Port-Hamiltonian Systems

van der Schaft, Arjan

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Chapter 6
Port-Hamiltonian Systems

As described in the previous Chaps. 3 and 4, (cyclo-)passive systems are defined by the existence of a storage function (nonnegative in case of passivity) satisfying the dissipation inequality with respect to the supply rate $s(u, y) = u^T y$. In contrast, port-Hamiltonian systems, the topic of the current chapter, are endowed with the property of (cyclo-)passivity as a *consequence* of their system formulation. In fact, port-Hamiltonian systems arise from first principles physical modeling. They are defined in terms of a *Hamiltonian* function together with two *geometric structures* (corresponding, respectively, to power-conserving interconnection and energy dissipation), which are such that the Hamiltonian function automatically satisfies the dissipation inequality.

6.1 Input-State-Output Port-Hamiltonian Systems

An important subclass of port-Hamiltonian systems, especially for control purposes, is defined as follows.

Definition 6.1.1 An *input-state-output port-Hamiltonian system* with n-dimensional state space manifold $\mathcal{X}$, input and output spaces $U = Y = \mathbb{R}^m$, and Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, is given as[1]

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \tag{6.1}$$

where the $n \times n$ matrices $J(x)$, $R(x)$ satisfy $J(x) = -J^T(x)$ and $R(x) = R^T(x) \geq 0$. By the properties of $J(x)$, $R(x)$, it immediately follows that

[1] As before, $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H.

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$$\begin{aligned}&\tfrac{dH}{dt}(x(t)) = \tfrac{\partial^T H}{\partial x}(x(t))\dot{x}(t) = \\ &-\tfrac{\partial^T H}{\partial x}(x(t))R(x(t))\tfrac{\partial H}{\partial x}(x(t)) + y^T(t)u(t) \leq u^T(t)y(t),\end{aligned} \tag{6.2}$$

implying, cf. Definition 4.1.1, *cyclo-passivity* and *passivity* if $H \geq 0$.

The Hamiltonian H is is equal to the *total stored* energy of the system, while $u^T y$ is the externally supplied power. In the definition of a port-Hamiltonian system, two geometric structures on the state space $\mathcal{X}$ play a role: the internal *interconnection* structure given by $J(x)$, which by skew-symmetry is *power-conserving*, and a *resistive* structure given by $R(x)$, which by nonnegativity is responsible for internal *dissipation* of energy. For a further discussion on the mathematical theory underlying these geometric structures, as well as the port-based modeling origins of port-Hamiltonian systems, we refer to the Notes at the end of this chapter.

A useful extension of Definition 6.1.1 to systems with *feedthrough terms* is given as follows.

Definition 6.1.2 An *input-state-output port-Hamiltonian system with feedthrough terms* is specified by an n-dimensional state space manifold $\mathcal{X}$, input and output spaces $U = Y = \mathbb{R}^m$, Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and dynamics

$$\begin{aligned}\dot{x} &= [J(x) - R(x)]\,\tfrac{\partial H}{\partial x}(x) + [G(x) - P(x)]\,u \\ y &= [G(x) + P(x)]^T\,\tfrac{\partial H}{\partial x}(x) + [M(x) + S(x)]\,u,\end{aligned} \tag{6.3}$$

where the matrices $J(x), M(x), R(x), P(x), S(x)$ satisfy the skew-symmetry conditions $J(x) = -J^T(x), M(x) = -M^T(x)$, and the nonnegativity condition

$$\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \geq 0, \quad x \in \mathcal{X} \tag{6.4}$$

In this case, the power balance (6.2) takes the following form (using skew-symmetry of $J(x), M(x)$, and exploiting the nonnegativity condition (6.4))

$$\begin{aligned}&\tfrac{d}{dt}H(x) = \tfrac{\partial^T H}{\partial x}(x)\left([J(x) - R(x)]\,\tfrac{\partial H}{\partial x}(x) + [G(x) - P(x)]\,u\right) = \\ &-\begin{bmatrix} \tfrac{\partial^T H}{\partial x}(x) & u^T \end{bmatrix}\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix}\begin{bmatrix} \tfrac{\partial H}{\partial x}(x) \\ u \end{bmatrix} + y^T u \leq u^T y\end{aligned} \tag{6.5}$$

leading to the same conclusion regarding (cyclo-)passivity as above.

Remark 6.1.3 Note that by (6.4) $P = 0$ whenever $S = 0$ (no feedthrough).

Both (6.3) and (6.5) correspond to a *linear* resistive structure. The extension to *nonlinear* energy dissipation is given next.

Definition 6.1.4 An *input-state-output port-Hamiltonian system with nonlinear resistive structure* is given as

$$\begin{aligned}\dot{x} &= J(x)z - \mathcal{R}(x, z) + g(x)u, \quad z = \tfrac{\partial H}{\partial x}(x) \\ y &= g^T(x)z\end{aligned} \tag{6.6}$$

where $J(x) = -J^T(x)$, and the *resistive mapping* $\mathcal{R}(x, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$z^T \mathcal{R}(x, z) \geq 0, \quad \text{for all } z \in \mathbb{R}^n, \; x \in \mathcal{X} \tag{6.7}$$

Remark 6.1.5 Geometrically $z = \frac{\partial H}{\partial x}(x) \in T_x^* \mathcal{X}$, with $T_x^* \mathcal{X}$, the co-tangent space of $\mathcal{X}$ at $x \in \mathcal{X}$, while $\dot{x} \in T_x \mathcal{X}$, the tangent space at $x \in \mathcal{X}$. Hence, the resistive mapping $\mathcal{R}$ is defined geometrically as a *vector bundle map* $\mathcal{R} : T^* \mathcal{X} \rightarrow T\mathcal{X}$.

Similarly to (6.2) we obtain

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -\frac{\partial^T H}{\partial x}(x)\mathcal{R}\left(x, \frac{\partial H}{\partial x}(x)\right) + y^T u \leq u^T y, \tag{6.8}$$

showing again (cyclo-)passivity. We leave the extension to systems with feedthrough terms to the reader.

Example 6.1.6 Consider a mass–spring–damper system (mass m, spring constant k, momentum p, spring extension q, external force F) subject to ideal Coulomb friction

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ c \operatorname{sign} \frac{p}{m} \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix}, \tag{6.9}$$

where sign is the multivalued function defined by

$$\operatorname{sign} v = \begin{cases} 1 & , \quad v > 0 \\ [-1, 1] & , \quad v = 0 \\ -1 & , \quad v < 0 \end{cases} \tag{6.10}$$

and $c > 0$ is a constant. This defines an input-state-output port-Hamiltonian system with nonlinear resistive structure defined by the multivalued function $c \operatorname{sign}$. Note that strictly speaking, this entails a further generalization of Definition 6.1.4 since the Coulomb friction mapping (6.10) is multivalued. The Hamiltonian $H(q, p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$ satisfies

$$\frac{d}{dt}H = -\frac{p}{m} \operatorname{sign} \frac{p}{m} + F\frac{p}{m} \leq F\frac{p}{m} \tag{6.11}$$

Example 6.1.7 The dynamics of a detailed-balanced mass action kinetics chemical reaction network can be written as, see the Notes at the end of this chapter for further information,

$$\begin{aligned} \dot{x} &= -Z\mathcal{L}\,\mathrm{Exp}\left(Z^T \mathrm{Ln}\, \tfrac{x}{x^*}\right) + S_b u \\ y &= S_b^T \mathrm{Ln}\, \tfrac{x}{x^*} \end{aligned} \tag{6.12}$$

where $x \in \mathbb{R}^n$ is the vector of chemical species concentrations, u is the vector of boundary fluxes, and y is the vector of boundary chemical potentials. Furthermore, x^* is a thermodynamic equilibrium, Z is the complex composition matrix, S_b specifies which are the boundary chemical species, and $\mathcal{L}$ is a symmetric Laplacian matrix (see Definition 4.4.6) on the graph of chemical complexes, with weights determined by the kinetic reaction constants. Exp and Ln denote the component-wise exponential and logarithm mappings, i.e., $(\mathrm{Exp}\,(x))_i = \exp x_i$, $(\mathrm{Ln}\,(x))_i = \ln x_i$, $i = 1, \ldots, n$. Similarly, $\frac{x}{x^*}$ denotes component-wise division of the vector x by the vector x^*. The Hamiltonian is given by the Gibbs' free energy, which (up to constants) is equal to

$$H(x) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{x_i^*} + \sum_{i=1}^{n} (x_i^* - x_i), \tag{6.13}$$

corresponding to the chemical potentials $z_i = \frac{\partial H}{\partial x_i}(x) = \ln \frac{x_i}{x_i^*}$. Since [299]

$$\gamma^T \mathcal{L}\,\mathrm{Exp}\,\gamma \geq 0 \tag{6.14}$$

for all vectors γ, this defines an input-state-output port-Hamiltonian system, with $J = 0$ and nonlinear resistive structure given by the mapping $z \mapsto Z^T \mathcal{L}\,\mathrm{Exp}\, Z^T z$.

Finally, a *linear* input-state-output port-Hamiltonian system with feedthrough terms is given by the following specialization of Definition 6.1.2

$$\begin{aligned} \dot{x} &= [J - R]\, Qx + [G - P]\, u \\ y &= [G + P]^T\, Qx + [M + S]\, u \end{aligned} \tag{6.15}$$

with quadratic Hamiltonian $H(x) = \frac{1}{2} x^T Q x$, $Q = Q^T$, and constant matrices J, M, R, P, S satisfying $J = -J^T$, $M = -M^T$ and

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0 \tag{6.16}$$

Since subtracting a constant from the Hamiltonian function H does not change the system, the condition $H \geq 0$ can be replaced by H being bounded from below. Hence based on (6.2), (6.5), (6.8), we can summarize the characterization of (cyclo-)passivity of input-state-output port-Hamiltonian systems as follows.

Proposition 6.1.8 *Any input-state-output port-Hamiltonian system given by one of the expressions (6.1), (6.3), (6.6), (6.15) is cyclo-passive, and passive if H is bounded from below, respectively, $Q \geq 0$. Furthermore, if the (nonlinear) resistive structure is absent, then the system is lossless in case H is bounded from below.*

In the modeling of physical systems, the port-Hamiltonian formulation directly follows from the physical structure of the system; see Sects. 6.2 and 6.3 and the Notes at the end of this chapter for further information. On the other hand, one may still wonder when the converse of Proposition 6.1.8 holds, i.e., when and how a passive system can be written as a port-Hamiltonian system. In the linear case, this question can be answered as follows. Consider the passive linear system (for simplicity without feedthrough terms)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6.17}$$

with *positive-definite* storage function $\frac{1}{2}x^T Qx$, i.e.,

$$A^T Q + QA \leq 0, \quad B^T Q = C, \quad Q > 0 \tag{6.18}$$

Now decompose AQ^{-1} into its skew-symmetric and symmetric part as

$$AQ^{-1} = J - R, \quad J = -J^T, R = R^T \tag{6.19}$$

Then $A^T Q + QA \leq 0$ implies $R \geq 0$, and $\dot{x} = Ax + Bu, y = Cx$, can be rewritten into port-Hamiltonian form $\dot{x} = (J - R)Qx + Bu, y = B^T Qx$. The same result can be shown to hold for linear passive systems with $Q \geq 0$ under the additional assumption $\ker Q \subset \ker A$. In this case, one defines F such that $A = FQ$, and factorizes F into its skew-symmetric and symmetric part.

On the other hand, since in general the storage matrix Q of a passive system is not unique, also the interconnection and resistive structure matrices J and R as obtained in the above port-Hamiltonian formulation are not unique. Hence, if Q is not unique then there exist essentially *different* port-Hamiltonian formulations of the same linear passive system $\dot{x} = Ax + Bu, y = Cx$.

For *nonlinear* systems, the conversion from passive to port-Hamiltonian systems is more subtle. For example,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{6.20}$$

is lossless with storage function $H \geq 0$ iff

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x) f(x) &= 0 \\ g^T(x) \frac{\partial H}{\partial x}(x) &= h(x) \end{aligned} \tag{6.21}$$

Nevertheless, the first equality in (6.21) does not imply that there exists a skew-symmetric matrix $J(x)$ such that $f(x) = J(x)\frac{\partial H}{\partial x}(x)$, as illustrated by the next example.

Example 6.1.9 Consider the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= x_1^2 x_2 \end{aligned} \tag{6.22}$$

which is lossless with respect to the storage function $H(x_1, x_2) = \frac{1}{2}x_1^2 x_2^2$. However, it is easy to see that there does not exist a 2×2 matrix $J(x) = -J^T(x)$, depending smoothly on $x = (x_1, x_2)$, such that

$$\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = J(x) \begin{bmatrix} x_1 x_2^2 \\ x_1^2 x_2 \end{bmatrix}$$

Hence, the system is *not* a port-Hamiltonian system with respect to $H(x_1, x_2)$.

6.2 Mechanical Systems

The port-Hamiltonian formulation of standard mechanical systems directly follows from classical mechanics. Consider as in Proposition 4.5.1, the Hamiltonian representation of fully actuated Euler–Lagrange equations in n configuration coordinates $q = (q_1, \ldots, q_n)$ given by the $2n$-dimensional system

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p), & p &= (p_1, \ldots, p_n) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) + u, & u &= (u_1, \ldots, u_n) \\ y &= \tfrac{\partial H}{\partial p}(q,p) \; (= \dot{q}), & y &= (y_1, \ldots, y_n) \end{aligned} \tag{6.23}$$

with u the vector of (generalized) external forces and y the vector of (generalized) velocities. The state space of (6.23) with local coordinates (q, p) is called the *phase space*. In most mechanical systems, the Hamiltonian $H(q, p)$ is the sum of a positive kinetic energy and a potential energy

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + P(q) \tag{6.24}$$

It was shown in Proposition 4.5.1 that along every trajectory of (6.23)

$$H(q(t_1), p(t_1)) = H(q(t_0), p(t_0)) + \int_{t_0}^{t_1} u^T(t) y(t) dt, \tag{6.25}$$

expressing that the increase in internal energy H equals the *work* supplied to the system ($u^T y$ is generalized force times generalized velocity, i.e., power). Hence, the system (6.23) is an input-state-output port-Hamiltonian system, which is lossless if H is bounded from below. The system description (6.23) can be further generalized to

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p), & (q,p) &= (q_1, \ldots, q_n, p_1, \ldots, p_n) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) + B(q)u, & u &\in \mathbb{R}^m \\ y &= B^T(q) \tfrac{\partial H}{\partial p}(q,p) \; (= B^T(q)\dot{q}), \; y \in \mathbb{R}^m, & & \end{aligned} \tag{6.26}$$

where $B(q)$ is an input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. If $m < n$, we speak of an *underactuated* mechanical system. Also for (6.26) we obtain the power balance

$$\frac{dH}{dt}(q(t), p(t)) = u^T(t)y(t) \tag{6.27}$$

A further generalization is obtained by extending (6.26) to input-state-output port-Hamiltonian systems

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u, \quad J(x) = -J^T(x),\ x \in \mathcal{X} \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{6.28}$$

where $\mathcal{X}$ is an $\bar{n}$-dimensional state space manifold, and $J(x)$ is state-dependent skew-symmetric matrix. Note that (6.23) and (6.26) correspond to the full rank and constant skew-symmetric matrix J given by

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{6.29}$$

Models (6.28) arise, for example, by *symmetry reduction* of (6.23) or (6.26). A classical example is Euler's equations for the dynamics of the angular velocities of a rigid body.

Example 6.2.1 (*Euler's equations*; *Example* 4.2.4 *continued*) Consider a rigid body spinning around its center of mass in the absence of gravity. In Example 4.2.4, we already encountered Euler's equations for the dynamics of the angular velocities. The Hamiltonian formulation is obtained by considering the body angular momenta $p = (p_x, p_y, p_z)$ along the three principal axes, and the Hamiltonian given by the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right), \tag{6.30}$$

where I_x, I_y, I_z are the principal moments of inertia. The vector p of angular momenta is related to the vector ω of angular velocities as $p = I\omega$, where I is the diagonal matrix with positive diagonal elements I_x, I_y, I_z. Euler's equations are now given as

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{J(p)} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} u \tag{6.31}$$

In the scalar input case, the last term bu denotes the torque around an axis with coordinates $b = (b_x\ b_y\ b_z)^T$, with corresponding collocated output given as

$$y = b_x \frac{p_x}{I_x} + b_y \frac{p_y}{I_y} + b_z \frac{p_z}{I_z}, \tag{6.32}$$

which is the velocity around the same axis $(b_x\ b_y\ b_z)^T$.

In many cases (including the one obtained by symmetry reduction from the canonical J given in (6.29)), the dependence of the matrix J on the state x will satisfy the *integrability* conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x) \frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x) \frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x) \frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \tag{6.33}$$

for $i, j, k = 1, \ldots, n$. These integrability conditions are also referred to as the *Jacobi-identity*. If these integrability conditions are met, we can construct by Darboux's theorem (see e.g., [347]), around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates

$$\tilde{x} = (q, p, s) = (q_1, \ldots, q_l, p_1, \ldots, p_k, s_1, \ldots s_l), \tag{6.34}$$

with $2k$ the rank of J and $n = 2k + l$, such that J in these coordinates takes the form

$$J = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.35}$$

The coordinates (q, p, s) are also called *canonical* coordinates, and J satisfying (6.33) is called a *Poisson structure matrix*. Otherwise, it is called an *almost-Poisson structure*.

Example 6.2.2 (*Example* 6.2.1 *continued*) It can be directly checked that the skew-symmetric matrix $J(p)$ defined in (6.31) satisfies the Jacobi-identity (6.33). This also follows from the fact that $J(p)$ is the canonical *Lie–Poisson structure* matrix on the dual of the Lie algebra $so(3)$ corresponding to the configuration space $SO(3)$ of the rigid body; see the Notes at the end of this chapter for further information.

The rest of this section will be devoted to mechanical systems with *kinematic constraints*, which is an important class of systems in applications (for example in robotics). Consider a mechanical system with n degrees of freedom, locally described by n configuration variables

$$q = (q_1, \ldots, q_n) \tag{6.36}$$

Expressing the kinetic energy as $\frac{1}{2}\dot{q}^T M(q) \dot{q}$, with $M(q) > 0$ being the generalized mass matrix, we define in the usual way the Lagrangian function

$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q)$, where P is the potential energy. Suppose now that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^T(q)\dot{q} = 0, \tag{6.37}$$

with $A(q)$ an $n \times k$ matrix of rank k everywhere. This means that there are k independent kinematic constraints. Classically, the constraints (6.37) are called *holonomic* if it is possible to find new configuration coordinates $\overline{q} = (\overline{q}_1, \dots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0\,, \tag{6.38}$$

in which case it is possible to eliminate the configuration variables $\overline{q}_{n-k+1}$, $\dots, \overline{q}_n$, since the kinematic constraints (6.38) are equivalent to the *geometric constraints* constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \dots, \overline{q}_n = c_n\,, \tag{6.39}$$

for constants $c_{n-k+1}, \dots, c_n$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the remaining configuration coordinates $(\overline{q}_1, \dots, \overline{q}_{n-k})$. If it is *not* possible to find coordinates $\overline{q}$ such that (6.38) holds (that is, if we are not able to *integrate* the kinematic constraints as above), then the kinematic constraints are called *nonholonomic*.

The equations of motion for the mechanical system with Lagrangian $L(q, \dot{q})$ and kinematic constraints (6.37) are given by the *constrained Euler–Lagrange equations*

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) &= A(q)\lambda + B(q)u, \quad \lambda \in \mathbb{R}^k, u \in \mathbb{R}^m \\ A^T(q)\dot{q} &= 0, \end{aligned} \tag{6.40}$$

where $B(q)u$ are the external forces applied to the system, for some $n \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces*. The Lagrange multipliers $\lambda(t)$ are uniquely determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ are satisfied for all t.

