Strategic Differentiation in Non-Cooperative Games on Networks (I)

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Abstract— In the existing models for finite non-cooperative games on networks, it is usually assumed that in each single round of play, regardless of the evolutionary update rule driving the dynamics, each player selects the same strategy against all of its opponents. When a selfish player can distinguish the identities of its opponents, this assumption becomes highly restrictive. In this paper, we introduce the mechanism of *strategic differentiation* through which a subset of players in the network, called *differentiators*, are able to employ different pure strategies against different opponents in their local game interactions. Within this new framework, we study the existence of pure Nash equilibria and finite-time convergence of differentiated myopic best response dynamics by extending the theory of potential games to non-cooperative games with strategic differentiation. Finally, we illustrate the effect of strategic differentiation on equilibrium strategy profiles by simulating a non-linear spatial public goods game and the simulation results show that depending on the position of differentiators in the network, the level of cooperation of the whole population at an equilibrium can be promoted or hindered. Our findings indicate that strategic differentiation may provide new ideas for solving the challenging free-rider problem on complex networks.

I. Introduction

Simple decisions or actions taken by interacting individuals can lead to surprisingly complex and unpredictable population-level outcomes. In particular, when individual decisions or actions are based on personal interest, the long run collective behavior, characterized by these selfish decisions, can be detrimental for the population as a whole. Mathematical models of such systems require trade-offs between the complexity of micro-dynamics and the accuracy with which the model can describe a macro-behavior. *Evolutionary game theory*, originally proposed as a dynamical model for Darwinian competition [1], has proven to be a valuable tool in providing mathematical models for such complex dynamical systems. In evolutionary game theory, the concept of *bounded rationality* [2] was proposed that supports the idea that in order to reach decisions players are satisfied with, they may rely on simple rules. In economic contexts evolutionary dynamics are, typically, driven by simple rational thinking, (e.g. *myopic best response*). Such dynamics have been studied extensively for finite non-cooperative games using potential functions [3] and Markov chain theory [4], [5]. Much literature has been developed on evolutionary games on networks that study the consequences of spatial or social population structure [6], [7] on the *evolutionary success* of

the population. These extended research efforts resulted in the identification of several mechanisms that help to explain the emergence of cooperation in *competitive* environments [8]. One such mechanism is known as *network reciprocity*: when a cooperator pays some cost that its neighbors can benefit from and a defector bears no costs, not creating any benefit for its neighbors, then cooperators can succeed by forming clusters in the network [8]. Evolutionary games on networks and the study of their evolutionary success have later been generalized to include *groupwise interactions* [9], and *multilayer* networks [10], [11]. An extensive review of these games can be found in [12].

A common assumption in the existing models for finite non-cooperative evolutionary games on networks is that players do not distinguish between their opponents. In some sense the opponents are *anonymous* and hence, there is no difference in the strategies employed against each of them. In real life competitive settings, in order to create a competitive advantage it is often crucial to identify the rivals [13]. And indeed, avoiding 'blindspots' in a *competitive decision process*, i.e. those decisions that require taking into account the decisions of competitors, is a major topic in the strategic decision making literature [14]. Thus, in such competitive environments, decision makers are likely to distinguish their opponents, and consequently they may employ different strategies against them. As a first contribution of this paper, we introduce the mechanism of *strategic differentiation* through which a subset of players in the network, called differentiators, are able to employ different pure strategies against different opponents. Within this new framework, strategic differentiation can be applied to both pairwise and groupwise games on networks. As a second contribution, we connect strategic differentiation to the theory of potential games and their generalizations and show that for the class of weighted potential games the effect of strategic differentiation on any network topology can be studied analytically using the potential function of the original game. Third, our results indicate that the effect of strategic differentiation on the equilibrium strategy profile can be profound and that the location of the differentiators in the network has a crucial effect on the evolutionary success of the non-cooperative evolutionary game. We believe that decision making processes of interacting individuals that can *identify and distinguish* their opponents can be modeled more accurately using the mechanism strategic differentiation compared to more traditional evolutionary network game models and that the framework and results presented in this paper can be useful in understanding such systems.

The paper is structured as follows. In Section II a short

overview is given on traditional non-cooperative games on networks with evolutionary game dynamics. In Section III the mechanism strategic differentiation is formally defined for non-cooperative games on networks. In Section IV myopic best response dynamics in games with strategic differentiation and their equilibria are formally defined. In Section V strategic differentiation is linked to potential games and their generalizations. In Section VI the effect of strategic differentiation on the equilibrium strategy profile is investigated via simulations.

II. NOTATIONS AND PRELIMINARIES

In this section we briefly define the notation to be used in the paper and introduce the existing framework of non-cooperative games on networks.

