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# A Feedback Control Algorithm to Steer Networks to a Cournot–Nash Equilibrium

Claudio De Persis  and Nima Monshizadeh 

**Abstract**—In this paper, we propose distributed feedback control that steers a dynamical network to a prescribed equilibrium corresponding to the so-called Cournot–Nash equilibrium. The network dynamics considered here are a class of passive nonlinear second-order systems, where production and demands act as external inputs to the systems. While productions are assumed to be controllable at each node, the demand is determined as a function of local prices according to the utility of the consumers. Using reduced information on the demand, the proposed controller guarantees the convergence of the closed-loop system to the optimal equilibrium point dictated by the Cournot–Nash competition.

**Index Terms**—Aggregative games, distributed control, internal model.

## I. INTRODUCTION

IN RECENT years, there has been a renewed interest in controllers that can steer a given network dynamical system to a steady state, which is optimal in a suitable economic sense, mostly motivated by research connected to power networks, where, for instance, given a certain demand, the problem of dynamically adjusting the generation in order to satisfy the demand while fulfilling an optimal criterion at steady state, for example, minimizing the generation costs or maximizing the social welfare, has been formulated and addressed with different approaches. Other application domains of interests are flow and heat networks [1], [2] as well as logistical systems [3], [4].

Generally speaking, the proposed approaches to dynamically control networks while fulfilling steady-state optimality criteria can be classified in two categories. One relies on primal–dual gradient algorithms, which solve the optimization problem, and apply online the computed control input to the physical network, possibly taking into account feedback signals coming from the network for improved robustness [5]–[8]. This approach returns control algorithms that can handle general convex objective

functions and constraints but typically require exactly knowing the demand in real time. This requirement can be alleviated at the expense of requiring more information about the network parameters [6], [7, Sec. IV-C].

The second category relies on internal-model-based controllers [9]–[11], popularized under distributed averaging proportional integral controllers [12], which can typically deal with linear-quadratic cost functions only, but can, on the other hand, tackle uncertainty in the power demand.

This paper aims at contributing to the second category of results allowing for an uncertain demand and considering an economic objective, which is different from the ones considered so far—economic optimal dispatch, implemented via a distributed [9], [10] or a semidecentralized control architecture [13]. In fact, our interest is to design controllers at the producers that aim at maximizing their profit according to a Cournot model of competition [14]. While the papers [9], [10], and [13] assume a demand that is constant over time, in this paper, we investigate whether the stability of a network is preserved when the demand changes as a function of some control parameters that can be regarded as “prices” (see [15] and references therein for the same demand function). Moreover, different from the welfare optimization problem, the local objective function of each player depends on the other players’ decision variables via the price. The Cournot model provides us with an economically accepted framework where a price-dependent demand function can be specified and used in the analysis. In electricity markets, the dynamic stability of real-time pricing influencing the consumers’ demand has been studied at least since [16]. In this paper, we focus on the close interaction between the pricing algorithms and the physics of a second-order nonlinear network representing a physical system.

Other works are available that have solved problems different from the economic optimal dispatch problem, for example, [17] and [18], where algorithms solving the optimal power generation model under a Bertrand model of competition and a primal–dual setting have been proposed. These papers propose saddle-point-based algorithms for modeling the Bertrand game of competition [17] and interconnect them in feedback with the dynamical model representing the physical network [18]. Being based on saddle-point dynamics, these algorithms require the knowledge in real time of the total demand, which is collected by a central aggregator.

Cournot models of competition and the resulting Cournot–Nash equilibrium [15], [19], [20] are very well-studied topics in game theory and its applications. Reference [21] provides

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a general framework where Nash equilibria are characterized as solutions to variational inequalities and Nash equilibrium-seeking algorithms are given. Reference [15] studies the existence of a generalized Nash equilibrium in a transmission electricity market with Cournot competition, where demand is modeled as a linear function of the price, generators maximize their profits, and the market maximizes some objective functions (social welfare, residual social welfare, and consumer surplus). Properties of Cournot equilibria in a Cournot oligopoly model under convex cost and inverse demand functions are provided in [20], where they are used to establish the efficiency of Cournot equilibria compared to the optimal social welfare problem. As a special case, the study is conducted for affine inverse demand functions, which are those relevant to our investigation.

In the context of game-theoretic equilibria seeking dynamics, pseudogradient dynamical algorithms that converge to Cournot–Nash equilibria have been extensively investigated in the literature [22]–[25]. In fact, the Cournot model of competition falls in the class of aggregative games, where players can be controlled to a prescribed game-theoretic equilibrium while fulfilling local and coupling constraints [25]. As a special case of noncooperative games over networks, network Cournot competition has been studied in [24]. Being based on interdependent Karush–Kuhn–Tucker (KKT) conditions, these algorithms implement projected dynamics that require the knowledge of the parameters defining the constraints, which, in the context of Cournot games, amount to the real-time measurement of the demand function.

What differentiates our results from the existing ones are three features. First, as highlighted before, we devise a feedback control algorithm that does not rely on the exact knowledge in real time of the demand in the network. Second, this algorithm is interconnected in a feedback loop with a given physical network, and the resulting closed-loop system is analyzed as a whole, showing that the physical network converges to an equilibrium at which the controller output is equal to the value of the Cournot–Nash equilibrium. Third, while the existing results on dynamic integral controllers focus on achieving social welfare optimization [9], [10], [13], we are interested in a controller that guarantees optimality in the Cournot–Nash sense. The latter is valued for providing good explanation of observed price variation, for it gives rise to a form of price that depends on the demand in an affine way, a well-accepted fact in economics [15], [20]. Compared with the existing literature, the analysis requires carefully crafting a control algorithm that *simultaneously* satisfies a suitable passivity property, ensures stability, guarantees convergence of generation, demand, and prices to the Cournot–Nash equilibrium.

Since the controller state variable has the interpretation of a price, we, at times, refer to the feedback control also as a pricing mechanism.

Although the motivation for this investigation was inspired by problems in power networks, in order to convey the results to an audience that is not necessarily interested in power networks, we decided to present our results for a class of nonlinear second-order passive dynamical networks [26], in which power networks fall after suitable modifications. The passivity property

plays an important role in the modeling, design, and analysis of many physical systems [27], including the ones that are of interest in this paper. Besides power networks, other examples of physical dynamics that can be modeled via the second-order dynamical network considered in this paper are: robotic networks [26] arising in formation control problems with disturbance rejection; and flow and hydraulic networks with frictionless pipes networks and electrical networks with inductive lines [2], [28].

The rest of this paper is organized as follows. In Section II, we recall basic concepts and results about the Cournot model of competition. Section III contains the main results of this paper, namely, the design and analysis of a distributed feedback controller steering the closed-loop system toward a Cournot–Nash equilibrium. A case study is discussed in Section IV. Finally, conclusions are drawn in Section V.

## II. COURNOT COMPETITION

In this section, we revisit a few results about the Cournot model of competition [14], [19], which are used to determine the optimal triple of supply, demand, and price. Although Cournot–Nash equilibria are a well-studied topic in economics, game theory, and control [20], [24], [25], we could not find the specific characterization of the Cournot–Nash equilibrium, optimal demand solution, and price function given in Theorem 1 in the existing literature. The theorem is the principal contribution of this section and is instrumental to formulate our main results in the next section, where a dynamic controller is designed to steer a given network to such an optimal Cournot triple.

In a model of Cournot competition,  $n$  producers produce a homogeneous good that is demanded by  $m$  consumers. The price  $p$  of the good is assumed to be determined by the good producers, whereas the consumers are price takers, a scenario motivated by having few producers and many consumers.

Each producer  $i$  aims at maximizing its profit  $\Pi_{gi} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ , given the production of other firms. The profit is defined by the objective function

$$\begin{aligned} \Pi_{gi}(P_{gi}, P_{-gi}) &= p(P_g)P_{gi} - C_{gi}(P_{gi}), \\ i \in \mathcal{I} &:= \{1, 2, \dots, n\} \end{aligned}$$

where  $P_{gi} \in \mathbb{R}_{\geq 0}$  is the good production by producer  $i$ ,  $P_g = \text{col}(P_{g1}, \dots, P_{gn}) \in \mathbb{R}^n$  is the vector of all productions,  $P_{-gi} \in \mathbb{R}^{n-1}$  is the vector obtained by removing the  $i$ th element of  $P_g$ ,  $p : \mathbb{R}^n \rightarrow \mathbb{R}$  is the price function, and  $C_{gi} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a function satisfying the following assumption [19].

**Assumption 1:** For each  $i$ , the cost function  $C_{gi}$  is convex, nondecreasing, and continuously differentiable for  $P_{gi} \in \mathbb{R}_{\geq 0}$ . Moreover,  $C_{gi}(0) = 0$ .

Given a price  $\bar{p} \in \mathbb{R}$ , each consumer  $j$  wants to maximize its utility, described by the function

$$\Pi_{dj}(P_{dj}, \bar{p}) = U_j(P_{dj}) - \bar{p}P_{dj}, \quad j \in \mathcal{J} := \{1, 2, \dots, m\}$$

where  $P_{dj} \in \mathbb{R}_{\geq 0}$  is the good demand by consumer  $j$ , and  $U_j : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a continuously differentiable function. To simplify the notation, in the sequel, we denote  $\bar{p}$  simply by  $p$ . Now, given

a price  $p \in \mathbb{R}$ , we consider the utility maximization problem

$$\sup_{P_{dj} \geq 0} \Pi_{dj}(P_{dj}, p). \quad (1)$$

**Assumption 2:** For each  $j \in \mathcal{J}$ , the utility function  $U_j : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is continuously differentiable and strictly concave. Moreover,  $U_j' : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  satisfies  $\lim_{P_{dj} \rightarrow +\infty} U_j'(P_{dj}) = -\infty$  and  $U_j'(0) > 0$ .