Defining as before (cf. (4.107)) the generalized momenta

$$p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = M(q)\dot{q}, \tag{6.41}$$

the constrained Euler–Lagrange equations (6.40) transform into *constrained Hamiltonian equations*

$$\begin{aligned}\dot q &= \frac{\partial H}{\partial p}(q,p)\\ \dot p &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u\\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p)\\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p)\end{aligned} \tag{6.42}$$

with $H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q)$ the total energy. Thus, the kinematic constraints appear as *algebraic constraints* on the phase space, and the *constrained* state space is given as the following subset of the phase space

$$\mathcal{X}_c = \left\{(q,p) \mid A^T(q)\frac{\partial H}{\partial p}(q,p) = 0\right\} \tag{6.43}$$

The algebraic constraints $A^T(q)\frac{\partial H}{\partial p}(q,p) = 0$ and constraint forces $A(q)\lambda$ can be *eliminated* in the following way. Since rank $A(q) = k$, there exists locally an $n \times (n-k)$ matrix $S(q)$ of rank $n-k$ such that

$$A^T(q)S(q) = 0 \tag{6.44}$$

Now define $\tilde p = (\tilde p^1, \tilde p^2) = (\tilde p_1, \ldots, \tilde p_{n-k}, \tilde p_{n-k+1}, \ldots, \tilde p_n)$ as

$$\begin{aligned}\tilde p^1 &:= S^T(q)p,\ \tilde p^1 \in \mathbb{R}^{n-k}\\ \tilde p^2 &:= A^T(q)p,\ \tilde p^2 \in \mathbb{R}^k\end{aligned} \tag{6.45}$$

It is readily checked that $(q,p) \mapsto (q, \tilde p^1, \tilde p^2)$ is a coordinate transformation. Indeed, by (6.44) the rows of $S^T(q)$ are orthogonal to the rows of $A^T(q)$. In the new coordinates the constrained system (6.42) takes the form [293], $*$ denoting unspecified elements,

$$\begin{aligned}\begin{bmatrix}\dot q\\ \dot{\tilde p}^1\\ \dot{\tilde p}^2\end{bmatrix} &= \begin{bmatrix}0_n & S(q) & *\\ -S^T(q) & \left(-p^T[S_i,S_j](q)\right)_{i,j} & *\\ * & * & *\end{bmatrix}\begin{bmatrix}\frac{\partial \tilde H}{\partial q}\\ \frac{\partial \tilde H}{\partial \tilde p^1}\\ \frac{\partial \tilde H}{\partial \tilde p^2}\end{bmatrix} +\\ &\quad \begin{bmatrix}0\\ 0\\ A^T(q)A(q)\end{bmatrix}\lambda + \begin{bmatrix}0\\ B_c(q)\\ \overline{B}(q)\end{bmatrix}u\\ A^T(q)\frac{\partial H}{\partial p} &= A^T(q)A(q)\frac{\partial \tilde H}{\partial \tilde p^2} = 0\end{aligned} \tag{6.46}$$

with $\tilde H(q,\tilde p)$ the Hamiltonian H expressed in the new coordinates $q, \tilde p$. Here S_i denotes the i-th column of $S(q)$, $i = 1, \ldots, n-k$, and $[S_i, S_j]$ is the *Lie bracket* of S_i and S_j, in local coordinates q given as (see e.g., [1, 233])

$$[S_i, S_j](q) \;=\; \frac{\partial S_j}{\partial q}(q)S_i(q) - \frac{\partial S_i}{\partial q}S_j(q) \tag{6.47}$$

with $\frac{\partial S_j}{\partial q}, \frac{\partial S_i}{\partial q}$ denoting the $n \times n$ Jacobian matrices.

The constraints $A^T(q)\frac{\partial H}{\partial p}(q,p) = 0$ are equivalently given as $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,\tilde{p}) = 0$, and by non-degeneracy of the kinetic energy $\frac{1}{2}p^T M^{-1}(q)p$ these equations can be solved for $\tilde{p}^2$. Since λ only influences the $\tilde{p}^2$-dynamics, the constrained dynamics is thus determined by the dynamics of q and $\tilde{p}^1$ alone (which together serve as coordinates for the constrained state space $\mathcal{X}_c$), given as

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{bmatrix} = J_c(q,\tilde{p}^1)\begin{bmatrix} \frac{\partial H_c}{\partial q}(q,\tilde{p}^1) \\ \frac{\partial H_c}{\partial \tilde{p}^1}(q,\tilde{p}^1) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c(q) \end{bmatrix} u \tag{6.48}$$

Here $H_c(q,\tilde{p}^1)$ equals $\tilde{H}(q,\tilde{p})$ with $\tilde{p}^2$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,\tilde{p}^1,\tilde{p}^2) = 0$, and where the skew-symmetric matrix $J_c(q,\tilde{p}^1)$ is given as the left-upper part of the structure matrix in (6.46), that is

$$J_c(q,\tilde{p}^1) = \begin{bmatrix} O_n & S(q) \\ -S^T(q) & \left(-p^T[S_i,S_j](q)\right)_{i,j} \end{bmatrix}, \tag{6.49}$$

where p is expressed as function of $q,\tilde{p}$, with $\tilde{p}^2$ eliminated from $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. Finally, in the coordinates $q,\tilde{p}$, the output map is given as

$$y = \begin{bmatrix} B_c^T(q) \; \overline{B}^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q,\tilde{p}^1) \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q,\tilde{p}^1) \end{bmatrix} \tag{6.50}$$

which reduces on the constrained state space $\mathcal{X}_c$ to

$$y = B_c^T(q)\frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q,\tilde{p}^1) \tag{6.51}$$

Summarizing, (6.48) and (6.51) define an *input-state-output port-Hamiltonian system* on $\mathcal{X}_c$, with Hamiltonian H_c given by the constrained total energy, and with structure matrix J_c given by (6.49).

The skew-symmetric matrix J_c defined on $\mathcal{X}_c$ is an *almost-Poisson structure* since it does not necessarily the integrability conditions (6.33). In fact, J_c satisfies the integrability conditions (6.33), and thus defines a Poisson structure on $\mathcal{X}_c$, if and only if the kinematic constraints (6.37) are *holonomic*. In fact, if the constraints are holonomic then the coordinates s as in (6.34) can be taken equal to the "integrated constraint functions" $\overline{q}_{n-k+1},\ldots,\overline{q}_n$ of (6.39).

Example 6.2.3 (*Rolling coin*) Let x, y be the Cartesian coordinates of the point of contact of a vertical coin with the plane. Furthermore, φ denotes the heading angle

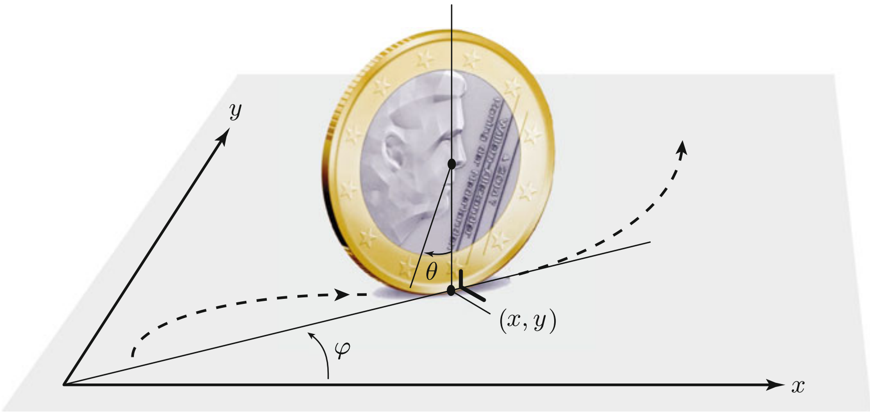


Fig. 6.1 The geometry of the rolling coin

of the coin on the plane, and θ the angle of Willem Alexander's head; cf. Fig. 6.1. With all constants set to unity, the constrained Lagrangian equations of motion are

$$\begin{aligned} \ddot{x} &= \lambda_1 \\ \ddot{y} &= \lambda_2 \\ \ddot{\theta} &= -\lambda_1 \cos\varphi - \lambda_2 \sin\varphi + u_1 \\ \ddot{\varphi} &= u_2 \end{aligned} \tag{6.52}$$

where u_1 is the control torque about the rolling axis, and u_2 the control torque about the vertical axis. The rolling constraints are

$$\dot{x} = \dot{\theta}\cos\varphi, \quad \dot{y} = \dot{\theta}\sin\varphi \tag{6.53}$$

(rolling without slipping). The energy is $H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$, and the kinematic constraints can be rewritten as $p_x = p_\theta \cos\varphi$, $p_y = p_\theta \sin\varphi$. Define according to (6.45) new p-coordinates

$$\begin{aligned} p_1 &= p_\varphi \\ p_2 &= p_\theta + p_x \cos\varphi + p_y \sin\varphi \\ p_3 &= p_x - p_\theta \cos\varphi \\ p_4 &= p_y - p_\theta \sin\varphi \end{aligned} \tag{6.54}$$

The constrained state space $\mathcal{X}_c$ is given by $p_3 = p_4 = 0$, and the dynamics on $\mathcal{X}_c$ is computed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} & & & & 0 & \cos\varphi \\ & & & & 0 & \sin\varphi \\ & & O_4 & & 0 & 1 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos\varphi & -\sin\varphi & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial x} \\ \frac{\partial H_c}{\partial y} \\ \frac{\partial H_c}{\partial \theta} \\ \frac{\partial H_c}{\partial \varphi} \\ \frac{\partial H_c}{\partial p_1} \\ \frac{\partial H_c}{\partial p_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6.55}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial p_1} \\ \frac{\partial H_c}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 \\ p_1 \end{bmatrix}$$

where $H_c(x, y, \theta, \varphi, p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{4}p_2^2$. (Note that $\frac{\partial H_c}{\partial p_2} = \frac{1}{2}p_2 = p_\theta$.) It can be verified that the structure matrix J_c in (6.55) does *not* satisfy the integrability conditions, in accordance with the fact that the rolling constraints (rolling without slipping) are *nonholonomic*.

6.3 Port-Hamiltonian Models of Electromechanical Systems

This section will contain a collection of characteristic examples of port-Hamiltonian systems arising in electromechanical systems, illustrating the use of port-Hamiltonian models for *multi-physics* systems. In most of the examples the interaction between the *mechanical* and the *electrical* part of the system will take place through the Hamiltonian function, which will depend in a non-separable way on state variables belonging to the mechanical and variables belonging to the electrical domain.

Example 6.3.1 (*Capacitor microphone* [230]) Consider the capacitor microphone depicted in Fig. 6.2.

The capacitance $C(q)$ of the capacitor is varying as a function of the displacement q of the right plate (with mass m), which is attached to a spring (with spring constant $k > 0$) and a damper (with constant $d > 0$), and affected by a mechanical force F (air pressure arising from sound). Furthermore, E is a voltage source. The equations of motion can be written as the port-Hamiltonian system

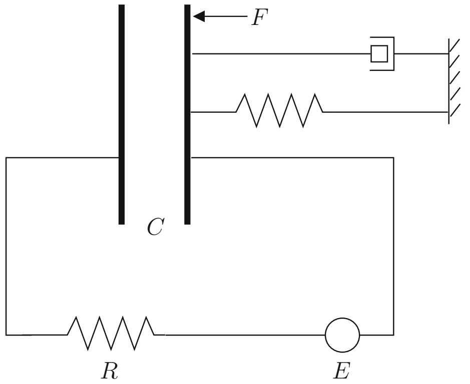


Fig. 6.2 Capacitor microphone

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1/R \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ 1/R \end{bmatrix} E \qquad (6.56)$$

$$y_1 = \frac{\partial H}{\partial p} = \dot{q}$$

$$y_2 = \frac{1}{R}\frac{\partial H}{\partial Q} = I$$

where p is the momentum, R the resistance of the resistor, I the current through the voltage source, and the Hamiltonian H is the total energy

$$H(q, p, Q) = \frac{1}{2m}p^2 + \frac{1}{2}k(q - \bar{q})^2 + \frac{1}{2C(q)}Q^2, \qquad (6.57)$$

with $\bar{q}$ denoting the rest length of the spring. Note that the electric energy $\frac{1}{2C(q)}Q^2$ not only depends on the electric charge Q, but also on the q-variable belonging to the mechanical part of the system. Furthermore

$$\frac{d}{dt}H = -c\dot{q}^2 - RI^2 + F\dot{q} + EI \leq F\dot{q} + EI, \qquad (6.58)$$

with $F\dot{q}$ the mechanical power and EI the electrical power supplied to the system. In the application as a microphone the voltage over the resistor will be used (after amplification) as a measure for the mechanical force F. Finally, we note that the same model can be used for an electrical *micro-actuator*. In this case, the system is controlled at its electrical side in order to produce a certain desired force at its mechanical side. This physical phenomenon of *bilateral operation* will be also evident in the following examples.

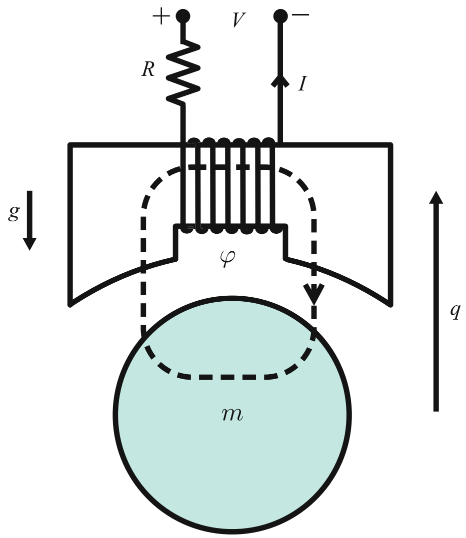


Fig. 6.3 Magnetically levitated ball

Example 6.3.2 (*Magnetically levitated ball*) Consider the dynamics of an iron ball that is levitated by the magnetic field of a controlled inductor as schematically depicted in Fig. 6.3. The port-Hamiltonian description of this system (with q the height of the ball, p the vertical momentum, and φ the magnetic flux linkage of the inductor) is given as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V$$
$$I = \frac{\partial H}{\partial \varphi} \tag{6.59}$$

Although at first instance the mechanical and the magnetic part of the system look decoupled, they are actually coupled via the Hamiltonian

$$H(q, p, \varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2L(q)}, \tag{6.60}$$

where the inductance $L(q)$ depends on the height q. In fact, the magnetic energy $\frac{\varphi^2}{2L(q)}$ depends both on the flux φ and the mechanical variable q. As a result, the right-hand side of the second equation (describing the evolution of the mechanical momentum variable p) depends on the magnetic variable φ, and conversely the right-hand side of the third equation (describing the evolution of the magnetic variable φ) depends on the mechanical variable q.

Example 6.3.3 (*Permanent magnet synchronous motor* [242]) A state vector for a permanent magnet synchronous motor (in rotating reference (dq) frame) is defined as

$$x = M \begin{bmatrix} i_d \\ i_q \\ \omega \end{bmatrix}, \qquad M = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & \frac{j}{n_p} \end{bmatrix} \tag{6.61}$$

composed of the magnetic flux linkages and mechanical momentum (with i_d, i_q being the currents, and ω the angular velocity), L_d, L_q stator inductances, j the moment of inertia, and n_p the number of pole pairs. The Hamiltonian $H(x)$ is given as $H(x) = \frac{1}{2}x^T M^{-1} x$. This leads to a port-Hamiltonian formulation with $J(x)$, $R(x)$ and $g(x)$ determined as

$$\begin{aligned} J(x) &= \begin{bmatrix} 0 & L_0 x_3 & 0 \\ -L_0 x_3 & 0 & -\Phi_{q0} \\ 0 & \Phi_{q0} & 0 \end{bmatrix}, \\ R(x) &= \begin{bmatrix} R_S & 0 & 0 \\ 0 & R_S & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{n_p} \end{bmatrix}, \end{aligned} \tag{6.62}$$

with R_S the stator winding resistance, Φ_{q0} a constant term due to interaction of the permanent magnet and the magnetic material in the stator, and $L_0 := L_d n_p / j$. The three inputs are the stator voltages $\left(v_d, v_q\right)^T$ and the (constant) load torque. Outputs are i_d, i_q, and ω. The system can also operate as a *dynamo*, converting mechanical power into electrical power.

Example 6.3.4 (*Synchronous machine*) The standard eight-dimensional model for the synchronous machine, as described, e.g., in [177], can be written in port-Hamiltonian form as (see [98] for details)

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_s \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} -R_s & 0_{33} & 0_{31} & 0_{31} \\ 0_{33} & -R_r & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & -d & -1 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix} + \\ &\quad \begin{bmatrix} I_3 & 0_{31} & 0_{31} \\ 0_{33} & e_1 & 0_{31} \\ 0_{13} & 0 & 1 \\ 0_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ V_f \\ \tau \end{bmatrix} \\ \begin{bmatrix} I_s \\ I_f \\ \omega \end{bmatrix} &= \begin{bmatrix} I_3 & 0_{33} & 0_{31} & 0_{31} \\ 0_{13} & e_1^T & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix}, \end{aligned} \tag{6.63}$$

where 0_{lk} denotes the $l \times k$ zero matrix, I_3 denotes the 3×3 identity matrix, and e_1 is the first basis vector of $\mathbb{R}^3$. This defines a port-Hamiltonian input-state-output system with Poisson structure matrix $J(x)$ given by the constant matrix

$$J = \begin{bmatrix} 0_{66} & 0_{62} \\ 0_{26} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{bmatrix}, \tag{6.64}$$

and resistive structure matrix $R(x)$, which is also constant, having diagonal blocks

$$R_s = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix}, \quad R_r = \begin{bmatrix} r_f & 0 & 0 \\ 0 & r_{kd} & 0 \\ 0 & 0 & r_{kq} \end{bmatrix}, \quad d, \quad 0, \tag{6.65}$$

denoting, respectively, the *stator resistances, rotor resistances*, and *mechanical friction*. The state variables x of the synchronous machine comprise of

- $\psi_s \in \mathbb{R}^3$, the *stator fluxes*,
- $\psi_r \in \mathbb{R}^3$, the *rotor fluxes*: the first one corresponding to the *field winding* and the remaining two to the *damper windings*,
- p, the *angular momentum* of the rotor,
- θ, the *angle* of the rotor.

Moreover, $V_s \in \mathbb{R}^3$, $I_s \in \mathbb{R}^3$ are the three-phase *stator terminal voltages and currents*, V_f, I_f are the rotor *field winding voltage and current*, and τ, ω are the mechanical *torque* and *angular velocity*.

The synchronous machine is designed depending on two possible modes of operation: synchronous generator or synchronous motor. In the first case, mechanical power is converted to electrical power (supplied to an electrical transmission network); see Fig. 6.4 for a schematic view. Conversely, in the synchronous motor case electrical power is drawn from the power grid in order to deliver mechanical power.

