dots, X_n]^{\tilde{H}} \to C(t)$ and the Picard–Vessiot field of L_{st} is $K := C(t)[X_1, \dots, X_n]/(f_1 - ev(f_1), \dots, f_N - ev(f_N))$.

Our "*Procedure*" 4.3 computing L_{st} works as follows. A set of homogeneous generators $f_1, \dots, f_N$ and their relations are taken (if possible) from the literature. The given H-invariant irreducible curve $Z \subset \mathbb{P}(C^n)$ with $C(Z)^H = C(t)$ effectively produces an essentially unique evaluation, see 4.6. From the explicit presentation of K one computes the derivation D on K extending $\frac{d}{dt}$. Then one obtains the monic operator $L \in C(t)[\frac{d}{dt}]$ of degree n with kernel $C\overline{X}_1 + \cdots + C\overline{X}_n$, where $\overline{X}_i$ denotes the image of X_i in K. Finally L_{st} is obtained by normalizing L such that its coefficient of $(\frac{d}{dt})^{n-1}$ is zero.

Our Procedure can be seen as the "opposite" of an algorithm, by M. van Hoeij and J.-A. Weil [11], which computes for a given differential operator, the associated evaluation of the invariants $C[X_1, ..., X_n]^G \to C(z)$.

Section 5. For order $n = 2$, we show how to obtain evaluations of the invariants and apply the Procedure to produce the known standard operators. For the group $G_{168} \subset \mathrm{PSL}(C^3)$ and the Klein curve $Z \subset \mathbb{P}^2_C$, a direct computation of the standard operator from its construction in §3 fails. However, evaluation and the Procedure produce the standard operator.

The LIST, copied from [18], contains all possibilities, determined by the Riemann Existence Theorem, of order 3 differential operators over $C(z)$ (up to equivalence) with group G_{168} and singular locus $\{0, 1, \infty\}$. In most of these cases *one does not know* a stable $Z \subset \mathbb{P}^2_C$ such that the normalization of Z/G_{168} has genus zero. The methods of [18] produced explicit third order equations for about half of the cases. For the same cases our new method of evaluation and the Procedure produces more easily the standard equations.

In [18] no standard equation for the group $H_{72}^{\mathrm{SL}_3}$ was found. Our new methods produce an equation.

In Section 6 standard equations for $A_5 \subset \mathrm{SL}_3$ are studied. Moreover, properties in relation with the preimage $A_5^{\mathrm{SL}_2} \subset \mathrm{SL}_2(C)$ of $A_5 \subset \mathrm{PSL}(C^2)$ and the lists of differential operators in [18] are discussed.

2. Objects associated to a differential operator L over $k = C(z)$ with finite differential Galois group

L has the form $d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_0$ with all $a_i \in C(z)$, $d_z = \frac{d}{dz}$ and all solutions are supposed to be algebraic over $C(z)$. Associated to L is:

(1) *The Picard–Vessiot field* $K \supset C(z)$ with its Galois group G.

(2) *The (contravariant) solution space* $V \subset K$ of L with the action of G on it. The image of $G \subset \mathrm{GL}(V)$ into $\mathrm{PGL}(V)$ will be denoted by G^{proj} and is called the *projective differential Galois group.*

(3) *The Picard–Vessiot curve* X_{pv} is the smooth, irreducible, projective curve over C with function field K. G acts on X_{pv} and there is an isomorphism $X_{pv}/G \cong \mathbb{P}^1_z$. Here $\mathbb{P}^1_z$ denotes the projective line with function field $C(z)$.

(4) *Evaluation of the invariants.* One considers a C-linear homomorphism $\phi : C[X_1, \ldots, X_n] \to K$ which sends the variables $X_1, \ldots, X_n$ to a basis of V. The C-linear action of G on $C[X_1, \ldots, X_n]$ is defined by the G-invariance of $CX_1 + \cdots + CX_n$ and the G-equivariance of ϕ. This makes G into a subgroup of $\mathrm{GL}(n, C)$. The homomorphism ϕ induces a homomorphism $ev : C[X_1, \ldots, X_n]^G \to K^G = C(z)$ which we will call the *evaluation of the invariants*. Write $C[X_1, \ldots, X_n]^G = C[f_1, \ldots, f_N]$ where $f_1, \ldots, f_N$ are homogeneous generators and ev maps each f_i to an element in $C(z)$.

Now suppose that the action of G on V is known and is *irreducible*, i.e., no proper linear subspace $\neq (0)$ of V is invariant under G. If we define the action of G on $CX_1 + \cdots + CX_n$ such that an equivariant ϕ with $\phi(CX_1 + \cdots + CX_n) = V$ exists, then this ϕ is unique up to multiplication by a scalar $c \in C^*$. As a consequence, *the evaluation map is unique* up to changing each $ev(f_i)$ into $c^{\deg f_i} ev(f_i)$ for all i.

(5) *The Fano curve.* $\mathbb{H} \subset \ker(\phi)$, the "homogeneous kernel", is the ideal generated by the homogeneous elements in $\ker(\phi)$. For $n = 2$ one has $\mathbb{H} = 0$. For notational reasons we will call $\mathbb{P}(V) = \mathbb{P}^1$ itself *the Fano curve* in this case.

Suppose that $n > 2$, then $\mathbb{H}$ defines an irreducible curve in $\mathbb{P}^{n-1}$, invariant under the action of G. Indeed, $\mathbb{H}$ is the homogeneous ideal induced by the kernel J of the corresponding homomorphism $C[\frac{X_2}{X_1}, \ldots, \frac{X_n}{X_1}] \to K$. It is a curve since K/C has transcendence degree 1. The curve in $\mathbb{P}^{n-1}$ defined by $\mathbb{H}$ will be denoted by X_{fano} and will be called *the Fano curve*. This curve was indeed considered by Fano in his 1900-paper [9]. We note that X_{fano} can have singularities. From the definition one sees that $C(X_{fano}) = C(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1})$, where $x_1, \ldots, x_n$ is a basis of $V \subset K$.
(6) *The Schwarz map.* The homomorphism $C[X_1, \ldots, X_n]/\mathbb{H} \to K$ induces a morphism of curves $X_{pv} \to X_{fano}$ which is G-equivariant. After dividing by G we obtain a multi-valued map $Schw : \mathbb{P}^1_z = X_{pv}/G \cdots \to X_{fano}$ called *the Schwarz map*. For $n = 2$ it is the well known classical Schwarz map.

After dividing by G we obtain $qSchw : \mathbb{P}^1_z = X_{pv}/G \to X_{fano}/G^{proj}$ which can be called *the quotient Schwarz map*. We note that X_{fano}/G^{proj} can have singularities. The relation between X_{pv} and X_{fano} is in general not obvious.

Lemma 2.1. *Suppose that $qSchw : \mathbb{P}^1_z = X_{pv}/G \to X_{fano}/G^{proj}$ is birational. Let $c(G) \subset G$ be the group of the multiples of the identity belonging to G. Since $c(G)$ acts trivially on the curve X_{fano}, the map $X_{pv} \to X_{fano}$ factors over $X_{pv}/c(G)$. The morphism $X_{pv}/c(G) \to X_{fano}$ is birational.*

Proof. One has $K = C(X_{pv}) \supset K^{c(G)} \supset C(X_{fano})$. The group $G^{proj} = G/c(G)$ acts faithfully on $K^{c(G)} = C(X_{pv}/c(G))$ and $(K^{c(G)})^{G^{proj}} = C(z)$. Since $C(X_{fano}) = C(\frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1})$, the group G^{proj} acts faithfully on $C(X_{fano})$. By assumption $C(X_{fano})^{G^{proj}} = C(z)$. Therefore $K^{c(G)} = C(X_{fano})$. □

3. Construction of a standard operator and pullbacks

Theorem 3.1. *Let the following data be given:*
(i) *a C-vector space V with $n := \dim V \geq 2$,*
(ii) *an irreducible finite subgroup $H \subset \mathrm{PSL}(V)$,*
(iii) *an irreducible H-invariant curve $Z \subset \mathbb{P}(V)$ such that the normalisation of Z/H has genus 0 and*
(iv) *a variable z such that $C(Z)^H = C(z)$.*

The data V, H, Z, z determine a differential operator

$$L_{st} = (\frac{d}{dz})^n + a_{n-2}(\frac{d}{dz})^{n-2} + \cdots + a_0 \in C(z)[\frac{d}{dz}] \text{ such that}$$

(a) *The C-vector space $W := \{f \in \overline{C(z)} \mid L_{st} f = 0\}$ has dimension n (i.e., all solutions are algebraic).*

(b) *Let $G \subset \mathrm{SL}(W)$ denote the differential Galois group of L_{st}. There is a C-linear isomorphism $\phi : W \to V$ such that the projective differential Galois group G^{proj} is mapped isomorphically to H and the Fano curve $X_{fano} \subset \mathbb{P}(W)$ of L_{st} is mapped isomorphically to $Z \subset \mathbb{P}(V)$.*

Remarks 3.2. (0). It is a standard fact that in (b) one has $G \subset \mathrm{SL}(W)$; see, e.g., [17, Exc. 1.35 5(b)-(c)].
(1). The operator L_{st} will be called the *standard operator* for the data V, H, Z, z. For $n = 2$, one has $Z = \mathbb{P}(V)$. Further Z/H is identified with $\mathbb{P}^1_z$ and so $C(Z/H) = C(z)$. One knows that the possibilities for H are D_n, A_4, S_4, A_5. The variable z is chosen such that $z = 0, 1, \infty$ are the branch of $Z \to Z/H$. Thus L_{st} depends essentially only on H.
(2). In the proof of Theorem 3.1 we will use a group $\tilde{H} \subset \mathrm{SL}(V)$ which maps surjectively to H and is minimal with respect to this property. The kernel of $\tilde{H} \to H$ has the form $\{\lambda \cdot \mathbf{1} |\ \lambda^m = 1\}$ for a certain divisor m of n.
(3). In the construction of L_{st} only the data V, N, Z, z are used. It can be shown that the operator L_{st} is actually determined by the properties (a) and (b) in Theorem 3.1.
(4). *The action of H on Z is faithful.* Indeed, since H is irreducible and Z is H-invariant, Z is not contained in a proper projective subspace of $\mathbb{P}(V)$. By induction on i, one finds for $i = 1, \ldots, n$, elements $z_0, \ldots, z_i \in Z$ such that $z_0, \ldots, z_i$ is not contained in a projective subspace of dimension $< i$. Further, for each j, one can replace z_j by infinitely many elements $\tilde{z}_j \in Z$ such that $z_0, \ldots, z_{j-1}, \tilde{z}_j, z_{j+1}, \ldots, z_n$ has the same property.