A. Notations and definitions

Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ denote an undirected network with node set $\mathcal{V}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighborhood of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\} \cup \{i\}$. The degree of player $i \in \mathcal{V}$ is denoted by $d_i = |\mathcal{N}_i| - 1$, where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. The neighborhood hypergraph of $\mathbb{G}$ is defined by $\mathbb{H} = (\mathbb{G}, \mathcal{I})$, where the hyperedge set $\mathcal{I}$ is a family of non-empty sets over $\mathcal{V}$, corresponding to the n neighborhoods in $\mathbb{G}$. For a vector $x \in \mathbb{R}^n$ we denote its ith element by x_i. For any $i = 1 \ldots n$, we write $x = (x_i, x_{-i})$ where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We denote the n-ary Cartesian product over the n sets $\mathcal{A}_1 \ldots, \mathcal{A}_n$ by $\times_{i=1}^{n} \mathcal{A}_i := \{(a_1 \ldots, a_n) : a_i \in \mathcal{A}_i \; \forall i = 1, \ldots, n\}$.

B. Finite Non-Cooperative Games on Networks

Non-cooperative evolutionary games on networks are defined by a network, a strategy space, payoff functions and strategy update dynamics. We now introduce these elements separately.

Network and finite strategy spaces: Consider an undirected network $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ whose nodeset $\mathcal{V} = \{1, \ldots, n\}$ corresponds to a finite set of players. Each player $i \in \mathcal{V}$ has a fixed and finite pure strategy set $\mathcal{X}_i$. The resulting strategy space is denoted by $\mathcal{X} = \times_{i \in \mathcal{V}} \mathcal{X}_i$. The strategy profile is indicated by $x = (x_1, \ldots, x_n) \in \mathcal{X}$, such that $x_i \in \mathcal{X}_i$ for all $i \in \mathcal{V}$.

Single round payoffs: In a single round of a *pairwise* non-cooperative game on a network, a player separately interacts with each neighbor. Let $\pi_{ij}(x_i, x_j)$ denote the payoff that player i obtains from strategy $x_i \in \mathcal{X}_i$ in the pairwise interaction against opponent $j \in \mathcal{N}_i$ with strategy $x_j \in \mathcal{X}_j$. The total payoff that player i obtains in a single round of play is given by a weighted sum of the local payoffs. That is,

$$\pi_i(x_i, x_{-i}) = \sum_{j \in \mathcal{N}_i \setminus \{i\}} w_{ij} \pi_{ij}(x_i, x_j), \qquad (1)$$

with $w_{ij} \in \mathbb{R}$ denoting the weight associated to the local interaction between i and j. We refer to a non-cooperative game with a payoff function of the form (1) as a *pairwise game on a network*. Alternatively, players may interact in groups with a size greater than two, and thus the local

interactions form a *multiplayer* game. In general, the payoffs of multiplayer games on networks cannot be represented by the corresponding sum of pairwise interactions and thus the local multiplayer game interactions are described by the neighborhood hypergraph of $\mathbb{G}$. Again, the total payoff that player i obtains in a single round of play is a weighted sum of the local payoffs,

$$\pi_i(x_i, x_{-i}) = \sum_{j \in \mathcal{N}_i} w_j \pi_{ij}(x_i, x_{-i}), \qquad (2)$$

with $w_j \in \mathbb{R}$ denoting a common weight that the players in $\mathcal{N}_j$ associate to the local multiplayer game. Note that in equation (2) the single round local payoffs depend on $|\mathcal{N}_j| \geq 2$ strategies and the network structure imposes an interdependence in the payoffs of players that are connected via an undirected path with the length two, sometimes referred to as the 2-hop neighbors. We refer to a non-cooperative game with payoff functions of the form (2) as a *groupwise game on network*. For both pairwise and groupwise games we indicate the combined payoff function by $\pi : \mathcal{X} \to \mathbb{R}^n$ that maps each strategy profile $x \in \mathcal{X}$ to a payoff vector $\pi(x) = [\pi_1(x), \ldots, \pi_n(x)]$.

Strategy update dynamics: We study evolutionary dynamics in which the players may change their strategy over time. For some time $t \in \mathbb{N}_0$, let $x^t \in \mathcal{X}$ indicate the strategy profile at time t. A strategy update rule $f : \mathcal{X} \times \mathbb{R}^n \to \mathcal{X}$ is a function that maps the current strategy profile x^t and payoff vector $\pi(x^t)$ to an updated strategy profile x^{t+1}. Update rules may be seen as a learning process or a simple rule of thumb that players use in an effort to reach decisions they are satisfied with. In this paper we assume the evolutionary dynamics driven by f are *asynchronous*. That is, for each $t = 0, 1, 2, 3, \ldots$, there exists a unique player $i \in \mathcal{V}$, referred to as the *unique deviator*, such that $x^{t+1} = (x_i^{t+1}, x_{-i}^t)$. A non-cooperative game on a network is then defined by the triplet $\Gamma = (\mathbb{G}, \mathcal{X}, \pi)$. When the evolution of the strategy profile changes according to the strategy update rule f, we denote the evolutionary game on the network by (Γ, f).

III. STRATEGIC DIFFERENTIATION

In this section we formally define the mechanism strategic differentiation and show how it can be incorporated into the existing framework of evolutionary games on networks.

