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BEARING RIGIDITY AND FORMATION STABILIZATION FOR MULTIPLE RIGID BODIES IN $SE(3)$

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Abstract. In this work, we first distinguish different notions related to bearing rigidity in graph theory and then further investigate the formation stabilization problem for multiple rigid bodies. Different from many previous works on formation control using bearing rigidity, we do not require the use of a shared global coordinate system, which is enabled by extending bearing rigidity theory to multi-agent frameworks embedded in the three dimensional *special Euclidean group* $SE(3)$ and expressing the needed bearing information in each agent's local coordinate system. Here, each agent is modeled by a rigid body with 3 DOFs in translation and 3 DOFs in rotation. One key step in our approach is to define the bearing rigidity matrix in $SE(3)$ and construct the necessary and sufficient conditions for infinitesimal bearing rigidity. In the end, a gradient-based bearing formation control algorithm is proposed to stabilize formations of multiple rigid bodies in $SE(3)$.

1. Introduction.

In the past decades, distance-based rigidity has been extensively studied both as a mathematical topic in graph theory [2] and an engineering problem in the application domains such as formations of multi-agent systems [1], mechanical structures [20], and biological materials [4]. The concept of rigidity of frameworks was defined in [3], while a closely related, but more restrictive concept, infinitesimal rigidity, was discussed in [15] using the infinitesimal motions of the framework. Sometimes, infinitesimal rigidity becomes handy since it can be checked by examining the rank of the framework's rigidity matrix. In [19], it was proved that rigidity and infinitesimal rigidity are equivalent when the configuration of the framework is *generic*. The work in [21] investigated how to construct an infinitesimally and minimally rigid framework based on Laman's theorem [17] and Henneberg's construction [16]. Necessary and sufficient conditions were given in [9] for a generically and globally rigid graph in the plane. A survey on rigidity graph theory for modelling autonomous formations has been provided in [1].

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Recently, as the dual of distance-based rigidity, bearing rigidity has been studied, for which multi-agent frameworks are constrained by the inter-agent bearings, angles or directions. When the multi-agent framework is implemented as a team of mobile vehicles and agents are equipped with sensors, there are mainly two different types of existing definitions for "bearing" according to how it can be obtained through sensing. The first is defined to be the angle between the sensing beam and the x-axis of the corresponding agent's local coordinate system [10, 6, 5, 7]. The second is represented in a global coordinate system by the unit vector starting from the corresponding sensing agent's position along the sensing beam [26, 25]. Correspondingly, there are research results for each of these two definitions and we give a brief review of some of them as follows.

Eren *et al.* [10, 11] used the first definition and established direction-based bearing rigidity. They made use of the fact that a bearing constraint can be transformed into a direction constraint on the direction in which an agent is confined to move once all agents know the global coordinate system. In [7], by using the first definition, *stiff rigidity* with mixed bearing and distance constraints was defined under the constraint that a global coordinate system was required to be shared. Zhao *et al.* [26] established the bearing rigidity using the second definition, in which a bearing-only formation control algorithm was designed to guarantee almost global convergence. In a follow-up work [25], translational and scaling formation maneuvering control was realized via a bearing-based approach, in which both bearing information and relative position information were required.

However, the angle- or direction-based bearing rigidity has only been defined in the 2-dimensional Euclidean space [10]. For higher dimensional Euclidean spaces, i.e., $\mathbb{R}^n, n \geq 3$, the established angle- and direction-based bearing rigidity does not work any more [13]. In comparison, the unit-vector bearing rigidity established in [26] works in an arbitrary dimensional Euclidean space. In [26], the unit-vector bearing can be obtained under the constraint that each agent has the knowledge of a global coordinate system; however, in many practical applications, this assumption is technically hard to be satisfied. For example, for spacecraft formations in deep space, it is difficult for satellites to get the knowledge of a common global coordinate system with respect to the earth due to extremely long distance [14]. By expressing unit-vector bearing in each agent's local coordinate system, bearing rigidity has been extended to those multi-agent frameworks that are embedded in the 2 dimensional *special Euclidean group* $SE(2) = \mathbb{R}^2 \times SO(1)$ [23]. Based on the developed bearing rigidity in $SE(2)$, a bearing-only formation control algorithm was designed in [24] for each agent with 2 DOFs in translation and 1 DOF in rotation. In [18], bearing rigidity has been further extended for multi-agent frameworks embedded in the three dimensional *special Euclidean group* $SE(3) = \mathbb{R}^3 \times SO(3)$.