### A. Utility Maximization

The utility maximization problem admits the following well-known solution (e.g., [16, Remark 1]).

**Lemma 1:** Let Assumption 2 hold. Then,  $P_{dj}$  is a solution to (1) if and only if

$$P_{dj} = \pi_j(p) \quad (2)$$

where  $\pi_j : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  is defined as

$$\pi_j(\lambda) = \begin{cases} (U_j')^{-1}(\lambda) & \text{if } \lambda < U_j'(0) \\ 0 & \text{if } \lambda \geq U_j'(0) \end{cases}. \quad (3)$$

**Proof:** The proof descends from the KKT conditions. ■

### B. Supply–Demand Matching

As will be discussed in Section III, we are interested in a supply–demand balancing condition, where the total generation is equal to the total demand, namely

$$\mathbb{1}^\top P_g = \mathbb{1}^\top P_d \quad (4)$$

where  $P_d = \text{col}(P_{d1}, \dots, P_{dm})$  and  $\mathbb{1}$  denotes the vector of all ones of suitable size. Under this balancing condition, the price can be written as a function of the total production, as stated next.

**Lemma 2:** Let Assumption 2 and the balance equation (4) hold, with  $\mathbb{1}^\top P_g \geq 0$ . Then, there exists a continuous function  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  satisfying

$$u(\mathbb{1}^\top P_g) = p(P_g) \quad (5)$$

with the following properties: i)  $u(0) > 0$ ; ii)  $u$  is strictly decreasing; and iii)  $\lim_{q \rightarrow +\infty} u(q) = -\infty$ .

**Proof:** See the Appendix. ■

**Example 1:** Consider the case of two consumers with linear-quadratic utility functions

$$U_j(P_{dj}) = -\frac{1}{2} P_{dj} Q_{dj} P_{dj} + b_{dj} P_{dj}$$

where  $Q_{dj}, b_{dj} > 0$ ,  $j = 1, 2$ . For each  $j$ , we have

$$\pi_j(\lambda) = \begin{cases} Q_{dj}^{-1}(b_{dj} - \lambda) & \text{if } \lambda < b_{dj} \\ 0 & \text{if } \lambda \geq b_{dj} \end{cases}. \quad (6)$$

Let  $b_{d1} < b_{d2}$ . Then

$$\pi(\lambda) = \begin{cases} \sum_{j=1,2} Q_{dj}^{-1}(b_{dj} - \lambda) & \text{if } \lambda \leq b_{d1} \\ Q_{d2}^{-1}(b_{d2} - \lambda) & \text{if } b_{d1} < \lambda \leq b_{d2} \\ 0 & \text{if } \lambda > b_{d2} \end{cases}$$

and for  $q = \pi(\lambda)$ ,  $q > 0$ , we have

$$\pi^{-1}(q) = \begin{cases} -Q_{d2}q + b_{d2} & \text{if } 0 < q \leq Q_{d2}^{-1}(b_{d2} - b_{d1}) \\ -\alpha q + \beta & \text{if } q \geq Q_{d2}^{-1}(b_{d2} - b_{d1}) \end{cases}$$

where

$$\beta = \frac{\sum_{j=1,2} \frac{b_{dj}}{Q_{dj}}}{\sum_{j=1,2} \frac{1}{Q_{di}}}, \quad \alpha = \frac{1}{\sum_{j=1,2} \frac{1}{Q_{di}}}.$$

It then follows that:

$$u(q) = \begin{cases} -Q_{d2}q + b_{d2} & \text{if } 0 \leq q \leq Q_{d2}^{-1}(b_{d2} - b_{d1}) \\ -\alpha q + \beta & \text{if } q \geq Q_{d2}^{-1}(b_{d2} - b_{d1}) \end{cases}.$$

In view of (2) and (5), we conclude that the optimal demands are given by

$$P_{dj} = \begin{cases} (U_j')^{-1}(u(\mathbb{1}^\top P_g)) & \text{if } u(\mathbb{1}^\top P_g) < U_j'(0) \\ 0 & \text{if } u(\mathbb{1}^\top P_g) \geq U_j'(0) \end{cases}. \quad (7)$$

By construction, the optimal demand  $P_d$  detailed above satisfies  $\mathbb{1}^\top P_d = \mathbb{1}^\top P_g$ .

### C. Profit Maximization: The Cournot–Nash Equilibrium

Motivated by the discussion before, we consider a Cournot game consisting of the set of producers  $\mathcal{I}$ , each one aiming at solving the maximization problem

$$\max_{P_{gi} \geq 0} \Pi_{gi}(P_{gi}, P_{-gi}) \quad (8)$$

with

$$\Pi_{gi}(P_{gi}, P_{-gi}) = p(P_g)P_{gi} - C_{gi}(P_{gi}). \quad (9)$$

More formally, we define the Cournot game as follows.

**Definition 1:** A Cournot game  $CG(\mathcal{I}, (\Pi_{gi}, i \in \mathcal{I}))$  consists of the following:

- 1) a set  $\mathcal{I}$  of producers (or players);
- 2) a strategy  $P_{gi} \in \mathbb{R}$  for each producer  $i \in \mathcal{I}$ ;
- 3) the convex and closed set  $\mathbb{R}_{\geq 0}^n$  of allowed strategies  $P_g = (P_{g1}, \dots, P_{gn})$ ;
- 4) a payoff function  $\Pi_{gi}(P_{gi}, P_{-gi})$ , where for  $P_g \in \mathbb{R}_{\geq 0}^n$ ,  $\Pi_{gi}(P_g)$  is continuous in  $P_g$  and concave in  $P_{gi}$  for each fixed  $P_{-gi}$ .

The Cournot–Nash equilibrium is defined next [19].

**Definition 2:** A Cournot–Nash equilibrium of the game  $CG(\mathcal{I}, (\Pi_{gi}, i \in \mathcal{I}))$  is a vector  $P_g^* \in \mathbb{R}_{\geq 0}^n$  that for each  $i \in \mathcal{I}$  satisfies

$$\Pi_{gi}(P_{gi}^*, P_{-gi}^*) \geq \Pi_{gi}(P_{gi}, P_{-gi}^*)$$

for all  $P_{gi} \in \mathbb{R}_{\geq 0}$ .

The existence of a Cournot–Nash equilibrium is a consequence of a well-known result on concave games due to [29]. Let us recall the definition of a concave game.

**Definition 3:** A concave game consists of the following:

- 1) a set  $\mathcal{I}$  of players;
- 2) a strategy  $x_i \in \mathbb{R}^{p_i}$  for each player  $i \in \mathcal{I}$ ;
- 3) the convex, closed, and bounded set  $R \subset \mathbb{R}^p$  of allowed strategies  $x = (x_1, \dots, x_n)$ , with  $p = \sum_{i=1}^n p_i$ ;



- 4) a payoff function  $\varphi_i(x_i, x_{-i})$  for each player  $i \in \mathcal{I}$ , where for  $x \in S$ ,  $\varphi_i(x)$  is continuous in  $x$  and concave in  $x_i$  for each fixed  $x_{-i}$ , with  $S = P_1 \times \cdots \times P_n$  and  $P_i$  is the projection of  $R$  on  $\mathbb{R}^{m_i}$ .

A difference between a Cournot and a concave game is the lack of a bounded set of bounded strategies for the Cournot game ( $\mathbb{R}_{\geq 0}^n$  is clearly unbounded). However, following [19, Prop. 2], it can be shown that solving (8) is equivalent to solving

$$\max_{0 \leq P_{gi} \leq \bar{P}_{gi}} \Pi_{gi}(P_{gi}, P_{-gi}) \quad (10)$$

with  $\bar{P}_{gi}$  being some positive constant. As a matter of fact,  $\Pi_{gi}(P_{gi}, P_{-gi}) = u(\mathbf{1}^\top P_g)P_{gi} - C_{gi}(P_{gi})$  is zero at  $P_{gi} = 0$ , and by Assumptions 1 and 2 and Lemma 2, there exists  $\bar{P}_{gi} > 0$  such that  $\Pi_{gi}(\bar{P}_{gi}, P_{-gi}) = 0$  and  $\Pi_{gi}(P_{gi}, P_{-gi}) < 0$  for  $P_{gi} \geq \bar{P}_{gi}$ . Hence, in the case of (10),  $R = S = [0, \bar{P}_{g1}] \times \cdots \times [0, \bar{P}_{gn}]$ . Moreover, the payoff function  $\Pi_{gi}(P_{gi}, P_{-gi})$  satisfies the properties of a concave game by Assumptions 1 and 2 and Lemma 2. Therefore, the game defined by (10) or, equivalently, by (8) is a concave game. It can then be concluded by [29, Th. 1] that a Cournot–Nash equilibrium exists.