A. Strategy spaces and payoffs in non-cooperative games with strategic differentiation

In a non-cooperative game on a network with strategic differentiation, a *differentiator* can employ a separate pure strategy for each neighbor; see figure 1 for an example of a pairwise game with a single differentiator. Let $\mathcal{D}$ be a non-empty subset of $\mathcal{V}$ denoting the set of differentiators in the network, and let $\mathcal{F} := \mathcal{V} \setminus \mathcal{D}$ denote the set of non-differentiators. In a groupwise game on a network, the strategy of a player $i \in \mathcal{D}$ is a *vector* $s_i \in \mathcal{S}_i := \mathcal{X}_i^{|\mathcal{N}_i|}$: one pure strategy can be chosen to employ in the multiplayer game against each closed neighborhood that the player belongs to. When the game interactions are pairwise, the

dimension of the strategy vector of player i is reduced by one because in this case players only interact with their d_i neighbors. For some $j \in \mathcal{N}_i$, we indicate by $s_{ij} \in s_i$ the strategy that player $i \in \mathcal{D}$ employs in the local pairwise (resp. groupwise) game played against j (resp. $\mathcal{N}_j$). Note that for all $i \in \mathcal{D}$, $j \in \mathcal{N}_i$ we assume that $s_{ij} \in \mathcal{X}_i$, i.e. each pure strategy employed by a differentiator is in their own pure strategy set. The strategy space of non-differentiators $j \in \mathcal{F}$ is indicated by $\mathcal{X}_{\mathcal{F}} := \times_{j \in \mathcal{F}} \mathcal{X}_j$. Without loss of generality, label differentiators by $\mathcal{D} = \{1, \ldots |\mathcal{D}|\}$ and the non-differentiators by $\mathcal{F} = \{|\mathcal{D} + 1|, \ldots, n\}$. Then the strategy space of the networked game with strategic differentiation is given by $\hat{\mathcal{S}} = \mathcal{S}_{\mathcal{D}} \times \mathcal{X}_{\mathcal{F}}$, with $\mathcal{S}_{\mathcal{D}} := \times_{i \in \mathcal{D}} \mathcal{S}_i$. In a game with strategic differentiation we denote the local payoff function for the interaction between i and (the neighbors of) $j \in \mathcal{N}_i$ by $u_{ij} : \hat{\mathcal{S}} \to \mathbb{R}$. Similarly, $u : \hat{\mathcal{S}} \to \mathbb{R}^n$ denotes the combined payoff vector of the game with strategic differentiation. For pairwise interactions the payoffs of a differentiator $i \in \mathcal{D}$ is given by,

$$u_i(\hat{s}_i, \hat{s}_{-i}) = \sum_{j \in \mathcal{N}_i \cap \mathcal{D} \setminus i} w_{ij} u_{ij}(s_{ij}, s_{ji}) + \sum_{h \in \mathcal{N}_i \cap \mathcal{F} \setminus i} w_{ih} u_{ih}(s_{ih}, x_h).$$

And the payoff of a non-differentiator $k \in \mathcal{F}$ is given by,

$$u_k(\hat{s}_k, \hat{s}_{-k}) = \sum_{l \in \mathcal{N}_k \cap \mathcal{D} \setminus i} w_{kl} u_{kl}(x_k, s_{lk}) + \sum_{v \in \mathcal{N}_k \cap \mathcal{F} \setminus i} w_{kv} u_{kv}(x_k, x_v).$$

For games with strategic differentiation and *groupwise* interactions, the payoffs are obtained using (2) with the strategy space $\hat{\mathcal{S}}$. We are now ready to formally define a non-cooperative game with strategic differentiation.

Definition 1 (Strategically differentiated game). *A non-cooperative game on a network with strategic differentiation is defined by the triplet $\Xi := (\mathbb{G}, \hat{\mathcal{S}}, u)$. If $\pi_{ij} = u_{ij}$ for all $(i, j) \in \mathcal{E}$, then Ξ is said to be the strategically differentiated version of $\Gamma = (\mathbb{G}, \mathcal{X}, \pi)$.*
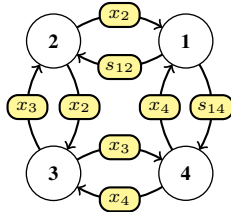

Fig. 1: *Graphical interpretation of a pairwise non-cooperative game on a network with strategic differentiation. The label of outgoing edges indicate the strategy played in the local pairwise interaction. In this example $\mathcal{D} = \{1\}$ and $\mathcal{F} = \{2, 3, 4\}$.*

IV. BEST RESPONSES AND NASH EQUILIBRIA IN STRATEGICALLY DIFFERENTIATED GAMES

Strategic differentiation requires a slight modification of the concept of a Nash equilibrium, best responses and my-

opic best response update dynamics for games on networks. These are provided in the following subsections.