Motivated by the existing results, in this paper we study bearing rigidity for multi-agent frameworks that are embedded in $SE(3)$, in which each agent is modeled by a rigid body with 3 DOFs in translation and 3 DOFs in rotation. Each agent can read its bearing measurements according to its *local* coordinate system. Correspondingly, the bearing rigidity matrix in $SE(3)$ is defined and the necessary and sufficient conditions for infinitesimal bearing rigidity are also derived. Moreover, the formation stabilization problem for multiple rigid bodies in $SE(3)$ is studied, for which the gradient-based bearing formation control algorithm is designed.

The rest of this paper is organized as follows. Section 2 introduces preliminaries. In Section 3, different definitions for bearing and bearing rigidity are listed for comparison. In Section 4, bearing rigidity in $SE(3)$ is defined, which is further used in Section 5 to develop control strategies for bearing formation stabilization in $SE(3)$.

2. **Preliminaries on rigid body and** *SE(3)*. In this paper, each agent is modeled by a rigid body with 3 DOFs in translation, 3 DOFs in rotation, and so total 6 DOFs. We label these agents by $1, \ldots, n$. Agent i's, $i \in \{1, \ldots, n\}$, position and attitude (represented by Euler angles) are respectively denoted by vectors in $\mathbb{R}^3$

$$p_i = [x_i, y_i, z_i]^T \tag{1}$$

$$\theta_i = [\alpha_i, \beta_i, \gamma_i]^T \tag{2}$$

in some reference coordinate system of choice. Then agent i's state is the vector in $\mathbb{R}^6$

$$x_i = [p_i^T \ \theta_i^T]^T. \tag{3}$$

To describe agent i's translational and rotational motions, we introduce the *special Euclidean group* $SE(3) = \mathbb{R}^3 \times SO(3)$, in which $SO(3)$ represents the three dimensional *special orthogonal group*.

$$SE(3) = \{A | A = \begin{bmatrix} R & r \\ 0_{1\times 3} & 1 \end{bmatrix}, R \in \mathbb{R}^{3\times 3}, r \in \mathbb{R}^3, R^T R = I_3, \det\, R = 1\} \tag{4}$$

where $R \in SO(3)$ is the rotation matrix which can be determined by the orientation of agent i's rigidly attached frame with respect to the reference coordinate system, det() denotes the determination of a square matrix, and r is the vector from the reference frame's origin to agent i's frame.

To establish the relationship between the rotation matrix $R(\theta_i)$ and the Euler angles α_i, β_i, and γ_i, we rotate the axes in sequence: $x - y - z$ to realize the orientation change from the reference coordinate system to agent i's rigidly attached frame. Then, the rotation matrix $R(\theta_i)$ can be written as

$$R(\theta_i) = R_z(\gamma_i) R_y(\beta_i) R_x(\alpha_i) =$$
$$\begin{bmatrix} \cos\gamma_i & \sin\gamma_i & 0 \\ -\sin\gamma_i & \cos\gamma_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_i & 0 & -\sin\beta_i \\ 0 & 1 & 0 \\ \sin\beta_i & 0 & \cos\beta_i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_i & \sin\alpha_i \\ 0 & -\sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{5}$$

where $R_z(\gamma_i)$, $R_y(\beta_i)$, $R_x(\alpha_i)$ represent the rotation matrices whose rotation axes are z, y, x and rotation angles are γ_i, β_i, α_i, respectively.

3. **Existing definitions for bearing and bearing rigidity.**

3.1. **Bearing defined by angle and direction in 2D** [10]**.** As shown in Figure 1, the bearing constraints for agents i and j, denoted by θ_{ij} and θ_{ji} respectively, are the angles between the beam connecting agents i and j and the x-axis of i's and j's local coordinate system respectively in the counterclockwise direction [10].