The *Hamiltonian H* (total stored energy of the synchronous machine) is the sum of the magnetic energy of the machine and the kinetic energy of the rotating rotor, given as the sum of the two nonnegative terms

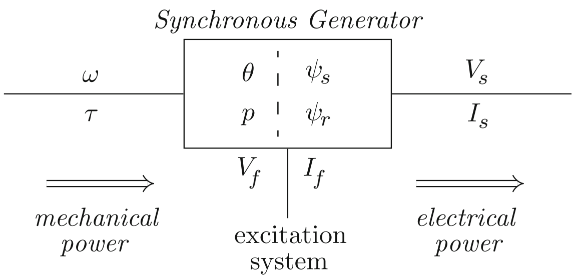


Fig. 6.4 The state and port variables of the synchronous generator

$$\begin{aligned} H(\psi_s, \psi_r, p, \theta) &= \tfrac{1}{2}\begin{bmatrix}\psi_s^T & \psi_r^T\end{bmatrix} L^{-1}(\theta) \begin{bmatrix}\psi_s \\ \psi_r\end{bmatrix} + \tfrac{1}{2J_r}p^2 \\ &= \text{ magnetic energy } H_m + \text{ kinetic energy } H_k, \end{aligned} \tag{6.66}$$

where J_r is the rotational inertia of the rotor, and $L(\theta)$ is an 6×6 *inductance* matrix. In the *round rotor* case (no saliency; cf. [177, 192])

$$L(\theta) = \begin{bmatrix} L_{ss} & L_{sr}(\theta) \\ L_{sr}^T(\theta) & L_{rr} \end{bmatrix} \tag{6.67}$$

where

$$L_{ss} = \begin{bmatrix} L_{aa} & -L_{ab} & -L_{ab} \\ -L_{ab} & L_{aa} & -L_{ab} \\ -L_{ab} & -L_{ab} & L_{aa} \end{bmatrix}, \; L_{rr} = \begin{bmatrix} L_{ffd} & L_{akd} & 0 \\ L_{akd} & L_{kkd} & 0 \\ 0 & 0 & L_{kkq} \end{bmatrix} \tag{6.68}$$

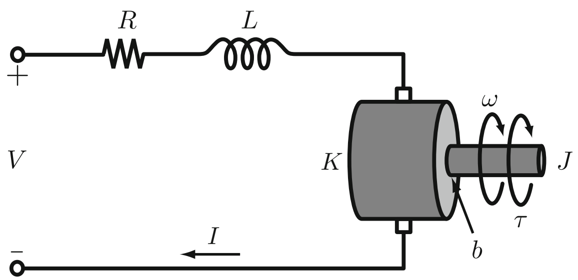
while

$$\begin{aligned} L_{sr}(\theta) = &\begin{bmatrix} \cos\theta & \cos\theta & -\sin\theta \\ \cos(\theta - \frac{2\pi}{3}) & \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) \\ \cos(\theta + \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \times \\ &\begin{bmatrix} L_{afd} & 0 & 0 \\ 0 & L_{akd} & 0 \\ 0 & 0 & L_{akq} \end{bmatrix} \end{aligned} \tag{6.69}$$

A crucial feature of the magnetic energy term H_m in the Hamiltonian H is its dependency on the mechanical rotor angle θ; see the formula (6.69) for $L_{sr}(\theta)$. This dependence is responsible for the interaction between the mechanical domain of the generator (the mechanical motion of the rotor) and the electromagnetic domain (the dynamics of the magnetic fields in the rotor and stator), and thus for the functioning of the synchronous machine as an energy-conversion device, transforming mechanical power into electrical power, or conversely (Fig. 6.4).

The synchronous machine is connected to its environment by three types of ports; see Fig. 6.4. In the case of operation as a synchronous *generator*, the scalar *mechanical port* with power variables τ, ω is to be interconnected to a *prime mover*, such as a

Fig. 6.5 DC motor



turbine. This port is also used for control purposes, e.g., via so-called *droop control*. Second, there are three *stator terminal ports*, with vectors of power variables V_s, I_s. Third, there is the port with scalar power variables V_f, I_f, which is responsible for the magnetization of the rotor, and which is controlled by an *excitation system*.

Example 6.3.5 (*DC motor*) The system depicted in Fig. 6.5 consists of five ideal modeling subsystems: an inductor L with state φ (flux), a rotational inertia J with state p (angular momentum), a resistor R and friction b, and a gyrator K. The Hamiltonian (corresponding to the linear inductor and inertia) reads as $H(p, \varphi) = \frac{1}{2L}\varphi^2 + \frac{1}{2J}p^2$. The linear resistive relations are $V_R = -RI$, $\tau_d = -b\omega$, with $R, b > 0$ and τ_d a damping torque. The equations of the gyrator (converting magnetic power into mechanical, and conversely) are

$$V_K = -K\omega, \quad \tau = KI \tag{6.70}$$

with K the gyrator constant. The subsystems are interconnected by the equations $V_L + V_R + V_K + V = 0$ (and equal currents), as well as $\tau_J + \tau_d + \tau = 0$ (with common angular velocity), leading to the port-Hamiltonian input-state-output system

$$\begin{aligned} \begin{bmatrix} \dot{\varphi} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} -R & -K \\ K & -b \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V \\ I &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix}. \end{aligned} \tag{6.71}$$

Note that, as in the case of the synchronous machine, the system can operate in two modes: either as a motor (converting electrical power into mechanical power) or as a dynamo (converting rotational motion and mechanical power into electrical current and power).

6.4 Properties of Port-Hamiltonian Systems

A crucial property of a port-Hamiltonian system is *cyclo-passivity*, and *passivity* if the Hamiltonian satisfies $H \geq 0$. Apart from this, the port-Hamiltonian formulation also reveals other structural properties. The first one is the existence of *conserved quantities*, which are determined by the structure matrices $J(x), R(x)$.

Definition 6.4.1 A *Casimir* function for an input-state-output port-Hamiltonian system (6.1) or (6.3) is any function $C : \mathcal{X} \to \mathbb{R}$ satisfying

$$\frac{\partial^T C}{\partial x}(x)\,[J(x) - R(x)] = 0, \quad x \in \mathcal{X} \tag{6.72}$$

It follows that for $u = 0$

$$\frac{d}{dt}C = \frac{\partial^T C}{\partial x}(x)\,[J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) = 0, \tag{6.73}$$

and thus a Casimir function is a *conserved quantity* of the system for $u = 0$, *independently* of the Hamiltonian H. Note furthermore that if $C_1, \ldots, C_r$ are Casimirs, then also the composed function $\Phi(C_1, \ldots, C_r)$ is a Casimir for any $\Phi : \mathbb{R}^r \to \mathbb{R}$. Finally, the existence of Casimirs $C_1, \ldots, C_r$ entails the following invariance property of the dynamics: any subset

$$\{x \mid C_1(x) = c_1, \ldots, C_r(x) = c_r\} \tag{6.74}$$

for arbitrary constants $c_1, \ldots, c_r$ is an invariant subset of the dynamics.

Proposition 6.4.2 *$C : \mathcal{X} \to \mathbb{R}$ is a Casimir function for (6.1) or (6.3) if and only if*

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0 \text{ and } \frac{\partial^T C}{\partial x}(x)R(x) = 0, \quad x \in \mathcal{X} \tag{6.75}$$

Proof The "if" implication is obvious. For the converse we note that (6.72) implies by skew-symmetry of $J(x)$

$$0 = \frac{\partial^T C}{\partial x}(x)\,[J(x) - R(x)]\,\frac{\partial C}{\partial x}(x) = -\frac{\partial^T C}{\partial x}(x)R(x)\frac{\partial C}{\partial x}(x), \tag{6.76}$$

and therefore, in view of $R(x) \geq 0$, $\frac{\partial^T C}{\partial x}(x)R(x) = 0$, and thus also $\frac{\partial^T C}{\partial x}(x)$ $J(x) = 0$. □

Hence, the vectors $\frac{\partial C}{\partial x}(x)$ of partial derivatives of the Casimirs C are contained in the intersection of the kernels of the matrices $J(x)$ and $R(x)$ for any $x \in \mathcal{X}$, implying that the maximal number of independent number of Casimirs is always bounded from above by dim $(\ker J(x) \cap \ker R(x))$. Equality, however, need not be true because of *lack of integrability* of $J(x)$ and/or $R(x)$; see Example 6.4.4 below and the Notes at the end of this chapter.

Example 6.4.3 (*Example* 6.2.1 *continued*) Consider Euler's equations for the angular momenta of a rigid body, with J being given by (6.31) and $R = 0$. It follows that $C(p_1, p_2, p_3) = p_x^2 + p_y^2 + p_z^2$ (the squared total angular momentum) is a Casimir function.

Example 6.4.4 (*Example* 6.2.3 *continued*) The pde's (6.72) for the existence of a Casimir function take the form

$$\begin{aligned} &\tfrac{\partial C}{\partial p_1} = \tfrac{\partial C}{\partial p_2} = \tfrac{\partial C}{\partial \phi} = 0 \\ &\tfrac{\partial C}{\partial x}\cos\phi + \tfrac{\partial C}{\partial y}\sin\phi + \tfrac{\partial C}{\partial \theta} = 0 \end{aligned} \tag{6.77}$$

This can be seen *not* to possess a non-trivial solution C, due to the non-holonomicity of the kinematic constraints.

The definition of Casimir C for (6.1) can be further strengthened by requiring that $\frac{d}{dt}C = 0$ for all input values u. This leads to the *stronger* condition

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \quad \frac{\partial^T C}{\partial x}(x)R(x) = 0, \quad \frac{\partial^T C}{\partial x}(x)g(x) = 0, \quad x \in \mathcal{X} \tag{6.78}$$

A second property of the dynamics of port-Hamiltonian systems, which is closely connected to the structure matrix $J(x)$ and its integrability conditions (6.33) is *volume-preservation*. Indeed, consider the case $R(x) = 0$, and let us assume that (6.33) is satisfied with rank $J(x) = \dim \mathcal{X} = n$, implying the existence of local coordinates (q, p) such that (see (6.35))

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \tag{6.79}$$

with $n = 2k$. Define the *divergence* of any set of differential equations

$$\dot{x}_i = X_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{6.80}$$

in a set of local coordinates $x_1, \ldots, x_n$ as

$$\operatorname{div}(X)(x) = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}(x) \tag{6.81}$$

Denote the solution trajectories of (6.80) from $x(0) = x_0$ by $x(t; x_0) = X^t(x_0),\ t \geq 0$. Then it is a standard fact that the maps $X^t : \mathbb{R}^n \to \mathbb{R}^n$ are *volume-preserving*, that is,

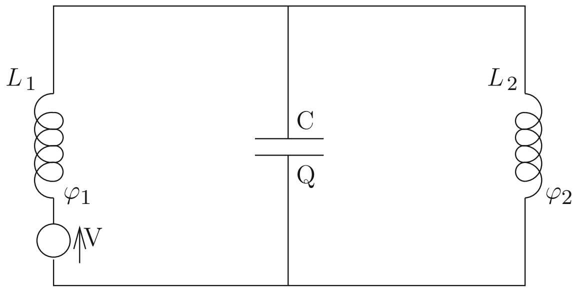
$$\det\left[\frac{\partial X^t}{\partial x}(x)\right] = 1, \qquad \text{for all } x, t \geq 0, \tag{6.82}$$

if and only if $\operatorname{div}(X)(x) = 0$ for all x. Returning to the Hamiltonian dynamics

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x), \tag{6.83}$$

with J given by (6.79) it is easily verified that the divergence in the (q, p)-coordinates is everywhere zero, and hence the solutions of (6.83) preserve the standard volume in (q, p)-space. In case rank $J(x) < \dim \mathcal{X}$ and there exist local coordinates (q, p, s) as in (6.35), then the divergence is still zero, and it follows that the Hamiltonian dynamics (6.83) preserves the standard volume in (q, p, s)-space, with the additional property that on any (invariant) level set

Fig. 6.6 LC-circuit



$$s_1 = c_1, \ldots, s_\ell = c_l \tag{6.84}$$

the volume in (q, p)-coordinates is preserved.

Example 6.4.5 (*LC-circuit*) Consider the LC-circuit (see Fig. 6.6) consisting of two inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear then $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, $H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2$ and $H_3(Q) = \frac{1}{2C}Q^2$. Furthermore, $V = u$ denotes a voltage source. Using Kirchhoff's current and voltage laws one immediately arrives at the port-Hamiltonian system formulation

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{6.85}$$

$$y = \frac{\partial H}{\partial \varphi_1} \qquad (= \text{current through first inductor})$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly, the matrix J is skew-symmetric, and since J is constant it trivially satisfies (6.33). The quantity $\varphi_1 + \varphi_2$ (total flux linkage) can be seen to be a Casimir function. The volume in $(Q, \varphi_1, \varphi_2)$-space is preserved.

Finally, let us comment on the implications of the port-Hamiltonian structure for the use of *Brockett's necessary condition* for asymptotic stabilizability. Loosely speaking, Brockett's necessary condition [51] tells us that a necessary condition for asymptotic stabilizability of a nonlinear system $\dot{x} = f(x, u), f(0, 0) = 0$, using *continuous* state feedback is that the image of the map $(x, u) \mapsto f(x, u)$, for x and u arbitrarily close to zero, should contain a neighborhood of the origin. Application to input-state-output port-Hamiltonian systems leads to the following necessary condition for asymptotic stabilizability.

Proposition 6.4.6 *Consider the input-state-output port-Hamiltonian system (6.3) with equilibrium x_0. A necessary condition for asymptotic stabilizability around x_0 is that for every $\varepsilon > 0$*

$$\cup_{\{x;||x-x_0||<\varepsilon\}} (\text{im}\,[J(x) - R(x)] + \text{im}\,[G(x) - P(x)]) = \mathbb{R}^n \tag{6.86}$$

As an application of this result, we note that the port-Hamiltonian system (6.48) arising from a mechanical system with kinematic constraints does never satisfy the necessary condition (6.86). Indeed, by the specific forms of J, $R = 0$, and g, see (6.48) and (6.49),

$$\text{im}\,[J(x) - R(x)] + \text{im}\,[G(x) - P(x)] \subset \text{im} \begin{bmatrix} S(q) \\ 0 \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \tag{6.87}$$

where rank $S(q) = n - k$, $q \in \mathcal{Q}$. After possibly reordering the rows of $S(q)$ we may without loss of generality assume that

$$S(q) = \begin{bmatrix} S_1(q) \\ S_2(q) \end{bmatrix} \tag{6.88}$$

with the $(n-k) \times (n-k)$ matrix $S_2(q)$ of full rank $n - k$ in a neighborhood of the equilibrium position vector of interest, and therefore the rows of S_1 depending on the rows of S_2. It follows that vectors of the form $\begin{bmatrix} * \\ 0 \end{bmatrix}$, with 0 the $(n-k)$-dimensional zero-vector, can not be in the image of $S(q)$, and hence not in $\text{im}\,[J(x) - R(x)] + \text{im}\,[G(x) - P(x)]$. Hence,

Corollary 6.4.7 *Mechanical systems with kinematic constraints (6.48) are not asymptotically stabilizable using continuous feedback.*

For *holonomic* kinematic constraints this is not surprising, since in this case we should first eliminate the conserved quantities $\bar{q}_{n-k+1}, \ldots, \bar{q}_n$ as in (6.39) from the system (6.48). However, since for *nonholonomic* kinematic constraints such an elimination is *not* possible, the above observation indeed entails an important obstruction for asymptotic stabilization[2] of mechanical systems with nonholonomic constraints.

For a further discussion of the dynamical properties of port-Hamiltonian systems, we refer to the extensive literature on this topic; see the references quoted in the Notes at the end of this chapter. Still another use of the port-Hamiltonian structure will be provided separately in the next section.

6.5 Shifted Passivity of Port-Hamiltonian Systems

In many cases of interest, the desired set-point of a port-Hamiltonian system is *not* equal to the minimum of the Hamiltonian function H (an equilibrium of the system for zero-input), but instead is a *steady-state* value corresponding to a nonzero constant

[2]However, asymptotic feedback stabilization using *discontinuous* or *time-varying* feedback may still be possible.

input. (We already encountered the same scenario in Chap. 4 in the context of passive systems.) This motivates the following developments.

Proposition 6.5.1 *Consider an input-state-output port-Hamiltonian system with feedthrough terms (6.3), together with a constant input $\bar{u}$ with corresponding steady-state $\bar{x}$ determined by*

$$0 = [J(\bar{x}) - R(\bar{x})] \frac{\partial H}{\partial x}(\bar{x}) + [G(\bar{x}) - P(\bar{x})]\, \bar{u} \tag{6.89}$$

Denote

$$\bar{y} = [G(\bar{x}) + P(\bar{x})]^T \frac{\partial H}{\partial x}(\bar{x}) + [M(\bar{x}) + S(\bar{x})]\, \bar{u} \tag{6.90}$$

Suppose we can find coordinates x in which the system matrices $J(x), M(x)$, $R(x), P(x), S(x), G(x)$ are all constant. Then the system can be rewritten as

$$\begin{aligned} \dot{x} &= [J - R]\, \tfrac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x) + [G - P]\,(u - \bar{u}) \\ y - \bar{y} &= [G + P]^T\, \tfrac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x) + [M + S]\,(u - \bar{u}) \end{aligned} \tag{6.91}$$

with respect to the shifted Hamiltonian[3] *defined as*

$$\widehat{H}_{\bar{x}}(x) := H(x) - \frac{\partial^T H}{\partial x}(\bar{x})(x - \bar{x}) - H(\bar{x}) \tag{6.92}$$

If H is convex in the coordinates x, then $\widehat{H}_{\bar{x}}$ has a minimum at $x = \bar{x}$ (with value 0), and the port-Hamiltonian system is passive with respect to the shifted supply rate $s(u, y) = (u - \bar{u})^T (y - \bar{y})$, with storage function $\widehat{H}_{\bar{x}}$.

Proof Observe that

$$\frac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(\bar{x}) \tag{6.93}$$

Adopting the shorthand notation $z = \frac{\partial H}{\partial x}(x)$ and $\bar{z} = \frac{\partial H}{\partial x}(\bar{x})$, we obtain

$$\begin{aligned} &\tfrac{d}{dt}\widehat{H}_{\bar{x}} = (z - \bar{z})^T\, [(J - R)z + (G - P)u] \\ &= (z - \bar{z})^T\, [(J - R)(z - \bar{z}) + (G - P)(u - \bar{u})] \\ &= -\left[(z - \bar{z})^T\ (u - \bar{u})^T\right] \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \begin{bmatrix} (z - \bar{z}) \\ (u - \bar{u}) \end{bmatrix} + (u - \bar{u})^T (y - \bar{y}) \\ &\leq (u - \bar{u})^T (y - \bar{y}), \end{aligned} \tag{6.94}$$

showing passivity with respect to the shifted supply rate $(u - \bar{u})^T (y - \bar{y})$. Finally, $\widehat{H}_{\bar{x}}(\bar{x}) = 0$ and convexity of H is equivalent to

[3]Note that the function $\widehat{H}_{\bar{x}}$ admits the following geometric interpretation. Consider the surface in $\mathbb{R}^{n+1}$ defined by H, and the tangent plane at the point $(\bar{x}, H(\bar{x}) \in \mathbb{R}^n$ to this surface. Then $\widehat{H}_{\bar{x}}(x)$ is the vertical distance above the point $x \in \mathbb{R}^n$ from this tangent plane to the surface.

$$H(x) \geq \frac{\partial^T H}{\partial x}(\bar{x})(x-\bar{x}) + H(\bar{x}), \text{ for all } x, \bar{x}, \tag{6.95}$$

implying that $\widehat{H}_{\bar{x}}(x) \geq 0, \ x \in \mathcal{X}$. □

Recall from Chap. 4, cf. (4.159), that the property of being passive with respect to the shifted supply rate $(u-\bar{u})^T(y-\bar{y})$ is referred to as *shifted passivity*. It follows from Proposition 6.5.1 that with *constant* system matrices J, M, R, P, S, G and a *convex* Hamiltonian H the input-state-output port-Hamiltonian system with feedthrough term is shifted passive with respect to *any* constant $\bar{u}$ for which there exists a steady-state $\bar{x}$ (and corresponding $\bar{y}$).

Remark 6.5.2 The function $\widehat{H}_{\bar{x}}(x)$, regarded as a function of x *and* $\bar{x}$, is known in convex analysis as the *Bregman divergence* or *Bregman distance*. It also appears as the *availability function* in thermodynamics (dating back to the classical work of Gibbs [113]), and was introduced in the present context in [146]. Note that the definition of $\widehat{H}_{\bar{x}}$ (as well as the notion of a convex function) depends on the choice of coordinates x for the state space $\mathcal{X}$.