A. Nash equilibrium of games with strategic differentiation

In finite games a pure best response for player i to a strategy profile $x \in \mathcal{X}$ is a pure strategy $\bar{x}_i \in \mathcal{X}_i$ such that no other pure strategy available to player i gives a higher payoff against $x \in \mathcal{X}$. This defines player $i's$ best response correspondence [15],

$$\beta_i(x, \pi) := \{\bar{x}_i \in \mathcal{X}_i : \pi_i(\bar{x}_i, x_{-i}) \geq \pi_i(x_i, x_{-i}) \, \forall x_i \in \mathcal{X}_i\}. \tag{3}$$

A strategy profile $x \in \mathcal{X}$ is a *pure Nash equilibrium* if for all $i \in \mathcal{V}$, x_i is a pure best response. In non-cooperative games on networks with strategic differentiation the players may employ a multitude of pure strategies. Based on the definition of a players best response correspondence (3), a best response of a differentiator is defined as follows.

Definition 2 (Differentiated Best Response). *For player $i \in \mathcal{D}$ the strategy $s_i \in \mathcal{S}_i$ is a strategically differentiated pure best response for $\hat{s} \in \hat{\mathcal{S}}$ if for all $s_{ik} \in s_i$*

$$s_{ik} \in \beta_{ik}(\hat{s}, u), \tag{4}$$

with $\beta_{ik}(\hat{s}, u) := \{x_i^* \in \mathcal{X}_i : u_{ik}(x_i^*, \hat{s}_{-i}) \geq u_{ik}(x_i, \hat{s}_{-i}) \, \forall x_i \in \mathcal{X}_i\}.$

Based on the definition of a differentiated best response, a Nash equilibrium in a strategically differentiated game is naturally defined as follows.

Definition 3 (Differentiated Pure Nash equilibrium). *A strategy profile $\hat{s}^* \in \hat{\mathcal{S}}$ is a differentiated pure Nash equilibrium of Ξ if for all $i \in \mathcal{F}$, $x_i^* \in \hat{s}^*$ is a pure best response and for all $j \in \mathcal{D}$, $s_j^* \in \hat{s}^*$ is a strategically differentiated pure best response.*

When $\mathcal{D} = \emptyset$ the original definition of a pure Nash equilibrium is recovered. Best responses of differentiators are thus vectors of pure strategies for which each element is *locally* optimal. Herein lies the main distinguishing feature of best replies in games without strategic differentiation: a best reply x_i^* over the aggregated payoff $\pi_i(x_i, x_{-i})$ might not optimize the payoffs of each separate local game with payoff π_{ij}. Hence, a strategically differentiated Nash equilibrium might contain strategies that are not present in the Nash equilibrium of the game without strategic differentiation.

B. Myopic best response dynamics in games with strategic differentiation

In non-cooperative games on networks with payoff functions of the form (1) or (2), the myopic best response of player i against strategy profile $x_{-i} \in \mathcal{X}_{-i}$ is given by

$$x_i^{t+1} \in \beta_i(x^t, \pi). \tag{5}$$

Let us consider such myopic best response dynamics in games with strategic differentiation: the strategy s_{ij} is chosen such that it maximizes u_{ij}, ceteris paribus.

Definition 4 (Differentiated Myopic Best Response).

$$s_{ij}^{t+1} \in \beta_{ij}(s^t, u) \qquad (6)$$

$$s_i^{t+1} = [s_{ij}^{t+1}, \ldots, s_{ij'}^{t+1}]^\top, \ j, j' \in \mathcal{N}_i. \qquad (7)$$

When all differentiators update their strategy according to the differentiated best response dynamics (6) and all non-differentiators according to (5), we indicate the evolutionary game with strategic differentiation by (Ξ, β).

The evolutionary dynamics (7) are an *unconstrained* version of the classic myopic best response dynamics (5) for non-cooperative games on networks in the sense that the local strategies are optimized over the local payoffs without requiring that the employed strategies are equal. It follows that when $\mathcal{D} = \mathcal{V}$, for innovative strategy update dynamics like myopic best response in which players can introduce new strategies into the network, the effect of the network structure on the equilibria is lost. In this case the equilibrium strategy profile of the networked game would correspond to a collection of separate Nash equilibria of the local games played on the network. When $\mathcal{D} \subset \mathcal{V}$ the network structure remains important to the evolutionary dynamics. Moreover, the differentiators may obtain an advantage over their opponents that are not able to differentiate their strategies because for each $x_i \in \mathcal{X}_i$, $\hat{s}_{-i} \in \hat{\mathcal{S}}_{-i}$, there exists $s_i \in \mathcal{S}_i$ such that $u_i(s_i, \hat{s}_{-i}) \geq u_i(x_i, \hat{s}_{-i})$. Hence, in terms of payoffs, players that differentiate their strategy rationally are always at least as successful as they would have been not differentiating their strategy. The benefit that differentiators can get over non-differentiators implies that especially for evolutionary update dynamics in which the most successful players are likely to be imitated, the existence of differentiators can have a significant impact on the evolution of the strategies in the network and the set of equilibria. We will investigate this effect in the simulation section.

V. POTENTIAL FUNCTIONS FOR NON-COOPERATIVE GAMES WITH STRATEGIC DIFFERENTIATION

In this section we describe conditions on the local game interactions of games on networks that ensure that their strategically differentiated versions have pure Nash equilibria and convergence of differentiated myopic best response dynamics is guaranteed. For this we apply the theory of potential games to strategically differentiated games. Consider the following definition derived from ordinal potential games [3].