**Proposition 1:** (see [29, Theor. 1]) Under Assumptions 1 and 2, there exists a Cournot–Nash equilibrium  $P_g^*$ .

#### D. Linear-Quadratic Utility and Cost Functions

In this section and for the remainder of this paper, we restrict the cost and utility functions of producers and consumers to linear-quadratic functions, namely

$$C_{gi}(P_{gi}) = \frac{1}{2}P_{gi}Q_{gi}P_{gi} + b_{gi}P_{gi}, \quad b_{gi}, Q_{gi} > 0 \quad (11)$$

$$U_j(P_{dj}) = -\frac{1}{2}P_{dj}Q_{dj}P_{dj} + b_{dj}P_{dj}, \quad b_{dj}, Q_{dj} > 0. \quad (12)$$

Analogous results can be obtained for more general convex cost functions, if the price function  $p(P_g)$  admits an affine form in (17). The restriction to linear-quadratic cost and utility functions is motivated by two reasons: to obtain more explicit expressions for optimal production and demand (see Theorem 1), and for technical reasons, namely, to prove stability of the overall system in the next section (see Remark 9).

Now, Lemma 1 is specialized as follows.

**Corollary 1:** The scalar  $P_{dj}$  is a solution to (1) with  $U_j$  as in (12) if and only if

$$P_{dj} = \begin{cases} Q_{dj}^{-1}(b_{dj} - p) & \text{if } p < b_{dj} \\ 0 & \text{if } p \geq b_{dj} \end{cases}. \quad (13)$$

We are particularly interested in the case where all producers and consumers enter the market, that is,  $P_{gi}, P_{dj} > 0$  for all  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$ . Conditions under which this case occurs are formalized next.

**Lemma 3:** Let the utility functions of consumers be given by (12) and consider the utility maximization problem (1). Then, the following statements are equivalent.

- 1) There exists  $P_d \in \mathbb{R}_{> 0}^n$  solution to (1).
- 2)  $p < \underline{b}_d$ , where  $\underline{b}_d := \min\{b_{dj} : j \in \mathcal{J}\}$ .
- 3) The vector  $P_d$  given by

$$P_d = Q_d^{-1}(b - \mathbf{1}p) \quad (14)$$

is the unique solution to (1), and  $p \neq \underline{b}_d$ .

**Proof:** See the Appendix. ■

We note that given the strictly positive demand (14) in Lemma 3, the expression can be inverted to obtain

$$p = \beta^* - \alpha^* \mathbf{1}^\top P_d \quad (15)$$

where

$$\beta^* := \frac{\sum_{j \in \mathcal{J}} \frac{b_{dj}}{Q_{dj}}}{\sum_{j \in \mathcal{J}} \frac{1}{Q_{dj}}} = \frac{\mathbf{1}^\top Q_d^{-1} b_d}{\mathbf{1}^\top Q_d^{-1} \mathbf{1}},$$

$$\alpha^* := \frac{1}{\sum_{j \in \mathcal{J}} \frac{1}{Q_{dj}}} = \frac{1}{\mathbf{1}^\top Q_d^{-1} \mathbf{1}} \quad (16)$$

and  $b_d := \text{col}(b_{d1}, \dots, b_{dn})$ ,  $Q_d = \text{diag}(Q_{d1}, \dots, Q_{dn})$ ,  $b_g := \text{col}(b_{g1}, \dots, b_{gn})$ ,  $Q_g = \text{diag}(Q_{g1}, \dots, Q_{gn})$ .

We turn now our attention to the producers. Let the price function in (8) admit the affine form

$$p(P_g) = \beta - \alpha \mathbf{1}^\top P_g \quad (17)$$

for some scalars  $\alpha, \beta > 0$ , which is motivated by Lemma 2 specialized to the case of linear-quadratic cost functions with strictly positive generation and demand. In the next section, scalars  $\alpha$  and  $\beta$  in (17) will be set to those in (16), as we are interested in the supply–demand matching condition (4). This will be made more explicit in Theorem 1.

Since Assumptions 1 and 2 are satisfied in the case of linear-quadratic functions, Proposition 1 holds and a Cournot–Nash equilibrium exists. Then, the computation of the Cournot–Nash equilibrium  $P_g^*$  descends from the optimization problem

$$P_{gi}^* \in \arg \max_{P_{gi} \geq 0} \Pi_{gi}(P_{gi}, P_{-gi}^*) \quad (18)$$

where in the view of (9), (11), and (17)

$$\Pi_{gi}(P_{gi}, P_{-gi}) = (\beta - \alpha P_{gi} - \alpha \mathbf{1}^\top P_{-gi})P_{gi} - \frac{1}{2}P_{gi}Q_{gi}P_{gi} - b_{gi}P_{gi}.$$

One could expect the presence of an upper bound on the production  $P_{gi}$  in the optimization problem (18). The reason for neglecting this is technical and is explained in Remark 1.

The conditions under which the parabola  $(\beta - \alpha P_{gi} - \alpha \mathbf{1}^\top P_{-gi})P_{gi} - \frac{1}{2}P_{gi}Q_{gi}P_{gi} - b_{gi}P_{gi}$  has a non-negative maximizer can be formalized as follows.

**Lemma 4:**  $P_{gi}$  is a solution to (18) if and only if

$$P_{gi} = \begin{cases} \frac{\gamma_i(P_{-gi})}{2\alpha + Q_{gi}} & \text{if } \gamma_i(P_{-gi}) > 0 \\ 0 & \text{if } \gamma_i(P_{-gi}) \leq 0 \end{cases} \quad (19)$$

with

$$\gamma_i(P_{-gi}) := \beta - b_{gi} - \alpha \sum_{j \neq i: P_{gj} > 0} P_{gj}.$$

The proof is straightforward and, thus, omitted. Recall that we are interested in the case where every producer contributes to a strictly positive production, that is,  $P_g \in \mathbb{R}_{> 0}^n$ . This brings us to the following lemma.

**Lemma 5:** The following statements are equivalent.

- 1) There exists a  $P_g \in \mathbb{R}_{> 0}^n$  solution to (18).

2) The vector  $P_g$  given by

$$P_g = (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta\mathbf{1} - b_g) \quad (20)$$

is the unique solution to (18) and  $\beta - \alpha\mathbf{1}^\top P_g \neq \bar{b}_g$ , where  $\bar{b}_g := \max_{i \in \mathcal{I}} b_{gi}$ .

3)  $(\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta\mathbf{1} - b_g) \in \mathbb{R}_{>0}^n$ .

4) The inequality

$$\mathbf{1}^\top (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta\mathbf{1} - b_g) < \frac{\beta - \bar{b}_g}{\alpha} \quad (21)$$

holds.

**Proof:** See the Appendix. ■

To summarize, Lemma 4 and Corollary 1 identify the optimal production and optimal demand, respectively, for the cost and utility functions (11) and (12). Lemma 3 characterizes the conditions under which the optimal demand of each consumer is strictly positive, and Lemma 5 provides equivalent conditions for strict positivity of optimal productions. Based on the aforementioned results, the following necessary and sufficient condition for the existence of an optimal triple  $(P_g^*, P_d^*, p^*) \in \mathbb{R}_{>0}^n \times \mathbb{R}_{>0}^m \times \mathbb{R}_{>0}$  can be given.

**Theorem 1:** Let the price function admit the affine form  $p(P_g) = \beta - \alpha\mathbf{1}^\top P_g$ , for some positive scalars  $\alpha, \beta > 0$ . For linear-quadratic functions (11), (12), let  $P_g^*$  denote the Cournot–Nash equilibrium solution to (18), and  $P_d^*$  the optimal demand solution to (1) computed with respect to the optimal price  $p^* := p(P_g^*)$ . Then, the following conditions are equivalent:

1)

$$\frac{\beta - \underline{b}_d}{\alpha} < \mathbf{1}^\top (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta\mathbf{1} - b_g) < \frac{\beta - \bar{b}_g}{\alpha}. \quad (22)$$

2)

$$P_g^* = (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta\mathbf{1} - b_g) \quad (23a)$$

$$P_d^* = Q_d^{-1}(b_d - \beta\mathbf{1} + \alpha\mathbf{1}\mathbf{1}^\top P_g^*) \quad (23b)$$

$$p^* = \beta - \alpha\mathbf{1}^\top P_g^* \quad (23c)$$

and  $\bar{b}_g \neq p^* \neq \underline{b}_d$ .

3)  $(P_g^*, P_d^*, p^*) \in \mathbb{R}_{>0}^n \times \mathbb{R}_{>0}^m \times \mathbb{R}_{>0}$ .

Moreover, in case  $\alpha = \alpha^*$  and  $\beta = \beta^*$  with  $\alpha^*$  and  $\beta^*$  as in (16), the balancing condition holds, namely

$$\mathbf{1}^\top P_g^* = \mathbf{1}^\top P_d^*. \quad (24)$$

**Proof:** By Lemmas 3 and 5, establishing the equivalence of the three statements is straightforward. Now suppose that  $\alpha = \alpha^*$  and  $\beta = \beta^*$ , then (23b) becomes

$$P_d^* = Q_d^{-1}(b_d - \beta^*\mathbf{1} + \alpha^*\mathbf{1}\mathbf{1}^\top P_g^*).$$

This yields

$$\mathbf{1}^\top P_d^* = \mathbf{1}^\top Q_d^{-1}b_d - \beta^*\mathbf{1}^\top Q_d^{-1}\mathbf{1} + \alpha^*\mathbf{1}^\top Q_d^{-1}\mathbf{1}\mathbf{1}^\top P_g^*.$$

The balancing condition (24) then follows from (16). ■

**Remark 1:** The vector  $P_g^*$  in (23a) can be rewritten as

$$\begin{aligned} P_g^* &= (\alpha I + Q_g)^{-1}(\beta\mathbf{1} - b_g - \alpha\mathbf{1}\mathbf{1}^\top P_g^*) \\ &= (\alpha I + Q_g)^{-1}(\mathbf{1}p^* - b_g). \end{aligned} \quad (25)$$

Hence, the triple  $(P_g^*, P_d^*, p^*)$  can be equivalently characterized by the implicit form

$$P_g^* = (\alpha I + Q_g)^{-1}(\mathbf{1}p^* - b_g) \quad (26a)$$

$$P_d^* = Q_d^{-1}(b_d - \mathbf{1}p^*) \quad (26b)$$

$$p^* = \beta - \alpha\mathbf{1}^\top P_g^*. \quad (26c)$$

We see from (26a) that the optimal generation vector  $P_g^*$  depends affinely on the optimal price  $p^*$ . This allows us to design the feedback controller that asymptotically converges to such a value in the next section. The presence of active constraints on  $P_g$  at the optimal solution would lead to a nonlinear expression of  $P_g^*$  as a function of the price and would prevent us from deriving a controller, for which analytical stability guarantees can be provided. In fact, the design of internal-model-based regulators in the presence of constraints is, to the best of the authors' knowledge, an open problem.