It was assumed in [13, 12] that in real implementations, the information of the global coordinate system (x_G, y_G) can be known by all agents by passing "heading" information (an angle ϕ_i between the x-axis of agent i's local coordinate system and the y-axis of the global coordinate system), which is shown in Figure 2 [10]. Therefore, once all agents know this piece of global information, the bearing information measured in their local coordinate systems (e.g., θ_{ij} and θ_{ji}) can be transformed into the bearing information measured in the global coordinate system, e.g., Θ_{ij} and Θ_{ji} in Figure 3. Thus, when all agents are able to obtain the information of

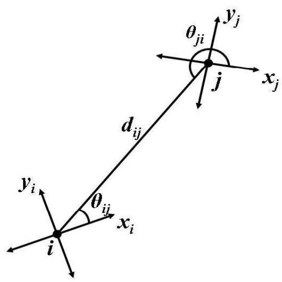

FIGURE 1

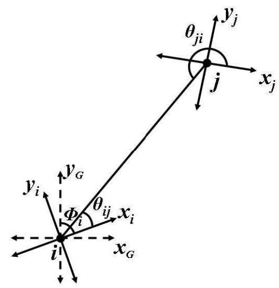

FIGURE 2

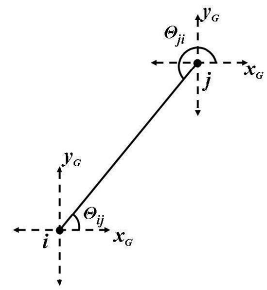

FIGURE 3

the global coordinate system, a bearing constraint can be written into the form of a direction constraint [13, 12, 11]. Based on the direction constraints, the bearing rigidity was established in [12, 11].

Definition 3.1. [12] For a framework[1] $\mathcal{G}_2(p)$ in $\mathbb{R}^2$, one embeds a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with the vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ to obtain the position of each $i \in \mathcal{V}$ as $p_i \in \mathbb{R}^2$. Then for a different embedded framework $\mathcal{G}_2(q)$ of $\mathcal{G}(\mathcal{V},\mathcal{E})$ with the vertex positions $q = [q_1^T, ..., q_n^T]^T, q_i \in \mathbb{R}^2, i \in \mathcal{V}$, we call $\mathcal{G}_2(q)$ a *parallel framework* associated with $\mathcal{G}_2(p)$ if

$$(p_i - p_j)^{\perp} \cdot (q_i - q_j) = 0, \forall (i,j) \in \mathcal{E} \tag{6}$$

where $\perp$ denotes the perpendicular of the corresponding vector.

Trivial parallel frameworks are the translations and scalings of the original framework. We consider the bearing constraints to be those constrained on the magnitudes of θ_{ij} as shown in Figure 3.

Definition 3.2. [12] (*Bearing rigidity in* $\mathbb{R}^2$) A framework $\mathcal{G}_2(p)$ with bearing constraints for $\theta_{ij}, \forall (i,j) \in \mathcal{E}$ is said to be *bearing rigid* if all parallel frameworks of $\mathcal{G}_2(p)$ are trivially parallel to $\mathcal{G}_2(p)$. Otherwise it is said to be *bearing flexible*.

3.2. **Bearing defined by unit vectors in a global coordinate system [26].** For an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, let n be its number of vertices and m the number of edges. The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. The position of each $i \in \mathcal{V}$ is given in the d-dimensional space as $p_i \in \mathbb{R}^d$, $d \geq 1$. A framework $\mathcal{G}_d(p)$ in $\mathbb{R}^d$ is a combination of a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with the vertex positions $p = [p_1^T, ..., p_n^T]^T, p_i \in \mathbb{R}^d, \forall i = 1, ..., n$. Bearing was defined in [26] by the unit vector starting from agent i's position along the sensing beam, i.e.,

$$g_{ij} = \frac{e_{ij}}{\|e_{ij}\|}, \forall (i,j) \in \mathcal{E}, \tag{7}$$

where $e_{ij} = p_j - p_i$, and $j \in \mathcal{N}_i$. We label all the edges in $\mathcal{E}$ by $1, \ldots, m$, and let e_k be the vector between the locations of edge k's two embedded vertices, and define $g_k = \frac{e_k}{||e_k||}$. Then, the bearing function $F_B : \mathbb{R}^{dn} \to \mathbb{R}^{dm}$ can be defined by

$$F_B(p) = [g_1^T, ..., g_m^T]^T \in \mathbb{R}^{dm} \tag{8}$$

[1] In rigidity graph theory, framework is usually defined as the combination of a graph and its realization in e.g. 2-dimensional Euclidean space (Definition 3.1), d-dimensional Euclidean space (Section 3.2), $SE(2)$ (Definition 3.6), and $SE(3)$ (Definition 4.1).

The bearing rigidity matrix is defined by the Jacobian of the bearing function

$$R(p) = \frac{\partial F_B(p)}{\partial p} \in \mathbb{R}^{dm \times dn} \tag{9}$$

For any nonzero vector $x \in \mathbb{R}^d$, define the orthogonal projection $P_x : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ by

$$P_x = I_d - \frac{x}{\|x\|}\frac{x^T}{\|x\|}, \tag{10}$$

where I_d denotes the $d \times d$ identity matrix.