Definition 5 (Differentiated ordinal potential game). Ξ *is a strategically differentiated ordinal potential game if there exists an ordinal potential function $P : \hat{\mathcal{S}} \to \mathbb{R}$ such that for all $x_j, x'_j \in \mathcal{X}_j$, $s_i, s'_i \in \mathcal{S}_i$, $\hat{s}_{-i} \in \hat{\mathcal{S}}_{-i}$ and $\hat{s}_{-j} \in \hat{\mathcal{S}}_{-j}$ the following holds:*

$\forall i \in \mathcal{D}:$
$$u_i(s_i, \hat{s}_{-i}) - u_i(s'_i, \hat{s}_{-i}) > 0 \Leftrightarrow P(s_i, \hat{s}_{-i}) - P(s'_i, \hat{s}_{-i}) > 0$$
$\forall j \in \mathcal{F}:$
$$u_j(x_j, \hat{s}_{-j}) - u_j(x'_j, \hat{s}_{-j}) > 0 \Leftrightarrow P(x_j, \hat{s}_{-j}) - P(x'_j, \hat{s}_{-j}) > 0.$$

Note that if $\mathcal{D} = \emptyset$, the original definition of an ordinal potential game introduced in [3] is recovered.

It is well known that every *finite* ordinal potential game has a pure Nash equilibrium. This property is generalized to strategically differentiated games in the following lemma.

Lemma 1. *Every finite differentiated ordinal potential game possesses a differentiated pure Nash equilibrium.*

Proof. The proof is omitted due to space limitations. □

One can show that if Ξ is a differentiated ordinal potential game, then the evolutionary game (Ξ, β) will always terminate in a differentiated Nash equilibrium. Instead we now focus on finding conditions on the local interactions in groupwise games on networks that ensure the convergence properties of Γ are preserved in its strategically differentiated version Ξ. This is especially useful when one already has a potential function for the original game on a network and is interested in comparing the behavior of the game with strategic differentiation. Before doing so, consider the following definition.

Definition 6 (Differentiated weighted potential games). Ξ *is a strategically differentiated weighted potential game if there exists a potential function $\bar{P} : \hat{\mathcal{S}} \to \mathbb{R}$ and weights $\alpha_i, \alpha_j \in \mathbb{R}_+$, such that for all $x_j, x'_j \in \mathcal{X}_j$, $s_i, s'_i \in \mathcal{S}_i$, $\hat{s}_{-i} \in \hat{\mathcal{S}}_{-i}$ and $\hat{s}_{-j} \in \hat{\mathcal{S}}_{-j}$ the following holds:*

$\forall i \in \mathcal{D}:$
$$u_i(s_i, \hat{s}_{-i}) - u_i(s'_i, \hat{s}_{-i}) = \alpha_i \left(\bar{P}(s_i, \hat{s}_{-i}) - \bar{P}(s'_i, \hat{s}_{-i}) \right)$$
$\forall j \in \mathcal{F}:$
$$u_j(x_j, \hat{s}_{-j}) - u_j(x'_j, \hat{s}_{-j}) = \alpha_j \left(\bar{P}(x_j, \hat{s}_{-j}) - \bar{P}(x'_j, \hat{s}_{-j}) \right).$$

Note that if $\mathcal{D} = \emptyset$, the original definition of a weighted potential game introduced in [3] is recovered.

The following result relates the fundamental properties of weighted potential games to their strategically differentiated version.

Theorem 1. *In Γ, if for all players $i \in \mathcal{V}$ there exists for each local payoff function $\pi_{ij} : \mathcal{X} \to \mathbb{R}$, $j \in \mathcal{N}_i$, a weighted potential function $\rho_j : \mathcal{X} \to \mathbb{R}$ with a common weight $\alpha_i \in \mathbb{R}_+$ for player i, then (Ξ, β) converges to a differentiated pure Nash equilibrium.*

Proof. For all $j \in \mathcal{F}$, let $\bar{s}_j := (x_j, \ldots, x_j) \in \mathcal{X}^{|\mathcal{N}_j|}$ such that each element in $\bar{s}_j$ is equal to $x_j \in \mathcal{X}_j$. For all differentiators $i \in \mathcal{D}$ let $\bar{s}_i := s_i$. Then, for all $j \in \mathcal{F}$ the payoff in the strategically differentiated game can be written as $u_j(x_j, \hat{s}_{-j}) = \sum_{l \in \mathcal{N}_j} w_l u_{jl}(\bar{s}_{jl}, \bar{s}_{-jl})$, where $\bar{s}_{-jl} := \{\bar{s}_{kl} \in \mathcal{X}_k : k \neq j \ \wedge \ k \in \mathcal{N}_l\}$. For differentiators $i \in \mathcal{D}$, the payoff in the strategically differentiated game is $u_i(s_i, \hat{s}_{-i}) = \sum_{k \in \mathcal{N}_i} w_k u_{ik}(\bar{s}_{ik}, \bar{s}_{-ik})$. By assumption, in any local game with the neighbors of some $j \in \mathcal{V}$, for all $i \in \mathcal{N}_j$ there exists a function $\rho_j : \mathcal{X} \to \mathbb{R}$ and weights $\alpha_i \in \mathbb{R}_+$, such that for every $x_i, x'_i \in \mathcal{X}_i$ and every $x_{-i} \in \mathcal{X}_{-i}$ the following equality holds, $\pi_{ij}(x_i, x_{-i}) - \pi_{ij}(x'_i, x_{-i}) = \alpha_i \left(\rho_j(x_i, x_{-i}) - \rho_j(x_i, x_{-i}) \right)$. For all