**Remark 2:** To give an interpretation to condition (22), we rewrite it in a different form. By Theorem 1, condition (22) can be rewritten as

$$\frac{\beta - \underline{b}_d}{\alpha} < \mathbf{1}^\top P_g^* < \frac{\beta - \bar{b}_g}{\alpha} \quad (27)$$

which is equivalent to  $\bar{b}_g < \beta - \alpha\mathbf{1}^\top P_g^* < \underline{b}_d$ , or also

$$C'_i(0) < p(P_g^*) < U'_i(0) \quad \forall i \in \mathcal{I}.$$

Noting that  $p(P_g)$  is monotonically decreasing, the lower bound yields  $C'_i(0) < p(0)$ . This means that the marginal costs of the producers are lower than the price at zero generation ( $P_{gi} = 0$ ). Therefore, the producers always benefit from providing nonzero amount of goods to the consumers. Analogously, the upper bound indicates that the marginal utility of each consumer at zero demand ( $P_{dj} = 0$ ) is higher than the eventual optimal price  $p^* = p(P_g^*)$  dictated by the consumers. Hence, under condition (22), it is always advantageous for consumers to enter the market and have a strictly positive demand.

### III. COURNOT–NASH OPTIMAL DYNAMICAL NETWORKS

In the previous section, we studied Cournot competition and characterized the Cournot–Nash equilibrium among producers and consumers. In this section, the Cournot model of competition is used to devise a feedback control algorithm that, when interconnected with a physical network, steers its state to a point prescribed by the Cournot–Nash equilibrium. In particular, we introduce a dynamical network whose output variables are affected by the cumulative effect of demand and generation mismatch. Using these variables as measurements, we propose a dynamic output feedback algorithm that steers the dynamical network to the Cournot–Nash optimal solution identified by the triple  $(P_g^*, P_d^*, p^*)$  with

$$P_g^* = (\alpha^*(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta^*\mathbf{1} - b_g) \quad (28a)$$

$$P_d^* = Q_d^{-1}(b_d - \beta^*\mathbf{1} + \alpha^*\mathbf{1}\mathbf{1}^\top P_g^*) \quad (28b)$$

$$p^* = \beta^* - \alpha^*\mathbf{1}^\top P_g^*. \quad (28c)$$

Note that the triple above is obtained from (23) by setting  $\alpha = \alpha^*$  and  $\beta = \beta^*$ . The reason why we are interested in the

latter choice is to ensure that the balancing condition (24) is met. Moreover, we require a reduced amount of information about the consumers to allow for the changing demand present in dynamic interactive markets. Finally, note that the feedback algorithm should be designed such that the stability of the physical system is not compromised.

### A. Network Dynamics

In this section, we provide the model of the physical network. The topology of such a network is represented by a connected and undirected graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  with a vertex set  $\mathcal{V} = \{1, \dots, n\}$ , and an edge set  $\mathcal{E}$  given by the set of unordered pairs  $\{i, j\}$  of distinct vertices  $i$  and  $j$ . The cardinality of  $\mathcal{E}$  is denoted by  $m$ .<sup>1</sup> The set of neighbors of node  $i$  is denoted by  $\mathcal{N}_i = \{j \in \mathcal{V} \mid \{i, j\} \in \mathcal{E}\}$ .

We consider a second-order consensus-based network dynamics of the form

$$m_i \ddot{x}_i + d_i \dot{x}_i - \sum_{j \in \mathcal{N}_i} \nabla H_{ij}(x_i - x_j) = u_i \quad (29)$$

where  $m_i, d_i \in \mathbb{R}_{>0}$  are constant,  $H_{ij} : \mathbb{R} \rightarrow \mathbb{R}$  is a continuously differentiable strictly convex function<sup>2</sup> with its minimum at the origin,  $\nabla H_{ij}$  denotes the partial derivative of  $H_{ij}$  with respect to its argument  $x_i - x_j$ ,  $x_i \in \mathbb{R}$  is the state associated with node  $i$ , and  $u_i \in \mathbb{R}$  is the input applied to the dynamics of the  $i$ th node. Note that as  $H_{ij}$  is strictly convex,  $\nabla H_{ij}$  is a strictly increasing function of  $x_i - x_j$ . The second-order dynamics (29) can represent different kinds of networks, including power networks, by appropriately choosing the function  $H_i$ , see, for example, [30] and [31], formation of mobile robots [26], and flow networks [2], [28]. Producers and consumers affect the dynamics (29) via

$$u_i = P_{gi} - P_{di}$$

where  $P_{gi}$  is the production and  $P_{di}$  is the aggregated demand at node  $i$  as before. This means that a mismatch between production and demand causes the node  $i$ 's state variable to drift away from its unforced behavior. Note that the number of producers and consumers here are considered to be the same, and we thus use the notation  $P_{di}$  rather than  $P_{dj}$ , which was used in the previous section.

Let  $R$  be the incidence matrix of the graph  $\mathcal{G}$ . Note that by associating an arbitrary orientation to the edges, the incidence matrix  $R \in \mathbb{R}^{n \times m}$  is defined elementwise as  $R_{ik} = 1$ , if node  $i$  is the sink of edge  $k$ ,  $R_{ik} = -1$ , if  $i$  is the source of edge  $k$ , and  $R_{ik} = 0$  otherwise. In addition,  $\ker R^\top = \text{im } \mathbf{1}$  for a connected graph  $\mathcal{G}$ . Then, (29) can be written in vector form as

$$\dot{x} = y \quad (30a)$$

$$M\dot{y} = -Dy - R\nabla H(R^\top x) + P_g - P_d \quad (30b)$$

<sup>1</sup>The integer  $m$  should not be confused with the number of consumers in the previous section, as the latter is equal to the number of producers in this section and is denoted by  $n$ .

<sup>2</sup>Locally, strictly convex functions can be analogously treated in the analysis, with the only difference being that the convergence result will become local in this case.

where  $x = \text{col}(x_i)$ ,  $M = \text{blockdiag}(m_i)$ ,  $D = \text{blockdiag}(d_i)$ ,  $P_g = \text{col}(P_{gi})$ ,  $P_d = \text{col}(P_{di})$ ,  $i \in \mathcal{V}$ . In addition,  $\text{col}(\nabla H_{ij})$  is denoted by  $\nabla H : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , where the edge ordering in  $\nabla H$  is the same as that of the incidence matrix  $R$ . It is easy to see that (30) has nonisolated equilibria for constant vectors  $P_g$  and  $P_d$ . In fact, given a solution  $(x, y)$  of (30),  $(x + c\mathbf{1}, y)$  is a solution to (30) as well, for any constant  $c \in \mathbb{R}$ . To avoid this complication, we perform a change of coordinates by defining

$$\zeta_i = x_i - x_n, \quad i = 1, \dots, n-1. \quad (31)$$

Let  $R_\zeta \in \mathbb{R}^{n-1} \times \mathbb{R}^m$  denote the incidence matrix with its  $n$ th row removed. Then, we have

$$R^\top x = R_\zeta^\top \zeta$$

where  $\text{col}(\zeta_i) = \zeta \in \mathbb{R}^{n-1}$ . Moreover, it holds that  $\zeta = E^\top x$ , where  $E^\top = [I_{n-1} \quad -\mathbf{1}_{n-1}]$ . Noting that  $R = ER_\zeta$ , and defining a function  $H_\zeta$  such that  $H(R_\zeta^\top \zeta) := H_\zeta(\zeta)$  and, thus,  $R_\zeta \nabla H(R_\zeta^\top \zeta) = \nabla H_\zeta(\zeta)$ , system (30) in the new coordinates reads as

$$\dot{\zeta} = E^\top y \quad (32a)$$

$$M\dot{y} = -Dy - E\nabla H_\zeta(\zeta) + P_g - P_d. \quad (32b)$$

**Remark 3:** The system above belongs to a class of dynamical networks given by

$$\dot{x} = f(x) + gu \quad (33a)$$

$$z = h(x) \quad (33b)$$

which are output-strictly shifted passive [27, Ch. 6.5], satisfying an equilibrium-observability property [32, Remark 6] and the inclusion

$$\{x^* \mid h(x^*) = 0\} \subseteq \{x^* \mid \mathbf{1}^\top g^+ f(x^*) = 0\}$$

with  $g^+$  being a left inverse of  $g$ . In (32),  $u = P_g - P_d$ ,  $x = \text{col}(\zeta, y)$ ,  $h(x) = y$ , and

$$f(x) = \begin{bmatrix} 0 & E^\top \\ -M^{-1}E & -M^{-1}D \end{bmatrix} \begin{bmatrix} \nabla H_\zeta(\zeta) \\ y \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

We have opted to consider system (32) rather than more general subclasses of (33) to keep the focus of this paper and provide more explicit results. The choice of the second-order consensus-based dynamics is motivated by applications in power networks, hydraulic networks, and robotics [2], [26], [28]. Although dynamic controllers for the network systems (32) or (33) have been studied before, for example, [4], there is no controller available that enforces a Cournot–Nash equilibrium in the closed-loop system and makes it attractive. The proposed controller establishes such a result.