Example 6.5.3 Consider the model of a power network formulated in Example 4.4.4; see (4.87). Identifying the Hamiltonian with the storage function already defined[4] in (4.88)

$$H(q,p) = \frac{1}{2}p^T J^{-1} p - \sum_{j=1}^{M} \gamma_j \cos q_j, \tag{6.96}$$

the system takes the port-Hamiltonian form

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \\ y &= \frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{6.97}$$

The steady-state $(\bar{q}, \bar{p})$ corresponding to constant input $\bar{u}$ is determined by

$$\begin{aligned} 0 &= D^T \frac{\partial H}{\partial p}(\bar{q}, \bar{p}) \\ 0 &= D \frac{\partial H}{\partial q}(\bar{q}, \bar{p}) + A \frac{\partial H}{\partial p}(\bar{q}, \bar{p}) + \bar{u} \end{aligned} \tag{6.98}$$

Assuming the graph to be *connected* the first equation leads to $\frac{\partial H}{\partial p}(\bar{q}, \bar{p}) = \mathbb{1}\omega_*$, with $\omega_* \in \mathbb{R}$ a common frequency deviation. Furthermore, by premultiplying the second equation by the row-vector $\mathbb{1}^T$ of all ones,

$$0 = \omega_* \sum_{i=1}^{N} A_i + \sum_{i=1}^{N} \bar{u}_i, \quad \bar{p} = J\mathbb{1}\omega_*, \tag{6.99}$$

[4] The matrix J in the Hamiltonian refers to the inertia of the generators; not to be confused with the Poisson structure.

determining ω_*, and thus $\bar{p}$, as a function of the total generated and consumed power $\sum \bar{u}_i$. Finally, the steady-state vector $\bar{q}$ of phase angle differences is determined by

$$0 = D\Gamma \operatorname{Sin} \bar{q} - A\mathbb{1}\frac{\sum_{i=1}^{N} \bar{u}_i}{\sum_{i=1}^{N} A_i} + \bar{u} \tag{6.100}$$

(Note that by boundedness of the mapping Sin this does not have a solution for large $\bar{u}$.) Defining the shifted Hamiltonian as in (6.92) yields

$$\begin{aligned} \widehat{H}_{(\bar{q},\bar{p})}(q,p) = \tfrac{1}{2}(p-\bar{p})^T J^{-1}(p-\bar{p}) - \textstyle\sum_{j=1}^{M}\gamma_j \cos q_j \\ -\textstyle\sum_{j=1}^{M}\gamma_j \sin \bar{q}_j (q_j - \bar{q}_j) + \sum_{j=1}^{M}\gamma_j \cos \bar{q}_j \end{aligned} \tag{6.101}$$

It follows that the system is shifted passive with respect to the shifted supply rate $(u-\bar{u})^T(y-\bar{y})$ and storage function $\widehat{H}_{(\bar{q},\bar{p})}$, with $\bar{y} = \frac{\partial H}{\partial p}(\bar{q},\bar{p}) = J^{-1}\bar{p} = \mathbb{1}\omega_*$.

If no coordinates exist in which the matrices $J(x), M(x), R(x), P(x), S(x), G(x)$ are all constant, the analysis for nonzero $\bar{u}$ becomes much harder. Define the combined interconnection and resistive structure matrix

$$K(x) := \begin{bmatrix} -J(x)+R(x) & -G(x)+P(x) \\ G^T(x)+P^T(x) & M(x)+S(x) \end{bmatrix} \tag{6.102}$$

Proposition 6.5.4 *Consider an input-state-output port-Hamiltonian system with feedthrough terms (6.3), and a steady-state triple $\bar{u}, \bar{x}, \bar{y}$. Then the system is shifted passive with storage function $\widehat{H}_{\bar{x}}$ if*[5]

$$\begin{bmatrix} \frac{\partial^T H}{\partial x}(x) - \frac{\partial^T H}{\partial x}(\bar{x}) & u^T - \bar{u}^T \end{bmatrix} \left(K(x)\begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ u \end{bmatrix} - K(\bar{x})\begin{bmatrix} \frac{\partial H}{\partial x}(\bar{x}) \\ \bar{u} \end{bmatrix} \right) \geq 0 \tag{6.103}$$

for all x, u.

Proof By direct computation of $\frac{d}{dt}\widehat{H}_{\bar{x}}$; see [97]. □

Furthermore, shifting with respect to constant inputs $\bar{u}$ can still be done if the input matrix $g(x)$ of the port-Hamiltonian system (6.1) satisfies the following *integrability condition* with respect to the combined geometric structure $J(x) - R(x)$. Assume that for each j-th column $g_j(x)$ of the input matrix $g(x)$ there exists a function $C_j : \mathcal{X} \to \mathbb{R}$ such that

$$g_j(x) = -\left[J(x) - R(x)\right]\frac{\partial F_j}{\partial x}(x), \quad j = 1, \ldots, m \tag{6.104}$$

Then for any constant $\bar{u}$, the dynamics of the port-Hamiltonian system (6.1) can be rewritten as

[5] Since $K(x) + K^T(x) \geq 0$ the condition (6.103) is automatically satisfied in case K does not depend on x.

$$\dot{x} = [J(x) - R(x)] \frac{\partial \widetilde{H}}{\partial x}(x) + g(x)(u - \bar{u}), \tag{6.105}$$

with

$$\widetilde{H}(x) := H(x) - \sum_{j=1}^{m} F_j(x)\bar{u}_j \tag{6.106}$$

(In case of constant matrices J, R, g one verifies that $\widetilde{H}(x) = \widehat{H}_{\bar{x}}$, where $\bar{x}$ is the steady-state corresponding to $\bar{u}$.) However, in general this does *not* imply passivity with respect to the shifted supply rate $(u - \bar{u})^T(y - \bar{y})$, with $\bar{y}$ the steady-state output value. Nevertheless, the condition *is* useful especially in case part of the inputs can be considered as constant "disturbances" $\bar{u}$, with remaining other inputs v corresponding to the dynamics

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + b(x)v + g(x)\bar{u} \tag{6.107}$$

for some input matrix $b(x)$. In this case, satisfaction of (6.104) allows one to rewrite the system as

$$\dot{x} = [J(x) - R(x)] \frac{\partial \widetilde{H}}{\partial x}(x) + b(x)v \tag{6.108}$$

which is port-Hamiltonian with respect to the inputs v and corresponding outputs $z = b^T(x)\frac{\partial \widetilde{H}}{\partial x}(x)$.

Finally, let us come back to the notion of the *steady-state input–output relation* as defined in Chap. 4, cf. (4.31). In the port-Hamiltonian case, one obtains the following result

Proposition 6.5.5 *Consider an input-state-output port-Hamiltonian system with feedthrough terms (6.3). Its steady-state input–output relation is given as*

$$\begin{aligned} \{(\bar{u}, \bar{y}) \mid \exists \bar{x} \text{ s.t. } 0 &= [J(\bar{x}) - R(\bar{x})]\tfrac{\partial H}{\partial x}(\bar{x}) + [G(\bar{x}) - P(\bar{x})]\bar{u}, \\ \bar{y} &= [G(\bar{x}) + P(\bar{x})]^T \tfrac{\partial H}{\partial x}(\bar{x}) + [M(\bar{x}) + S(\bar{x})]\bar{u}\,\} \end{aligned} \tag{6.109}$$

In particular, if $[J(\bar{x}) - R(\bar{x})]$ is invertible, the steady-state input–output relation is given as the graph of the mapping (from $\bar{u}$ to $\bar{y}$)

$$\bar{y} = -[G(\bar{x}) + P(\bar{x})]^T \left(J(\bar{x}) - R(\bar{x})\right)^{-1} [G(\bar{x}) - P(\bar{x})]\bar{u} + [M(\bar{x}) + S(\bar{x})]\bar{u} \tag{6.110}$$

which is linear in case the matrices J, R, G, P, M, S are all constant.

Note that the matrix in (6.110) is equal to the Schur complement of the matrix $K(\bar{x})$ defined in (6.102) with respect to its left-upper block. Since the symmetric part of $K(\bar{x})$ is ≥ 0, this Schur complement inherits the same positivity property.

Proposition 6.5.5 can be extended to input-state-output port-Hamiltonian systems with *nonlinear resistive structure* as in Definition 6.1.4. For example, the steady-state input–output relation corresponding to the port-Hamiltonian system

$$\dot{x} = -\mathcal{R}(z) + u, \; y = z, \quad z = \frac{\partial H}{\partial x}(x) \tag{6.111}$$

with the nonlinear resistive mapping $\mathcal{R}$ satisfying (6.7), is given by

$$\{(\bar{u}, \bar{y}) \mid \bar{u} = \mathcal{R}(\bar{y})\} \tag{6.112}$$

provided for each $\bar{u}$ there exists a steady-state $\bar{x}$ such that $\bar{u} = \mathcal{R}(\frac{\partial H}{\partial x}(\bar{x}))$.

6.6 Dirac Structures

In Chap. 3, the definition of *dissipativity* was extended to differential-algebraic equation (DAE) systems $F(\dot{x}, x, w) = 0$, with w denoting the vector of external variables (inputs *and* outputs).

Similarly, in this and the next section we show how the definition of input-state-output port-Hamiltonian systems can be extended to the DAE case. This extension is crucial from a modeling point of view, since first principles modeling of physical systems often leads to DAE systems. This stems from the fact that in many modeling approaches the system under consideration is naturally regarded as obtained from interconnecting simpler subsystems. These interconnections often give rise to algebraic constraints between the state space variables of the subsystems; thus leading to DAE systems.

The key to define port-Hamiltonian DAE systems is the geometric notion of a *Dirac structure*, formalizing the concept of a power-conserving interconnection, and generalizing the notion of an (almost-)Poisson structure matrix $J(x)$ as encountered before.

Let us return to the basic setting of passivity (see Chap. 2), starting with a finite-dimensional linear space and its dual, with the duality product defining power. Thus, let $\mathcal{F}$ be an ℓ-dimensional linear space, and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{E} := \mathcal{F}^*$. We call $\mathcal{F}$ the space of *flows* f, and $\mathcal{E}$ the space of *efforts* e. On the product space $\mathcal{F} \times \mathcal{E}$, *power* is defined by

$$< e \mid f >, \quad (f, e) \in \mathcal{F} \times \mathcal{E}, \tag{6.113}$$

where $< e \mid f >$ denotes the *duality product*, that is, the linear function $e \in \mathcal{E} = \mathcal{F}^*$ acting on $f \in \mathcal{F}$.

Remark 6.6.1 Recall from Chap. 2 that if $\mathcal{F}$ is endowed with an *inner-product* structure $<,>$, then $\mathcal{E} = \mathcal{F}^*$ can be *identified* with $\mathcal{F}$ in such a way that $< e \mid f >= < e, f >, \; f \in \mathcal{F}, \; e \in \mathcal{E} \simeq \mathcal{F}$.

Example 6.6.2 Let $\mathcal{F}$ be the space of generalized *velocities*, and $\mathcal{E} = \mathcal{F}^*$ the space of generalized *forces*, then $< e \mid f >$ is mechanical power. Similarly, let $\mathcal{F}$ be the space of *currents*, and $\mathcal{E} = \mathcal{F}^*$ be the space of *voltages*, then $< e \mid f >$ is electrical power.

In multi-body systems one considers the space of *twists* $\mathcal{F} = se(3)$ (the Lie algebra of the matrix special Euclidian group $SE(3)$), with $\mathcal{E} = \mathcal{F}^* = se^*(3)$ the space of *wrenches*.

As already introduced in Sect. 2.4, there exists on $\mathcal{F} \times \mathcal{E}$ a canonically defined symmetric bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1 \mid f_2 > + < e_2 \mid f_1 > \tag{6.114}$$

for $f_i \in \mathcal{F},\ e_i \in \mathcal{E}, i = 1, 2$. Now consider a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E} \tag{6.115}$$

and its orthogonal companion $\mathcal{D}^{\perp\!\!\perp}$ with respect to the bilinear form $\ll, \gg$ on $\mathcal{F} \times \mathcal{E}$, defined as

$$\mathcal{D}^{\perp\!\!\perp} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid \ll (f, e), (\tilde{f}, \tilde{e}) \gg = 0 \text{ for all } (\tilde{f}, \tilde{e}) \in \mathcal{D}\} \tag{6.116}$$

Clearly, if $\mathcal{D}$ has dimension d, then the subspace $\mathcal{D}^{\perp\!\!\perp}$ has dimension $2 \dim \mathcal{F} - d$ (since $\ll, \gg$ is a non-degenerate form on $\mathcal{F} \times \mathcal{E}$, and furthermore $\dim \mathcal{F} \times \mathcal{E} = 2 \dim \mathcal{F}$).

Definition 6.6.3 A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a (constant) Dirac structure if

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\perp} \tag{6.117}$$

It immediately follows that the dimension of any Dirac structure $\mathcal{D}$ is equal to $\dim \mathcal{F}$. Furthermore, let $(f, e) \in \mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$. Then by (6.114)

$$0 = \ll (f, e), (f, e) \gg = 2 < e \mid f > = 0 \tag{6.118}$$

Hence, a Dirac structure $\mathcal{D}$ defines a *power-conserving* relation between the variables $(f, e) \in \mathcal{F} \times \mathcal{E}$. Conversely, we obtain

Proposition 6.6.4 *Let $\mathcal{F}$ be a finite-dimensional linear space. Then $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure if and only if $< e \mid f > = 0$ for all $(f, e) \in \mathcal{D}$, and $\mathcal{D}$ is a maximal subspace with this property. In particular, for any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ satisfying $< e \mid f > = 0$ for all $(f, e) \in \mathcal{D}$ we have $\dim \mathcal{D} \leq \dim \mathcal{F}$, while $\mathcal{D}$ satisfying $< e \mid f > = 0$ for all $(f, e) \in \mathcal{D}$ is a Dirac structure if and only if $\dim \mathcal{D} = \dim \mathcal{F}$.*

Proof First, consider any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ satisfying $< e \mid f > = 0$ for all $(f, e) \in \mathcal{D}$. Let $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$. Then also $(f_1 + f_2, e_1 + e_2) \in \mathcal{D}$, and thus

$$\begin{aligned} 0 = &< e_1 + e_2 \mid f_1 + f_2 > = \\ &< e_1 \mid f_2 > + < e_2 \mid f_1 > + < e_1 \mid f_1 > + < e_2 \mid f_2 > = \\ &< e_1 \mid f_2 > + < e_2 \mid f_1 > = \ll (f_1, e_1), (f_2, e_2) \gg \end{aligned} \tag{6.119}$$

Hence, $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. In view of (6.118), we have thus proved that $< e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$ if and only if $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. Furthermore, $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$ implies $\dim \mathcal{D} \leq \dim \mathcal{D}^{\perp\!\!\perp} = 2 \dim \mathcal{F} - \dim \mathcal{D}$, and hence $\dim \mathcal{D} \leq \dim \mathcal{F}$. Conversely, if $\dim \mathcal{D} = \dim \mathcal{F}$ then $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$, and $\mathcal{D}$ is a Dirac structure. Hence, we have proved the second claim, and the "only if" direction of the first claim using (6.118). For the "if" direction of the first claim we use again that $< e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$ implies $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. Now suppose that $\mathcal{D} \subsetneq \mathcal{D}^{\perp\!\!\perp}$. Then we can non-trivially *extend* $\mathcal{D}$ to a subspace $\mathcal{D}'$ such that $\mathcal{D}' \subset \mathcal{D}'^{\perp\!\!\perp}$, and thus $\mathcal{D}$ is not maximal. □

Remark 6.6.5 The condition $\dim \mathcal{D} = \dim \mathcal{F}$ is intimately related to the statement that a physical interconnection can *not* determine at the same time both the flow and effort (e.g., current *and* voltage, or velocity *and* force).

Constant Dirac structures admit different *matrix representations*.

Proposition 6.6.6 *Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, with $\dim \mathcal{F} = \ell$, be a constant Dirac structure. Take linear coordinates for $\mathcal{F}$ and dual coordinates for $\mathcal{E} = \mathcal{F}^*$, resulting in $\mathcal{F} \simeq \mathbb{R}^m \simeq \mathcal{E}$. Then $\mathcal{D}$ can be represented in any of the following ways.*

1. *(Kernel and Image representation)*

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0\} \tag{6.120}$$

   for $\ell \times \ell$ matrices[6] F and E satisfying

$$\begin{aligned} (i)\;\; & EF^T + FE^T = 0 \\ (ii)\; & \operatorname{rank}\, [F \,\vdots\, E] = \ell \end{aligned} \tag{6.121}$$

   Equivalently in image representation,

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid \exists \lambda \in \mathbb{R}^\ell \; s.t. \, f = E^T \lambda, \;\; e = F^T \lambda\} \tag{6.122}$$

   Conversely, for any $\ell \times \ell$ matrices F and E satisfying (6.121)*, the subspaces* (6.120) *and* (6.122) *are Dirac structures.*
2. *(Constrained input–output representation)*

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid \exists \lambda \; s.t. \, f = Je + G\lambda, \;\; G^T e = 0\} \tag{6.123}$$

   for an $\ell \times \ell$ skew-symmetric matrix J, and a matrix G such that $\operatorname{im} G = \{f \mid (f, 0) \in \mathcal{D}\}$*. Furthermore,* $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$*. Conversely, for any G and skew-symmetric J the subspace* (6.123) *is a Dirac structure.*
3. *(Hybrid input–output representation).*
   Let $\mathcal{D}$ be given as in (6.120). Suppose $\operatorname{rank} F = \ell^1 \leq \ell$*. Select ℓ^1 independent*

[6] We may also allow F and E to be $l' \times l$ matrices with $l' \geq l$, and satisfying (6.121). This is called a *relaxed* kernel representation.

columns of F, and group them into a matrix F^1. *Write (possibly after permutations)* $F = [F^1 \vdots F^2]$, *and correspondingly* $E = [E^1 \vdots E^2]$, $f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}$, $e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}$. *Then the matrix* $[F^1 \vdots E^2]$ *can be shown to be invertible, and*

$$\mathcal{D} = \left\{ \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, \begin{bmatrix} e^1 \\ e^2 \end{bmatrix} \mid \begin{bmatrix} f^1 \\ e^2 \end{bmatrix} = J \begin{bmatrix} e^1 \\ f^2 \end{bmatrix} \right\} \tag{6.124}$$

where $J := -[F^1 \vdots E^2]^{-1}[F^2 \vdots E^1]$ *is skew-symmetric. Conversely, for any skew-symmetric J the subspace (6.124) is a Dirac structure.*

4. *(Canonical coordinate representation)*
There exist linear coordinates (q, p, r, s) *for* $\mathcal{F}$ *such that in these coordinates and dual coordinates for* $\mathcal{E} = \mathcal{F}^*$, $(f, e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in \mathcal{D}$ *if and only if*

$$\begin{aligned} f_q &= e_p, \; f_p = -e_q \\ f_r &= 0, \quad e_s = \; 0 \end{aligned} \tag{6.125}$$

Proof (1) It is directly checked that (6.122) defines a Dirac structure. Since by (6.121) im $\begin{bmatrix} E^T \\ F^T \end{bmatrix} = \ker [F \vdots E]$, also (6.120) defines the same Dirac structure. Conversely, any ℓ-dimensional subspace $\mathcal{D}$ can be written as $\mathcal{D} = \text{im} \begin{bmatrix} E^T \\ F^T \end{bmatrix}$ for some $\ell \times \ell$ matrices F, E satisfying rank $[F \vdots E] = \ell$. If $\mathcal{D}$ is a Dirac structure then $0 = e^T f = (F^T \lambda)^T E^T \lambda = \lambda^T F E^T \lambda$ for all $\lambda \in \mathbb{R}^\ell$. This is equivalent to $EF^T + FE^T = 0$.
(2) Consider $\mathcal{D}$ given by (6.123) with $J = -J^T$. Then $e^T f = e^T(Je + G\lambda) = e^T Je + e^T G\lambda = (G^T e)^T \lambda = 0$. Hence, $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. Let now $(\tilde{f}, \tilde{e})$ be such that $0 = \ll (f, e), (\tilde{f}, \tilde{e}) \gg$ for all $(f, e) \in \mathcal{D}$, i.e., $f = Je + G\lambda$, $G^T e = 0$. Then

$$0 = e^T \tilde{f} + \tilde{e}^T f = e^T \tilde{f} + \tilde{e}^T (Je + G\lambda)$$

for all λ and e with $G^T e = 0$. First take $e = 0$. Then $0 = \tilde{e}^T G\lambda$ for all λ, implying that $G^T \tilde{e} = 0$. Hence, $0 = e^T \tilde{f} + \tilde{e}^T Je = e^T(\tilde{f} - J\tilde{e})$ for all e with $G^T e = 0$, implying that $\tilde{f} = J\tilde{e} + G\tilde{\lambda}$, for some $\tilde{\lambda}$. Thus $\mathcal{D}^{\perp\!\!\perp} \subset \mathcal{D}$, and therefore $\mathcal{D}^{\perp\!\!\perp} = \mathcal{D}$. On the other hand, take any Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$. Define the following subspace of $\mathcal{E}$

$$\mathcal{E}_{\mathcal{D}} = \{ e \in \mathcal{E} \mid \exists f \text{ s.t. } (f, e) \in \mathcal{D} \} \tag{6.126}$$