the non-differentiators $l \in \mathcal{N}_j \cap \mathcal{F}$, it follows that for any $x_l, x_l' \in \mathcal{X}_l$, $\bar{s}_{-lj} \in \mathcal{X}_{-lj}$, $u_{lj}(x_l, \bar{s}_{-lj}) - u_{lj}(x_l', \bar{s}_{-lj}) = \alpha_l \left(\rho_j(\bar{s}_{lj}, \bar{s}_{-lj}) - \rho_j(\bar{s}_{lj}', \bar{s}_{-lj}) \right)$. Similarly for the differentiators $i \in \mathcal{D} \cap \mathcal{N}_j$, since $\bar{s}_{ij} \in \mathcal{X}_i$ and $\bar{s}_{-ij} \in \mathcal{X}_{-ij}$, from the existence of the local weighted potential function ρ_j it follows that for all $\bar{s}_{ij}, \bar{s}_{ij}' \in \mathcal{X}_i$, $\bar{s}_{-ij} \in \mathcal{X}_{-ij}$, $u_{ij}(\bar{s}_{ij}, \bar{s}_{-ij}) - u_{ij}(\bar{s}_{ij}', \bar{s}_{-ij}) = \alpha_i \left(\rho_j(\bar{s}_{ij}, \bar{s}_{-ij}) - \rho_j(\bar{s}_{ij}', \bar{s}_{-ij}) \right)$. Let $\bar{s}_{\{\mathcal{N}_l\}} := \{ \bar{s}_{kl} \in \mathcal{X}_k : k \in \mathcal{N}_l \}$ denote the set of strategies employed by the players in the local interaction with the closed neighborhood of l. The difference in the payoff of a unique deviator $i \in \mathcal{V}$ switching from strategy $\bar{s}'$ to $\bar{s} \neq \bar{s}'$ is given by

$$u_i(\bar{s}_i, \bar{s}_{-i}) - u_i(\bar{s}_i', \bar{s}_{-i}) = \alpha_i \sum_{j=1}^{n} w_j \left(\rho_j(\bar{s}_{\{\mathcal{N}_j\}}) - \rho_j(\bar{s}_{\{\mathcal{N}_j\}}') \right),$$

Where we have used the fact that when the unique deviator is not a member of some closed neighborhood $\mathcal{N}_h$, then $\rho_h(s_{\{\mathcal{N}_h\}}) - \rho_h(s_{\{\mathcal{N}_h\}}') = 0$. This implies that $\bar{P} = \sum_{j \in \mathcal{V}} w_j \rho_j(\bar{s}_{\mathcal{N}_j})$, with weights α_i is a weighted potential function for Ξ, and thus Ξ is a strategically differentiated weighted potential game. The convergence of the differentiated myopic best response then follows from the argument used in traditional potential games. Clearly, since $\bar{\mathcal{S}}$ is finite, $\bar{P}$ is bounded. Moreover it is *increasing* along the trajectory generated by asynchronous myopic best responses of non-differentiators and asynchronous differentiated myopic best responses of differentiators. This implies convergence of the differentiated myopic best response strategy update dynamics to a differentiated pure Nash equilibrium. $\square$

The proof of Theorem 1 can be easily adjusted to show that the same statement holds for strategically differentiated pairwise games on networks with $w_{ij} = w_{ji}$ for all $(i,j) \in \mathcal{E}$. The following corollary of Theorem 1 applies to the class of *exact* potential games in which $\alpha_i = 1$ for all $i \in \mathcal{V}$.

Corollary 1. *If the local groupwise interactions of Γ are potential games, then (Ξ, β) converges to a strategically differentiated Nash equilibrium.*

Remark 1. *Theorem 1 and its corollary hold because there always exists a weighted potential function for payoffs that are a linear combination of local payoffs obtained from either potential games or weighted potential games with the same strategy sets and fixed weight vectors [16]. Hence, conditions on the local game interactions extend to the entire network game and its strategically differentiated version. This linear combination property does not hold for ordinal potential games [3]. Thus, assuming that the entire network game Γ is an ordinal potential game may not be sufficient for convergence of its strategically differentiated version (Ξ, β). Up to now we have not been able to find conditions on ordinal potential games that ensure that their differentiated versions share their fundamental properties.*

VI. THE FREE-RIDER PROBLEM AND STRATEGIC DIFFERENTIATION WITH IMITATION DYNAMICS

When the individuals of a group tend to be selfish the possibility to profit from others naturally results in trying to balance out one's own efforts and rewards. The *free-rider* problem describes the situation in which a good or service becomes under-provided, or even depleted, as a results of selfish individuals profiting from a good without contributing to it. Here, we seek to determine via simulations how strategic differentiation can result in more desirable outcomes in which contributions to a good are preserved in the long run.