As a result of the change of coordinates, network (32) now has at most one equilibrium, for given constant vectors  $P_g$  and  $P_d$ , and we have the following lemma.

**Lemma 6:** Let  $P_g = \bar{P}_g$  and  $P_d = \bar{P}_d$  for some constant vectors  $\bar{P}_g, \bar{P}_d \in \mathbb{R}^n$ . Then, the point  $(\bar{\zeta}, \bar{y})$  is an equilibrium

of (32) if and only if

$$\bar{y} = \mathbf{1}y^*, \quad y^* = \frac{\sum_{i \in \mathcal{V}} (\bar{P}_{gi} - \bar{P}_{di})}{\sum_{i \in \mathcal{V}} D_i} \quad (34)$$

$$\nabla H_{\zeta}(\bar{\zeta}) = E^+ \left( I_n - \frac{D\mathbf{1}\mathbf{1}^\top}{\mathbf{1}^\top D\mathbf{1}} \right) (\bar{P}_g - \bar{P}_d) \quad (35)$$

where  $E^+ = (E^\top E)^{-1} E^\top$ . Moreover, the equilibrium, if it exists, is unique.

**Proof:** The proof is analogous to similar ones given in the context of power networks [9], [10], [13], and is provided in the Appendix. ■

## B. Dynamic Pricing Mechanism

Next, we seek for a dynamic feedback (pricing mechanism) that steers the physical network to an asymptotically stable equilibrium, while guaranteeing the convergence of the production and the demand to the Cournot–Nash solution. In particular, we are interested in regulating the production  $P_g$  to  $P_g^*$ , the demand  $P_d$  to  $P_d^*$ , and attain the optimal price  $p^*$  given by (28). Note that setting the generation and production to constant vectors as  $P_g = P_g^*$  and  $P_d = P_d^*$  is undesirable since it requires complete information of the entire network and utility functions.

Recall that in the Cournot model, consumers are price takers, meaning that they optimize their utility functions given a price. Consistent with (26b), we consider the demand as

$$P_{di}(t) = Q_{di}^{-1}(b_{di} - p_i(t)) \quad (36)$$

where  $p_i(t)$  can be interpreted as a *momentary* or *estimated* price for the  $i$ th consumer at time  $t$ . In vector form, this is written as  $P_d = Q_d^{-1}(b_d - p(t))$ , with  $p(t) = \text{col}(p_i(t))$ .

Next, looking at the expression of  $y^*$  in Lemma 6, we notice that the deviation from the supply–demand matching condition (4) is reflected on the steady-state value of the state variable  $y$ . In fact, (4) holds if and only if  $y^* = 0$ . This motivates the implementation of a negative feedback from  $y$  to  $P_g$  in the controller. Moreover, in order to ensure optimality, we rely on a communication layer next to the physical network that appropriately distributes the information on the local price estimations  $p_i$  over the entire network. The topology of this communication layer is modeled via an undirected connected graph  $\mathcal{G}_c(\mathcal{V}, \mathcal{E}_c)$ , and the set of neighbors of the node  $i$  is denoted by  $\mathcal{N}_i^c$ . We stress that the “physical” network (29) and the communication network modeled by  $\mathcal{G}_c$  are, in general, distinct, although they share the same set of nodes.

Inspired by the aforementioned remarks, the following distributed controller (pricing mechanism) is proposed:

$$\tau_i \dot{p}_i = -k_i y_i - Q_{di}^{-1} y_i - \sum_{j \in \mathcal{N}_i^c} \rho_{ij} (p_i - p_j) \quad (37a)$$

$$P_{gi} = k_i (p_i - b_{gi}) \quad (37b)$$

where  $\tau_i > 0$  is the time constant,  $\rho_{ij} > 0$  indicates the weight of the communication at each link, and the constant parameter  $k_i > 0$  will be specified later.

**Remark 4:** In this control, the agents only share their state variable  $p_i$  and not the parameters of the cost and utility func-

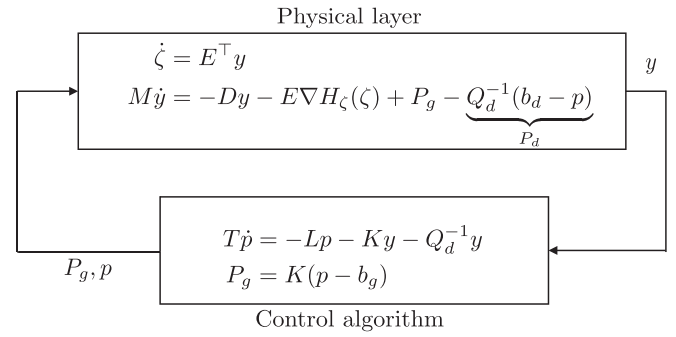


Fig. 1. Interconnection of the physical layer and the control/pricing algorithm.

tions, which are sensitive information in a competitive market. Rather than the distributed controllers (37), the literature on aggregative games suggests a one-to-all controller, which collects and processes a weighted average of the measurements and sends back the producers a control signal [25]. Such a controller could be an alternative to (37), which can be designed and analyzed following the results of this paper and [13], but this alternative is not investigated any further here.

Let  $T = \text{diag}(\tau_i)$ ,  $K = \text{diag}(k_i)$ , and the weighted Laplacian matrix of  $\mathcal{G}_c$  be denoted by  $L$ . Then, the overall closed system admits the following state-space representation:

$$\dot{\zeta} = E^\top y \quad (38a)$$

$$M\dot{y} = -Dy - E\nabla H_{\zeta}(\zeta) + K(p - b_g) - Q_d^{-1}(b_d - p) \quad (38b)$$

$$T\dot{p} = -Lp - Ky - Q_d^{-1}y. \quad (38c)$$

For an illustration of the interconnection of the control/pricing algorithm and the physical layer, see Fig. 1.

The result in the following characterizes the static properties of the closed-loop system (38).

**Lemma 7:** The point  $(\bar{\zeta}, \bar{y}, \bar{p})$  is an equilibrium of (38) if and only if  $\bar{y} = 0$

$$\bar{p} = \mathbf{1}_n q, \quad q = \frac{\mathbf{1}^\top K b_g + \mathbf{1}^\top Q_d^{-1} b_d}{\mathbf{1}^\top K \mathbf{1} + \mathbf{1}^\top Q_d^{-1} \mathbf{1}} \quad (39)$$

and  $\bar{\zeta}$  satisfies

$$\nabla H_{\zeta}(\bar{\zeta}) = E^+ (\bar{P}_g - \bar{P}_d) \quad (40)$$

with

$$\bar{P}_g = K(\bar{p} - b_g), \quad \bar{P}_d = Q_d^{-1}(b_d - \bar{p}).$$

The equilibrium, if it exists, is unique. Moreover, if  $K = (\alpha^* I_n + Q_g)^{-1}$ , then  $(\bar{P}_g, \bar{P}_d, \bar{p}) = (P_g^*, P_d^*, p^*)$  given by (28).

**Proof:** See the Appendix. ■

**Remark 5:** In case of linear dynamics, namely,  $2H_{\zeta}(\zeta) = \zeta^\top R_{\zeta} W R_{\zeta}^\top \zeta$ ,  $W > 0$ , the vector  $\bar{\zeta}$  is explicitly obtained as

$$\bar{\zeta} = (R_{\zeta} W R_{\zeta}^\top)^{-1} E^+ (\bar{P}_g - \bar{P}_d).$$

Lemma 7 imposes the following assumption.

**Assumption 3:** There exists  $\bar{\zeta} \in \mathbb{R}^{n-1}$  such that (40) is satisfied.



**Remark 6:** As evident from (35), the condition in Assumption 3 is a consequence of the agents' dynamics (29), rather than the choice of the controller. In case the graph  $\mathcal{G}$  is a tree, the incidence matrix has full-column rank and Assumption 3 is always satisfied with

$$\bar{\zeta} = R_{\zeta}^{-\top} \nabla H^{-1}((R^{\top} R)^{-1} R^{\top} (\bar{P}_g - \bar{P}_d))$$

where  $\nabla H(\nabla H^{-1}(x)) = x$ ,  $\forall x \in \mathbb{R}^m$ .

The next theorem provides the main result of this section, which validates the proposed feedback algorithm.

**Theorem 2:** Let Assumption 3 hold. Then, the equilibrium  $(\bar{\zeta}, \bar{y}, \bar{p})$  of (38) is asymptotically stable. Moreover, for  $K = (\alpha^* I_n + Q_g)^{-1}$ , the vector  $(P_g, P_d, p)$ , with  $P_g$  defined as in (37b) and  $P_d$  as in (36), asymptotically converges to the optimal Cournot–Nash solution  $(P_g^*, P_d^*, p^*)$ , the latter given by (28).