Suppose that $h \in H$ acts as identity on Z. Then $h \in \mathrm{PGL}(V)$ has a diagonal matrix with respect to a basis of V corresponding to any sequence $z_0, \ldots, z_{j-1}, \tilde{z}_j, z_{j+1}, \ldots, z_n$. This implies that $h = 1$. □

Proof. *We start the construction of L_{st}.* The above data yield inclusions $C(z) = C(Z)^H \subset C(Z) \subset \overline{C(z)}$. The variable z is given in the data and the embedding $C(Z) \subset \overline{C(z)}$ is unique up to an automorphism of $C(Z)$ over $C(z)$, i.e., an element of H. We would like to identify V with the solution space in $\overline{C(z)}$ of the standard operator to be constructed. However, V does not lie in $C(Z)$.

One chooses any $\ell \in V,\ \ell \neq 0$. For any $v \in V$ one considers the restriction of the rational function $\frac{v}{\ell}$ on $\mathbb{P}(V)$ to Z (this makes sense because Z is not contained in the hypersurface $\ell = 0$). Write $\frac{V}{\ell}$ for the functions on Z obtained in this way, so $\frac{V}{\ell} \subset C(Z)$. The C-vector space $\frac{V}{\ell}$ is not invariant under H, or what is the same, it is not invariant under π. The following lemma is the *key ingredient* of the construction.

Lemma 3.3. *There exists an element $f \in \overline{C(z)}^*$ such that $f\frac{V}{\ell}$ is invariant under π. The canonical map $\mathbb{P}(V) \to \mathbb{P}(f\frac{V}{\ell})$, given by $v \mapsto f \cdot \frac{v}{\ell}$ is equivariant for the action of π.*

Proof. The group $\tilde{H}$ is supposed to have the properties of Remarks 3.2. For each $\sigma \in H$, one denotes by $\tilde{\sigma}$ an element in $\tilde{H}$ with image σ. Now $\sigma(\frac{V}{\ell}) = \frac{V}{\tilde{\sigma}\ell} = \frac{\ell}{\tilde{\sigma}\ell} \cdot \frac{V}{\ell}$. The term $\frac{\ell}{\tilde{\sigma}\ell}$ depends in general on the choice of $\tilde{\sigma}$. But $(\frac{\ell}{\tilde{\sigma}\ell})^m$ depends only on σ and $\sigma \mapsto (\frac{\ell}{\tilde{\sigma}\ell})^m$ is a 1-cocycle. By Hilbert 90, there is an element $f \in C(Z)$ such that $\frac{\sigma f}{f} \cdot (\frac{\ell}{\tilde{\sigma}\ell})^m = 1$ for all $\sigma \in H$.

For the case $m = 1$ we conclude that $f\frac{V}{\ell} \subset C(Z)$ is invariant under H (and thus also under π). For the case $m > 1$ we *claim* that the equation $T^m - f$ is irreducible over $C(Z)$. Assuming this claim, the field $C(Z)(f_m)$ with $f_m^m = f$ is a Galois extension of $C(z)$ since for every $\sigma \in H$ one has $\frac{\sigma f}{f}$ is an mth power in $C(Z)$. We may embed $C(Z)(f_m)$ into $\overline{C(z)}$ and conclude that $f_m\frac{V}{\ell}$ is invariant under π.

Now we prove the claim. If the equation $T^m - f$ is reducible over $C(Z)$, then there exists a proper divisor d of m and an element $g \in C(Z)$ with $g^d = f$. The expression $E(\tilde{\sigma}) := \frac{\sigma g}{g} \cdot (\frac{\ell}{\tilde{\sigma}\ell})^{m/d}$ has the property $E(\tilde{\sigma})^d = 1$. One can consider for each $\sigma \in H$ the elements $\tilde{\sigma} \in \tilde{H}$ such that $E(\tilde{\sigma}) = 1$. This defines a proper subgroup of $\tilde{H}$ which has image H. This contradicts the assumptions on $\tilde{H}$.

The last statement of the lemma follows from $\sigma(f\frac{v}{\ell}) = \frac{\sigma f}{f} \cdot \frac{\ell}{\sigma\ell} \cdot f\frac{\sigma v}{\ell}$. $\square$

The monic operator L of order n over $\overline{C(z)}$, defined by $\ker(L, \overline{C(z)}) = W := f \cdot \frac{V}{\ell}$ has its coefficients in $C(z)$, since W is invariant under π. This operator L is not yet unique since we have made choices for ℓ and f.

The standard operator L_{st} is defined to be the operator of the form $L_{st} = (\frac{d}{dz})^n + 0 \cdot (\frac{d}{dz})^{n-1} + \cdots$, obtained from the above L by a shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$ for suitable $a = \frac{h'}{h}$ with $h \in \overline{C(z)}^*$.

We finish the proof of Theorem 3.1 by stating the following properties:

(1). L_{st} does not depend on the choices of ℓ and f in Lemma 3.3.

(2). The solution space of L_{st} has the form $g \cdot W$ for certain $g \in \overline{C(z)}^*$.

(3). Let $G \subset \mathrm{SL}(g \cdot W)$ denote the differential Galois group of L_{st}. From $g \cdot W = g \cdot f \cdot \frac{V}{\ell}$ one obtains a natural identification of the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(g \cdot W)$ and after this identification one has $G^{proj} = H$ and the Fano curve of L_{st} is Z.

Statement (1) follows easily from Lemma 3.4 and Observation 3.5, part (1). Statements (2) and (3) follow from the construction of L_{st}. $\square$

Lemma 3.4. *Let L_1, L_2 be monic differential operators over $C(z)$ such that all their solutions are algebraic. Let $V_1, V_2 \subset \overline{C(z)}$ denote the two solution spaces. The following are equivalent:*

(a). *L_1 is obtained from L_2 by a shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$ for some element $a \in C(z)$.*

(b). *There exists $f \in \overline{C(z)}^*$ such that $V_2 = fV_1$.*

Proof. (a)⇒(b). Let L_1 be obtained from L_2 by the shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$. One writes $a = \frac{f'}{f}$ with f in some differential field containing $\overline{C(z)}$. One finds $V_2 = fV_1$. Since $V_1, V_2 \subset \overline{C(z)}$ one actually has $f \in \overline{C(z)}^*$.

(b)⇒(a). If $V_2 = fV_1$, then clearly $L_1 = f^{-1} \circ L_2 \circ f$. Since $f^{-1} \circ \frac{d}{dz} \circ f = \frac{d}{dz} + \frac{f'}{f}$, one has that L_1 is obtained from L_2 by the shift $\frac{d}{dz} \mapsto \frac{d}{dz} + \frac{f'}{f}$. Note that $\frac{f'}{f} \in C(z)$ since L_1 and L_2 are both defined over $C(z)$. $\square$

Observations 3.5. (1). For general monic differential operators L_1, L_2 of order n, property (a) of Lemma 3.4 is called *projective equivalence*. If both L_1 and L_2 have the form $d_z^n + 0 \cdot d_z^{n-1} + \cdots$, then projective equivalence implies equality.
(2). The implication (b)⇒(a) in Lemma 3.4 holds for general differential operators. However (a)⇒(b) is in general false since the equation $f' = af$ with $a \in C(z)$, need not have a solution on $\overline{C(z)}^*$.
(3). For differential modules M_1, M_2 there is a somewhat different notion of projective equivalence defined by: there is a 1-dimensional module E such that $M_1 \otimes E \cong M_2$.
(4). *Projective equivalence* of subgroups $G_1, G_2 \subset \mathrm{GL}(V)$ means that $G_1^{proj} = G_2^{proj} \subset \mathrm{PGL}(V)$. Projective equivalence of operators implies projective equivalence of their differential Galois groups but the converse is false. □

Definition 3.6. Consider a homomorphism $\phi : C(z)[\frac{d}{dz}] \to C(x)[\frac{d}{dx}]$ of the form: $z \mapsto \phi(z) \in C(x) \setminus C$ and $\frac{d}{dz} \mapsto \frac{1}{\phi(z)'}(\frac{d}{dx} + b)$ with $b \in C(x)$. Let $L \in C(z)[\frac{d}{dz}]$. A *weak pullback* of L is an operator of the form $a \cdot \phi(L)$ with $a \in C(x)^*$. The restriction of ϕ to $C(z) \to C(x)$ is called the pullback function.

Proposition 3.7. *Let $L \in C(s)[\frac{d}{ds}]$ be an operator of order n such that all solutions are algebraic and let $M \subset \overline{C(s)}$ denote its solution space. The differential Galois group G of L is a subgroup of* $\mathrm{GL}(M)$.

Suppose that $G^{proj} \subset \mathrm{PSL}(M)$ is irreducible. According to §2, part (5) *and* (6), *L determines some $X_{fano} \subset \mathbb{P}(M)$ and $C(X_{fano})^{G^{proj}}$ is a subfield of $C(s)$. Choose z such that $C(z) = C(X_{fano})^{G^{proj}}$.*

Then L is a weak pullback of the standard operator L_{st} determined by the data M, $H = G^{proj} \subset \mathrm{PSL}(M)$ and $Z = X_{fano} \subset \mathbb{P}(M)$ and the variable z.

Proof. By construction the standard operator L_{st} has solution space $h \cdot W$ for some $h \in \overline{C(z)}^*$, where $W = f\frac{M}{m}$ for suitable $f \in \overline{C(z)}^*$ and $m \in M,\ m \neq 0$. Further, the inclusion $C(z) = C(X_{fano}/G^{proj}) \subset C(s)$ determines the pullback function. Using 3.4 and 3.6 one verifies that this pullback function applied to L_{st} produces L. □

Remark 3.8. A standard differential equation for given $H \subset \mathrm{PSL}(V)$, Fano curve $Z \subset \mathbb{P}(V)$ and variable z can be a proper pullback of another standard equation. This occurs essentially only when H is a proper subgroup of a finite automorphism group of (the desingularization of) Z.

Example 3.9. A calculation of the standard operator L_{st}, using the above construction, is possible. One has to compute the f in Lemma 3.3 and one has to compute the derivation on $C(Z)[f]$ in order to compute the monic differential operator L with solution space $f\frac{V}{\ell} \subset C(Z)[f]$. Further a computation of a generator of $C(Z)^{G^{proj}}$ is needed. However for the case $n = 2$ the calculation is well known ([1,3]) and rather easy. We illustrate this for the case $H = A_4 \subset \mathrm{PSL}_2$ and its preimage $\tilde{H} = A_4^{\mathrm{SL}_2}$ in SL_2.