It can be checked that

$$\mathcal{E}_{\mathcal{D}}^{\perp} = \{ f \in \mathcal{F} \mid (f, 0) \in \mathcal{D} \}, \tag{6.127}$$

where $^\perp$ denotes orthogonality with respect to the duality product $< \mid >$. Furthermore, select any subspace $\bar{\mathcal{E}}_{\mathcal{D}}$ complementary to $\mathcal{E}_{\mathcal{D}} \subset \mathcal{E}$, i.e.,

$$\mathcal{E} = \mathcal{E}_{\mathcal{D}} \oplus \bar{\mathcal{E}}_{\mathcal{D}}$$

Define any matrix G such that $\mathcal{E}_{\mathcal{D}} = \ker G^T$. Define the linear map $J : \mathcal{E} \to \mathcal{F}$ as follows. Define J to be zero on $\bar{\mathcal{E}}_{\mathcal{D}}$. In view of (6.127), there exists for any $e \in \mathcal{E}_{\mathcal{D}}$ a unique $f \in \bar{\mathcal{E}}_{\mathcal{D}}^{\perp}$ such that $(f, e) \in \mathcal{D}$. Define $Je = f$. Since $(f, e) \in \mathcal{D}$ we have $e^T f = 0$, implying skew-symmetry of J. It is readily checked that $\mathcal{D}$ is given as in (6.123).
(3) By skew-symmetry of J it directly follows that $\mathcal{D}$ defined by (6.124) is a Dirac structure. With regard to all remaining statements, see [47].
(4) See [72]. □

Remark 6.6.7 One may also convert any matrix representation into any other one. For example, start from a Dirac structure $\mathcal{D}$ given in constrained input–output representation (6.123). Define $G^{\perp}$ as a matrix of maximal rank such that $G^{\perp} G = 0$ and with independent rows. Then $\mathcal{D}$ is equivalently given in kernel representation as

$$\mathcal{D} = \{(f, e) \mid \begin{bmatrix} -G^{\perp} \\ 0 \end{bmatrix} f + \begin{bmatrix} G^{\perp} J \\ G^T \end{bmatrix} e = 0\} \tag{6.128}$$

Example 6.6.8 The combination of Kirchhoff's current and voltage laws for an electrical circuit constitute an example of a constrained input–output representation (6.123) of a Dirac structure. Let $\mathcal{F}$ be the space of currents I through the edges of the circuit graph, and $\mathcal{E} = \mathcal{F}^*$ the space of voltages V across the edges. Let D be the $N \times M$ incidence matrix of the circuit graph (N nodes/vertices, M branches/edges). Then Kirchhoff's current and voltage laws define the Dirac structure

$$\mathcal{D} := \{(I, V) \in \mathbb{R}^M \times \mathbb{R}^M \mid DI = 0, \exists \lambda \in \mathbb{R}^N \text{ s.t. } V = D^T \lambda\}, \tag{6.129}$$

which is in constrained input–output representation (6.123), with $J = 0$ and $G = D^T$. Defining a matrix E such that $\operatorname{im} D^T = \ker E$ one obtains the relaxed kernel representation $DI = 0, EV = 0$.

Given a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, one can define the following subspaces of $\mathcal{F}$, respectively, $\mathcal{E}$,

$$\begin{aligned} \mathcal{G}_0 &:= \{f \in \mathcal{F} \mid (f, 0) \in \mathcal{D}\} \\ \mathcal{G}_1 &:= \{f \in \mathcal{F} \mid \exists e \in \mathcal{E} \text{ s.t. } (f, e) \in \mathcal{D}\} \\ \mathcal{P}_0 &:= \{e \in \mathcal{E} \mid (0, e) \in \mathcal{D}\} \\ \mathcal{P}_1 &:= \{e \in \mathcal{E} \mid \exists f \in \mathcal{F} \text{ s.t. } (f, e) \in \mathcal{D}\} \end{aligned} \tag{6.130}$$

It can be readily checked that

$$\begin{aligned} \mathcal{P}_0 = \mathcal{G}_1^{\perp} &:= \{e \in \mathcal{E} \mid < e \mid f >= 0, \ \forall f \in \mathcal{G}_1\} \\ \mathcal{P}_1 = \mathcal{G}_0^{\perp} &:= \{e \in \mathcal{E} \mid < e \mid f >= 0, \ \forall f \in \mathcal{G}_0\} \end{aligned} \tag{6.131}$$

With $\mathcal{D}$ expressed in kernel/image representation (6.120), (6.122) one obtains

$$\begin{aligned} \mathcal{G}_1 &= \text{im } E^T, \quad \mathcal{P}_0 = \ker E \\ \mathcal{P}_1 &= \text{im } F^T, \quad \mathcal{G}_0 = \ker F \end{aligned} \tag{6.132}$$

The subspace $\mathcal{G}_1$ expresses the set of *admissible flows* f, and $\mathcal{P}_1$ the set of *admissible efforts* e. The first subspace will turn out to be instrumental in the determination of the *Casimirs* of a port-Hamiltonian DAE system in the next section, and the second subspace in the characterization of its *algebraic constraints*.

Another key property of Dirac structures is the fact that the *composition* of Dirac structures is again a Dirac structure. In the next section, this will lead to the fundamental property that any power-conserving interconnection of port-Hamiltonian DAE systems defines another port-Hamiltonian DAE system. We will start by showing that the composition of two Dirac structures is again a Dirac structure. This readily implies that the power-conserving interconnection of any number of Dirac structures is a Dirac structure.

Thus let us consider a Dirac structure $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{E}_1 \times \mathcal{E}_2$, and another Dirac structure $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{E}_2 \times \mathcal{E}_3$. The space $\mathcal{F}_2$ is the space of *shared flow* variables, and $\mathcal{E}_2$ is the space of *shared effort* variables; see Fig. 6.7.

Consider the interconnection equations (the minus sign included for a consistent power flow convention)

$$f_A = -f_B \in \mathcal{F}_2, \quad e_A = e_B \in \mathcal{E}_2 \tag{6.133}$$

Then the *composition* $\mathcal{D}_A \circ \mathcal{D}_B$ of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ is defined as

$$\begin{aligned} \mathcal{D}_A \circ \mathcal{D}_B := \Big\{ & (f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_3 \times \mathcal{E}_3 \mid \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2 \\ & \text{s.t. } (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B \Big\} \end{aligned} \tag{6.134}$$

The next theorem is proved in [63].

Theorem 6.6.9 *Let* $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$ *and* $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$ *be Dirac structures. Then* $\mathcal{D}_A \circ \mathcal{D}_B \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_3 \times \mathcal{E}_3$ *is a Dirac structure.*

(We refer to the next Sect. 6.7, see in particular (6.165), how this extends to the composition of *multiple* Dirac structures.) The following explicit expression can be given for the composition of two Dirac structures in terms of their kernel/image representation.

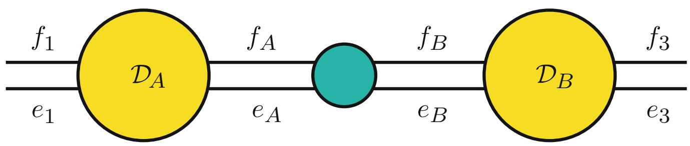


Fig. 6.7 The composition of $\mathcal{D}_A$ and $\mathcal{D}_B$

Proposition 6.6.10 *Consider Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$, $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$, given in combined kernel representation*

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0 \tag{6.135}$$

Then define

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix} \tag{6.136}$$

and let L_A, L_B be matrices such

$$L = \begin{bmatrix} L_A & L_B \end{bmatrix}, \qquad \ker L = \mathrm{im}\ M$$

Then a relaxed kernel representation of $\mathcal{D}_A \circ \mathcal{D}_B$ is obtained by premultiplying (6.135) *by the matrix L, resulting in*

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0$$

In many cases of interest, the notion of a *constant* Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, with $\mathcal{F}$ and $\mathcal{E} = \mathcal{F}^*$ linear spaces, is not sufficient for modeling purposes. We already observed this for input-state-output port-Hamiltonian systems, where the matrices J, R, P, S, M in (6.3) were allowed to be *state-dependent*. Furthermore, in many examples the state space $\mathcal{X}$ is not a linear space, but instead a manifold. In particular, this often occurs for 3-D mechanical systems. In such cases, the notion of a constant Dirac structure given in Definition 6.6.3 needs to be extended to the following definition of *Dirac structures on manifolds*.

Definition 6.6.11 Let $\mathcal{X}$ be a manifold. A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is a vector sub-bundle of the Whitney sum[7] $T\mathcal{X} \oplus T^*\mathcal{X}$ such that

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$$

is for every $x \in \mathcal{X}$ a constant Dirac structure as before.

Simply put, a Dirac structure on a manifold $\mathcal{X}$ is point-wise (for every $x \in \mathcal{X}$) a constant Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$.

Most of the preceding theory concerning constant Dirac structures can be extended to Dirac structures on manifolds. In particular, the kernel and image, constrained

[7] The Whitney sum of two vector bundles with the same base space is defined as the vector bundle whose fiber above each element of this common base space is the product of the fibers of each individual vector bundle.

input–output, and hybrid input–output representation of Proposition 6.6.6 carry over to the case of a Dirac structure on a manifold; the difference being that the matrices involved may be *depending on* x, and that the representations may exist only *locally* on the state space manifold $\mathcal{X}$.

In particular, given a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$ and any point $x_0 \in \mathcal{X}$ there exists a coordinate neighborhood of x_0 such that for x within this coordinate neighborhood

$$\mathcal{D}(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid F(x)f + E(x)e = 0\} \tag{6.137}$$

for $\ell \times \ell$ matrices $F(x)$ and $E(x)$ satisfying

$$E(x)F^T(x) + F(x)E^T(x) = 0, \quad \text{rank}[F(x) \vdots E(x)] = \ell, \tag{6.138}$$

or equivalently,

$$\mathcal{D}(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid f = E^T(x)\lambda,\ e = F^T(x)\lambda, \lambda \in \mathbb{R}^\ell\} \tag{6.139}$$

Conversely, for any $\ell \times \ell$ matrices $F(x)$ and $E(x)$ satisfying (6.138), the subspaces (6.137) and (6.139) define locally a Dirac structure on $\mathcal{X}$.

Furthermore, $\mathcal{D}$ may be locally represented as

$$\mathcal{D}(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid \exists\lambda \text{ s.t. } f = J(x)e + G(x)\lambda,\ G^T(x)e = 0\}, \tag{6.140}$$

for an $\ell \times \ell$ skew-symmetric matrix $J(x)$, and a matrix $G(x)$. Conversely, for any $G(x)$ and skew-symmetric $J(x)$ (6.140) defines locally a Dirac structure.

Finally, starting from (6.137) we may locally split the flows f and e, and correspondingly $F(x)$, $E(x)$, in such a way that

$$\mathcal{D}(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} = \left\{ \left[\begin{matrix} f^1 \\ f^2 \end{matrix}\right], \left[\begin{matrix} e^1 \\ e^2 \end{matrix}\right] \mid \left[\begin{matrix} f^1 \\ e^2 \end{matrix}\right] = J(x) \left[\begin{matrix} e^1 \\ f^2 \end{matrix}\right] \right\} \tag{6.141}$$

where $J(x) := -[F(x)^1 \vdots E(x)^2]^{-1}[F(x)^2 \vdots E(x)^1]$ is skew-symmetric. Conversely, for any skew-symmetric $J(x)$ as above (6.141) defines locally a Dirac structure.

On the other hand, the canonical coordinate representation (6.125) is *not* always possible for a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{X}$. In fact, analogously to the *integrability conditions* (6.33) characterizing $J(x)$ to be a *Poisson structure* for which canonical coordinates as in (6.35) can be found, one can formulate integrability conditions on $\mathcal{D}$ which (together with a constant rank assumption) are necessary and sufficient for the local existence of canonical coordinates representing $\mathcal{D}$ as in (6.125). We refer to the Notes at the end of this chapter for further information.

The subspaces G_0, G_1, P_0, P_1 defined for a constant Dirac structure in (6.130) generalize for a Dirac structure $\mathcal{D}$ on $\mathcal{X}$ to the *distributions*, respectively, *co-distributions*, on $\mathcal{X}$

$$\begin{aligned}
\mathcal{G}_0(x) &:= \{f \in T_x\mathcal{X} \mid (f, 0) \in \mathcal{D}(x)\} \\
\mathcal{G}_1(x) &:= \{f \in T_x\mathcal{X} \mid \exists e \in T_x^*\mathcal{X} \text{ s.t. } (f, e) \in \mathcal{D}(x)\} \\
\mathcal{P}_0(x) &:= \{e \in T_x^*\mathcal{X} \mid (0, e) \in \mathcal{D}(x)\} \\
\mathcal{P}_1(x) &:= \{e \in T_x^*\mathcal{X} \mid \exists f \in T_x\mathcal{X} \text{ s.t. } (f, e) \in \mathcal{D}(x)\}
\end{aligned} \tag{6.142}$$

The integrability of the distributions G_0, G_1 and co-distributions P_0, P_1 on $\mathcal{X}$ is *implied* by the integrability of the Dirac structure $\mathcal{D}$; see again the Notes at the end of this chapter.

Also, the theory regarding composition of constant Dirac structures can be extended to Dirac structures on manifolds; we refer to the next section for the appropriate setting.

6.7 Port-Hamiltonian DAE Systems

From a network modeling perspective (see also the Notes at the end of this chapter), lumped-parameter physical systems are naturally described by a set of ideal *energy-storing* elements, a set of *energy-dissipating* or *resistive* elements, and a set of *external ports* by which interaction with the environment can take place. All of them are interconnected to each other by a *power-conserving interconnection*, see Fig. 6.8.

This power-conserving interconnection includes ideal *power-conserving* elements such as (in the electrical domain) transformers, gyrators, or (in the mechanical domain) transformers, kinematic pairs, and kinematic constraints. Power-conserving elements do not store energy, nor dissipate energy, but instead *route* the energy flow.

Associated with the energy-storing elements are state variables $x_1, \ldots, x_n$, being coordinates for some n-dimensional state space manifold $\mathcal{X}$, and a total energy $H : \mathcal{X} \to \mathbb{R}$. The power-conserving interconnection is formalized by a *Dirac structure*

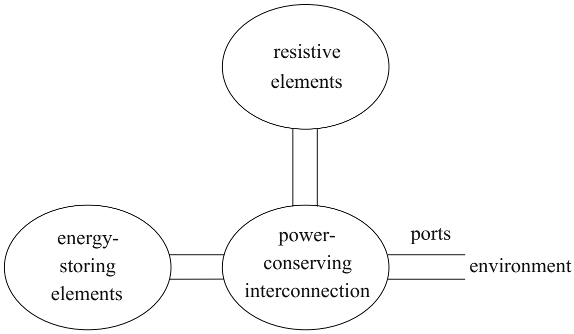


Fig. 6.8 Port-Hamiltonian DAE system

relating the flows and efforts of the energy-storing, energy-dissipating elements and external ports

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P, \quad x \in \mathcal{X}, \tag{6.143}$$

where $(f_S, e_S) \in T_x\mathcal{X} \times T_x^*\mathcal{X}$ are the flows and efforts of the energy-storing elements, $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$ are the flows and efforts of the energy-dissipating elements, and finally $(f_P, e_P) \in \mathcal{F}_P \times \mathcal{E}_P$ are the flows and efforts of the external ports.

Remark 6.7.1 Geometrically, $\mathcal{D}$ is a Dirac structure on the manifold $\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_P$, which is invariant under translation along F_R, F_P directions, and therefore only depending on $x \in \mathcal{X}$. See also [41, 218].

In the case of a *linear* state space $\mathcal{X}$ and a *constant* Dirac structure $\mathcal{D}$, the expression (6.143) simplifies to

$$\mathcal{D} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P \tag{6.144}$$

where $\mathcal{F}_S = \mathcal{X}, \mathcal{E}_S = \mathcal{X}^*$. The equally dimensioned vectors of flow variables and effort variables of the energy-storing elements are given as

$$\dot{x}(t) = \frac{dx}{dt}(t), \quad \frac{\partial H}{\partial x}(x(t)), \quad t \in \mathbb{R}, \tag{6.145}$$

which are equated with f_S, e_S by[8]

$$\begin{aligned} f_S &= -\dot{x} \\ e_S &= \tfrac{\partial H}{\partial x}(x) \end{aligned} \tag{6.146}$$

Furthermore, f_R, e_R are related by a (static) *energy-dissipating* (resistive) relation, which can be any subset $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$, satisfying the property

$$e_R^T f_R \leq 0, \quad \text{for all } (f_R, e_R) \in \mathcal{R} \tag{6.147}$$

This leads to the following.

Definition 6.7.2 A port-Hamiltonian DAE system is defined by a Dirac structure $\mathcal{D}$ as in (6.143), a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and an energy-dissipating relation $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ satisfying (6.147). The dynamics is given by the requirement that for all $t \in \mathbb{R}$

$$\begin{gathered} \left(-\tfrac{dx}{dt}(t), \tfrac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_P(t), e_P(t)\right) \in \mathcal{D}(x(t)) \\ (f_R(t), e_R(t)) \in \mathcal{R} \end{gathered} \tag{6.148}$$

It is directly verified that this definition includes the definitions of input-state-output port-Hamiltonian systems as given before, cf. (6.1), (6.3), (6.6), (6.15), as special

[8] The minus sign is inserted in order to have a consistent power flow convention.

cases (with inputs u and outputs y given by (f_P, e_P). However, in general Definition 6.7.2 entails *algebraic constraints* on the state variables x.

By the power-conservation property of a Dirac structure (6.118) and (6.147), any port-Hamiltonian DAE system satisfies the energy-balance

$$\begin{aligned} \frac{dH}{dt}(x(t)) &= < \frac{\partial H}{\partial x}(x(t)) \mid \dot{x}(t) >= \\ &= e_R^T f_R(t) + e_P^T(t) f_P(t) \leq e_P^T(t) f_P(t), \end{aligned} \tag{6.149}$$

as was the case for input-state-output port-Hamiltonian systems. Thus port-Hamiltonian DAE systems are cyclo-passive with respect to the supply rate $e_P^T f_P$, and passive if H is bounded from below.

The *algebraic constraints* that are present in a port-Hamiltonian DAE system are determined by the distribution P_1 defined by $\mathcal{D}$ (cf. (6.142)), as well as by the Hamiltonian H. In fact, the condition

$$\left(\frac{\partial H}{\partial x}(x), e_R, e_P \right) \in P_1(x), \quad x \in \mathcal{X}, \tag{6.150}$$

may entail algebraic constraints on the state x.

On the other hand, the *Casimir functions* $C : \mathcal{X} \to \mathbb{R}$ of the port-Hamiltonian DAE system (6.148) are determined by the distribution G_1. Indeed, $\frac{dC}{dt} = \frac{\partial^T C}{\partial x}(x)\dot{x} = 0$ if and only $\frac{\partial^T C}{\partial x}(x) f_S = 0$ for all f_S for which there exists f_R, f_P such that $(f_S, f_R, f_P) \in G_1(x)$. Furthermore, C is a Casimir for $f_P = 0$ if and only if $\frac{\partial^T C}{\partial x}(x) f_S = 0$ for all f_S for which there exists f_R such that $(f_S, f_R, 0) \in G_1(x)$.