**Proof:** See the Appendix. ■

The above-mentioned result shows that generation, demand, and price converge to the Cournot–Nash solution *asymptotically*, implying that demand and supply balance can be violated during the transient. Since the demand is not precisely known, it cannot be compensated by the generation in real time and instead some time is required for the integral action to provide the right amount of generation. In practice, for instance, in power networks (see, e.g., [6] and the references therein for considerations about primary control), imbalance of the network over short periods of time is a widely accepted event. In the current setting, the transient behavior of system (38) and the magnitude of the momentary power imbalance can be partially regulated by tuning the parameters of the proposed feedback control algorithm (37); however, providing analytical measures is difficult.

**Remark 7:** By Theorem 2, the controller (37) with  $k_i = (\alpha^* + Q_{gi})^{-1}$  steers the network to the Cournot–Nash optimal solution. Note that with the exception of the parameter  $\alpha^*$ , the  $i$ th controller uses only the local variables at node  $i$  together with the communicated variables  $p_i - p_j$  of the neighboring nodes in the communication graph. If the parameter  $\alpha^*$  is not precisely known, then  $k_i$  is set to  $(\hat{\alpha}_i^* + Q_{gi})^{-1}$ , where  $\hat{\alpha}_i^*$  is an approximation of  $\alpha^*$  at node  $i$ . This approximation will shift the equilibrium of the closed-loop system away from the one associated with the Cournot–Nash solution. However, by Theorem 2, asymptotic stability will not be jeopardized, local price variables will synchronize, and the vector  $(P_g, P_d, p)$  will converge to the point  $(\bar{P}_g, \bar{P}_d, \bar{p})$  given in Lemma 7. The investigation of how far the equilibrium is from the Cournot–Nash equilibrium in the presence of uncertainty on  $\alpha^*$  is left for future research.

**Remark 8:** When  $\alpha^*$  is not precisely known, another possibility is to estimate the parameter offline, as utility functions are not frequently changing. To this end, one can implement a distributed algorithm such as

$$\dot{\chi}_{ij} = \hat{\alpha}_i - \hat{\alpha}_j, \quad \{i, j\} \in E_c \quad (41a)$$

$$\dot{\hat{\alpha}}_i = \frac{1}{n} - Q_{di}^{-1} \hat{\alpha}_i - \sum_{j \in \mathcal{N}_i^c} \kappa_{ij} \chi_{ij}, \quad i \in \mathcal{V} \quad (41b)$$

which requires local parameter  $Q_{di}$ , communicated variables  $\chi_{ij}$ , and assumes that each controller is aware of the total number of participating agents, namely,  $n$ . It is easy to see that  $\hat{\alpha}$

TABLE I  
SIMULATION PARAMETERS

Areas	1	2	3	4
$M_i$	5.22	3.98	4.49	4.22
$D_i$	1.60	1.22	1.38	1.42
$Q_{gi}$	1.50	4.50	3.00	6.00
$b_{gi}$	0.60	1.05	1.50	2.70
$Q_{di}$	1.50	2.25	3.60	6.00
$b_{di}$	6.00	5.00	7.00	8.00
$\tau_i$	2.00	3.00	3.00	1.50

asymptotically converges to  $\alpha^* = (\mathbb{1}^{\top} Q_d^{-1} \mathbb{1})^{-1}$ . While it is difficult to provide analytical guarantees for the online use of this estimator in the controller (37), our numerical investigation in Section IV validates the stability and performance of such a scheme [see (43)].

**Remark 9:** While one can suggest heuristic modifications in the proposed controller to incorporate more general cost and utility functions, providing analytical guarantees on the stability of the overall closed-loop systems turns out to be a very difficult problem. Due to the very same challenge, the cost functions in [9], [10], and [12] in the context of optimal distributed integral control of networks have been restricted to quadratic or linear-quadratic functions.

## IV. CASE STUDY

We illustrate the proposed pricing mechanism on a specific example of a network with producers and consumers, namely, a four-area power network [10]; see [33] on how a four-area network equivalent can be obtained for the IEEE New England 39-bus system or the South Eastern Australian 59-bus. The power network model we consider here is given by the so-called swing equation [31], and is mathematically equivalent to the dynamics in (32), under the assumption that voltages are constant and the frequency dynamics is decoupled from the reactive power flow. In this case,  $\zeta$  is the vector of phase angles measured with respect to the phase angle of a reference bus (area 4),  $y$  is the vector of frequency deviations from the nominal frequency (50/60 Hz), and the diagonal matrices  $M$  and  $D$  collect the inertia and damping constants. The vectors  $P_g$  and  $P_d$  denote the vector of generation and demand as before. The numerical values of the system parameters are provided in Table I. The physical and communication graphs, namely,  $\mathcal{G}$  and  $\mathcal{G}_c$ , are depicted in Fig. 2, where the solid and dotted edges denote the transmission lines and communication links, respectively.

For each  $\{i, j\} \in \mathcal{E}$ , the (locally convex) function  $H_{ij}$  in (29) is given by  $-|B_{ij}|V_i V_j \cos(x_i - x_j)$ , where  $B_{ij} < 0$  is the susceptance of the line  $\{i, j\}$ , and  $V_i$  and  $x_i$  are the voltage magnitude and voltage phase angle at the  $i$ th area (bus). In (32), this yields the expression

$$\nabla H_{\zeta} = R_{\zeta} W \sin(R_{\zeta}^{\top} \zeta)$$

where  $W = \text{diag}(w_k)$ , with  $w_k := B_{ij} V_i V_j$ ,  $k \sim \{i, j\}$ , and  $\sin(\cdot)$  is interpreted elementwise.

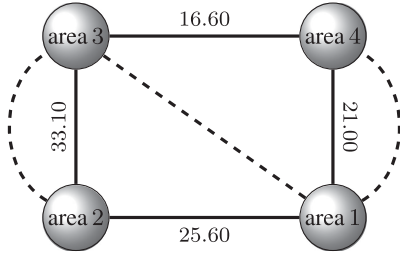


Fig. 2. Solid lines denote the transmission lines, and the dashed lines depict the communication links. The edge weights indicate the susceptance of the transmission lines.

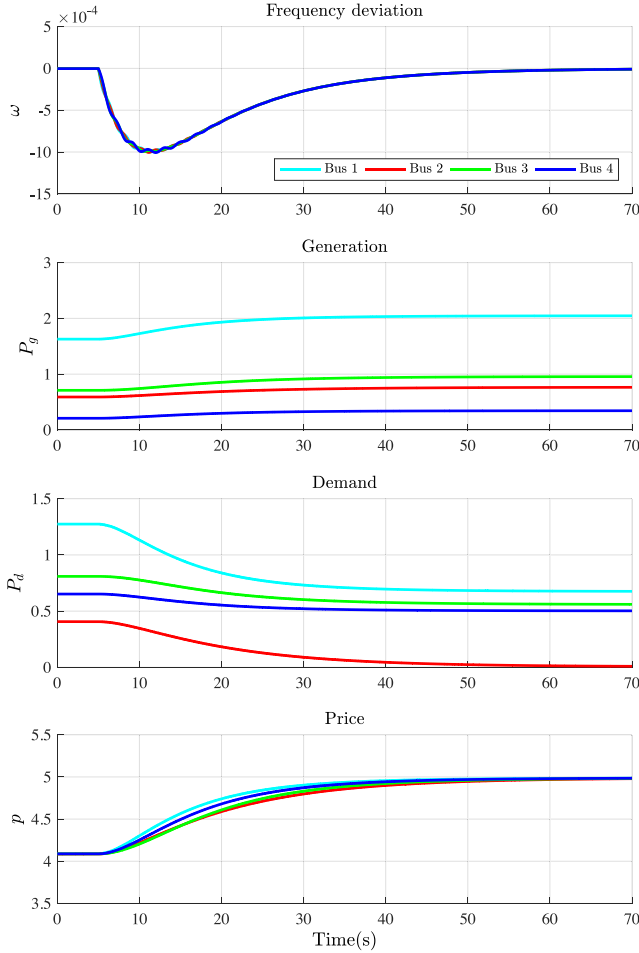


Fig. 3. Numerical simulation of the closed-loop system (38).

We consider linear-quadratic cost and utility functions given by (11) and (12), with the parameters provided in Table 1.

We consider the distributed controller in (37), and we set  $\rho_{ij} = 1$ , for each  $\{i, j\} \in \mathcal{E}_c$ . The closed-loop system (38) is initially at steady state. At time  $t = 5$  s, we modify the utility functions by increasing  $b_d$  by 25%, which results in a step in the demand. The response of the closed-loop system to this change is shown in Fig. 3, where the values are in per unit with respect to a base power of 1000 MVA. As can be seen in the figure, at steady state, the frequency is regulated to its nominal value, which indicates that the matching condition (4)

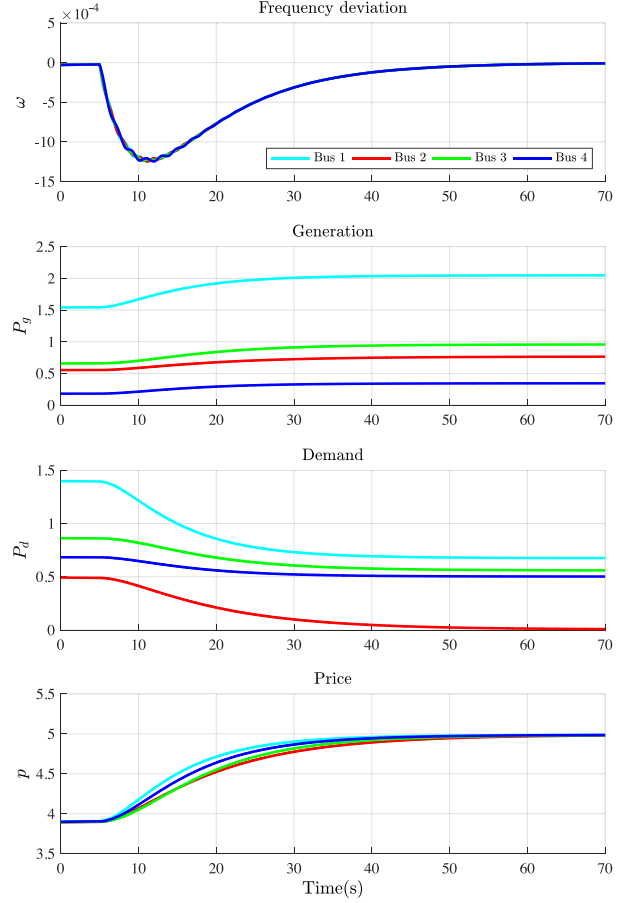


Fig. 4. Numerical simulation of the closed-loop system (43).