For $Z = \mathbb{P}^1$ we use homogeneous coordinates x, y and the function field is $C(t)$ with $t = \frac{y}{x}$. According to [5], the invariants under the action of $A_4^{\mathrm{SL}_2}$ are generated by: $Q_3 = xy(x^4 - y^4)$, $Q_4 = (x^4 + \sqrt{-12}x^2y^2 + y^4) \cdot (x^4 - \sqrt{-12}x^2y^2 + y^4)$, $Q_6 = (x^4 + \sqrt{-12}x^2y^2 + y^4)^3 + (x^4 - \sqrt{-12}x^2y^2 + y^4)^3$. There is one relation $Q_6^2 - Q_3^4 - 4Q_4^3 = 0$. The field of the homogeneous invariants of degree zero is generated over C by $\frac{Q_6}{Q_3^2}$ and $\frac{Q_4^3}{Q_3^4}$ and there is one relation $(\frac{Q_6}{Q_3^2})^2 = 1 + 4\frac{Q_4^3}{Q_3^4}$. Hence we can take $z = \frac{Q_6}{Q_3^2}$ where x, y in this expression is replaced by x, tx. This expresses z as rational function in t of degree 12. Thus $\frac{dt}{dz}$ is also known.

Now $V = Cx + Cy$, take $\ell = x$, then $\frac{V}{\ell} = C1 + Ct$. Then $f \in C(t)$ should satisfy $(\frac{x}{\tilde{\sigma}x})^2 = \frac{f}{\sigma f}$. An explicit choice for f turns out to be $\frac{1}{t'}$ where $t' := \frac{dt}{dz}$. Then the Picard–Vessiot field is $C(t)[\sqrt{t'}]$. The operator that we want to compute has solution space $C\frac{1}{\sqrt{t'}} + C\frac{t}{\sqrt{t'}}$. This leads to the standard operator for case A_4. The other standard operators for $n = 2$ can be computed in a similar way. This "classical" calculation fails for $n > 2$ and one needs the new method "evaluation of invariants and Procedure" (see §4,3). This new method will be applied in §5 for another computation of the standard operators for $n = 2$ and for cases with $n > 2$.

Observation 3.10. *The singular points of the standard equations.* Let $L \in C(z)[\frac{d}{dz}]$ be an operator of order n such that its Picard–Vessiot field is a finite extension of $C(z)$. The singular points of L, which are not apparent, are the branch points of $X_{pv} \to \mathbb{P}_z^1$. Indeed, suppose that $z = 0$ is not a branch point, then the solutions of L live at any point p above $z = 0$. The fraction field of $\widehat{O}_{X_{pv},p}$ can be identified with $C((z))$ and contains n independent solutions of L. It follows that the singularity is at most apparent.

In the special case L_{st} and $G = G^{proj} = H$, one can identify X_{pv} with the normalization $\tilde{Z}$ of $Z \subset \mathbb{P}(V)$ and the non apparent singular points are the branch points of $\tilde{Z} \to \mathbb{P}_z^1$. In the general case, the cyclic extension $X_{pv} \to \tilde{Z}$ can be responsible for more singularities of L_{st}.

4. Compoint's theorem and evaluation of invariants

Notation and assumptions:

Suppose that the differential equation $y' = Ay$ over $k = C(z)$ has a reductive differential Galois group $G \subset \mathrm{GL}_n(C)$. The differential algebra $R := k[\{X_{i,j}\}, \frac{1}{D}]$ (with $D = \det(X_{i,j})$) is defined by $(X'_{i,j}) = A \cdot (X_{i,j})$.

Let I be a maximal differential ideal in R and K the Picard–Vessiot field obtained as field of fractions of R/I.

$\mathrm{GL}_n(C)$ acts on the $C(z)$-algebra R by sending the matrix of variables $(X_{i,j})$ to the matrix $(X_{i,j}) \cdot g$ for any $g \in \mathrm{GL}_n(C)$. Then G is identified with the $g \in \mathrm{GL}_n(C)$ such that $gI = I$.

The algebra of invariants $C[\{X_{i,j}\}]^G$ is generated over C by homogeneous elements $f_1, \ldots, f_N$ (since G is reductive). The natural map $R \to K$ induces a homomorphism $ev_e : C[\{X_{i,j}\}]^G \to C(z)$ which is called the *evaluation of the invariants*.

Theorem 4.1 *(E. Compoint 1998). The ideal $I \subset R$ generated by the elements $\{f_1 - ev_e(f_1), \ldots, f_N - ev_e(f_N)\}$ is a maximal differential ideal.*

The proof of Compoint's theorem, [6], has been simplified in [3] and Theorem 4.1 is almost identical to the formulation in [3]. Note that although [3] formulates the result for $C = \mathbb{C}$, the argument is completely algebraic hence the result holds for any C. We will apply Compoint's theorem for the case of finite differential Galois groups. Moreover we will need a formulation in terms of differential operators (or scalar differential equations).

Notation and assumptions:

Let $L = (\frac{d}{dz})^n + a_{n-1}(\frac{d}{dz})^{n-1} + \cdots + a_1\frac{d}{dz} + a_0$ over $C(z)$ have a finite differential Galois group G and Picard–Vessiot field $K \subset \overline{C(z)}$.

Consider the homomorphism $\phi : R_0 = C(z)[X_1, \ldots, X_n] \to K$ which sends $X_1, \ldots, X_n$ to a basis of the solution space of L in K. G acts $C(z)$-linear on R_0 by a C-linear action on $CX_1 + \cdots + CX_n$ which coincides with the action of G (or of π) on the solution space of L.

The restriction of ϕ to $C[X_1, \ldots X_n]^G \to C(z)$ is also called *the evaluation of the invariants* and denoted by *ev* (see also §2). Write $C[X_1, \ldots, X_n]^G = C[\phi_1, \ldots, \phi_r]$ for certain homogeneous elements ϕ_k.

Corollary 4.2. *The kernel of $\phi : C(z)[X_1, \ldots, X_n] \to K$ is generated by the elements $\{\phi_1 - ev(\phi_1), \ldots, \phi_r - ev(\phi_r)\}$.*

Proof. Write again $R_0 = C(z)[X_1, \ldots, X_n]$ and $R := C(z)[\{X_i^j\}_{i=1,\ldots,n}^{j=0,\ldots,n-1}]$ where X_i^j denotes formally the jth derivative of X_i (all i, j). The map $\phi : R_0 \to K$ has a unique extension $\phi_e : R \to K$ defined by $\phi_e(X_i^j) = \phi(X_i)^{(j)}$ (all i, j). The restriction of ϕ to $R_0^G \to C(z)$ is called ev and the restriction of ϕ_e to $R^G \to C(z)$ is called ev_e.

By Compoint's theorem, the ideal $\ker(\phi_e) \subset R$ is generated by the set $\{F - ev_e(F) \mid F \in R^G\}$. We want to prove that the ideal $\ker(\phi) \subset R_0$ is generated by $\{F - ev(F) \mid F \in R_0^G\}$. We will construct a $C(z)$-algebra homomorphism $\Psi : R \to R_0$ which has the following properties: $\Psi(r) = r$ for $r \in R_0$; $\Psi(X_i^0) = X_i$; $\phi \circ \Psi = \phi_e$ and Ψ is G-equivariant.

Consider an element $\xi \in \ker \phi$. Then also $\xi \in \ker \phi_e$ and ξ is a finite sum $\sum c(F) \cdot (F - ev_e(F))$ with $F \in R^G$ and $c(F) \in R$. Applying Ψ to this expression yields $\xi = \sum \Psi(c(F)) \cdot (\Psi(F) - \Psi(ev_e(F))$. Since Ψ is G-equivariant $\Psi(F) \in R_0^G$. Moreover $\Psi(ev_e(F)) = ev(\Psi(F))$. This implies that ξ lies in the ideal generated by the $\{F - ev(F) \mid F \in R_0^G\}$ in the ring R_0.

Construction of Ψ. Define a C-linear derivation $E : R_0 \to R_0$ by $E(z) = 1$ and, for $i = 1, \ldots, n$, $E(X_i) \in R_0$ has the property that $\phi E(X_i) = \phi(X_i)'$. We note that E exists since the map $\phi : R_0 \to K$ is surjective. Then $D := \frac{1}{\#G} \sum_{g \in G} gEg^{-1} : R_0 \to R_0$ is a C-linear derivation with $D(z) = 1$, $\phi(D(X_i)) = \phi(X_i)'$ for all i and D is G-equivariant.

Define the $C(z)$-algebra homomorphism $\Psi : R \to R_0$ by $\Psi(X_i^j) = D^j(X_i)$ for all i, j. The first two properties of Ψ are obvious. Further $\phi(\Psi(X_i^j)) = \phi(D^j(X_i)) = \phi(X_i)^{(j)}$ (for all i, j; the case $j = 1$ given earlier implies the general case) and so $\phi \circ \Psi = \phi_e$. Finally Ψ is G-equivariant because D is G-equivariant and the actions of G on the vector spaces $CX_1^j + \cdots + CX_n^j$, for $j = 0, \ldots, n-1$, are identical. □

The explicit description of the kernel of ϕ given in Corollary 4.2 provides an important step in the computation of a standard operator, as will now be explained.

Procedure 4.3. *Constructing the differential operator from an evaluation.* Let an irreducible finite group $G \subset \mathrm{GL}(C^n)$ be given. The group G acts on $C[X_1, \ldots, X_n]$ by identifying C^n with $\sum CX_j$. Suppose that $C[X_1, \ldots, X_n]^G = C[f_1, \ldots, f_N]$ with known homogeneous elements $f_1, \ldots, f_N$.

Consider a C-algebra homomorphism $h : C[X_1, \ldots, X_n]^G \to C(z)$ such that the image of h generates the field $C(z)$ over C. We will call such h again *an evaluation of the invariants*. The aim is to compute a differential operator $L = d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1 d_z + a_0$ over $C(z)$ that induces the group G and such that the evaluation ev defined above Corollary 4.2 is equal to h.

The $C(z)$-algebra $R := C(z)[x_1, \ldots, x_n] = C(z)[X_1, \ldots, X_n]/I$, where $I = (f_1 - h(f_1), \ldots, f_N - h(f_N))$, has finite dimension over $C(z)$ (by observing that $C(z)[X_1, \ldots, X_n]$ is finite over $C(z)[f_1, \ldots, f_N]$).

(a). *We assume that R is a field.* We note that in the opposite case, R cannot be a Picard–Vessiot field for a suitable operator over $C(z)$. The action of G on $Cx_1 + \ldots + Cx_n$ is induced by the irreducible action of G on $CX_1 + \cdots + CX_n$. Hence either $Cx_1 + \ldots + Cx_n = (0)$ or it is isomorphic to $CX_1 + \cdots + CX_n$. The assumption that the image of h generates $C(z)$ implies that $Cx_1 + \cdots + Cx_n \neq 0$, hence it has dimension n.