Definition 6.7.2 is a geometric, coordinate-free, definition. *Equational representations* of port-Hamiltonian DAE systems are obtained by choosing a coordinate representation of the Dirac structure $\mathcal{D}$ as in (6.143). In case the Dirac structure $\mathcal{D}$ is given in *kernel representation*

$$\begin{aligned} \mathcal{D}(x) = \{(f_S, f_R, f_P, e_S, e_R, e_P) \mid & F_S(x) f_S + E_S(x) e_S + \\ & F_R(x) f_R + E_R(x) e_R + F_P(x) f_P + E_P(x) e_P = 0\} \end{aligned} \tag{6.151}$$

for matrices $F_S(x), E_S(x), F_R(x), E_R(x), F_P(x), E_P(x)$ satisfying

$$\begin{aligned} &(i) \;\; E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0 \\ &(ii) \; \text{rank} \left[F_S \vdots F_R \vdots F_P \vdots E_S \vdots E_R \vdots E_P \right] = \dim \mathcal{F} \end{aligned} \tag{6.152}$$

this leads to the following specification of algebraic constraints and Casimirs. With respect to the algebraic constraints, we notice that

$$e_S \in \text{im } F_S^T(x), \tag{6.153}$$

implying the algebraic constraints

$$\frac{\partial H}{\partial x}(x) \in \text{im } F_S^T(x) \tag{6.154}$$

With respect to the Casimirs we notice that

$$f_S \in \text{im } E_S^T(x), \tag{6.155}$$

implying that $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function if and only if $\frac{dC}{dt}(x(t)) = \frac{\partial^T C}{\partial x}(x(t))$ $\dot{x}(t) = 0$ for all $\dot{x}(t) \in \text{im } E_S^T(x(t))$. Hence, C is a Casimir of the port-Hamiltonian DAE system (6.148) if and only if it satisfies the set of pde's

$$E_S(x)\frac{\partial C}{\partial x}(x) = 0, \quad x \in \mathcal{X} \tag{6.156}$$

Finally, C is a Casimir function for $f_P = 0$ if and only if $\frac{\partial^T C}{\partial x}(x)E_S(x)\lambda = 0$ for all λ such that $E_P^T(x)\lambda = 0$. As a result, C is a Casimir function for $f_P = 0$ if and only if it satisfies the conditions

$$E_S(x)\frac{\partial C}{\partial x}(x) \in \text{im } E_P(x), \quad x \in \mathcal{X} \tag{6.157}$$

Example 6.7.3 Consider the LC-circuit of Example 6.4.5 without voltage source ($V = 0$), and where the two inductors are replaced by two capacitors with charges Q_1, Q_2, and dually the capacitor is replaced by an inductor with flux linkage φ. This does not change the Dirac structure (determined by Kirchhoff's current and voltage laws). However, while the original LC-circuit has a *Casimir* $\varphi_1 + \varphi_2$, in the present LC-circuit there is the *algebraic constraint*

$$\frac{\partial H}{\partial Q_1}(Q_1, Q_2, \varphi) + \frac{\partial H}{\partial Q_2}(Q_1, Q_2, \varphi) = 0 \tag{6.158}$$

constraining the state variables Q_1, Q_2.

Example 6.7.4 The constrained Hamiltonian equations (6.42) can be viewed as a port-Hamiltonian DAE system, with respect to the Dirac structure $\mathcal{D}$ given in constrained input–output representation (6.123) as

$$\begin{aligned}
\mathcal{D} = \{(f_S, f_P, e_S, e_P) \mid 0 = A^T(q)e_S,\ e_P = B^T(q)e_S,\\
-f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_P,\ \lambda \in \mathbb{R}^k\}
\end{aligned} \tag{6.159}$$

The kinematic constraints correspond to the following algebraic constraints on the state variables (q, p)

$$0 = A^T(q)\frac{\partial H}{\partial p}(q, p), \tag{6.160}$$

while the Casimir functions C for $u = 0$ are determined by the equations

$$\frac{\partial^T C}{\partial q}(q)\dot{q} = 0, \quad \text{for all } \dot{q} \text{ satisfying } A^T(q)\dot{q} = 0 \tag{6.161}$$

Hence, finding a Casimir function amounts to (partially) *integrating* the kinematic constraints $A^T(q)\dot{q} = 0$. In particular, if the kinematic constraints are *holonomic*, and thus can be expressed as in (6.38), then $\bar{q}_{n-k+1}, \ldots, \bar{q}_n$ generate all the Casimir functions.

The results concerning *composition* of Dirac structures as treated in the previous Sect. 6.6 imply that any power-conserving interconnection of port-Hamiltonian systems is *again a port-Hamiltonian system*. Indeed, let us consider k port-Hamiltonian DAE systems specified by Dirac structures

$$\mathcal{D}_i(x_i) \subset T_{x_i}\mathcal{X}_i \times T^*_{x_i}\mathcal{X}_i \times \mathcal{F}^i_R \times \mathcal{E}^i_R \times \mathcal{F}^i_P \times \mathcal{E}^i_P, \quad x_i \in \mathcal{X}_i, \quad i = 1, \ldots, k \tag{6.162}$$

together with Hamiltonians and energy-dissipating relations

$$H_i : \mathcal{X}_i \to \mathbb{R}, \quad \mathcal{R}_i \subset \mathcal{F}^i_R \times \mathcal{E}^i_R, \quad i = 1, \ldots, k \tag{6.163}$$

Furthermore, define an *interconnection Dirac structure*

$$\mathcal{D}_I \subset \mathcal{F}^1_P \times \mathcal{E}^1_P \times \cdots \times \mathcal{F}^k_P \times \mathcal{E}^k_P \times \mathcal{F}^e_P \times \mathcal{E}^e_P \tag{6.164}$$

with $\mathcal{F}^e_P, \mathcal{E}^e_P$ spaces of external flows and efforts. $\mathcal{D}_I$ specifies the way the flows and efforts f^i_P, e^i_P of the composing systems are connected to each other and to the new external flows and efforts f^e_P, e^e_P in a power-conserving manner. The *composition* through the shared flows and efforts in $\mathcal{F}^1_P \times \mathcal{E}^1_P \times \cdots \times \mathcal{F}^k_P \times \mathcal{E}^k_P$ defines a new Dirac structure

$$(\mathcal{D}_1(x_1) \times \cdots \times \mathcal{D}_k(x_k)) \circ \mathcal{D}_I \subset T_x\mathcal{X} \times T^*_x\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}^e_P \times \mathcal{E}^e_P \tag{6.165}$$

(note that this amounts to the composition of *two* Dirac structures), where

$$x \in \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k, \; \mathcal{F}_R := \mathcal{F}^1_R \times \cdots \times \mathcal{F}^k_R, \; \mathcal{E}_R := \mathcal{E}^1_R \times \cdots \times \mathcal{E}^k_R \tag{6.166}$$

As a result, the interconnected system is again a port-Hamiltonian DAE system on the product state space $\mathcal{X}$ with Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ given as $H(x) = H_1(x_1) +$

$\cdots + H_k(x_k)$, and with energy-dissipating relation $\mathcal{R}$ given as the direct product of $\mathcal{R}_1, \ldots, \mathcal{R}_k$.

Example 6.7.5 (*PID control*) Consider the standard *Proportional–Integral-Derivative* (PID) controller

$$y_c = k_P u_c + k_I \int u_c dt + k_D \dot{u}_c \tag{6.167}$$

for certain positive constants k_P, k_I, k_D. Trivially rewriting (6.167) as

$$k_D \dot{u}_c = -k_P u_c - k_I \int u_c dt + y_c, \tag{6.168}$$

and defining $\xi = \int u_c dt$ (or equivalently $\dot{\xi} = u_c$) and $\eta = k_D u_c$, the PID-controller can be formulated as the linear input-state-output[9] port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -k_P \end{bmatrix} \begin{bmatrix} k_I \xi \\ \frac{\eta}{k_D} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_c \\ u_c &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} k_I \xi \\ \frac{\eta}{k_D} \end{bmatrix} \end{aligned} \tag{6.169}$$

with Hamiltonian $H_c(\xi, \eta) = \frac{1}{2} k_I \xi^2 + \frac{1}{2k_D} \eta^2$.

Considering any plant input-state-output port-Hamiltonian system as in (6.1)

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x) u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \tag{6.170}$$

the closed-loop system arising from standard feedback $u = -y_c, u_c = y$ with the PID-controller is given by the port-Hamiltonian DAE system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x) u \\ \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -k_P \end{bmatrix} \begin{bmatrix} k_I \xi \\ \frac{\eta}{k_D} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ 0 &= g^T(x) \frac{\partial H}{\partial x}(x) - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} k_I \xi \\ \frac{\eta}{k_D} \end{bmatrix} \end{aligned} \tag{6.171}$$

with total Hamiltonian $H(x) + \frac{1}{2} k_I \xi^2 + \frac{1}{2k_D} \eta^2$. This port-Hamiltonian system is in constrained input–output representation (6.123), with u acting as a vector of Lagrange multipliers.

[9] Note that y_c serves as an input to (6.169) and u_c as an output, contrary to the intuitive use of a PID-controller, where u_c equals the output of the *plant* system and $-y_c$ is the input applied to the plant system. This is of course caused by the fact that the *D*-action involves a differentiation.

We close this section by indicating two direct extensions from the input-state-output port-Hamiltonian case to the DAE case. The first concerns *shifted passivity*. Consider a port-Hamiltonian DAE system for which we can find coordinates in which the Dirac structure is *constant*. Furthermore, assume that the resistive structure $\mathcal{R}$ is *linear*. A *steady-state* $\bar{x}$, corresponding to steady-state values $\bar{f}_R, \bar{e}_R, \bar{f}_P, \bar{e}_P$, is characterized by

$$\left(0, \frac{\partial H}{\partial x}(\bar{x}), \bar{f}_R, \bar{e}_R, \bar{f}_P, \bar{e}_P\right) \in \mathcal{D}, \quad (\bar{f}_R, \bar{e}_R) \in \mathcal{R} \tag{6.172}$$

Using now the linearity of $\mathcal{D}$ and $\mathcal{R}$, we can subtract (6.172) from (6.148), so as to obtain

$$\begin{aligned} &(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)) - \frac{\partial H}{\partial x}(\bar{x}), \\ &\qquad f_R(t) - \bar{f}_R, e_R(t) - \bar{e}_R, f_P(t) - \bar{f}_P, e_P(t) - \bar{e}_P) \in \mathcal{D} \\ &(f_R(t) - \bar{f}_R, e_R(t) - \bar{e}_R) \in \mathcal{R} \end{aligned} \tag{6.173}$$

Similar to Proposition 6.5.1, this defines a *shifted* port-Hamiltonian system with respect to the same Dirac structure $\mathcal{D}$ and resistive structure $\mathcal{R}$, and with Hamiltonian given by the shifted Hamiltonian function $\widehat{H}_{\bar{x}}$, and shifted external port variables $f_P - \bar{f}_P, e_P - \bar{e}_P$.

The second extension concerns the notion of *steady-state input–output relation* (4.31). For a port-Hamiltonian DAE system, Σ this relation is given as

$$\begin{aligned} \Sigma_{ss} = \{(\bar{f}_P, \bar{e}_P) \mid \exists \bar{x}, \bar{f}_R, \bar{e}_R \text{ such that} \\ \left(0, \tfrac{\partial H}{\partial x}(\bar{x}), \bar{f}_R, \bar{e}_R, \bar{f}_P, \bar{e}_P\right) \in \mathcal{D}(\bar{x}), (\bar{f}_R, \bar{e}_R) \in \mathcal{R}\} \end{aligned} \tag{6.174}$$

It directly follows that $\bar{e}_P^T \bar{f}_P \geq 0$ for all $(\bar{f}_P, \bar{e}_P) \in \Sigma_{ss}$.

6.8 Port-Hamiltonian Network Dynamics

Section 4.4 already presented a treatment of passive network systems. In this section we will go one step further, by identifying large classes of network systems as port-Hamiltonian systems, where the Dirac structure of the network system is determined by the *network interconnection structure*.

Let us start with some basic notions regarding graphs, extending the background already provided in Sect. 4.4. Like in Sect. 4.4 "graph" throughout means "directed graph." Given a graph, we define its *vertex space* Λ_0 as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the examples, $\mathcal{R}$ will be mostly $\mathcal{R} = \mathbb{R}$ in which case Λ_0 can be identified with $\mathbb{R}^N$. Furthermore, we define the *edge space* Λ_1 as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R}$. Again, if $\mathcal{R} = \mathbb{R}$ then Λ_1 can be identified with $\mathbb{R}^M$. The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0, respectively, by Λ^1. The duality pairing between $f \in \Lambda_0$ and $e \in \Lambda^0$ is given as

$$< f \mid e >= \sum_{v \in \mathcal{V}} < f(v) \mid e(v) > , \tag{6.175}$$

where $< \mid >$ on the right-hand side denotes the duality pairing between $\mathcal{R}$ and $\mathcal{R}^*$. A similar expression holds for $f \in \Lambda_1$ and $e \in \Lambda^1$ (with summation over the edges).

The incidence matrix D of the graph induces a linear map $\widehat{D}$ from the edge space to the vertex space as follows. Define $\widehat{D} : \Lambda_1 \to \Lambda_0$ as the linear map with matrix representation $D \otimes I$, where $I : \mathcal{R} \to \mathcal{R}$ is the identity map and $\otimes$ denotes the Kronecker product. $\widehat{D}$ will be called the *incidence operator*. For $\mathcal{R} = \mathbb{R}$ the incidence operator reduces to the linear map given by the matrix D itself, in which case we will throughout use D *both for the incidence matrix and for the incidence operator.* The adjoint map of $\widehat{D}$ is denoted as

$$\widehat{D}^* : \Lambda^0 \to \Lambda^1,$$

and is called the *coincidence* operator. For $\mathcal{R} = \mathbb{R}^3$ the coincidence operator is given by $D^T \otimes I_3$, while for $\mathcal{R} = \mathbb{R}$ the coincidence operator is simply given by the transposed matrix D^T, and we will throughout use D^T *both for the co-incidence matrix and for the coincidence operator.*

In order to define *open* network systems we will identify a subset $\mathcal{V}_b \subset \mathcal{V}$ of *boundary vertices*. The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are called the *internal vertices* of the graph.

The splitting of the vertices into internal and boundary vertices induces a splitting of the vertex space and its dual, given as

$$\Lambda_0 = \Lambda_{0i} \oplus \Lambda_{0b}, \;\; \Lambda^0 = \Lambda^{0i} \oplus \Lambda^{0b}, \tag{6.176}$$

where Λ_{0i} is the vertex space corresponding to the internal vertices and Λ_{0b} the vertex space corresponding to the boundary vertices. Consequently, the incidence operator $\widehat{D} : \Lambda_1 \to \Lambda_0$ splits as

$$\widehat{D} = \widehat{D}_i \oplus \widehat{D}_b, \tag{6.177}$$

with $\widehat{D}_i : \Lambda_1 \to \Lambda_{0i}$ and $\widehat{D}_b : \Lambda_1 \to \Lambda_{0b}$. For $\mathcal{R} = \mathbb{R}$ we will simply write

$$D = \begin{bmatrix} D_i \\ D_b \end{bmatrix} \tag{6.178}$$

Furthermore, we define the *boundary space* Λ_b as the linear space of all functions from the set of boundary vertices $\mathcal{V}_b$ to the linear space $\mathcal{R}$. Note that the boundary space Λ_b is equal to the linear space Λ_{0b}, and that the linear mapping $\widehat{D}_b$ can be also regarded as a mapping $\widehat{D}_b : \Lambda_1 \to \Lambda_b$. The dual space of Λ_b will be denoted as Λ^b. The elements $f_b \in \Lambda_b$ are called the *boundary flows* and the elements $e^b \in \Lambda^b$ the *boundary efforts*.

A paradigmatic example of a port-Hamiltonian network system is a *mass–spring–damper system*. Let us start with mass–spring systems, as already considered in Chap. 4, Example 4.4.5, as an example of a *passive* network system. Any mass–spring system is modeled by a graph $\mathcal{G}$ with N vertices corresponding to the masses and M edges corresponding to the springs, specified by an incidence matrix D. For ease of notation, consider first the situation that the mass–spring system is located in one-dimensional space $\mathcal{R} = \mathbb{R}$, and the springs are scalar. A vector in the vertex space Λ_0 then corresponds to the vector p of the scalar momenta of all N masses, i.e., $p \in \Lambda_0 = \mathbb{R}^N$. Furthermore, a vector in the dual edge space Λ^1 will correspond to the total vector q of extensions of all M springs, i.e., $q \in \Lambda^1 = \mathbb{R}^M$.

Next ingredient is the Hamiltonian $H : \Lambda^1 \times \Lambda_0 \to \mathbb{R}$, which is the sum of the kinetic and potential energies of each mass and spring. In the absence of boundary vertices the dynamics of the mass–spring system is described as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}, \tag{6.179}$$

defined with respect to the constant Poisson structure on the linear state space $\Lambda^1 \times \Lambda_0$ given by the skew-symmetric matrix

$$J := \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix} \tag{6.180}$$

implicitly already encountered in Sect. 4.4; see (4.83).

The inclusion of boundary vertices, and thereby of external interaction, can be done in different ways. The first option is to associate *boundary masses* to the boundary vertices. We are then led to the port-Hamiltonian input-state-output system

$$\begin{aligned} \dot{q} &= D^T \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -D \frac{\partial H}{\partial q}(q,p) + E f_b \\ e^b &= E^T \frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{6.181}$$

Here E is a matrix with as many columns as there are boundary vertices; each column consists of zeros except for exactly one 1 in the row corresponding to the associated boundary vertex. Furthermore $f_b \in \Lambda_b$ are the external *forces* exerted (by the environment) on the boundary masses, and $e_b \in \Lambda^b$ are the *velocities* of these boundary masses.

A second possibility is to regard the boundary vertices as being *massless*. In this case, we obtain the port-Hamiltonian input-state-output system (with p now denoting the vector of momenta of the masses associated to the *internal* vertices)

$$\begin{aligned}\dot{q} &= D_i^T \frac{\partial H}{\partial p}(q,p) + D_b^T e^b \\ \dot{p} &= -D_i \frac{\partial H}{\partial q}(q,p) \\ f_b &= D_b \frac{\partial H}{\partial q}(q,p)\end{aligned} \tag{6.182}$$

with $e^b \in \Lambda^b$ the velocities of the massless boundary vertices, and $f_b \in \Lambda_b$ the forces at the boundary vertices as *experienced* by the environment. Note that in this second case the external velocities e^b of the boundary vertices can be considered to be *inputs* to the system and the forces f_b to be *outputs*; in contrast to the previously considered case (boundary vertices corresponding to boundary masses), where the forces f_b are inputs and the velocities e^b the outputs of the system.

For a *mass–spring–damper system* the edges will correspond partly to springs, and partly to dampers. This corresponds to an incidence matrix

$$D = \begin{bmatrix} D_s & D_d \end{bmatrix}, \tag{6.183}$$

where the columns of D_s reflect the spring edges and the columns of D_d the damper edges. For the case *without* boundary vertices the dynamics of a mass–spring–damper system with linear dampers takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} \tag{6.184}$$

with R the diagonal matrix of damping coefficients. In the presence of boundary vertices, we may again distinguish between *massless* boundary vertices, with inputs being the boundary velocities and outputs the boundary (reaction) forces, and *boundary masses*, in which case the inputs are the external forces and the outputs the velocities of the boundary masses.

The formulation of mass–spring–damper systems in $\mathcal{R} = \mathbb{R}$ directly extends to $\mathcal{R} = \mathbb{R}^3$ using the incidence operator $\widehat{D} = D \otimes I_3$ as defined before. Furthermore, the set-up can be extended [298] to *multi-body systems* and *spatial mechanisms* (networks of rigid bodies in $\mathbb{R}^3$ related by joints) by considering the linear space $\mathcal{R} := \mathrm{se}^*(3)$, the dual of the Lie algebra of the Lie group $SE(3)$ describing the position of a rigid body in $\mathbb{R}^3$. Finally we note that other examples like hydraulic networks are analogous to mass–spring–damper system; see e.g., [287].

Remark 6.8.1 The example of a power network given in Example 4.4.4 defines a port-Hamiltonian system which is similar to a mass–spring–damper system, with the difference that in this case the *dampers* (corresponding to A) are associated to the *vertices* of the graph and the edges correspond to the transmission lines with potential energies $-\gamma_j \cos q_j, j = 1, \ldots, M$.