is satisfied. The local prices converge to the same value, which identifies the market clearing price  $p^*$ . As desired, the triple  $(P_g, P_d, p)$  converges to the Cournot–Nash optimal solution

$$P_g^* = \begin{bmatrix} 2.05 \\ 0.77 \\ 0.96 \\ 0.34 \end{bmatrix}, \quad P_d^* = \begin{bmatrix} 1.67 \\ 0.56 \\ 1.04 \\ 0.83 \end{bmatrix}, \quad p^* = 4.99. \quad (42)$$

Next, we consider the case where the parameter  $\alpha^*$  is unknown and is identified in real time by the estimator (41). This results in the closed-loop dynamics

$$\dot{\zeta} = E^\top y \quad (43a)$$

$$M\dot{y} = -Dy - E \nabla H_\zeta(\zeta) + (\hat{\alpha}I_n + Q_g)^{-1}(p - b_g) - Q_d^{-1}(b_d - p) \quad (43b)$$

$$T\dot{p} = -Lp - (\hat{\alpha}I_n + Q_g)^{-1}y - Q_d^{-1}y \quad (43c)$$

$$\dot{\chi} = R_c^\top \hat{\alpha} \quad (43d)$$

$$\dot{\hat{\alpha}} = \frac{1}{n}\mathbf{1} - Q_d^{-1}\hat{\alpha} - R_c\chi \quad (43e)$$

where  $R_c$  denotes the incidence matrix of the communication graph, and we have set  $\kappa_{ij} = 1$  for simplicity. The system is

initially at steady state. At time  $t = 5$  s, we modify as before the utility functions by increasing  $b_d$  by 25%. At the same time, we decrease the elements of  $Q_d$  by 20%, which modifies the actual value of  $\alpha$  according to (16). For a better comparison to the system without the estimator, the initial value of  $Q_d$  is chosen such that its new value will be equal to the one provided in Table I. The response of the closed-loop system (43) is illustrated in Fig. 4. As can be seen from the figure, frequency is regulated to its nominal value and the triple  $(P_g, P_d, p)$  converges again to the one given by (42). This means that the controller (37) equipped with the estimator (41) is able to steer the network to the Cournot–Nash optimal solution. Note that compared to Fig. 3, the transient performance is only slightly degraded.

## V. CONCLUSION

We have proposed a distributed feedback algorithm that steers a dynamical network to a prescribed equilibrium corresponding to the so-called Cournot–Nash equilibrium. We characterized this equilibrium for linear-quadratic utility and cost functions, and specified the algebraic conditions under which the production and the demand are strictly positive for all agents. For a class of passive nonlinear second-order systems, where production and demands act as external inputs to the systems, we propose a control algorithm (pricing mechanism) that guarantees the convergence of the closed-loop system to the optimal equilibrium point associated with the previously characterized Cournot–Nash equilibrium. Considering a different type of competition such as Bertrand and Stackelberg games as well as a thorough comparison of the equilibrium points resulting from competitive games against those obtained from a social welfare problem is of interest for future research. The Cournot game belongs to the class of aggregative games [25]. Using the passivity property of projected pseudogradient algorithms highlighted in [34] and game equilibrium seeking integral control for aggregative games [35], one could consider the feedback interconnection of the network dynamics with projected dynamical systems [36], possibly enabling the extension of the results in this paper to a more general case.

## APPENDIX

**Proof of Lemma 2:** In view of Assumption 2, the functions  $\pi_j(\lambda)$  in (2) are continuous with a range equal to the whole  $\mathbb{R}_{\geq 0}$ . Each function  $\pi_j(\lambda)$  is strictly decreasing on the interval  $(-\infty, U'_j(0))$  and identically zero on the interval  $[U'_j(0), +\infty)$ . Define the function  $\pi(\lambda) := \sum_{j \in \mathcal{J}} \pi_j(\lambda)$ . This is a continuous function with a range equal to the whole  $\mathbb{R}_{\geq 0}$ . Moreover, it is strictly decreasing on  $(-\infty, \max_j U'_j(0))$  and identically zero on the interval  $[\max_j U'_j(0), +\infty)$ . Let  $q := \pi(\lambda)$ . Then, the function  $\pi$  can be inverted, on the interval where  $q \in \mathbb{R}_{> 0}$ , to obtain  $\lambda = \pi^{-1}(q)$ . Now we define  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  as

$$u(q) = \begin{cases} \pi^{-1}(q) & \text{if } q > 0 \\ \pi^{-1}(0^+) & \text{if } q = 0 \end{cases} \quad (44)$$

where  $\lim_{q \rightarrow 0^+} \pi^{-1}(q) =: \pi^{-1}(0^+)$ . By construction,  $u$  is a continuous function, which is strictly decreasing and satisfies

the properties i)–iii) of the statement. Moreover, by substituting  $q = \mathbf{1}^\top P_g$  in (44), using (2) together with the balancing condition (4), and noting the monotonicity of  $u$ , we obtain equality (5). ■

**Proof of Lemma 3:** From Corollary 1, it follows that the first two statements are equivalent, and they imply the third statement. It remains to show that 3)  $\Rightarrow$  2). Now, suppose that the third statement holds. From  $P_d = Q_d^{-1}(b - \mathbf{1}p)$ , using again Corollary 1, we obtain that  $p \leq b_{dj}$  for all  $j \in \mathcal{J}$ . The condition  $p \neq \underline{b}_d$  yields  $p < \underline{b}_d$ . ■

**Proof of Lemma 5:** 1)  $\Rightarrow$  2) Suppose that the first statement holds. Then, by Lemma 4, we have

$$P_g = (2\alpha I + Q_g)^{-1}(\beta \mathbf{1} - b_g + \alpha(I - \mathbf{1}\mathbf{1}^\top)P_g). \quad (45)$$

This is equivalent to  $\alpha P_g + \alpha \mathbf{1}\mathbf{1}^\top P_g + Q_g P_g = \beta \mathbf{1} - b_g$ , and to  $P_g = (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta \mathbf{1} - b_g)$ . Hence, (20) is obtained. This also shows the uniqueness of the solution. Now, note that  $(\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)P_g = \beta \mathbf{1} - b_g$ , which is equivalent to  $(\alpha I + Q_g)P_g = \mathbf{1}(\beta - \alpha \mathbf{1}^\top P_g) - b_g$ . Elementwise, this can be written as  $(\alpha + Q_{gi})P_{gi} = \beta - \alpha \mathbf{1}^\top P_g - b_{gi}$ . Since  $P_{gi} > 0$  for all  $i$ , we obtain that  $\beta - \alpha \mathbf{1}^\top P_g > \bar{b}_g$ .

2)  $\Rightarrow$  3) Next, suppose that the second statement of the lemma holds. Then, using the same chain of equivalences as above, we obtain (45). Therefore, by Lemma 4, we must have that  $\gamma_i(P_{-gi}) > 0$  for all  $i \in \mathcal{I}$ . Without loss of generality, assume that  $b_{g1} = \bar{b}_g$ . Suppose by contradiction that  $\gamma_1(P_{-g1}) \leq 0$ . Then, we have  $P_{g1} = 0$  and, thus,  $\beta - \bar{b}_g - \alpha \mathbf{1}^\top P_g = 0$ . This contradicts the inequality in the second statement of the lemma, which completes this part of the proof.

3)  $\Rightarrow$  1) Now, let the third statement hold, and set

$$\hat{P}_g := (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta \mathbf{1} - b_g), \quad \hat{P}_g \in \mathbb{R}_{> 0}^n. \quad (46)$$

Again, the vector  $\hat{P}_g$  can be written in analogy to (45) as  $\hat{P}_{gi} = (2\alpha + Q_{gi})^{-1}(\beta - b_{gi} - \alpha \mathbf{1}^\top \hat{P}_{-gi})$  for every component  $i \in \mathcal{I}$ . For every  $i \in \mathcal{I}$ , since  $\hat{P}_{gi} > 0$ , then  $(2\alpha + Q_{gi})^{-1}(\beta - b_{gi} - \alpha \mathbf{1}^\top \hat{P}_{-gi}) > 0$ , which is equivalent to say that  $\gamma_i(\hat{P}_{-gi}) > 0$ . Hence, the vector  $\hat{P}_g$  satisfies (19), and is therefore a solution to (18), belonging to the interior of the positive orthant.