The derivation $\frac{d}{dz}$ has a unique extension to R which we call $\tilde{D}$. There is a unique operator $\tilde{L} := \tilde{D}^n + a_{n-1}\tilde{D}^{n-1} + \cdots + a_1\tilde{D} + a_0$ (with all $a_i \in R$) having kernel the n-dimensional vector space $Cx_1 + Cx_2 + \cdots + Cx_n$. By uniqueness and the G-invariance of $Cx_1 + Cx_2 + \cdots + Cx_n$, the operator $\tilde{L}$ is G-invariant and therefore $a_{n-1}, \ldots, a_0 \in R^G = C(z)$. Then $\tilde{L}$ is the differential operator associated to the evaluation h of the invariants.

In order to find $\tilde{L}$ one needs to compute the $\tilde{D}^j x_i$. This is done as follows.

(b). By assumption R is a finite field extension of $C(z)$ and I is a maximal ideal of $C(z)[X_1, \ldots, X_n]$, which is the coordinate ring of the nonsingular variety $\mathbb{A}^n$ over $C(z)$. The well known Jacobian criterion for smoothness implies that the unit ideal of $C(z)[X_1, \ldots, X_n]$ is generated by I and the determinants $\det\left(\frac{\partial(f_j - h(f_j))}{\partial X_i}\right)_{i=1,\ldots,n}^{j \in J}$ where J ranges over the subsets of $\{1, \ldots, N\}$ with $\#J = n$.

Since the elements $h(f_j)$ belong to $C(z)$ we have $\frac{\partial(f_j - h(f_j))}{\partial X_i} = \frac{\partial f_j}{\partial X_i}$. After renumbering we may suppose that $DET = \det\left(\frac{\partial f_j}{\partial X_i}\right)_{i=1,\ldots,n}^{j=1,\ldots,n}$ is non zero. Then $df_1 \wedge \cdots \wedge df_n = DET \cdot dX_1 \wedge \cdots \wedge dX_n$. Thus for $\sigma \in G \subset \mathrm{GL}(C^n)$ one has $\sigma(DET) = \det(\sigma)^{-1} \cdot DET$.

Since G is finite, there exists an integer $m \geq 1$ with $DET^m \in C[X_1, \ldots, X_n]^G$ and $h(DET^m) \in C(z)$. Since R is a field, it follows that $h(DET^m) \neq 0$.

The extension $\tilde{D}$ of $\frac{d}{dz}$ on R lifts to a derivation D on $C(z)[X_1, \ldots, X_n]$ with $D(z) = 1$ and such that $D(I) \subset I$. The lift D is not unique since one can add to each $D(X_i)$ any element in the ideal I.

The condition $D(I) \subset I$ with $I = (f_1 - h(f_1), \ldots, f_N - h(f_N))$ can be rewritten as the following explicit formula

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} \cdot D(X_j) \equiv h(f_i)' \mod I, \text{ for } i = 1, \ldots, N.$$

Since we have assumed that R is a field, $h(DET^m) \neq 0$ and this suffices for the computation of the vector $(DX_1, \ldots, DX_n)^t$ satisfying the equation

$$\left(\frac{\partial f_i}{\partial X_j}\right)(DX_1, \ldots, DX_n)^t = (h(f_1)', \ldots, h(f_n)')^t.$$

Then $D(f_i - h(f_i)) \in I$ for all $i = 1, \ldots, N$ and $D(I) \subset I$. One then computes formulas for D^i, $i = 0, \ldots, n$. From this one deduces a linear combination $L := d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1 d_z + a_0$ such that $L(x_i) = 0$ for all i. This relation is unique since we have assumed that R is a field and we know that $x_1, \ldots, x_n$ are C-linearly independent. It follows that L is G-invariant and all $a_j \in R^G = C(z)$. We conclude:
L is the differential operator associated to the evaluation h. □

Remarks 4.4. (1). We briefly explain why 4.3 is called “Procedure” rather than “Algorithm”. A successful application of the Procedure depends on properties of the evaluation h of the invariants. If the R in 4.3 is known to be a field, then L exists. If h is known to be the evaluation of an operator L, then the Procedure computes L up to (projective) equivalence. For some choices of h the operator L does not exist. It can happen, in the case that R is not a field, that L exists but has a differential Galois group which is a proper subgroup of G (see 4.5).
(2). Suppose that the evaluation of the invariants h produces the operator L. For the change of h into h_λ, given by $h_\lambda(f_i) = \lambda^{\deg f_i} h(f_i)$ for all i and fixed λ such that a power $\lambda^m \in C(z)^*$ for some integer $m \geq 1$, Procedure computes a new operator, namely $\lambda L \lambda^{-1}$. Thus if $L = d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1 d_z + a_0$, then the new operator is obtained from L by the shift $d_z \mapsto d_z - \frac{\lambda'}{\lambda}$ (note that $\frac{\lambda'}{\lambda} \in C(z)$). The evaluations h and h_λ are called *essentially the same*.

Examples 4.5. In some cases, the ideal I of Procedure 4.3 is not a maximal ideal of $C(z)[X_1, \ldots, X_n]$ and therefore R is not be field. We consider, as in Example 3.9, $G = A_4^{\mathrm{SL}_2}$ and $C[x, y]^G = C[Q_3, Q_4, Q_6]$ with the relation $Q_6^2 - Q_3^4 - 4Q_4^3 = 0$. For the evaluations $h_1 : (Q_3, Q_4, Q_6) \mapsto (z, 0, z^2)$ and $h_2 : (Q_3, Q_4, Q_6) \mapsto (0, z^2, 2z^3)$ the above ideal I is not maximal. In both cases, R is a product of a number of copies of the field $C(z)$.

Procedure 4.3 applied to the evaluation h_1 leads to the *first* order differential operator $d_z - \frac{1}{6z}$ instead of a second order operator. This is in accordance with the observation that for suitable $x_0, y_0 \in C^*, x_0 \neq y_0$ one has $Q_3(x_0 z^{1/6}, y_0 z^{1/6}) = z,\ Q_4(x_0 z^{1/6}, y_0 z^{1/6}) = 0,\ Q_6(x_0 z^{1/6}, y_0 z^{1/6}) = z^2$. Moreover the differential Galois group is C_6, the cyclic group of order 6, which can be seen as a subgroup of $A_4^{\mathrm{SL}_2}$.

The Procedure does not produce an operator for h_2. Indeed, Q_3 is a product of six linear forms in the two C-linearly independent solutions x, y and $h_2(Q_3) = 0$ contradicts this linear independence. □

For the existence and construction of an evaluation from a G-invariant curve Z with $C(Z)^G = C(z)$ we will use the following lemma and its proof.

Lemma 4.6. *Let A be a finitely generated graded C-algebra. Assume that A is a domain. Let $A_{((0))}$ denote the subfield of the field of fractions $A_{(0)}$ of A consisting of the homogeneous elements of degree 0.*

Assume that $A_{((0))} = C(z)$. Then there exists a C-algebra homomorphism $h : A \to C[z]$ such that h induces the identification $A_{((0))} = C(z)$.

Proof. Write $A = C[f_1, \ldots, f_r]$ where the $f_1, \ldots, f_r$ are homogeneous elements of degrees $d_1, \ldots, d_r \in \mathbb{Z}_{>0}$. Let $v(i) = (v(i)_1, \ldots, v(i)_r)$ for $i = 1, \ldots, r-1$ denote free generators of $\{(n_1, \ldots, n_r) \in \mathbb{Z}^r \mid \sum n_i d_i = 0\}$. We may and will suppose that the matrix $\{v(i)_j\}_{i,j=1}^{r-1}$ is invertible. Let $m \in \mathbb{Z}_{\neq 0}$ be its determinant.

The elements $\{f_1^{v(i)_1} \cdots f_r^{v(i)_r} \mid i = 1, \ldots, r-1\}$ generate the field $A_{((0))} = C(z)$ over C and thus we can identify $f_1^{v(i)_1} \cdots f_r^{v(i)_r}$ with some $\alpha_i \in C(z)$. First we define $\tilde{h} : A \to \overline{C(z)}$ by $\tilde{h}(f_r) = 1$ and the $\tilde{h}(f_1), \ldots, \tilde{h}(f_{r-1})$ are such that $\tilde{h}(f_1)^{v(i)_1} \cdots \tilde{h}(f_{r-1})^{v(i)_{r-1}} = \alpha_i$ for $i = 1, \ldots, r-1$. One observes that the $\tilde{h}(f_i)$ are Laurent polynomials in $\alpha_1^{1/m}, \ldots, \alpha_{r-1}^{1/m}$. Thus the expressions $\tilde{h}(f_i)$ have the form $R(z) \cdot (z - a_1)^{n_1/m} \cdots (z - a_s)^{n_s/m}$ with $R \in C(z)$, certain distinct $a_1, \ldots, a_s \in C$ and certain integers $n_i \in \{0, \ldots, m-1\}$.

The algebraic relations between the $f_1, \ldots, f_r$ are generated by homogeneous relations. Hence for any expression $\lambda \in \overline{C(z)}^*$ we can consider the C-algebra homomorphism h given as $h(f_i) = \lambda^{d_i} \tilde{h}(f_i)$ for $i = 1, \ldots, r$. Using the shape of the $\tilde{h}(f_i)$ one observes that for suitable λ all $\lambda^{d_i} \tilde{h}(f_i) \in C[z]$. Thus the required h exists and can be seen to be unique (up to constants) under the condition that $\sum_{i=1}^r \deg h(f_i)$ is minimal.

We recall that h and $\tilde{h}$ are "essentially the same" according to 4.4. □

Corollary 4.7. *Let be given an irreducible finite group $G \subset \mathrm{SL}(V)$ and an irreducible G-invariant curve $Z \subset \mathbb{P}(V)$ such that the function field of Z/G is $C(z)$. Lemma 4.6 produces an evaluation of the invariants $h : C[V]^G \to C[z]$ which induces the identification of the function field of Z/G with $C(z)$.*

This evaluation h is essentially the same as the evaluation of the invariants induced by the standard operator L_{st} for the data G and Z (see §3).

Proof. Let $M \subset C[V]$ be the homogeneous prime ideal of $Z \subset \mathbb{P}(V)$. Then $M \cap C[V]^G$ defines the curve Z/G and the homogeneous algebra of Z/G is $A := C[V]^G/(M \cap C[V]^G)$. Now one applies Lemma 4.6 to A. The last statement follows from the unicity of h up to a change $h(f_i) \mapsto \lambda^{\deg f_i} h(f_i)$ for $i = 1, \ldots, r$. □

5. Computations with Procedure 4.3

In Sections 5.1-5.3 we present in concrete cases the differential operator obtained from Procedure 4.3. The evaluations used are computed as in the proof of Lemma 4.6.