Remark 6.8.2 Note the slight discrepancy of the role of the flows f and efforts e with respect to the definition of a port-Hamiltonian DAE systems given in the previous Sect. 6.7. Indeed, in Sect. 6.7 flows and efforts are used interchangeably, with the exception of the flows and efforts f_S, e_S corresponding to the energy-storing elements, which (cf. (6.146)) are given by $f_S = -\dot{x}$ and $e_S = \frac{\partial H}{\partial x}(x)$. In the current network setting, the flows f are elements of the spaces Λ_0, Λ_1, and the efforts e of their *dual* spaces Λ^0, Λ^1. In particular, the flows in Λ_1 correspond to the classical [211] *through* variables, and the efforts in Λ^1 to the *across* variables.

The port-Hamiltonian formulation of the dynamics (6.184) leads to the following *stability analysis*. Without loss of generality[10], we throughout assume that the graph is connected, or equivalently, see Sect. 4.4, $\ker D_s^T \cap \ker D_d^T = \text{span } \mathbb{1}$, where $\mathbb{1}$ is the vector of all ones. We start with the following proposition regarding the *equilibria*.

Proposition 6.8.3 *Consider the dynamics (6.184). Its set of equilibria $\mathcal{E}$ is given as*

$$\mathcal{E} = \{(q, p) \in \Lambda^1 \times \Lambda_0 \mid \frac{\partial H}{\partial q}(q, p) \in \ker D_s, \frac{\partial H}{\partial p}(q, p) \in \text{span } \mathbb{1}\} \tag{6.185}$$

Proof (q, p) is an equilibrium whenever

$$D_s^T \frac{\partial H}{\partial p}(q, p) = 0, \; D_s \frac{\partial H}{\partial q}(q, p) + D_d R D_d^T \frac{\partial H}{\partial p}(q, p) = 0 \tag{6.186}$$

Premultiplication of the second equation by the row-vector $\frac{\partial^T H}{\partial p}(q, p)$, making use of the first equation, yields $\frac{\partial^T H}{\partial p}(q, p) B_d R B_d^T \frac{\partial H}{\partial p}(q, p) = 0$, or equivalently $D_d^T \frac{\partial H}{\partial p}(q, p) = 0$, which implies $D_s \frac{\partial H}{\partial q}(q, p) = 0$. Hence, $\frac{\partial H}{\partial p}(q, p) \in \ker D_s^T \cap \ker D_d^T = \text{span } \mathbb{1}$. □

In other words, for (q, p) to be an equilibrium, the elements of the vector of velocities $\frac{\partial H}{\partial p}(q, p)$ should be equal to each other, whereas $\frac{\partial H}{\partial q}(q, p)$ should be in the space $\ker D_s$ of *cycles* of the subgraph of masses and springs (resulting in zero net spring forces applied to the masses at the vertices).

Similarly, the *Casimirs* are computed as follows.

Proposition 6.8.4 *The Casimir functions of (6.184) are functions $C(q, p)$ satisfying*

$$\frac{\partial C}{\partial p}(q, p) \in \text{span } \mathbb{1}, \; \frac{\partial C}{\partial q}(q, p) \in \ker D_s \tag{6.187}$$

Proof The function $C(q, p)$ is a Casimir if

$$\begin{bmatrix} \frac{\partial C}{\partial q}(q, p) & \frac{\partial C}{\partial p}(q, p) \end{bmatrix} \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} = 0, \tag{6.188}$$

or equivalently (see Proposition 6.4.2)

[10] Since otherwise the same analysis can be performed on each connected component of the graph.

$$\frac{\partial^T C}{\partial p}(q,p)D_s = 0, \ \frac{\partial^T C}{\partial q}(q,p)D_s^T = 0, \ \frac{\partial^T C}{\partial p}(q,p)D_d R D_d^T = 0 \tag{6.189}$$

Post-multiplication of the third equation by $\frac{\partial C}{\partial p}(q,p)$, making use of the first equation, gives the result. □

Therefore, all Casimir functions can be expressed as functions of the *linear* Casimir functions

$$C(q,p) = \mathbb{1}^T p, \quad C(q,p) = k^T q, \ k \in \ker D_s \tag{6.190}$$

This implies that starting from an arbitrary initial position $(q_0, p_0) \in \Lambda^1 \times \Lambda_0$ the solution of the mass–spring–damper system (6.184) will be contained in the affine space

$$\mathcal{A}_{(q_0,p_0)} := \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ker \mathbb{1}^T \end{bmatrix} + \begin{bmatrix} \operatorname{im} D_s^T \\ 0 \end{bmatrix} \tag{6.191}$$

i.e., for all t the difference $q(t) - q_0$ remains in the space im D_s^T of *co-cycles* of the mass–spring graph, while $\mathbb{1}^T p(t) = \mathbb{1}^T p_0$.

Under generically fulfilled conditions on the Hamiltonian $H(q,p)$, each affine space $\mathcal{A}_{(q_0,p_0)}$ will intersect the set of equilibria $\mathcal{E}$ in a *single* point (q_∞, p_∞), which qualifies as the point of asymptotic convergence starting from (q_0, p_0). For simplicity, consider *linear* mass–spring–damper systems, corresponding to a quadratic Hamiltonian function

$$H(q,p) = \frac{1}{2} q^T K q + \frac{1}{2} p^T G p, \tag{6.192}$$

where K is the positive diagonal matrix of spring constants, and G is the positive diagonal matrix of reciprocals of the masses. In this case, the set of equilibria is given as $\mathcal{E} = \{(q,p) \in \Lambda^1 \times \Lambda_0 \mid Kq \in \ker B_s, Gp \in \text{span } \mathbb{1}\}$, while indeed it is easily seen that for each (q_0, p_0) there exists a *unique* point $(q_\infty, p_\infty) \in \mathcal{E} \cap \mathcal{A}_{(q_0,p_0)}$. In fact, q_∞ is given by the spring graph co-cycle/cycle decomposition

$$q_0 = v_0 + q_\infty, \quad v_0 \in \operatorname{im} D_s^T \subset \Lambda^1, Kq_\infty \in \ker D_s \subset \Lambda_1 \tag{6.193}$$

Furthermore, p_∞ is uniquely determined by

$$Gp_\infty \in \text{span } \mathbb{1}, \quad \mathbb{1}^T p_\infty = \mathbb{1}^T p_0 \tag{6.194}$$

This leads to the following asymptotic stability theorem. First note that

$$\begin{aligned} \frac{d}{dt} H(q,p) &= -\frac{\partial^T H}{\partial p}(q,p) D_d R D_d^T \frac{\partial H}{\partial p}(q,p) \\ &= -p^T G D_d R D_d^T G p \leq 0 \end{aligned} \tag{6.195}$$

Theorem 6.8.5 *Consider a linear mass–spring–damper system with* $H(q,p) = \frac{1}{2}q^T Kq + \frac{1}{2}p^T Gp$*, where* K *and* G *are diagonal positive matrices. Then for every* (q_0, p_0)*, there exists a unique equilibrium point* $(q_\infty, p_\infty) \in \mathcal{E} \cap \mathcal{A}_{(q_0,p_0)}$*, determined by (6.193), (6.194). Define the spring Laplacian matrix* $L_s := D_s K D_s^T$*. Then for every* (q_0, p_0) *the following holds: the trajectory starting from* (q_0, p_0) *converges asymptotically to* (q_∞, p_∞) *if and only if the largest* GL_s*-invariant subspace contained in* $\ker D_d^T$ *is equal to* span $\mathbb{1}$.

The condition that the largest GL_s-invariant subspace contained in $\ker D_d^T$ is equal to span $\mathbb{1}$ amounts to *pervasive damping*: the influence of the dampers spreads through the whole system.

Another feature of the dynamics of the mass–spring–damper system (6.184) is its *robustness* with regard to constant external (disturbance) forces. Indeed, consider a mass–spring–damper system with boundary masses and general Hamiltonian $H(q,p)$, subject to *constant* forces $\bar{f}_b$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} \bar{f}_b, \tag{6.196}$$

where we *assume*[11] the existence of a $\bar{q}$ such that

$$D_s \frac{\partial H}{\partial q}(\bar{q}, 0) = E\bar{f}_b \tag{6.197}$$

The shifted Hamiltonian $\widehat{H}_{(\bar{q},0)}(q,p) := H(q,p) - (q-\bar{q})^T \frac{\partial H}{\partial q}(\bar{q},0) - H(\bar{q},0)$ as defined before in (6.92) satisfies

$$\frac{d}{dt}\widehat{H}_{(\bar{q},0)}(q,p) = -\frac{\partial^T H}{\partial p}(q,p) D_d R D_d^T \frac{\partial H}{\partial p}(q,p) \leq 0 \tag{6.198}$$

Specializing to a quadratic Hamiltonian $H(q,p) = \frac{1}{2}q^T Kq + \frac{1}{2}p^T Gp$ one obtains $\widehat{H}_{(\bar{q},0)}(q,p) = \frac{1}{2}(q-\bar{q})^T K(q-\bar{q}) + \frac{1}{2}p^T Gp$, leading to the following analog of Theorem 6.8.5.

Proposition 6.8.6 *Consider a linear mass–spring–damper system (6.196) with constant external disturbance* $\bar{f}_b$ *and Hamiltonian* $H(q,p) = \frac{1}{2}q^T Kq + \frac{1}{2}p^T Gp$*, where* K *and* G *are diagonal positive matrices. and with* im $E \subset$ im D_s. *The set of steady states is given by* $\bar{\mathcal{E}} = \{(q,p) \in \Lambda^1 \times \Lambda_0 \mid D_s Kq = E\bar{f}_b, Gp \in \text{span } \mathbb{1}\}$. *For every* (q_0, p_0) *there exists a unique equilibrium point* $(\bar{q}_\infty, p_\infty) \in \bar{\mathcal{E}} \cap \mathcal{A}_{(q_0,p_0)}$. *Here* p_∞ *is determined by (6.194), while* $\bar{q}_\infty = \bar{q} + q_\infty$*, with* $\bar{q}$ *such that* $D_s K\bar{q} = E\bar{f}_b$ *and* q_∞ *the unique solution of (6.193) with* q_0 *replaced by* $q_0 - \bar{q}$. *Furthermore, for each*

[11] If the mapping $q \mapsto \frac{\partial H}{\partial q}(q,0)$ is surjective, then there exists for every $\bar{f}_b$ such a $\bar{q}$ if and only if im $E \subset$ im D_s.

(q_0, p_0) *the trajectory starting from* (q_0, p_0) *converges asymptotically to* $(\bar{q}_\infty, p_\infty)$ *if and only if the largest* GL_s*-invariant subspace contained in* $\ker D_d^T$ *is equal to* span $\mathbb{1}$.

Note that the above proposition has the classical interpretation in terms of robustness of *integral control* with regard to constant disturbances: the springs act as integral controllers which counteract the influence of the unknown external force $\bar{f}_b$ so that the vector of velocities $M^{-1}p$ will still converge to span $\mathbb{1}$.

An alternative to the above formulation of mass–spring–damper systems is to consider instead of the spring extensions q the configuration vector $q_c \in \Lambda^0 =: Q_c$ describing the *positions* of the masses. For ordinary springs, the relation between $q_c \in \Lambda^0$ and $q \in \Lambda^1$ describing the extensions of the springs is given as $q = D^T q_c$. Hence, the energy can be also expressed as the function H_c of (q_c, p) defined as

$$H_c(q_c, p) := H(D^T q_c, p) \tag{6.199}$$

It follows that the dynamics of the mass–spring–damper system is alternatively given by the following Hamiltonian equations in the state variables q_c, p

$$\begin{aligned} \dot{q}_c &= \frac{\partial H_c}{\partial p}(q_c, p) \\ \dot{p} &= -\frac{\partial H_c}{\partial q_c}(q_c, p) - D_d R D_d^T \frac{\partial H_c}{\partial p}(q_c, p) + E f_b \\ e^b &= E^T \frac{\partial H_c}{\partial p}(q_c, p) \end{aligned} \tag{6.200}$$

What is the relation with the formulation given before? It turns out that this relation is precisely given by the standard procedure of *symmetry reduction* of a Hamiltonian system. Indeed, since $\mathbb{1}^T D = 0$ the Hamiltonian function $H_c(q_c, p)$ given in (6.199) is *invariant* under the action of the group $\mathbb{R}$ acting on the phase space $\Lambda^0 \times \Lambda_0 \simeq \mathbb{R}^{2N}$ by the symplectic group action

$$(q_c, p) \mapsto (q_c + \alpha \mathbb{1}, p), \quad \alpha \in \mathbb{R} \tag{6.201}$$

From standard reduction theory of Hamiltonian dynamics with symmetries, see e.g., [179, 197], it thus follows that we may factor out the configuration space $Q_c := \Lambda^0$ to the *reduced configuration space*

$$Q := \Lambda^0 / \mathbb{R} \tag{6.202}$$

Using the identification $Q := \Lambda^0/\mathbb{R} \simeq D^T \Lambda^0 \subset \Lambda^1$ the *reduced state space* of the mass–spring–damper system is given by im $D^T \times \Lambda_0$, with im $D^T \subset \Lambda^1$, and the Hamiltonian equations (6.200) on $\Lambda^0 \times \Lambda_0$ reduce to the port-Hamiltonian equations (6.184) on im $D^T \times \Lambda_0 \subset \Lambda^1 \times \Lambda_0$ as before.

The above example of a mass–spring–damper system on a graph can be generalized as follows. First note that a mass–spring–damper system with additive Hamiltonian H given by (6.192) can be also interpreted as the *interconnection* of port-Hamiltonian systems $\dot{p}_i = u_i^0,\ y_i^0 = \frac{\partial H_i^0}{\partial p_i}(p_i)$ corresponding to the masses (index i ranging over the vertices), port-Hamiltonian systems $\dot{q}_j = u_j^1,\ y_j^1 = \frac{\partial H_j^1}{\partial q_j}(q_j)$ corresponding to the springs (with j ranging over the spring edges), and static port-Hamiltonian systems $y_k^1 = r_j u_k^1, r_k > 0$, corresponding to dampers (with index k ranging over the damper edges), via the interconnection equations

$$u^v = -Dy^b,\ u^b = D^T y^v \tag{6.203}$$

Here the superscripts ^{v}, ^{b}, again refer to inputs and outputs of the port-Hamiltonian systems associated to, respectively, the vertices and edges (branches). In the same way we can therefore consider *arbitrary* port-Hamiltonian systems with scalar inputs and outputs associated with the vertices and the edges, interconnected by (6.203). Like in the general theory of interconnection of port-Hamiltonian systems this again defines a port-Hamiltonian DAE system, with Dirac structure determined by the Dirac structures of the port-Hamiltonian systems associated to the vertices and to the edges, and by the interconnection (6.203).

Remark 6.8.7 Similar to the second scenario considered for passive systems in Sect. 9.94, we may also consider the interconnection of single-input single-output port-Hamiltonian systems associated to the vertices of a graph by the interconnection $u = -Ly + e$, cf. (4.91), where L is a *balanced* Laplacian matrix. Decomposing L into its symmetric and skew-symmetric part we then obtain an interconnected port-Hamiltonian system with extra energy-dissipating terms corresponding to the symmetric part of L.

Remark 6.8.8 Another paradigmatic example of port-Hamiltonian systems on graphs are *RLC-electrical circuits*. In this case, all the energy-storing and energy-dissipating elements are associated to the *edges* of the circuit graph. This leads to the consideration of the *Kirchhoff–Dirac structure* defined as

$$\begin{aligned}\mathcal{D}_K := \{(f_1, e^1, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ D_i f_1 = 0, D_b f_1 = f_b, \exists\, e^{0i} \in \Lambda^{0i} \text{ s.t. } e^1 = -D_i^T e^{0i} - D_b^T e^b\}\end{aligned} \tag{6.204}$$

capturing Kirchhoff's current and voltage laws. The port-Hamiltonian formulation of the electrical circuit is obtained by supplementing the Kirchhoff–Dirac structure by energy-storage relations corresponding to either capacitors or inductors, and by energy-dissipating relations corresponding to the resistors [297].

6.9 Scattering of Port-Hamiltonian Systems

In Sects. 2.4 and 3.4, we already introduced the scattering transformation from *flow and effort* vectors f, e to *wave vectors* v, z. Thus, let $\mathcal{F}$ be an ℓ-dimensional linear space of flows, and consider the canonically defined symmetric bilinear form, cf. (6.114), on $\mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, given as

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1 \mid f_2 > + < e_2 \mid f_1 > \tag{6.205}$$

for $f_i \in \mathcal{F}, e_i \in \mathcal{E}^*, i = 1, 2$. Furthermore, as in Sect. 2.4, let $\mathcal{V} \subset \mathcal{F} \times \mathcal{E}$ be any ℓ-dimensional *positive* space of $\ll, \gg$, and $\mathcal{Z} \subset \mathcal{F} \times \mathcal{E}$ an ℓ-dimensional *negative* space of $\ll, \gg$, which is *orthogonal* (in the sense of $\ll, \gg$) to $\mathcal{V}$. This means that

$$\mathcal{F} \times \mathcal{E} = \mathcal{V} \oplus \mathcal{Z} \tag{6.206}$$

Now, consider a constant Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, that is

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp} \tag{6.207}$$

with $\perp\!\!\!\perp$ denoting orthogonal companion with respect to $\ll, \gg$. It follows that $\ll, \gg$ is *zero* when restricted to $\mathcal{D}$, and thus

$$\mathcal{D} \cap \mathcal{V} = 0, \quad \mathcal{D} \cap \mathcal{Z} = 0 \tag{6.208}$$

This implies that the Dirac structure $\mathcal{D}$ can be represented as the *graph* of an *invertible* linear map $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$, that is,

$$\mathcal{D} = \{(f, e) = v + z \mid z = \mathcal{O}v\}, \tag{6.209}$$

where $v + z \in \mathcal{V} \oplus \mathcal{Z}$ is the *scattering representation* of $(f, e) \in \mathcal{F} \times \mathcal{E}$ with respect to the scattering subspaces $\mathcal{V}, \mathcal{Z}$.

Furthermore, for any $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$, with scattering representation $v_1 + z_1$, respectively, $v_2 + z_2$, we obtain by (2.35) and (6.207)

$$0 = < e_1 \mid f_2 > + < e_2 \mid f_1 > = < v_1, v_2 >_{\mathcal{V}} - < z_1, z_2 >_{\mathcal{Z}}, \tag{6.210}$$

where $<, >_{\mathcal{V}}$ and $<, >_{\mathcal{Z}}$ are the inner-products on $\mathcal{V}$, respectively, $\mathcal{Z}$, induced from $\ll, \gg$; see Sect. 2.4, Eq. (2.35). This implies that

$$< z_1, z_2 >_{\mathcal{Z}} = < \mathcal{O}v_1, \mathcal{O}v_2 >_{\mathcal{Z}} = < v_1, v_2 >_{\mathcal{V}} \tag{6.211}$$

for all $v_1, v_2 \in \mathcal{V}$. Hence, the linear map $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$ is an *inner-product preserving* map from $\mathcal{V}$, with inner product $<, >_{\mathcal{V}}$, to $\mathcal{Z}$ with inner-product $<, >_{\mathcal{Z}}$. Conversely, let $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$ be an inner-product preserving map. If we now *define* $\mathcal{D}$ by (6.209), then by (6.210) and (6.211)

$$0 = < v_1, v_2 >_{\mathcal{V}} - < z_1, z_2 >_{\mathcal{Z}} = < e_1 \mid f_2 > + < e_2 \mid f_1 > ,$$

and thus $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. Furthermore, because $\dim \mathcal{D} = \ell$, we conclude $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$, implying that $\mathcal{D}$ is a Dirac structure. *Hence constant Dirac structures $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ are in one-to-one correspondence with inner-product preserving linear maps $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$.* This leads to the following definition.