All simulations are conducted on a *linear public goods game* in which players need to determine whether or not to contribute to a public good that their opponents can profit from. This decision is modeled by the pure strategy set $\mathcal{X}_i = \{0,1\}$ for all $i \in \mathcal{V}$. A differentiator $i \in \mathcal{D}$ may choose to contribute to one good but withhold from contributing to another. Hence, $s_i \in \{0,1\}^{|d_i+1|}$. For some $j \in \mathcal{N}_i$, when $s_{ij} = 1$ (resp. $s_{ij} = 0$) player i is *cooperating* (resp. *defecting*) in the local game against the group of opponent players in $\mathcal{N}_j$. Let $c_{ij} \in \mathbb{R}_{>0}$ denote the contribution of a cooperating player i in the game local game against the players in $\mathcal{N}_j$. In the linear public goods game, the contributions get multiplied by an enhancement factor $r \in [1, n]$, which can seen as a benefit-to-cost ratio of the local interaction. The payoff that player i obtains in the local game with the neighborhood of $j \in \mathcal{N}_i$ is, $u_{ij}(s_{ij}, \hat{s}_{-i}) = r(\sum_{l \in \mathcal{N}_j} s_{lj} c_{lj} + s_{jj} c_{jj})/(d_j+1) - c_{ij} s_{ij}$, and the aggregated payoff of player i is, $u_i(\hat{s}_i, \hat{s}_{-i}) = \sum_{j \in \mathcal{N}_i} u_{ij}(s_{ij}, \hat{s}_{-i})$. This model is well established in the field of economics, sociology and evolutionary biology and captures the free-rider problem because defectors can benefit from contributions of cooperators [9], [17].

In all simulations we start with 50% cooperators which are randomly assigned to the nodes on the network. The local contributions are determined by a players degree: for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, $c_{ij} = \frac{1}{d_i+1}$. Hence, the total contributions that a player can make is $\sum_{j \in \mathcal{N}_i} c_{ij} = 1$. This corresponds to a set-up known as *fixed costs per individual* [9]. The total number of contributions in an equilibrium strategy profile $\hat{s}$ is determined as,

$$0 \leq \sum_{i \in \mathcal{D}} \sum_{j \in \mathcal{N}_i} c_{ij} s_{ij} + \sum_{h \in \mathcal{F}} \sum_{l \in \mathcal{N}_h} c_h x_h \leq n,$$

where $c_h \in \mathbb{R}_{>0}$ is the contribution of a cooperating non-differentiator $h \in \mathcal{F}$.

In the simulations the players update their strategies according to an *imitation* process in which each differentiator updates their strategy in a local game by imitating the strategy employed in the local game by the player with the highest payoff. The players who do not differentiate update their strategy according to the well established unconditional imitation update rule [6]. We now formally define the strategy update rule for differentiators used in the simulation.

Definition 7 (Differentiated unconditional imitation). *Given some current state $\hat{s}^t$, in the local groupwise game with players $\mathcal{N}_j$, a differentiator $i \in \mathcal{D}$ updates according to*

$s_{ij}^{t+1} \in \mathcal{I}_j^t$, *with*

$$\mathcal{I}_j^t := \left\{ s_{lj}^t \; : \; u_l(\hat{s}^t) = \max_{k \in \mathcal{N}_j} u_k(\hat{s}^t) \right\}.$$

When $|\mathcal{I}_j^t| > 1$ and $\bar{s}_{ij}^t \notin \mathcal{I}_j^t$, we assume a player uniformly selects a random strategy from $\mathcal{I}_j^t$. When $\bar{s}_{ij}^t \in \mathcal{I}_j^t$ we apply the convention that a player does not switch strategies. This imitation process is based on the widely accepted unconditional imitation strategy update rule [18]. For these imitation based dynamics the effect of differentiators located on high degree nodes in the network is remarkable. For a *neutral* benefit to cost ratio ($r = 1$), increasing the number of differentiators tends to increase the level of cooperation in an equilibrium strategy profile, see figure 2. When there are only 4 differentiators located at high degree nodes almost half of the players cooperate in the equilibrium strategy profile. This behavior was consistent for different activation sequences. Such levels of cooperation cannot be seen without strategic differentiation. Next to network reciprocity, the concept of *false strategy attribution* seems to be crucial for large scale cooperation in the equilibrium of a social dilemma with imitation based strategy update dynamics and strategic differentiation. In these games, the players who differentiate can obtain high payoffs when they defect against some of their cooperating opponents. Other players in the network observe these high payoffs and imitate the strategy that the differentiator employs in *their* local game. False strategy attribution occurs when that local strategy happens to be cooperation: a defecting neighbor of the differentiator is then likely to switch to cooperation, even though the differentiator obtained the payoffs mostly by defecting. This suggests that when the number of differentiators increases, not the payoff parameters, but the initial strategy profile and the network structure become determinative for the frequency of the strategies in equilibrium. Indeed, whe observed that when there are many differentiators in the network the influence of the benefit to cost ratio r on the total contributions in equilibrium is suppressed. This effect is similar to *topological enslavement* [11] seen in evolutionary games on multiplex networks in which hubs dominate the game dynamics. When the differentiators are placed on low degree nodes, these effects are mitigated.