5.1. Finite subgroups of SL_2

For finite subgroups of SL_2 and their invariants we use the notations and equations from [5]. The standard equations for the subgroups D_n, A_4, S_4, A_5 of PSL_2 are classical and well known, see for instance [1,2].
(1). The group $D_n^{\mathrm{SL}_2}$ of order $4n$ is generated by $\left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with $\zeta = e^{2\pi i/2n}$. The semi-invariants are generated by $f_3 = xy$, $f_{12} = x^{2n} + y^{2n}$, and $f_{13} = x^{2n} - y^{2n}$. The invariants have generators

$$F_1 = f_3 f_{13},\ F_2 = f_{12},\ F_3 = f_3^2 \text{ and relation } F_1^2 - F_2^2 F_3 + 4F_3^{n+1} = 0.$$

This leads to the following evaluations.

If n is odd, $A_{((0))}$ is generated by $\frac{F_1}{F_3^{(n+1)/2}}, \frac{F_2}{F_3^n}$ with relation $(\frac{F_1}{F_3^{(n+1)/2}})^2 = \frac{F_2}{F_3^n} - 4$. Define z by $\frac{F_1}{F_3^{(n+1)/2}} = 2iz$. This gives $\tilde{h}\colon (F_1, F_2, F_3) \mapsto (2iz, 2i(z^2-1)^{1/2}, 1)$ and $h\colon (F_1, F_2, F_3) \mapsto (2iz(z^2-1)^{(n+1)/2}, 2i(z^2-1)^{(n+1)/2}, (z^2-1))$.

For $2|n$, generators of $A_{((0))}$ are $\frac{F_1^2}{F_3^{n+1}}, \frac{F_2}{F_3^{n/2}}$ satisfying $\frac{F_1^2}{F_3^{n+1}} = (\frac{F_2}{F_3^{n/2}})^2 - 4$. Corresponding evaluations are $\tilde{h}\colon (F_1, F_2, F_3) \mapsto (2(z^2-1)^{1/2}, 2z, 1)$ and $h\colon (F_1, F_2, F_3) \mapsto (2(z^2-1)^{1+n/2}, 2z(z^2-1)^{n/2}, (z^2-1))$.

For all n the differential operator is $L = d_z^2 + \frac{z}{z^2-1} d_z - \frac{1}{4n^2(z^2-1)}$. This becomes, after the transformation $z \mapsto 2z-1$, *the standard equation*

$$(\frac{d}{dz})^2 + \frac{3}{16z^2} + \frac{3}{16(z-1)^2} - \frac{n^2+2}{8n^2 z(z-1)} \quad \text{for } D_n.$$

(2). For the group $A_4^{\mathrm{SL}_2}$, we continue the discussion from Example 3.9. Generators for the invariants are the homogeneous polynomials Q_3, Q_4, Q_6 of degrees $6, 8, 12$ with relation $Q_6^2 = Q_3^4 + 4Q_4^3$. Here $A_{((0))} = C(\frac{Q_6}{Q_3^2}, \frac{Q_4^3}{Q_3^4}) = C(z)$ with $z = \frac{Q_6}{Q_3^2}$. This leads to the evaluations $\tilde{h}\colon (Q_3, Q_4, Q_6) \mapsto (1, (\frac{z^2-1}{4})^{1/3}, z)$ and $h\colon (Q_3, Q_4, Q_6) \mapsto ((\frac{z^2-1}{4})^2, (\frac{z^2-1}{4})^3, (\frac{z^2-1}{4})^4 z)$.

The differential operator is $d_z^2 + \frac{27z^2+101}{144(z^2-1)^2}$. This becomes after $z \mapsto 2z-1$ *the standard equation*

$$(\frac{d}{dz})^2 + \frac{3}{16z^2} + \frac{2}{9(z-1)^2} - \frac{3}{16z(z-1)} \quad \text{for } A_4.$$

(3). The group $S_4^{\mathrm{SL}_2}$ has ring of invariants $A := C[F_1, F_2, F_3]$ with generators F_j of degrees $12, 8, 18$, respectively. One finds $A_{((0))} = C(\frac{F_2^3}{F_1^2}, \frac{F_3^2}{F_1^3})$ with relation $\frac{F_2^3}{F_1^2} = \frac{F_3^2}{F_1^3} + 108$. Put $\tilde{h}\colon (F_1, F_2, F_3) \mapsto (1, 3 \cdot 2^{2/3} z, 2 \cdot 3^{3/2}(z^3-1)^{1/2})$ and $h\colon (F_1, F_2, F_3) \mapsto (2^2 3^3 (z^3 - 1)^3, 2^2 3^3 z(z^3-1)^2, 2^4 3^6 (z^3-1)^5)$.
The differential operator is $d_z^2 + \frac{(7z^3+101)z}{64(z^2+z+1)^2(z-1)^2}$. The equation has 4 singular points and is a pullback of the standard equation. Note that our choice of evaluation is not 'minimal', i.e., the map induced by $\tilde{h}$ and h from $A_{((0))}$ to $C(z)$ has image $C(z^3)$. The operator is a pullback of *the standard operator*

$$(\frac{d}{dz})^2 + \frac{3}{16z^2} + \frac{2}{9(z-1)^2} - \frac{101}{576z(z-1)} \quad \text{for } S_4.$$

(4). The group $A_5^{\mathrm{SL}_2}$ has ring of invariants $A := C[f_9, f_{10}, f_{11}]$, generators of degree $30, 20, 12$, respectively, with relation $f_9^2 + f_{10}^3 - 1728 f_{11}^5 = 0$. In the present case $A_{((0))} = C(\frac{f_9^2}{f_{11}^5}, \frac{f_{10}^3}{f_{11}^5})$ with $\frac{f_9^2}{f_{11}^5} = -\frac{f_{10}^3}{f_{11}^5} + 1728$. This leads to the evaluations $\tilde{h}\colon (f_9, f_{10}, f_{11}) \mapsto (-12^{3/2}(z-1)^{1/2}, 12 \cdot z^{1/3}, 1)$ and

$$h\colon (f_9, f_{10}, f_{11}) \mapsto (12^9 z^{10}(z-1)^8, -12^6 z^7 (z-1)^5, -12^3 z^4 (z-1)^3).$$

The differential operator is $d_z^2 + \frac{864z^2-989z+800}{3600z^2(z-1)^2}$. After $z \mapsto 1-z$, this becomes *the standard operator*

$$(\frac{d}{dz})^2 + \frac{3}{16z^2} + \frac{2}{9(z-1)^2} - \frac{611}{3600z(z-1)} \quad \text{for } A_5.$$

5.2. $G = G_{168} \subset \mathrm{SL}_3$

5.2.1. Computation of the differential equation related to Klein's quartic

For the unique simple group $G \subset \mathrm{SL}(3, \mathbb{C})$ of order 168 we use notations and formulas of [2, p. 50]. Here $C[X_1, X_2, X_3]^G = C[F_4, F_6, F_{14}, F_{21}]/(rel)$, where F_4, F_6, F_{14}, F_{21} are of degrees $4, 6, 14, 21$. The Klein quartic $Z \subset \mathbb{P}^2$ is given by $F_4 = 0$ with $F_4 := 2(X_1X_2^3 + X_2X_3^3 + X_3X_1^3)$.

Unlike the case of finite irreducible subgroups of SL_2 (compare Example 3.9), a direct computation of the standard operator for these data with the methods of Section 3 meets difficulties. How to compute $f \in C(Z)^*$ such that $\frac{\sigma(f)}{f} = \frac{\sigma X_1}{X_1}$ for all $\sigma \in G$? How to compute the derivatives w.r.t. $d_t = \frac{d}{dt}$ of a basis of the solution space $W = < f, fX_2/X_1, fX_3/X_1 >$?

We now use the methods of Section 4. The graded algebra of Z/G is

$$\{C[X_1,X_2,X_3]/(F_4)\}^G = C[F_6,F_{14},F_{21}]/(F_{21}^2-4F_{14}^3-54F_6^7);$$

the field $A_{((0))}=C(Z/G)$ equals $C(\frac{F_{21}^6}{F_6^{21}},\frac{F_{14}^3}{F_6^7})$ with relation $\frac{F_{21}^6}{F_6^{21}}=(4\frac{F_{14}^3}{F_6^7}+54)^3$. Hence $A_{((0))}=C(t)$ with $t=\frac{F_{14}^3}{F_6^7}$. A resulting evaluation is

$$h\colon (F_4,F_6,F_{14},F_{21})\mapsto (0,t^2(4t+54)^3,t^5(4t+54)^7,t^7(4t+54)^{11}).$$

Now Procedure 4.3 leads to an operator S_0 with singularities $t=0,-\frac{27}{2},\infty$. Its local exponents are $1,2/3,1/3||1,1/2,3/2||-3/7,-5/7,-6/7$.

The change $t=-\frac{27}{2}z$ (hence $d_t=-\frac{2}{27}d_z$) moves the singularities to $0,1,\infty$, with the same local exponents. The corresponding operator is

$$S_1 := d_z^3+\frac{1}{z}d_z^2+\frac{72z^2+61z+56}{252z^2(z-1)^2}d_z-\frac{6480z^3+3945z^2+13585z-5488}{24696z^3(z-1)^3}.$$

The conjugate $S_2:=z^{-1}(z-1)^{-1}S_1z(z-1)$ has the "classical" local exponents and coincides with the formulas in the literature [12,23,18]:

$$S_2=d_z^3+\frac{7z-4}{z(z-1)}d_z^2+\frac{2592z^2-2963z+560}{252z^2(z-1)^2}d_z+\frac{\frac{72\cdot 11}{7^3}z-\frac{40805}{24696}}{z^2(z-1)^2}.$$

5.2.2. The Hessian of the Klein quartic

The Hessian is the G-invariant curve $Z\subset\mathbb{P}^2$ with equation $F_6=0$. The graded algebra of Z/G is $C[F_4,F_{14},F_{21}]/(F_{21}^2-4F_{14}^3+8F_{14}F_4^7)$ and $C(Z)^G=C(t)$ with $t=\frac{F_{14}^2}{F_4^7}$. A resulting evaluation is

$$h\colon (F_4,F_6,F_{14},F_{21})\mapsto (t^3(t-2)^2,0,t^{11}(t-2)^7,2t^{16}(t-2)^{11}).$$

Procedure 4.3 then yields (after a change of variables) the operator

$$d_z^3+\frac{3(3z-2)}{2z(z-1)}d_z^2+\frac{3(116z-35)}{112z^2(z-1)}d_z+\frac{195}{2744z^2(z-1)}.$$

5.2.3. More third order operators with group $G=G_{168}$

The third order operators over $C(z)$, or more precisely, the differential modules of dimension 3, with singular points $0,1,\infty$ and differential Galois group G are classified in [18], using the "transcendental" Riemann–Hilbert correspondence. Each case is given by a branch type $[e_0,e_1,e_\infty]$ and a choice of one of the two irreducible characters χ_2,χ_3 of dimension 3. The LIST is:

$[2,3,7]$, *1 case*, $g=3$; $[2,4,7]$, *1 case*, $g=10$; $[2,7,7]$, *1 case*, $g=19$;
$[3,3,4]^*$, *2 cases*, $g=8$; $[3,3,7]$, *1 case*, $g=17$; $[3,4,4]^*$, *1 case*, $g=15$;

$[3,4,7]^*$, *2 cases,* $g = 24$*;* $[3,7,7]$*, 2 cases,* $g = 33$*;* $[4,4,4]^*$*, 2 cases,* $g = 22$*;* $[4,4,7]$*, 1 case,* $g = 31$*;* $[4,7,7]^*$*, 2 cases,* $g = 40$*;* $[7,7,7]$*, 1 case,* $g = 49$*.*
For many cases in LIST these data lead to a computation of the third order operator. The cases where this fails are indicated by a $*$.