Based on the defined unit-vector bearing and bearing function in the d-dimensional space, the bearing rigidity was established in [26].

Definition 3.3. [26] Consider two frameworks $\mathcal{G}_d(p)$ and $\mathcal{G}_d(p')$ with the same graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. They are said to be *bearing equivalent* if

$$P_{p_i - p_j}(p'_i - p'_j) = 0, \forall (i, j) \in \mathcal{E}. \tag{11}$$

Definition 3.4. [26] Consider two frameworks $\mathcal{G}_d(p)$ and $\mathcal{G}_d(p')$ with the same graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. They are said to be *bearing congruent* if

$$P_{p_i - p_j}(p'_i - p'_j) = 0, \forall i, j \in \mathcal{V}. \tag{12}$$

Definition 3.5. (*Bearing rigidity in* $\mathbb{R}^d$) A framework $\mathcal{G}_d(p)$ is *bearing rigid* if there exists a constant $\epsilon > 0$ such that any framework $\mathcal{G}_d(p')$ that is bearing equivalent to $\mathcal{G}_d(p)$ and satisfies $\|p' - p\| < \epsilon$ is also bearing congruent to $\mathcal{G}_d(p)$.

3.3. **Bearing defined by unit vectors in** *SE(2)* [23, 24]. As has been introduced in the preliminaries, $SE(2) = \mathbb{R}^2 \times SO(1)$ is the two dimensional *Special Euclidean Group*, which involves 2 DOFs in translation and 1 DOF in rotation. We denote agent i's state by the vectors in $\mathbb{R}^3$

$$x_i = [p_i^T \ \theta_i]^T. \tag{13}$$

A configuration in $SE(2)$ is denoted by $x = [x_1^T, ..., x_n^T]^T \in \mathbb{R}^{3n}$. Note that the configuration defined in this subsection consists of both positions and attitudes, which is different from the discussion in subsections 3.1 and 3.2.

Definition 3.6. An $SE(2)$ *framework* $\mathcal{G}_2(x)$ is composed of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a configuration with $x = [x_1^T, ..., x_n^T]^T$.

Then, the bearing vector $b_{ij} \in \mathbb{R}^2$ between agents i and j can be described in agent i's local coordinate system

$$b_{ij} = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \frac{p_j - p_i}{||p_j - p_i||} = T(\theta_i)\frac{p_j - p_i}{||p_j - p_i||} \tag{14}$$

where $T(\theta_i)$ is the rotation matrix determined by the orientation of agent i's rigidly attached frame with respect to the reference coordinate system. Then, the bearing function in $SE(2)$ is

$$b_{\mathcal{G}}(x) = diag\{\frac{T(\theta_i)}{||p_j - p_i||}\}Hp \tag{15}$$

where $H = \bar{H} \otimes I_2 \in \mathbb{R}^{2m\times 2n}$ and $\bar{H} \in \mathbb{R}^{m\times n}$ is the incidence matrix of the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, in which $[H]_{ki} = 1$ if vertex i is the head of edge k, $[H]_{ki} = -1$ if vertex i is the tail of edge k, and $[H]_{ki} = 0$ otherwise. Since the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is undirected, it is irrelevant how the directions of the edges are defined in $\bar{H}$. diag$\{\cdots\}$ stands for the block-diagonal matrix. Here $x = [x_1^T, ..., x_n^T]^T \in \mathbb{R}^{3n}$, $p = [p_1^T, ..., p_n^T]^T \in \mathbb{R}^{2n}$, and $\theta = [\theta_1, ..., \theta_n]^T \in \mathbb{R}^n$.

Definition 3.7. (*Bearing rigidity in* $SE(2)$) For an $SE(2)$ framework $\mathcal{G}_2(x)$ with the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and the configuration $x=[x_1^T,...,x_n^T]^T$, the framework $\mathcal{G}_2(x)$ is *bearing rigid* in $SE(2)$ if there exists a neighborhood $\mathcal{S}$ of x such that

$$b_K^{-1}(b_K(x))\cap\mathcal{S}=b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(x))\cap\mathcal{S} \tag{16}$$

where K represents the complete graph with the vertex set $\mathcal{V}$.