Definition 6.9.1 Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ be a Dirac structure, and let $(\mathcal{V}, \mathcal{Z})$ be a pair of scattering subspaces. The map $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$ satisfying (6.209) is called the *scattering representation* of $\mathcal{D}$.

A matrix representation of the scattering representation $\mathcal{O}$ of a Dirac structure $\mathcal{D}$ is obtained as follows. Consider a basis $a_1, \ldots, a_\ell$ for $\mathcal{F}$ and dual basis $a_1^*, \ldots, a_\ell^*$ for $\mathcal{E}$, together with the resulting scattering transformation as in (2.40). Furthermore, corresponding to this basis let $\mathcal{D}$ be given in kernel representation as

$$\mathcal{D} = \{(f, e) \mid Ff + Ee = 0\}, \tag{6.212}$$

with F, E square $\ell \times \ell$ matrices satisfying

$$EF^T + FE^T = 0, \quad \text{rank}[F \vdots E] = \ell \tag{6.213}$$

Proposition 6.9.2 *Let the Dirac structure $\mathcal{D}$ be given by* (6.212). *The matrix representation of its scattering representation $\mathcal{O} : \mathcal{V} \to \mathcal{Z}$ is the orthonormal matrix*

$$\mathcal{O} = (F - E)^{-1}(F + E) \tag{6.214}$$

Proof $\mathcal{D}$ is equivalently given in image representation as $\mathcal{D} = \{(f, e) | f = E^T\lambda, e = F^T\lambda, \lambda \in \mathbb{R}^\ell\}$. The coordinate relation between $(f, e) \in \mathcal{F} \times \mathcal{E}$ and its scattering representation $v + z$ is given as (cf. (2.41))

$$\begin{aligned} v &= \tfrac{1}{\sqrt{2}}(f + e) \\ z &= \tfrac{1}{\sqrt{2}}(-f + e) \end{aligned} \tag{6.215}$$

Thus in scattering representation $\mathcal{D}$ is given as

$$\mathcal{D} = \left\{ v + z \mid v = \frac{1}{\sqrt{2}}(E^T + F^T)\lambda, z = \frac{1}{\sqrt{2}}(-E^T + F^T)\lambda, \lambda \in \mathbb{R}^\ell \right\} \tag{6.216}$$

We claim that $E^T + F^T$ is invertible. Indeed, suppose $x \in \ker\,(E^T + F^T)$, that is, $E^T x = -F^T x$. Since by (6.213) $EF^T x + FE^T x = 0$ for all x, this implies $EE^T x = -EF^T x = FE^T x = -FF^T x$, and thus

$$[EE^T + FF^T]x = 0, \tag{6.217}$$

which in view of rank $[F \vdots E] = \ell$ implies $x = 0$. Hence, $E^T + F^T$ and $F + E$ are invertible. Therefore

$$\mathcal{D} = \{(v, z) \mid z = (F^T - E^T)(F^T + E^T)^{-1} v\} \tag{6.218}$$

Similarly, it follows that $-E^T + F^T$ and thus $F - E$ are invertible. Comparing with (6.209) we conclude that $\mathcal{O} = (F^T - E^T)(F^T + E^T)^{-1}$. Finally, adding, respectively, subtracting, $EF^T + FE^T = 0$ to the expression $FF^T + EE^T$ yields the equality

$$(F + E)(F^T + E^T) = (F - E)(F^T - E^T) \tag{6.219}$$

and thus $\mathcal{O}$ is also expressed as in (6.214). Furthermore, (6.219) implies

$$\begin{aligned}\mathcal{O}\mathcal{O}^T &= (F - E)^{-1}(F + E)(F^T + E^T)(F^T - E^T)^{-1} \\ &= (F - E)^{-1}(F - E)(F^T - E^T)(F^T - E^T)^{-1} = I_\ell,\end{aligned}$$

showing that O is orthonormal. □

Example 6.9.3 Let the Dirac structure $\mathcal{D}$ be given by a skew-symmetric matrix J, that is, $\mathcal{D} = \{(f, e) \mid f = Je,\ J = -J^T\}$. Then the scattering representation of $\mathcal{D}$ is the orthonormal matrix

$$\mathcal{O} = (I + J)^{-1}(I - J) \tag{6.220}$$

(known as the *Cayley transform* of J).

Remark 6.9.4 The same result holds for Dirac structures on a manifold $\mathcal{X}$. In this case, the Dirac structure is represented by an orthonormal matrix $\mathcal{O}(x)$ depending on $x \in \mathcal{X}$ (where also the scattering subspaces $\mathcal{V}$ and $\mathcal{Z}$ may depend on x). In particular, the scattering representation of the Dirac structure defined as the graph of $J(x) = -J^T(x)$ is $\mathcal{O}(x) = (I + J(x))^{-1}\,(I - J(x))$.

A special type of Dirac structures (called 0- and 1-*junctions*) are defined as follows

$$\begin{aligned}\mathcal{D}_0 &= \{(f, e) \in \mathbb{R}^\ell \times \mathbb{R}^\ell \mid f_1 + \cdots + f_\ell = 0,\ e_1 = \cdots = e_\ell\} \\ \mathcal{D}_1 &= \{(f, e) \in \mathbb{R}^\ell \times \mathbb{R}^n \mid e_1 + \cdots + e_\ell = 0,\ f_1 = \cdots = f_\ell\}\end{aligned} \tag{6.221}$$

Using scattering representations they can be characterized as follows.

Proposition 6.9.5 *Scattering representations $\mathcal{O}_0, \mathcal{O}_1$ of $\mathcal{D}_0, \mathcal{D}_1$ are given by*

$$\mathcal{O}_0 = \frac{2}{\ell}\mathbb{I}_\ell - I_\ell, \quad \mathcal{O}_1 = -\frac{2}{\ell}\mathbb{I}_\ell + I_\ell \tag{6.222}$$

where $\mathbb{I}_\ell$ denotes the $\ell \times \ell$ matrix filled with ones and I_ℓ is the $\ell \times \ell$ identity matrix. Moreover, $\mathcal{O}_0$ and $\mathcal{O}_1$ are the only orthonormal $\ell \times \ell$ matrices that have equal diagonal elements and equal off-diagonal elements.

Proof With respect to the last claim note that $\mathcal{O} = a\mathbb{I}_\ell + bI_\ell$ is orthonormal if and only $\ell a + 2b = 0$ and $b^2 = 1$. The case $b = 1$ gives $\mathcal{O}_1 = -\frac{2}{\ell}\mathbb{I}_\ell + I_\ell$, while $b = -1$ yields $\mathcal{O}_0 = \frac{2}{\ell}\mathbb{I}_\ell - I_\ell$. The rest follows by direct computation. □

Similarly to Sect. 3.4, let us finally apply scattering to a standard input-state-output port-Hamiltonian form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned} \tag{6.223}$$

Consider a scattering representation of $(f_P, e_P) = (u, y)$ (but *not* of (f_S, e_S)), defined as

$$\begin{aligned} v &= \tfrac{1}{\sqrt{2}}(u + y) \\ z &= \tfrac{1}{\sqrt{2}}(-u + y) \end{aligned} \tag{6.224}$$

The inverse of this transformation is $u = \frac{1}{\sqrt{2}}(v - z)$, $y = \frac{1}{\sqrt{2}}(v + z)$, which by substitution in (6.223) yields

$$\begin{aligned} \dot{x} &= \left[J(x) - R(x) - g(x)g^T(x)\right]\frac{\partial H}{\partial x}(x) + \sqrt{2}g(x)v \\ z &= \sqrt{2}g^T(x)\frac{\partial H}{\partial x}(x) - v \end{aligned} \tag{6.225}$$

Note that, compared with (6.223), artificial *energy dissipation* has been inserted in two ways: (i) by an extra resistive structure matrix $g(x)g^T(x) \geq 0$, (ii) by a negative unity feedthrough from v to z.

Finally, *composition* of Dirac structures takes the following form in scattering formulation. Consider two Dirac structures $\mathcal{D}_A, \mathcal{D}_B$ as in Theorem 6.6.9, composed by setting

$$f_A = -f_B \in \mathcal{F}, \quad e_A = e_B \in \mathcal{E} \tag{6.226}$$

Now consider scattering representations $(f_A, e_A) = v_A + z_A$ and $(f_B, e_B) = v_B + z_B$ with respect to the *same* scattering subspaces $\mathcal{V}, \mathcal{Z} \subset \mathcal{F} \times \mathcal{E}$. Then (6.226) becomes

$$\begin{aligned} z_A &= v_B \\ z_B &= v_A \end{aligned} \tag{6.227}$$

expressing that the outgoing wave vector for $\mathcal{D}_A$ equals the incoming wave vector for $\mathcal{D}_B$, and conversely. Hence, the composition of $\mathcal{D}_A, \mathcal{D}_B$ is seen to correspond to the configuration depicted in Fig. 6.9, known as the *Redheffer star product* [259] of the orthonormal matrices $\mathcal{O}_A$ and $\mathcal{O}_B$. This is formulated in the next proposition.

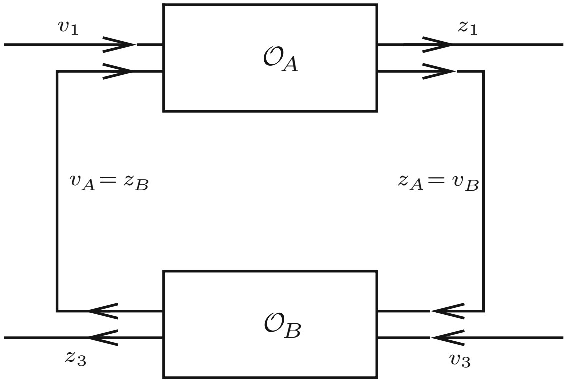


Fig. 6.9 Redheffer star product of $\mathcal{O}_A$ and $\mathcal{O}_B$

Proposition 6.9.6 *Let the orthonormal mappings $\mathcal{O}_A$ and $\mathcal{O}_B$ be scattering representations of $\mathcal{D}_A$ and $\mathcal{D}_B$ with respect to the same scattering subspaces. Then the scattering representation of $\mathcal{D}_A \circ \mathcal{D}_B$ is given by $\mathcal{O}_A \star \mathcal{O}_B$, with $\star$ denoting the Redheffer star product.*

Remark 6.9.7 Since $\mathcal{D}_A \circ \mathcal{D}_B$ is a Dirac structure it directly follows that the Redheffer star product of the orthonormal mappings $\mathcal{O}_A$ and $\mathcal{O}_B$ is again an orthonormal mapping.

6.10 Notes for Chapter 6

1. Port-Hamiltonian systems were originally introduced (under the slightly different name of *port-controlled Hamiltonian systems*) in Maschke & van der Schaft [201, 202], Maschke, van der Schaft & Breedveld [203] and van der Schaft & Maschke [294, 295].
2. A broad coverage of port-Hamiltonian systems and the background theory of port-based modeling, including application areas, can be found in [93] and the references quoted therein. The port-Hamiltonian formulation of bond-graph models is described in Golo, van der Schaft, Breedveld, Maschke [115]. A recent introductory survey of port-Hamiltonian systems theory, emphasizing new developments, is [291].
3. For port-Hamiltonian systems (6.1) two geometric structures play a role: (i) an (almost-)Poisson structure determined by the skew-symmetric matrix $J(x)$, (ii) the singular Riemannian metric determined by the symmetric positive semi-definite matrix $R(x)$. For some results and ideas on the interplay between these two structures, and its consequences for the resulting dynamics we refer to Morrison [223], and the references quoted therein. Similar structures have been used

in the description of thermodynamical systems, see e.g., Öttinger [246] and the references therein.

4. One can also define a bracket with respect to the *combined* structure $F(x) := J(x) - R(x)$, called the *Leibniz bracket*; see e.g., Ortega & Planas-Bielsa [236].
5. The formulation of detailed-balanced mass action kinetics chemical reaction networks as in Example 6.1.7 can be found in van der Schaft, Rao & Jayawardhana [299]; see also [258, 301] for the generalization to complex-balanced reaction networks. The port-Hamiltonian formulation was emphasized in van der Schaft, Rao & Jayawardhana [300].
6. A broader discussion about the obstruction to a port-Hamiltonian formulation indicated in Example 6.1.9 can be found in [213].
7. The formulation of autonomous Hamiltonian dynamics with regard to a Poisson structure which not necessarily has full rank, is standard in the literature on geometric mechanics, see e.g., Marsden & Ratiu [197], Olver [235].
8. The dual to any Lie algebra is endowed with a canonical Poisson structure, see e.g., Weinstein [347], Marsden & Ratiu [197]. For instance the Poisson structure given in Example 6.2.2 for the Hamiltonian formulation of Euler's equations is the Lie–Poisson structure on $\mathrm{so}^*(3)$.
9. For an in-depth treatment of mechanical systems with kinematic constraints, including the constrained Euler–Lagrange equations, see e.g., Bloch [43], Bullo & Lewis [53], and the older reference Neimark & Fufaev [230]. For a classical survey on the kinematic model, regarding the admissible velocities as being directly controlled, we refer to Kolmanovsky & McClamroch [167].
10. The introduction of the new "momentum" variables $\tilde{p}$ in (6.45) is close to the classical use of quasi-coordinates (see e.g., Steigenberger [326] for a survey).
11. The description in Sect. 6.2 of mechanical systems with kinematic constraints as port-Hamiltonian systems defined with respect to the almost-Poisson structure J_c on the constrained state space $\mathcal{X}_c$ given by (6.48) and (6.51) is taken from van der Schaft & Maschke [293], where also the result can be found that J_c is a Poisson structure (i.e., satisfying the Jacobi-identity (6.33)) if and only if the kinematic constraints are holonomic. See also van der Schaft & Maschke [295] and Dalsmo & van der Schaft [78]. For a survey on almost-Poisson structures in nonholonomic mechanics see Cantrijn, de Leon & de Diego [60].
12. The port-Hamiltonian formulation of the classical eight-dimensional model of the synchronous generator, see e.g., Kundur [177], in Example 6.3.4 is taken from Shaik Fiaz, Zonetti, Ortega, Scherpen & van der Schaft [98]; see also van der Schaft & Stegink [303].
13. Hamiltonian functions involving two kinds of state variables in a non-separable way not only show up in multi-physics systems, as illustrated in Sect. 6.3, but also in "cyber-physical systems" such as *variable impedance control*. In its

most simple form a variable impedance controller is defined by a (virtual) linear spring with energy $H(q) = \frac{1}{2}kq^2$, where we regard, next to the extension q of the spring, also the spring "constant" k as a state variable whose value may change in time. This leads to the consideration of the port-Hamiltonian system with inputs u_1, u_2 and outputs y_1, y_2 given as

$$\begin{bmatrix} \dot{q} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} kq \\ \frac{1}{2}q^2 \end{bmatrix} \tag{6.228}$$

Here the port (u_1, y_1) corresponds to interaction with the environment (defining an impedance k), while the port (u_2, y_2) defines a control port, regulating the value of the impedance k based on the output $y_2 = \frac{1}{2}q^2$, possibly modulated by information about other variables in the total system. In robotics, this basic idea is referred to as *variable stiffness control*; see e.g., [354] for a survey.

14. An extensive treatment of Casimir functions for autonomous Hamiltonian dynamics as discussed in Sect. 6.4 can be found e.g., in Marsden & Ratiu [197] and Olver [235]. For the Energy-Casimir method see e.g., Marsden & Ratiu [197] and the references quoted in there. Here also the close connection with *symmetries* can be found. Solving the pde's (9.92) involves integrability conditions on the structure matrices $J(x)$ and $R(x)$. In particular, if $J(x)$ is a Poisson structure (i.e., satisfying the Jacobi-identity), then there always exist r independent solutions $C_1, \ldots, C_r$ of the pde's $\frac{\partial^T C}{\partial x}(x)J(x) = 0$, with $r = \dim \ker J(x)$.
15. System theoretic properties of the closely related class of *input–output Hamiltonian systems* introduced in Brockett [50] are investigated e.g., in van der Schaft [269], Crouch & van der Schaft [73], Nijmeijer & van der Schaft [233] (Chap. 12).
16. A subclass of port-Hamiltonian systems, called *reciprocal port-Hamiltonian systems* can be converted into a *gradient system* [71] formulation (with respect to an indefinite Hessian Riemannian metric); cf. van der Schaft [284] and van der Schaft & Jeltsema [291].
17. A systematic treatment of port-Hamiltonian systems with *switching* structure matrices (with applications to switching electrical circuits or mechanical systems) can be found e.g., in Escobar, van der Schaft & Ortega [95], van der Schaft & Camlibel [290], Valentin, Magos & Maschke [340]; see also van der Schaft & Jeltsema [291].
18. The property that a system is shifted passive with respect to *any* constant $\bar{u}$ and corresponding steady-state $\bar{x}$, cf. Sect. 6.5 and Proposition 6.5.1, was coined as *equilibrium independent passivity* in Arcak, Meissen & Packard [11].
19. Proposition 6.5.4 is due to Ferguson, Middleton & Donaire [97].
20. Example 6.5.3 is taken from Bürger & De Persis [54]; see also Arcak [10], van der Schaft & Stegink [303].

21. The construction of the modified Hamiltonian $\widetilde{H}$ in (6.106) can be found in [199].

22. The definition of Dirac structure was originally intended as a generalization of both Poisson and symplectic structures; cf. Courant [72], Dorfman [85]. The name apparently originates from the concept of the *Dirac bracket* as appearing for Hamiltonian systems with constraints in Dirac [81, 82]. The kernel, image and constrained input–output representations of Dirac structures can be found in Dalsmo & van der Schaft [78], see also Courant [72]. The hybrid input–output representation is due to Bloch & Crouch [47]. See also van der Schaft [282] for a survey.

23. The proof of Theorem 6.6.9, as well as of Proposition 6.6.10, can be found in Cervera, van der Schaft & Banos [63] using ideas from Narajanan [228]; see also van der Schaft [281].

24. The definition of port-Hamiltonian systems with respect to Dirac structures was first given in van der Schaft & Maschke [294], and further developed in van der Schaft & Maschke [189, 292]; see also Bloch & Crouch [47] for the use of Dirac structures in the modeling of general LC circuits. For a treatment of constrained mechanical systems in this context, see Maschke & van der Schaft [206].

25. Integrability of Dirac structures (generalizing the Jacobi-identity for Poisson structures) is treated in Courant [72], Dorfman [85]. See also Merker [218] and the references quoted therein for further developments. For applications of the integrability of Dirac structures to properties of port-Hamiltonian DAE systems, including the connection to integrability of kinematic constraints, we refer to Dalsmo & van der Schaft [78]; see also van der Schaft & Jeltsema [291] and the references quoted therein. Necessary and sufficient conditions for the integrability of composed Dirac structures are obtained in Blankenstein & van der Schaft [40].

26. A further treatment of port-Hamiltonian DAE systems and their equational representations can be found in van der Schaft [286].

27. Section 6.8 is largely based on [298]. The port-Hamiltonian modeling of general *LC* circuits can be found in Maschke, van der Schaft & Breedveld [205], Maschke & van der Schaft [207]. See also Blankenstein [39]. The formulation of *RLC*-circuits alluded to in Remark 6.8.8 can be found in van der Schaft & Maschke [297]; with the notion of Kirchhoff–Dirac structure in (6.204) given in van der Schaft & Maschke [298].

28. The scattering representation of Dirac structures as dealt with in Sect. 6.9 can be found in Cervera, van der Schaft & Banos [63]. The proof of Proposition 6.9.2 is based on ideas from Courant [72].

29. Proposition 6.9.5 is originally due to Hogan & Fasse [128].

30. Many of the definitions and results in this chapter can be extended to *distributed-parameter* port-Hamiltonian systems; see e.g., van der Schaft [296], Duindam, Macchelli, Stramigioli & Bruyninckx [93], van der Schaft & Jeltsema [291], and the references quoted therein.
31. For a theory of *symmetries* of port-Hamiltonian systems, and the resulting reduction and existence of Casimirs, see e.g., van der Schaft [280], Blankenstein & van der Schaft [41], Merker [218].
32. An extension of the port-Hamiltonian formalism to thermodynamical systems can be found in Eberard, Maschke & van der Schaft [94].