In general the Fano curve corresponding to an element in LIST is not explicitly known. If one can identify for an item in LIST the G-invariant (Fano) curve $Z \subset \mathbb{P}^2$, this results in an evaluation and via Procedure 4.3 in a computation of the desired differential operator. $[2,3,7],[2,4,7]$ in LIST correspond to $F_4 = 0$ and $F_6 = 0$. [2] considered smooth G-invariant $Z \subset \mathbb{P}^2$ with quotient of genus 0 and did not find new examples.

We extend his search and consider the (singular) curves $aF_4^3 + F_6^2 = 0$.
If such a curve $Z = Z_a$ leads to an evaluation $\tilde{h}$ with $\tilde{h}(F_4) = 1$ and $\tilde{h}(F_6) = \lambda$ (so $\lambda^2 = -a$) and $\tilde{h}(F_{14}) = t$, then

$$h(F_{21})^2 = 4t^3 - 44\lambda t^2 + (126\lambda^4 + 68\lambda^2 - 8)t + 54\lambda^7 - 938\lambda^5 + 172\lambda^3 - 8\lambda.$$

The discriminant of this polynomial in t equals $-64(27\lambda^2-2)^3(\lambda^2+2)^4$, so $\lambda = (-2)^{1/2}$ and $\lambda = (2/27)^{1/2}$, or $a = 2, a = -2/27$ are special. Note that if the discriminant is nonzero then the quotient map from Z_a would have at least 5 branch points. Both special cases lead to quotient maps with exactly 3 branch points. In fact $Z_{-2/27}$ is birational to the Klein quartic (of genus 3), and Z_2 is birational to the curve given by $F_6 = 0$.

For $\lambda = (-2)^{1/2}$ one finds $\tilde{h}(F_{21}) = 2(-t+9\sqrt{-2})(t+7\sqrt{-2})^{1/2}$ and for $\lambda = (2/27)^{1/2}$ we have $\tilde{h}(F_{21}) = \frac{-2\sqrt{3}}{243}(27t+\sqrt{6})(-27t+35\sqrt{6})^{1/2}$. Using 4.3 the corresponding operators are found. The operators have three singular points and the solutions are generalized hypergeometric functions. We remark that the above "Fricke pencil" Z_a was also studied by M. Kato (see [13, Prop. 2.3]), using Schwarz maps. In a rather different way than our's he found the two special cases as well as the corresponding third order differential operators.

5.2.4. Computing the evaluation for differential operators in LIST

An element in LIST is given by a topological covering of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ with group $G = G_{168}$, produced by a triple $g_0, g_1, g_\infty \in G$ satisfying $g_0 g_1 g_\infty = 1$ and generating G. One may hope that from a given triple one can read off a part of an evaluation h of the operator, namely the orders of the functions $h(F_4), h(F_6), h(F_{14}), h(F_{21})$ at the points $0, 1, \infty$.

In a number of cases knowledge of these orders together with the relation between the four invariants suffices to compute a suitable h.

We illustrate this for the item $[2,4,7]$ *in LIST*:
Let x, y, z denote a basis of solutions for the differential equation we try to compute. As F_4, F_6, F_{14}, F_{21} are explicit expressions in x, y, z, and one has (by [18, §5.2]) lower bounds $-\frac{1}{2}, -\frac{3}{4}, \frac{8}{7}$ for the local exponents at $t = 0, 1, \infty$, one deduces

$$(h(F_4), h(F_6), h(F_{14}), h(F_{21})) = \left(\frac{f_4}{t^2(t-1)^3}, \frac{f_6}{t^3(t-1)^4}, \frac{f_{14}+g_{14}t}{t^7(t-1)^{10}}, \frac{f_{21}(t+2400)}{t^{10}(t-1)^{15}}\right)$$

for constants $f_4, f_6, f_{14}, g_{14}, f_{21}$ (unique up to an appropriate scaling). The relation between the F_j's yields $(f_4, f_6, f_{14}, g_{14}, f_{21}) = (\frac{-7}{4}, \frac{-3}{4}, \frac{-149}{8}, \frac{1}{4}, \frac{1}{8})$.

Evaluations for several items in LIST. The same idea used for $[2,4,7]$ above, results in evaluations for various other items in LIST. The next table presents the results. The first row gives the branch type and the rational functions $h(F_4), h(F_6), h(F_{14}), h(F_{21})$. The second row lists the local exponents at $0, 1, \infty$ and the accessory parameter μ (see [18, § 5.1]). The operator is uniquely determined by these data.

- $[2,3,7]$ $\quad 0, \dfrac{-3^3}{t^3(t-1)^4}, \dfrac{2^2 3^8}{t^7(t-1)^9}, \dfrac{2^3 3^{12}}{t^{10}(t-1)^{14}}$
  $-\dfrac{1}{2}, 0, \dfrac{1}{2} \mid\mid -\dfrac{1}{3}, -\dfrac{1}{3}, 0 \mid\mid \dfrac{8}{7}, \dfrac{9}{7}, \dfrac{11}{7} \mid\mid \dfrac{12293}{24696}.$
- $[2,4,7]$ $\quad \dfrac{-7}{4t^2(t-1)^3}, -\dfrac{3}{4t^3(t-1)^4}, -\dfrac{(-149+2t)}{8t^7(t-1)^{10}}, \dfrac{(t+2400)}{8t^{10}(t-1)^{14}}$
  $-\dfrac{1}{2}, 0, \dfrac{1}{2} \mid\mid -\dfrac{3}{4}, -\dfrac{1}{4}, 0 \mid\mid \dfrac{8}{7}, \dfrac{9}{7}, \dfrac{11}{7} \mid\mid \dfrac{5273}{10976}.$
- $[2,7,7]$ $\quad \dfrac{14}{t^2(t-1)^3}, \dfrac{3}{t^2(t-1)^5}, \dfrac{4(-294+294t+t^2)}{t^6(t-1)^{12}}, \dfrac{8(t-2)(t^2-9604t+9604)}{t^9(t-1)^{18}}$
  $-\dfrac{1}{2}, 1, \dfrac{1}{2} \mid\mid -\dfrac{6}{7}, -\dfrac{5}{7}, -\dfrac{3}{7} \mid\mid \dfrac{8}{7}, \dfrac{9}{7}, \dfrac{11}{7} \mid\mid \dfrac{1045}{686}.$
- $[3,3,7]$ $\quad 0, -\dfrac{2^4 3^3}{t^4(t-1)^4}, \dfrac{2^{12} 3^8}{t^9(t-1)^9}, \dfrac{2^{17} 3^{12}(1-2t)}{t^{14}(t-1)^{14}}$
  $-\dfrac{2}{3}, -\dfrac{1}{3}, 0 \mid\mid -\dfrac{2}{3}, -\dfrac{1}{3}, 0 \mid\mid \dfrac{9}{7}, \dfrac{11}{7}, \dfrac{15}{7} \mid\mid 0.$
- $[3,7,7]$ $\quad 0, \dfrac{3^3}{t^4(t-1)^5}, \dfrac{3^8(9t-8)}{t^9(t-1)^{12}}, \dfrac{3^{12}(27t^2-36t+8)}{t^{14}(t-1)^{18}}$
  $-\dfrac{2}{3}, -\dfrac{1}{3}, 0 \mid\mid -\dfrac{6}{7}, -\dfrac{5}{7}, -\dfrac{3}{7} \mid\mid \dfrac{10}{7}, \dfrac{13}{7}, \dfrac{19}{7} \mid\mid \dfrac{830}{1029}.$
- $[4,4,7]$ $\quad \dfrac{-14}{t^3(t-1)^3}, \dfrac{-12}{t^4(t-1)^4}, \dfrac{2^5(8t^2-8t-147)}{t^{10}(t-1)^{10}}, \dfrac{2^9(2t-1)(4t^2-4t+2401)}{t^{15}(t-1)^{15}}$
  $-\dfrac{3}{4}, -\dfrac{1}{4}, 0 \mid\mid -\dfrac{3}{4}, -\dfrac{3}{4}, 0 \mid\mid \dfrac{9}{7}, \dfrac{11}{7}, \dfrac{15}{7} \mid\mid 0.$
- $[7,7,7]$ $\quad \dfrac{16}{t^3(t-1)^3}, \dfrac{512t^2-512t+5}{16t^5(t-1)^5}, \dfrac{P_6(t)}{2^8 t^{12}(t-1)^{12}}, \dfrac{(2t-1)P_8(t)}{2^{12} t^{18}(t-1)^{18}}$
  $-\dfrac{6}{7}, -\dfrac{5}{7}, -\dfrac{3}{7} \mid\mid -\dfrac{6}{7}, -\dfrac{5}{7}, -\dfrac{3}{7} \mid\mid \dfrac{9}{7}, \dfrac{11}{7}, \dfrac{29}{7} \mid\mid 0$
  where $P_6(t) = 2^{20}t^5(t-3) + 2^{15}t^3(385t-610) + 2^7 t(74441t-457) + 1$ and $P_8(t) = 2^{29}t^7(t-4) - 2^{23}t^5(8869t-27503) - 2^{15}t^3(7074623t-2338174) + 2^8 t(1963429t-5413) - 1$.

Remarks 5.1. (1). For the remaining items in LIST the approach above does not determine h (up to equivalence).

(2). For the case $[3,4,4]$ a *choice* of h leads to a proper subgroup of G (compare [18, §8.2.1 part (6)]).