The set of infinitesimal motions is characterized by the null-space of the Jacobian of the relative bearing vector $b_{\mathcal{G}}(x)$ with respect to the configuration x.

Definition 3.8. An $SE(2)$ framework $\mathcal{G}_2(x)$ is *infinitesimally bearing rigid* if $\mathcal{D}[B_{\mathcal{G}}(x)]=\mathcal{D}[B_K(x)]$, where $\mathcal{D}[\cdot]$ denotes the null-space of a matrix, and $B_{\mathcal{G}}(x)=\begin{bmatrix}B_{\mathcal{G}}(p) & B_{\mathcal{G}}(\theta)\end{bmatrix}$, $B_K(x)=\begin{bmatrix}B_K(p) & B_K(\theta)\end{bmatrix}$, $B_{\mathcal{G}}(p)=\frac{\partial b_{\mathcal{G}}(x)}{\partial p}$, and $B_{\mathcal{G}}(\theta)=\frac{\partial b_{\mathcal{G}}(x)}{\partial \theta}$.

To summarize the above existing definitions for bearing and bearing rigidity, we give the following table:

Definitions for bearing	Measurement variable	Rigidity
Angle in 2D space	θ_{ij}	Parallel bearing rigidity
Unit vector in a global frame	$\frac{p_j-p_i}{\|\|p_j-p_i\|\|}$	Bearing rigidity in $\mathbb{R}^d$
Unit vector in $SE(2)$	$T(\theta_i)\frac{p_j-p_i}{\|\|p_j-p_i\|\|}$	Bearing rigidity in $SE(2)$

TABLE 1. Comparison of different definitions for bearing and bearing rigidity.

4. Bearing rigidity in *SE(3)*.

In the following section, we provide our new definition of bearing rigidity which relies only on the local information of each agent in the framework.

4.1. Definition of bearing in *SE(3)*.

In this section, we define the bearing to be the unit vector in the three dimensional *Special Euclidean group* $SE(3)$. Each agent is modeled by a rigid body with 3 DOFs in translation and 3 DOFs in rotation. Let $p_i\in\mathbb{R}^3$ denote the agent i' position from the reference frame's origin to agent i's rigidly attached frame's origin. $R(\theta_i)\in SO(3)$ is the rotation matrix which describes the relative orientation of the agent i's frame with respect to the reference frame.

Then, for the sensing beam between agents i and j, the relative bearing b_{ij} sensed in agent i's local coordinate frame can be described by.

$$b_{ij}=R(\theta_i)b_{ij}^G=R_z(\gamma_i)R_y(\beta_i)R_x(\alpha_i)\frac{p_j-p_i}{\|p_j-p_i\|}\in\mathbb{S}^2 \tag{17}$$

where $\mathbb{S}^2$ represents the 2D manifold on the unit sphere in $\mathbb{R}^3$, and b_{ij}^G is the unit vector of the sensing beam expressed in the global coordinate frame. Note that the local bearing information b_{ij} can be measured in agent i's local coordinate frame via onboard cameras [22]. Write (17) in its column vector form

$$b_{\mathcal{G}}(x)=diag\{\frac{R(\theta_i)}{\|p_j-p_i\|}\}Hp \tag{18}$$

where $H=\bar{H}\otimes I_3\in\mathbb{R}^{3m\times 3n}$.

4.2. **Definition of bearing rigidity matrix in** *SE(3)*. Denote $p = [p_1^T, ..., p_n^T]^T \in \mathbb{R}^{3n}$, $\theta = [\theta_1^T, ..., \theta_n^T]^T \in \mathbb{R}^{3n}$, $x = [x_1^T, ..., x_n^T]^T \in \mathbb{R}^{6n}$, and $b_{\mathcal{G}} = [b_1^T, ..., b_m^T]^T \in \mathbb{R}^{3m}$, where b_k represents the bearing measurement on the kth directed edge in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The bearing rigidity matrix is defined by the Jacobian of the relative bearing vector $b_{\mathcal{G}}(x)$ with respect to the configuration x. Since each agent has 3 DOFs in translation and 3 DOFs in rotation, we define the bearing translational rigidity matrix $B_{\mathcal{G}}(p) \in \mathbb{R}^{3m\times 3n}$ and bearing rotational rigidity matrix $B_{\mathcal{G}}(\theta) \in \mathbb{R}^{3m\times 3n}$ by

$$B_{\mathcal{G}}(p) = \frac{\partial b_{\mathcal{G}}(x)}{\partial p} \tag{19}$$

$$B_{\mathcal{G}}(\theta) = \frac{\partial b_{\mathcal{G}}(x)}{\partial \theta} \tag{20}$$