VII. Conclusions

We have shown how the framework of evolutionary games on networks can be extended to include a subset of players that are able to employ different strategies against different opponents. If the local games in the network admit a weighted potential function, then convergence of the strategically differentiated version with myopic best response dynamics is guaranteed. For imitation strategy update dynamics the topology of the network, the existence and location of differentiators in the networks can crucially alter the strategy profile at an equilibrium of groupwise games. When differentiators are plentiful the equilibrium strategy profile becomes less sensitive to changes in the values of the payoff parameters. The effect of strategic differentiation on different
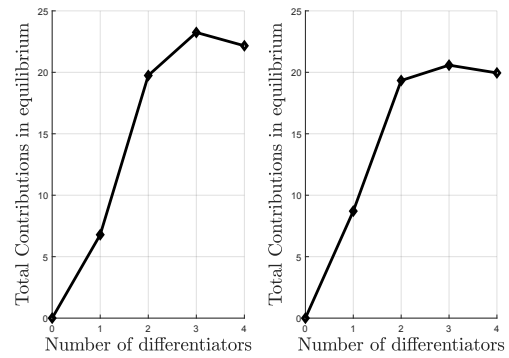

Fig. 2: *Each plot is obtained by averaging 10 trajectories of differentiated unconditional imitation. The horizontal axis shows the number of differentiators. The vertical axis is the total number of contributions in the equilibrium strategy profile. The two figures correspond to simulations on different networks generated by a preferential attachment process. In both cases, $n = 50$ and $r = 1$.*

types of myopic strategy update rules and the extension to continuous games remain interesting open problems.

References

[1] J. M. Smith and G. R. Price, "The logic of animal conflict," *Nature*, vol. 246, no. 5427, p. 15, 1973.

[2] H. A. Simon, *Models of bounded rationality: Empirically grounded economic reason.* MIT press, 1997, vol. 3.

[3] D. Monderer and L. S. Shapley, "Potential games," *Games and economic behavior*, vol. 14, no. 1, pp. 124–143, 1996.

[4] H. P. Young, *Individual strategy and social structure: An evolutionary theory of institutions.* Princeton University Press, 2001.

[5] J. R. Marden and J. S. Shamma, "Revisiting log-linear learning: Asynchrony, completeness and payoff-based implementation," *Games and Economic Behavior*, vol. 75, no. 2, pp. 788–808, 2012.

[6] J. Riehl, P. Ramazi, and M. Cao, "A survey on the analysis and control of evolutionary matrix games," *Annual Reviews in Control*, 2018.

[7] B. Allen and M. A. Nowak, "Games on graphs," *EMS surveys in mathematical sciences*, vol. 1, no. 1, pp. 113–151, 2014.

[8] M. A. Nowak, "Five rules for the evolution of cooperation," *science*, vol. 314, no. 5805, pp. 1560–1563, 2006.

[9] F. C. Santos, M. D. Santos, and J. M. Pacheco, "Social diversity promotes the emergence of cooperation in public goods games," *Nature*, vol. 454, no. 7201, p. 213, 2008.

[10] Z. Wang, A. Szolnoki, and M. Perc, "Self-organization towards optimally interdependent networks by means of coevolution," *New Journal of Physics*, vol. 16, no. 3, p. 033041, 2014.

[11] K.-K. Kleineberg and D. Helbing, "Topological enslavement in evolutionary games on correlated multiplex networks," *New Journal of Physics*, vol. 20, no. 5, p. 053030, 2018.

[12] M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, and Y. Moreno, "Evolutionary dynamics of group interactions on structured populations: a review," *Journal of the royal society interface*, vol. 10, no. 80, p. 20120997, 2013.

[13] M. Bergen and M. A. Peteraf, "Competitor identification and competitor analysis: a broad-based managerial approach," *Managerial and decision economics*, vol. 23, no. 4-5, pp. 157–169, 2002.

[14] E. J. Zajac and M. H. Bazerman, "Blind spots in industry and competitor analysis: Implications of interfirm (mis) perceptions for strategic decisions," *Academy of Management Review*, vol. 16, no. 1, pp. 37–56, 1991.

[15] J. W. Weibull, *Evolutionary game theory.* MIT press, 1997.

[16] G. Facchini, F. Van Megen, P. Borm, and S. Tijs, "Congestion models and weighted bayesian potential games," *Theory and Decision*, vol. 42, no. 2, pp. 193–206, 1997.

[17] M. van Veelen and M. A. Nowak, "Multi-player games on the cycle," *Journal of Theoretical Biology*, vol. 292, pp. 116–128, 2012.

[18] M. A. Nowak and R. M. May, "Evolutionary games and spatial chaos," *Nature*, vol. 359, no. 6398, p. 826, 1992.