(3). The items $[3, 3, 7]$ and $[3, 7, 7]$ correspond to weak pullbacks of the "standard" case $[2, 3, 7]$, with pullback functions $\phi(t) = 4t(t + 1) + 1$ and $\phi(t) = -\frac{(27t^2-36t+8)^2}{t-1}$, respectively.
(4). The Fano curve for $[2, 4, 7]$ is given by the equation $-\frac{7}{54}F_6^3 - \frac{1}{8}F_4F_{14} + F_4^3F_6 = 0$ and it has genus 10. $[2, 7, 7]$ and $[4, 4, 7]$ are weak pullbacks of the "standard" case $[2, 4, 7]$, with pullback functions $\phi(t) = \frac{-(t-1)^2}{4(t-1)}$ and $\phi(t) = (2t - 1)^2$, respectively. The Fano curves for these two cases can be obtained from the above one by the same pullbacks.

5.3. $H = H_{72}^{\mathrm{SL}_3} \subset \mathrm{SL}_3$

The group $H = H_{72}^{\mathrm{SL}_3} \subset \mathrm{SL}_3$ has order 216 and together with its invariants it is described in [5, p. 59] and in [19, Thm. 4]. We use the latter and write

$$P = xyz,\ Q = x^3y^3 + x^3z^3 + y^3z^3,\ S = x^3 + y^3 + z^3,$$

$$F_1 = S^2 - 12Q, F_2 = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3), F_3 = S^4 + 216P^3S,$$

$$F_4 = (S^2 - 18P^2 - 6PS)^2.$$

The algebra of invariants is $C[x, y, z]^H = C[F_1, F_2, F_3, F_4]$ with relation $(432F_2^2+3F_1F_3-F_1^3)^2 - 4(F_4^3 - 3F_4^2F_3 + 3F_4F_3^2) = 0$.

Let $Z \subset \mathbb{P}^2$ be given by $F_1 = 0$. The graded algebra of Z/H is $A = C[F_2, F_3, F_4]/(216^2F_2^4 - F_4^3 + 3F_4^2F_3 - 3F_4F_3^2))$. Then $A_{((0))} = C(Z)^H$ equals $C(\frac{F_2^4}{F_3^3}, \frac{F_4}{F_3})$ and $216^2\frac{F_2^4}{F_3^3} = (\frac{F_4}{F_3})^3 - 3(\frac{F_4}{F_3})^2 + 3\frac{F_4}{F_3}$, so $A_{((0))} = C(\frac{F_4}{F_3})$. This yields the evaluation $(F_1, F_2, F_3, F_4) \mapsto (0, \sqrt[4]{\frac{t^3-3t^2+3t}{6^6}}, 1, t)$ which by scaling simplifies to $(F_1, F_2, F_3, F_4) \mapsto (0, t^3 - 3t^2 + 3t, 36t(t^2 - 3t + 3), 36t^2(t^2 - 3t + 3))$. From this, Procedure 4.3 yields the differential operator

$$d_t^3 + \frac{5t^3 - 15t^2 + 15t - 6}{(t^3 - 3t^2 + 3t)(t - 1)}d_t^2 + \frac{(160t^3 - 480t^2 + 480t - 117)(t - 1)}{48(t^3 - 3t^2 + 3t)^2}d_t - \frac{(160t^3 - 480t^2 + 480t - 189)(t - 1)^3}{432(t^3 - 3t^2 + 3t)^3}.$$

One observes that $t = 1$ is an apparent singularity and that ∞ and the three zeros of $t^3 - 3t^2 + 3t$ are the singular points.

At present no differential equation over $C(t)$ with three singularities and Galois group H seems to be known. Another differential equation of order 3 with Galois group H was found by M. van Hoeij, see [10, § 2].

6. Differential equations for $A_5 \subset \mathrm{SL}_3$

Consider $A_5 \subset \mathrm{SL}_3(\mathbb{Q}(\zeta_5)) \subset \mathrm{SL}_3(\mathbb{C})$ (with $\zeta_5 = e^{2\pi i/5}$) the group with generators

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^{-1} \end{pmatrix}, \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\ 1 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2} \end{pmatrix}.$$

This inclusion corresponds to the irreducible character χ_2 of dimension 3 for A_5. The other irreducible character χ_3 of dimension 3 is obtained via the automorphism $\zeta_5 \mapsto \zeta_5^2$ of $\mathbb{Q}(\zeta_5)/\mathbb{Q}$. Denote by $A_5^{\mathrm{SL}_2} \subset \mathrm{SL}_2(\mathbb{C})$ the preimage of A_5 under the "symmetric square map" $\mathrm{SL}_2 \to \mathrm{SL}_3 \colon A \mapsto \mathrm{sym}^2 A$. The group $A_5^{\mathrm{SL}_2}$ (called the 'icosahedral group', of order 120) has two irreducible characters of dimension 2 and their second symmetric powers are the above 3-dimensional characters χ_2, χ_3 of A_5. The following proposition may be known, but we found no proof in the literature. The argument presented here relates to [16, Theorem 2.1]. We also offer a second proof using the solution of an embedding problem.

Proposition 6.1. (Comparing differential modules for A_5 and $A_5^{\mathrm{SL}_2}$).
(1). *Suppose that the 3-dimensional differential module M over $\mathbb{C}(z)$ has differential Galois group A_5. Then there is a 2-dimensional differential module N with differential Galois group $A_5^{SL_2}$ such that $\mathrm{sym}^2 N$ is isomorphic to M.*
(2). *The module N is unique up to tensoring with a 1-dimensional module D such that $D^{\otimes 2}$ is the trivial differential module* $\mathbf{1}$.

First proof of (1). The action of A_5 on the solution space W of M induces an action on $\mathrm{sym}^2 W$. It has an invariant line (which can be interpreted as an invariant quadratic form on W); this corresponds to a 1-dimensional submodule T of $\mathrm{sym}^2(M)$. In terms of a basis of M, a nonzero element of T is a nondegenerate quadratic form. This form has a nontrivial zero over $\mathbb{C}(z)$ since the latter is a C_1-field. Hence there is a basis x_1, x_2, x_3 of M such that T is generated by $x_1x_3 - x_2^2$. Moreover T is the trivial module since A_5 is simple. For some $q \in \mathbb{C}(z)^*$ one has $\partial(q(x_1x_3 - x_2^2)) = 0$.

The equation $\frac{1}{q}\partial(q(x_1x_3 - x_2^2)) = 0$ implies that the matrix A of ∂ w.r.t. the basis x_1, x_2, x_3 of M has the form $\begin{pmatrix} a_1 & b_1 & 0 \\ 2b_3 & \frac{-q'}{2q} & 2b_1 \\ 0 & b_3 & -a_1 - \frac{q'}{q} \end{pmatrix}$. Since A_5 is simple, $\det M = \mathbf{1}$. This implies that the equation $y' = tr(A)y$ has a solution in $\mathbb{C}(z)$. Therefore $q^{-3/2} \in \mathbb{C}(z)$ and thus q is a square.

After changing the basis of M one has $q = 1$. Now $\partial(x_1x_3 - x_2^2) = 0$ implies that the matrix of ∂ with respect to the basis $\{x_1, x_2, x_3\}$ has the form $\begin{pmatrix} 2a & b & 0 \\ 2c & 0 & 2b \\ 0 & c & -2a \end{pmatrix}$ for certain a, b, c. Consider the 2-dimensional module N and a basis y_1, y_2 such that the matrix of ∂ is $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Then $\text{sym}^2(N)$ has on basis $x_1 = y_1^2,\ x_2 = y_1y_2,\ x_3 = y_2^2$ the above matrix. Thus $\text{sym}^2 N \cong M$. The differential Galois group $G \subset \text{SL}_2$ of N has the property that its image $\text{sym}^2(G)$ in SL_3 equals A_5. Hence the action of G on $\mathbb{P}^1$ is that of A_5 and so $G = A_5^{\text{SL}_2}$.

Second proof of (1). The equivalence of Tannaka categories $\text{Diff}_{\overline{k}/k} \to Repr_\pi$, with $k = \mathbb{C}(z)$, $\pi = \text{Gal}(\overline{k}/k)$, translates 6.1(1) into solvability of the embedding problem for $1 \to \{\pm 1\} \to A_5^{\text{SL}_2} \to A_5 \to 1$, namely:

> *Any continuous surjective homomorphism $\pi \to A_5$ lifts to a continuous surjective homomorphism $\pi \to A_5^{SL_2}$.*

According to [15, Thm. 1.10(a)], this holds for $k = \mathbb{C}(z)$.

Proof of (2). Let the 1-dimensional module D satisfy $D^{\otimes 2} = \mathbf{1}$. Then $\text{sym}^2(D \otimes N) \cong D^{\otimes 2} \otimes \text{sym}^2(N) = M$. This proves one implication for part (2) of 6.1. Using the equivalence of Tannaka categories, part (2) translates into:
Given (for $i = 1, 2$) surjective continuous homomorphisms $\rho_i \colon \pi \to A_5^{\text{SL}_2}$ with $\text{sym}^2(\rho_1) \cong \text{sym}^2(\rho_2)$. Then a continuous homomorphism $\chi \colon \pi \to \{\pm 1\} \subset A_5^{\text{SL}_2}$ exists such that $\rho_1 \cong \rho_2 \otimes \chi$.

Let $can \colon \text{SL}_2 \to \text{PSL}_2$ be the canonical map. The assumption on ρ_1, ρ_2 is equivalent to $can \circ \rho_1 \cong can \circ \rho_2$. Since $A_5 \subset \text{PSL}_2$ is unique up to conjugation, we may suppose $can \circ \rho_1(g) = can \circ \rho_2(g)$ for every $g \in \pi$. Hence $\rho_1(g) = \chi(g)\rho_2(g)$ for some continuous homomorphism $\chi : \pi \to \{\pm 1\}$. □

Let $L_{st, A_5^{\text{SL}_2}}$ denote the standard second order operator for $A_5^{\text{SL}_2}$ with local exponents $1/4, 3/4 || 1/3, 2/3 || -2/5, -3/5$ (see §5.1 (4)). Define the standard operator L_{st, A_5} to be the second symmetric power of $L_{st, A_5^{\text{SL}_2}}$.

Proposition 6.2. *Every third order operator L over $C(z)$ with differential Galois group A_5 is equivalent to a weak pullback of L_{st, A_5}.*

Proof. This follows from Proposition 6.1 and Klein's theorem for second order equations with group $A_5^{\text{SL}_2}$. Indeed, the given L equals $\text{sym}^2(L_2)$ for a second order operator L_2 with differential Galois group $A_5^{\text{SL}_2}$. According to Klein's theorem L_2 is a weak pullback of the standard operator $L_{st, A_5^{\text{SL}_2}}$. Taking symmetric squares the result follows. □

Remarks 6.3. *Differential operators for A_5, $A_5^{\text{SL}_2}$ and the data of* [18]. (1). [18] lists all third order differential operators (up to equivalence) with differential Galois group A_5 and singular points $0, 1, \infty$. The branch types are $[2, 3, 5], [2, 5, 5], [3, 3, 5], [3, 5, 5](1)$, $[3, 5, 5](2), [5, 5, 5]$. For each type there are two differential modules; one for each of the

3-dimensional irreducible characters χ_2, χ_3. The genera for the Picard–Vessiot fields are $0, 4, 5, 9, 9, 13$.