Using (17) and (19), one has

$$\frac{\partial b_{ij}}{\partial p_i} = -R(\theta_i)\frac{g(b_{ij}^G)}{||p_j - p_i||} = -\frac{g(b_{ij})R(\theta_i)}{||p_j - p_i||} \tag{21}$$

$$\frac{\partial b_{ij}}{\partial p_j} = R(\theta_i)\frac{g(b_{ij}^G)}{||p_j - p_i||} = \frac{g(b_{ij})R(\theta_i)}{||p_j - p_i||} \tag{22}$$

where $g(b_{ij}) = I_3 - \frac{b_{ij}b_{ij}^T}{\|b_{ij}\|^2}$.

Combining (17) and (20), one obtains

$$\begin{aligned}
\frac{\partial b_{ij}}{\partial \theta_i} &= [\frac{\partial b_{ij}}{\partial \alpha_i}, \frac{\partial b_{ij}}{\partial \beta_i}, \frac{\partial b_{ij}}{\partial \gamma_i}] \\
&= [R_z(\gamma_i)R_y(\beta_i)\frac{\partial R_x(\alpha_i)}{\partial \alpha_i}b_{ij}^G, R_z(\gamma_i)\frac{\partial R_y(\beta_i)}{\partial \beta_i}R_x(\alpha_i)b_{ij}^G, \frac{\partial R_z(\gamma_i)}{\partial \gamma_i}R_y(\beta_i)R_x(\alpha_i)b_{ij}^G] \\
&= [R_z(\gamma_i)R_y(\beta_i)\frac{\partial R_x(\alpha_i)}{\partial \alpha_i}, R_z(\gamma_i)\frac{\partial R_y(\beta_i)}{\partial \beta_i}R_x(\alpha_i), \frac{\partial R_z(\gamma_i)}{\partial \gamma_i}R_y(\beta_i)R_x(\alpha_i)](I_3 \otimes b_{ij}^G) \\
&= R^r(\theta_i)(I_3 \otimes b_{ij}^G)
\end{aligned} \tag{23}$$

$$\frac{\partial b_{ij}}{\partial \theta_j} = [\frac{\partial b_{ij}}{\partial \alpha_j}, \frac{\partial b_{ij}}{\partial \beta_j}, \frac{\partial b_{ij}}{\partial \gamma_j}] = 0 \tag{24}$$

where $R^r(\theta_i) \in \mathbb{R}^{3\times 9}$ is an auxiliary partial derivative rotational matrix, and $\otimes$ represents the Kronecker product [8]. According to the above partial derivative of the bearing b_{ij} with respect to position p_i and attitude θ_i, one gets

$$B_{\mathcal{G}}(p) = diag\{\frac{R(\theta_i)g(b_{ij}^G)}{\|p_j - p_i\|}\}H \tag{25}$$

$$B_{\mathcal{G}}(\theta) = diag\{R^r(\theta_i)(I_3 \otimes b_{ij}^G)\}E \tag{26}$$

where $E = \bar{E} \otimes I_3 \in \mathbb{R}^{3m\times 3n}$ and $\bar{E} \in \mathbb{R}^{m\times n}$ is defined according to the rule that $[E]_{ki} = 1$ if vertex i is the head of edge k, and $[E]_{ki} = 0$ otherwise. The bearing translational rigidity matrix $B_{\mathcal{G}}(p)$ and bearing translational rigidity matrix $B_{\mathcal{G}}(\theta)$ will be used to check the infinitesimally bearing rigidity in $SE(3)$.

Now we give the definition for $SE(3)$ frameworks.

Definition 4.1. An $SE(3)$ *framework* is denoted by $\mathcal{G}_3(x)$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph, and $x = [x_1^T, ..., x_n^T]^T \in \mathbb{R}^{6n}$ is a configuration of n agents in $SE(3)$, in which each agent is modeled by a rigid body with 3 DOFs in translation described by $p_i \in \mathbb{R}^3$ and 3 DOFs in rotation described by $\theta_i \in \mathbb{R}^3$.

4.3. **Definition of bearing rigidity in SE(3).** We will give the definitions for bearing rigidity, and infinitesimal bearing rigidity in this subsection.