3)  $\Leftrightarrow$  4) To complete the proof of the lemma, it suffices to show that the last two statements of the lemma are equivalent, namely

$$\hat{P}_g \in \mathbb{R}_{> 0}^n \Leftrightarrow \mathbf{1}^\top (\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta \mathbf{1} - b_g) \leq \frac{\beta - \bar{b}_g}{\alpha}$$

where  $\hat{P}_g$  is given by (46). From (46), we have  $(\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)\hat{P}_g = \beta \mathbf{1} - b_g$ , which is equivalent to  $(\alpha I + Q_g)\hat{P}_g = \mathbf{1}(\beta - \alpha \mathbf{1}^\top \hat{P}_g) - b_g$ . Elementwise, this can be written as  $(\alpha + Q_{gi})\hat{P}_{gi} = \beta - \alpha \mathbf{1}^\top \hat{P}_g - b_{gi}$ . Therefore,  $\hat{P}_{gi} > 0$  if and only if  $\beta - \alpha \mathbf{1}^\top \hat{P}_g > \bar{b}_g$ . By replacing  $\hat{P}_g$  with  $(\alpha(I + \mathbf{1}\mathbf{1}^\top) + Q_g)^{-1}(\beta \mathbf{1} - b_g)$ , we see that the latter inequality is equivalent to (21). ■

**Proof of Lemma 6:** Suppose that  $(\bar{\zeta}, \bar{y})$  is an equilibrium of (32). Then

$$0 = E^\top \bar{y} \quad (47a)$$

$$0 = -D\bar{y} - E \nabla H_\zeta(\bar{\zeta}) + \bar{P}_g - \bar{P}_d. \quad (47b)$$

Hence, we find that  $y = \mathbf{1}_n \hat{y}$  for some  $\hat{y} \in \mathbb{R}$ , and

$$0 = -D\mathbf{1}\hat{y} - E \nabla H_\zeta(\bar{\zeta}) + \bar{P}_g - \bar{P}_d. \quad (48)$$

By multiplying both sides of the equality above from the left by  $\mathbf{1}^\top$ , we find  $\hat{y} = y^*$ , where the latter is given by (34). By replacing the expression of  $y^*$  back to equality (48), and noting that  $E$  has full-column rank, equality (35) is obtained.

Conversely, assume that a point  $(\bar{\zeta}, \bar{y})$  satisfies (34) and (35). Clearly,  $E^\top \bar{y} = 0$ . Moreover, note that

$$\left( I_n - \frac{D\mathbf{1}\mathbf{1}^\top}{\mathbf{1}^\top D\mathbf{1}} \right) (\bar{P}_g - \bar{P}_d) \in (\text{im } \mathbf{1}_n)^\perp = \text{im } E.$$

Hence, multiplying both sides of (35) from the left-hand side by  $E$  gives

$$E \nabla H_\zeta(\bar{\zeta}) = \left( I_n - \frac{D\mathbf{1}\mathbf{1}^\top}{\mathbf{1}^\top D\mathbf{1}} \right) (\bar{P}_g - \bar{P}_d).$$

By the definition of  $\bar{y}$  in (34), the equality above can be written as (47b) and, therefore,  $(\bar{\zeta}, \bar{y})$  is an equilibrium of (32). For uniqueness of the equilibrium, it suffices to show that

$$\nabla H_\zeta(\bar{\zeta}) = \nabla H_\zeta(\tilde{\zeta}) \text{ for some } \bar{\zeta}, \tilde{\zeta} \in \mathbb{R}^{n-1} \implies \bar{\zeta} = \tilde{\zeta}.$$

The equality  $\nabla H_\zeta(\bar{\zeta}) - \nabla H_\zeta(\tilde{\zeta}) = 0$  is equivalent to

$$R_\zeta \nabla H(R_\zeta^\top \bar{\zeta}) - R_\zeta \nabla H(R_\zeta^\top \tilde{\zeta}) = 0.$$

Multiplying both sides of the equality above from the left-hand side by  $(\bar{\zeta} - \tilde{\zeta})^\top$  returns

$$(R_\zeta^\top \bar{\zeta} - R_\zeta^\top \tilde{\zeta})^\top (\nabla H(R_\zeta^\top \bar{\zeta}) - \nabla H(R_\zeta^\top \tilde{\zeta})) = 0.$$

By strict convexity of  $H$ , we find that  $R_\zeta^\top \bar{\zeta} = R_\zeta^\top \tilde{\zeta}$ . The fact that  $R_\zeta$  has full-row rank yields  $\bar{\zeta} = \tilde{\zeta}$ . ■

**Proof of Lemma 7:** Suppose that  $(\bar{\zeta}, \bar{y}, \bar{p})$  is an equilibrium of (38). Then

$$0 = E^\top \bar{y} \quad (49a)$$

$$0 = -D\bar{y} - E \nabla H_\zeta(\bar{\zeta}) + K(\bar{p} - b_g) - Q_d^{-1}(b_d - \bar{p}) \quad (49b)$$

$$0 = -L\bar{p} - K\bar{y} - Q_d^{-1}\bar{y}. \quad (49c)$$

By the first equality, we have  $\bar{y} = \mathbf{1}y^*$  for some  $y^* \in \mathbb{R}$ . Substituting this into (49c), and multiplying both sides of (49c) from the left-hand side by  $\mathbf{1}^\top$ , we obtain that  $y^* = 0$ . Hence, (49c) results in  $\bar{p} = \mathbf{1}q$  for some  $q \in \mathbb{R}$ . The fact that  $q$  is given by (39) and that (40) holds, suitable algebraic manipulations follow analogous to the proof of Lemma 6. The converse result as well as uniqueness of the equilibrium also follows analogous to Lemma 6.

By (39), we have

$$(1 + \alpha^* \mathbf{1}^\top K \mathbf{1})q = \beta^* + \alpha^* \mathbf{1}^\top K b_g$$

where  $\alpha^*$  and  $\beta^*$  are given by (16). The equality above can be written as  $q = \beta^* - \alpha^* \mathbf{1}^\top K(\mathbf{1}q - b_g)$ , which yields

$$q = \beta^* - \alpha^* \mathbf{1}^\top \bar{P}_g. \quad (50)$$

Equivalently, we have

$$K^{-1}K(\mathbf{1}q - b_g) + b_g = \mathbf{1}\beta^* - \alpha^* \mathbf{1}\mathbf{1}^\top \bar{P}_g$$

and hence

$$(\alpha^* \mathbf{1}\mathbf{1}^\top + K^{-1})\bar{P}_g = \mathbf{1}\beta^* - b_g.$$

By setting  $K = (\alpha^* I_n + Q_g)^{-1}$ , the equality above returns  $\bar{P}_g = P_g^*$ , with  $P_g^*$  given by (23a). Then, by comparing (50) to (23c), we find that  $q = p^*$ . Finally

$$\bar{P}_d = Q_d^{-1}(b_d - \mathbf{1}q) = Q_d^{-1}(b_d - \mathbf{1}\beta^* + \alpha^* \mathbf{1}\mathbf{1}^\top P_g^*) = P_d^*$$

where the second equality follows from (50), and the last one from (23b). ■

**Proof of Theorem 2:** The proof is based on the interconnection of shifted passive systems in Fig. 1, with the interconnection ports  $(u, y)$ ,  $u = P_g - P_d$ . To prove asymptotic stability, we consider the Lyapunov function candidate (see also [10], [13], and [37, Ch. 6.5])

$$V = \frac{1}{2}(y - \bar{y})^\top M(y - \bar{y}) + \frac{1}{2}(p - \bar{p})^\top T(p - \bar{p}) + \mathcal{H}(\zeta)$$

where  $\mathcal{H}$  takes the form of the Bregman distance between  $\zeta, \bar{\zeta}$  associated with the *distance-generating* function  $H(\zeta)$  and the point  $\bar{\zeta}$ , namely

$$\mathcal{H}(\zeta) = H(\zeta) - H(\bar{\zeta}) - (\zeta - \bar{\zeta})^\top \frac{\partial H}{\partial \zeta} \Big|_{\bar{\zeta}}.$$

Since  $H$  is strictly convex, the Bregman distance  $\mathcal{H}$  is non-negative and is equal to zero whenever  $\zeta = \bar{\zeta}$ . Then, clearly the function  $V$  has a strict minimum at  $(\bar{\zeta}, \bar{y}, \bar{p})$ . Computing the time derivative of  $V$  along the solutions of (38) yields

$$\dot{V} = -(y - \bar{y})^\top D(y - \bar{y}) - (p - \bar{p})^\top L(p - \bar{p})$$

where we have used (49) together with the fact that

$$\frac{\partial \mathcal{H}}{\partial \zeta} = \frac{\partial H}{\partial \zeta} - \frac{\partial H}{\partial \zeta} \Big|_{\bar{\zeta}}.$$

Since  $V$  is positive definite and  $\dot{V}$  is nonpositive, we conclude that the solutions of (38) are bounded. By invoking LaSalle's invariance principle, on the invariant set, we have  $y = \bar{y}$ ,  $Lp = L\bar{p}$ . Noting that  $\bar{y} = 0$  and  $L\bar{p} = 0$ , we find that each point on the invariant set is an equilibrium of (38). By Lemma 7, the equilibrium is unique and, therefore, the invariant set comprises only the equilibrium point  $(\bar{\zeta}, 0, \bar{p})$  given by (39) and (40). By continuity, the vectors  $P_g$  and  $P_d$  asymptotically converge to  $\bar{P}_g = K(\bar{p} - b_g)$  and  $\bar{P}_d = Q_d^{-1}(b_d - \bar{p})$ , respectively. For  $K = (\alpha^* I_n + Q_g)^{-1}$ , by Lemma 7, we have  $(\bar{P}_g, \bar{P}_d, \bar{p}) = (P_g^*, P_d^*, p^*)$ . ■



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