One discovers, by comparing the genera, that each A_5 case is the second symmetric power of two, three or four second order equations with group $A_5^{\mathrm{SL}_2}$ and singularities $0, 1, \infty$ (also given as a list in [18]). This is explained by Proposition 6.1 and the observation that there are three 1-dimensional modules D with $D^{\otimes 2} = \mathbf{1}$ and singular points $0, 1, \infty$. Namely $D = C(z)e$ with $\partial e = ae$ and $a \in \{\frac{1}{2z}, \frac{1}{2(z-1)}, \frac{1}{2z(z-1)}\}$.
(2). Comparing the local exponents for $A_5^{\mathrm{SL}_2}$ and A_5 in both lists of [18] one further discovers that only for the two cases of [3,3,5] the operator L_3 is a second symmetric power. In all other cases the module M is a $\mathrm{sym}^2(N)$ but this does not hold for the operators.

Examples 6.4. *Invariants, evaluations and differential operators for A_5.*
(1). Generators for the ring $C[x, y, z]^{A_5}$ are, according to [5],

$$F_2 = x^2 + yz,\ F_6 = 8x^4yz - 2x^2y^2z^2 - x(y^5 + z^5) + y^3z^3; F_{10} = 320x^6y^2z^2 - 160x^4y^3z^3 + 20x^2y^4z^4 + 6y^5z^5 - 4x(y^5 + z^5)(32x^4 - 20x^2yz + 5y^2z^2) + y^{10} + z^{10}; F_{15}.$$

There is one relation (which determines F_{15} up to sign)

$$F_{15}^2 + 1728F_6^5 - F_{10}^3 - 720F_2F_6^3F_{10} + 80F_2^2F_6F_{10}^2 - 64F_2^3(-F_{10}F_2 + 5F_6^2)^2 = 0.$$

(2). *The evaluation for the A_5-invariant curve $Z \subset \mathbb{P}^2$ given by $F_2 = 0$.*

The graded algebra for Z/A_5 is $A = C[F_6, F_{10}, F_{15}]/(F_{15}^2 + 1728F_6^5 - F_{10}^3)$, hence $A_{((0))} = C(Z)^{A_5}$ equals $C(\frac{F_{15}^2}{F_6^5}, \frac{F_{10}^3}{F_6^5})$ with $\frac{F_{15}^2}{F_6^5} + 1728 - \frac{F_{10}^3}{F_6^5} = 0$. This leads to the evaluations $\tilde{h}\colon (F_2, F_6, F_{10}, F_{15}) \mapsto (0, 1, t^{1/3}, (t-1728)^{1/2})$ and $h\colon (F_2, F_6, F_{10}, F_{15}) \mapsto (0, t^4(t-1728)^3, t^7(t-1728)^5, t^{10}(t-1728)^8)$.

The third order differential operator deduced from this has three singular points $0, 1728, \infty$. Scaling moves the singular points to $0, 1, \infty$ and then conjugation with the function $(t-1)^{-1/2}t^{-1/3}$ results in an operator L_c with the required local exponents: $L_c = L_{st,A_5} = \mathrm{sym}^2(L_{st,A_5^{\mathrm{SL}_2}})$.

The operator L_c has to be *equivalent* to one of the two operators for A_5 with branch type $[2, 3, 5]$ in [18]. Explicitly, L_c is equivalent to L_u, the one with local data $-1, -1/2, 1/2||-2/3, -1/3, 0||6/5, 9/5, 2||\mu = 43/225$. Below are formulas for L_c, for $\tilde{L}_u$ obtained from L_u by $t \mapsto 1-t$, and for operators L, L' satisfying $\tilde{L}_uL = L'L_c$ (implying $\tilde{L}_u$ and L_c define the same differential module, hence L_c and L_u are equivalent).

$$L_c = d_t^3 + \frac{3(2t-1)}{t(t-1)}d_t^2 + \frac{6264t^2 - 6389t + 800}{900t^2(t-1)^2}d_t + \frac{1728t - 989}{1800t^2(t-1)^2},$$

$$\tilde{L}_u = d_t^3 + \frac{8t-4}{t(t-1)}d_t^2 + \frac{12744t^2 - 13169t + 2000}{900t^2(t-1)^2}d_t + \frac{7776t^2 - 12683t + 4457}{1800t^2(t-1)^3},$$

$$L = (t^2 - t)d_t^2 + (\frac{14t}{5} - \frac{4}{3})d_t + \frac{48t - 49}{60(t-1)},$$

$$L' = (t^2 - t)d_t^2 + (\frac{54t}{5} - \frac{16}{3})d_t + \frac{1440t^2 - 1453t + 280}{60t(t-1)}.$$

(3). *Comparing evaluations for* $A_5^{\mathrm{SL}_2}$ *and* A_5.

Consider a second order operator L_2 with differential Galois group $A_5^{\mathrm{SL}_2}$ and Picard–Vessiot field K^+. The third order operator $L_3 := \mathrm{sym}^2(L_2)$ has differential Galois group A_5 and Picard–Vessiot field $K = (K^+)^Z$, where Z is the center of $A_5^{\mathrm{SL}_2}$. The evaluation for L_2 is deduced from a homomorphism $h_1 : C(z)[X, Y] \to K^+$ which sends X, Y to a basis of solutions of L_2. The evaluation for L_3 is deduced from a homomorphism $h_2 : C(z)[X_1, X_2, X_3] \to K$ which sends X_1, X_2, X_3 to a basis of solutions for L_3. We may suppose that X_1, X_2, X_3 are mapped to $h_1(X)^2, h_1(XY), -h_1(Y^2)$. It follows that $F_2 = X_1X_3 + X_2^2$ lies in the kernel of the evaluation $h_2 : C[X_1, X_2, X_3] \to K$. Hence the evaluation for L_3 is induced by an evaluation for L_2. □

Examples 6.5. *Operators for other evaluations for* $A_5 \subset \mathrm{SL}_3$. An evaluation with nonzero image of F_2 to 0 is, after scaling, given by $(F_2, F_6, F_{10}, F_{15}) \mapsto (1, a, b, w)$ with $a, b \in C(z)$ not both constant. The assumption $w = 0$ leads to a contradiction. The relation between the invariants therefore implies $w^2 \in C(z)^*$. This leads to $h\colon (F_2, F_6, F_{10}, F_{15}) \mapsto (w^2, w^6a, w^{10}b, w^{16})$. The evaluation induces an A_5-invariant curve $Z \subset \mathbb{P}^2$ such that the normalisation of Z/A_5 has genus 0. We discuss a family of such curves Z.

Consider the A_5-invariant curve Z given by $-\lambda F_2^3 + F_6 = 0$. Then an evaluation as above with $a = \lambda \in C, b = z$ requires $w^2 \in C[z]$ to be a polynomial of degree 3. The singular points of the third order operator are included in the union of $\{0, \infty\}$ and the zeros of w^2. To have at most 3, the discriminant $-4096\lambda^5(\lambda-1)^2(27\lambda-32)^3$ of w^2 needs to vanish. This results in the cases:

(1). $\lambda = 1$. The curve given by $-F_2^3 + F_6 = 0$ is reducible; the defining polynomial has factors x and $x + y + z$ and a third one defining an irreducible rational curve of degree 4. The associated operator

$$L = d_t^3 + \frac{3(7t^2 - 147t + 676)}{2(t-4)(3t-37)(t-8)}d_t^2 + \frac{3(149t^2 - 3367t + 13584)}{100(3t-37)(t-8)(t-4)^2}d_t - \frac{3(t-29)}{200(3t-37)(t-8)(t-4)^2)}$$

factors. The two right hand factors are $L_2 = d_t^2 + \frac{t-6}{t^2-12t+32}d_t - \frac{1}{100(t^2-12t+32)}$ and $L_1 = d_t + \frac{1}{2(t-4)}$. A basis of solutions for L_2 is $(t - 6 \pm \sqrt{t^2 - 12t + 32})^{1/10}$, hence the Galois group of L_2 is the dihedral group D_{10} (of order 20). One checks that the solution $\sqrt{t-4}$ of L_1 is in the Picard–Vessiot field of L_2. Hence the Galois group of L is the subgroup D_{10} of A_5.

(2). $\lambda = 0$. The curve given by $F_6 = 0$ has genus 4 and is a Galois covering of $\mathbb{P}^1_z$ with group A_5 ramified over the points $0, -64, \infty$. The corresponding order three differential operator, with indeed Galois group A_5, is

$$d_t^3 + \frac{7t+256}{2t(t+64)}d_t^2 + \frac{149t+1024}{100t^2(t+64)}d_t - \frac{1}{200t^2(t+64)}.$$

(3). $\lambda = \frac{32}{27}$. We give some details for this interesting example.

The curve $-\frac{32}{27}F_2^3 + F_6 = 0$ has genus 0 and has 10 singular points (all over the cyclotomic field $\mathbb{Q}(\zeta_5)$). The curve is parametrized by $[-5s^3 : s^6 + 3s : 3s^5 - 1]$ and has function field $C(s)$. One has $C(s)^{A_5} = C(z)$ where z equals

$$\begin{aligned}(s^{60} &+ 2388s^{55} + 326394s^{50} - 8825700s^{45} + 117672975s^{40} + 83075976s^{35} + 380773868s^{30}\\ &-83075976s^{25} + 117672975s^{20} + 8825700s^{15} + 326394s^{10} - 2388s^5 + 1)\\ &/288s^5(s^2+s-1)^5(s^4+2s^3+4s^2+3s+1)^5(s^4-3s^3+4s^2-2s+1)^5.\end{aligned}$$

The evaluation obtained from this leads, using Procedure 4.3, the operator

$$\begin{aligned}L = d_z^3 &+ \frac{81(567z-4864)}{2(81z-1024)(81z-448)}d_z^2 + \frac{19683(4023z-46592)}{100(81z-448)(81z-1024)^2}d_z\\ &- \frac{531441}{200(81z-448)(81z-1024)^2}\end{aligned}$$

with Picard–Vessiot field $C(s)$ and basis of solutions $\{-5s^3,\ s^6+3s,\ 3s^5-1\}$. *The operator L is equivalent to the standard one, but is not itself a symmetric square.* Indeed, considering the degrees in s, no quadratic relation over C between the three solutions exists. The branch points of $C(s) \supset C(z)$ are $z = \frac{448}{81}, \frac{1024}{81}, \infty$, and the ramification type is $[2, 3, 5]$.

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