Definition 4.2. (*Bearing rigidity in* $SE(3)$) An $SE(3)$ framework $\mathcal{G}_3(x)$ is *bearing rigid in* $SE(3)$ if there exists a neighborhood $\mathcal{S}$ in $SE(3)$ of x such that

$$b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(x)) \cap \mathcal{S} = b_K^{-1}(b_K(x)) \cap \mathcal{S} \tag{27}$$

where K represents the complete graph with the vertex set $\mathcal{V}$.

The set $b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(x)) \subset SE(3)^n$ contains x and all its possible transformations induced by the graph $\mathcal{G}$. Thus, $b_{\mathcal{G}}^{-1}(b_{\mathcal{G}}(x)) - b_K^{-1}(b_K(x))$ is the set of all the possible transformations of x constrained by $\mathcal{G}$ that are not admissible by K[18].

The set of infinitesimal motions is characterized by the null-space of the Jacobian of the relative bearing vector $b_{\mathcal{G}}(x)$ with respect to the configuration x.

Definition 4.3. An $SE(3)$ framework $\mathcal{G}_3(x)$ is *infinitesimally bearing rigid* in $SE(3)$ if $\mathcal{D}[B_{\mathcal{G}}(x)] = \mathcal{D}[B_K(x)]$, where $B_{\mathcal{G}}(x) = \begin{bmatrix} B_{\mathcal{G}}(p) & B_{\mathcal{G}}(\theta) \end{bmatrix}$, and $B_K(x) = \begin{bmatrix} B_K(p) & B_K(\theta) \end{bmatrix}$.

Proposition 1. *An* $SE(3)$ *framework* $\mathcal{G}_3(x)$ *is infinitesimally bearing rigid if and only if* $Rank[B_{\mathcal{G}}(x)] = 6n - 7$.

Proof. The trivial infinitesimal bearing rigid motions for an $SE(3)$ framework include three categories, namely translation with 3 DOFs, rotation with 3 DOFs, and scaling with 1 DOF. Thus, the rank of the bearing rigidity matrix for an infinitesimal bearing rigid framework is the total degrees of freedom, i.e., $6n$, minus the degrees of freedom of trivial infinitesimal bearing rigid motions, i.e., 7. □

5. Bearing formation stabilization in SE(3).

5.1. **Control algorithm design.** In this section, we aim at designing a bearing formation control algorithm for multi-agent frameworks such that all agents can converge to the desired positions that satisfy all the bearing constraints.

The dynamics of each rigid body in $SE(3)$ include the position dynamics and attitude dynamics [24], which can be described by

$$\dot{p}_i = R(\theta_i)^T v_i^b \tag{28}$$

$$\dot{\theta}_i = R_b^g(\theta_i)\omega_i^b \tag{29}$$

where v_i^b and ω_i^b are the position and attitude control inputs to be designed, $R_b^g(\theta_i)$ is rigid body i's rotation kinematic matrix which can be written as

$$R_b^g(\theta_i) = \begin{bmatrix} \cos\gamma_i/\cos\beta_i & -\sin\gamma_i/\cos\beta_i & 0 \\ \sin\gamma_i & \cos\gamma_i & 0 \\ -\cos\gamma_i \sin\beta_i/\cos\beta_i & \sin\gamma_i\sin\beta_i/\cos\beta_i & 1 \end{bmatrix} \tag{30}$$

Our bearing formation control objective is

$$\lim_{t\to\infty}(b_{ij}(t) - b_{ij}^*) = 0, \forall(i,j) \in \mathcal{E} \tag{31}$$

where b_{ij}^* is the desired bearing between agent i and agent j which is described in agent i's local coordinate system. We assume that the $SE(3)$ framework is infinitesimally bearing rigid. Then, there will be a realization of $b_{ij}^*, \forall(i,j) \in \mathcal{E}$ in the $SE(3)$ framework.

To achieve the control objective, we use the gradient-based control. First, we define the potential function

$$J(x) = 1/2\|b_{\mathcal{G}} - b^*_{\mathcal{G}}\|^2. \tag{32}$$

Taking the partial derivative of the potential function along the position and attitude yields the position and attitude control inputs:

$$\dot{p} = diag\{R(\theta_i)^T\}v^b = -k\nabla_p J(x) = -k\{B_{\mathcal{G}}(p)\}^T(b_{\mathcal{G}} - b^*_{\mathcal{G}}) \tag{33}$$

$$\dot{\theta} = diag\{R^g_b(\theta_i)\}\omega^b = -k\nabla_\theta J(x) = -k\{B_{\mathcal{G}}(\theta)\}^T(b_{\mathcal{G}} - b^*_{\mathcal{G}}) \tag{34}$$

where k is a positive constant. Applying the above computation to each agent yields

$$\dot{p}_i = R(\theta_i)^T v^b_i = -k\sum_{(i,j)\in\mathcal{E}} \frac{[g(b_{ij})R(\theta_i)]^T}{||p_j - p_i||}(b_{ij} - b^*_{ij}) + k\sum_{(j,i)\in\mathcal{E}} \frac{[g(b_{ji})R(\theta_j)]^T}{||p_i - p_j||}(b_{ji} - b^*_{ji}) \tag{35}$$

$$\dot{\theta}_i = R^g_b(\theta_i)\omega^b_i = -k\sum_{(i,j)\in\mathcal{E}} [R^r(\theta_i)(I_3 \otimes b^G_{ij})]^T(b_{ij} - b^*_{ij}) \tag{36}$$

Because the distance information is unavailable, we modify the above bearing formation control algorithm to

$$v^b_i = -k\{\sum_{(i,j)\in\mathcal{E}} g(b_{ij})^T(b_{ij} - b^*_{ij}) - \sum_{(j,i)\in\mathcal{E}} R(\theta_i)R(\theta_j)^T g(b_{ji})^T(b_{ji} - b^*_{ji})\} \tag{37}$$

$$\omega^b_i = -k\{R^g_b(\theta_i)\}^{-1}\sum_{(i,j)\in\mathcal{E}} [R^r(\theta_i)(I_3 \otimes b^G_{ij})]^T(b_{ij} - b^*_{ij}) \tag{38}$$

5.2. **Stability analysis.** To show the stability, we define the Lyapunov candidate function

$$V = 1/2\|b_{\mathcal{G}} - b^*_{\mathcal{G}}\|^2 \tag{39}$$

where $b_{\mathcal{G}} = [b^T_1, ..., b^T_m]^T \in \mathbb{R}^{3m}$, and $b^*_{\mathcal{G}} = [b^{*T}_1, ..., b^{*T}_m]^T \in \mathbb{R}^{3m}$.

Taking the derivative of the Lyapunov candidate function V_1 yields

$$\dot{V} = (b_{\mathcal{G}} - b^*_{\mathcal{G}})^T(\frac{\partial b_{\mathcal{G}}}{\partial p}\dot{p} + \frac{\partial b_{\mathcal{G}}}{\partial \theta}\dot{\theta}) = -k(b_{\mathcal{G}} - b^*_{\mathcal{G}})^T Q(b_{\mathcal{G}} - b^*_{\mathcal{G}}) \tag{40}$$

where $Q = B_{\mathcal{G}}(p)\text{diag}(\|p_j - p_i\|)B^T_{\mathcal{G}}(p) + B_{\mathcal{G}}(\theta)B_{\mathcal{G}}(\theta)^T$. When Q is positive semi-definite, one can get $\dot{V} \leq 0$[24]. It follows that $V(t)$ is bounded, i.e., $\|b_{ij} - b^*_{ij}\|$ is bounded. According to (37), one has that $\|v^b_i\|$ is bounded, which implies that $\|p_i\|$ is always bounded. It follows then that the closed-loop system is asymptotically stable. In addition, according to [24], one knows that when the framework is infinitesimally bearing rigid, the desired local bearing constraints are realizable. According to [23], one gets that the relative rotation information $R(\theta_i)R(\theta_j)^T$ in the position control input (37) can be estimated by using the local bearing information b_{ij}. This completes the proof of the asymptotic stability of the closed-loop system.

6. **Conclusions and future work.** In this work, the existing definitions for bearing rigidity are compared and the bearing rigidity and formation stabilization are further studied for multi-agent frameworks embedded in the three dimensional *special Euclidean group* $SE(3)$. Each agent obtains the bearing measurements in its local coordinate system. The bearing rigidity matrix in $SE(3)$ is defined and the necessary and sufficient conditions for the infinitesimal bearing rigidity are constructed. Each agent is characterized by a rigid body with 3 DOFs in translation and 3 DOFs in rotation. Moreover, a gradient-based bearing formation control algorithm is designed to stabilize the formations of multiple rigid bodies in $SE(3)$. In the proposed bearing attitude formation control algorithm, $R^r(\theta_i)$ and $R_b^g(\theta_i)$ are used, which in our future work, will be estimated using estimators or taken as uncertainties for the controller to be adaptive for.

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