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On Persistency of Excitation and Formulas for Data-driven Control

C. De Persis and P. Tesi

Abstract—In a paper by Willems and coworkers it was shown that persistently exciting data could be used to represent the input-output trajectory of a linear system. Inspired by this fundamental result, we derive a parametrization of linear feedback systems that paves the way to solve important control problems using data-dependent Linear Matrix Inequalities only. The result is remarkable in that no explicit system's matrices identification is required. The examples of control problems we solve include the state feedback stabilization and the linear quadratic regulation problems. We also extend the stabilization problem to the case of output feedback control design.

I. INTRODUCTION

Learning from data is essential to every area of science. It is the core of statistics and artificial intelligence, and is becoming ever more prevalent also in the engineering domain. Control engineering is one of the fields where learning from data is now considered as a prime issue.

Learning from data is actually not novel in control theory. System identification [1] is one of the major developments of this paradigm, where modeling based on first principles is replaced by data-driven learning algorithms. Prediction error, maximum likelihood as well as subspace methods [2] are all data-driven techniques which can be now regarded as standard for what concerns modeling. The learning-from-data paradigm has been widely pursued also for control design purposes. A main question is how to design control systems directly from process data with no intermediate system identification step. Besides their theoretical value, answers to this question could have a major practical impact especially in those situations where identifying a process model can be difficult and time consuming, for instance when data are affected by noise. Despite many developments in this area, data-driven control is not yet well understood even if we restrict the attention to linear dynamics, which contrasts the achievements obtained in system identification. A major challenge is how to incorporate data-dependent stability and performance requirements in the control design procedure.

Contributions to data-driven control can be traced back to the pioneering work by Ziegler and Nichols [3], direct adaptive control [4] and neural networks [5] theories. Since then, many remarkable techniques have been developed. We mention unfalsified control [6], iterative feedback tuning [7], and virtual reference feedback tuning [8]. This topic is now attracting more and more researchers, and the considered

problems range from PID-like control design [9] to model reference [10], [11], [12], predictive [13], [14], robust and optimal control design [15], [16], [17], [18], [19], [20], the latter being one of the most frequently considered problems. The corresponding techniques are also quite varied, ranging from dynamics programming to optimization techniques and algebraic methods. We refer the interested reader to the survey [21] for more references on data-driven control methods.

Paper contribution. Despite the differences, all the aforementioned contributions are centred around the question of how one can replace process models with process data. For linear systems, there is actually a fundamental result which answers this question, proposed by Willems *et al.* [22]. Roughly, this result stipulates that the whole set of trajectories that a linear system can generate can be represented by a finite set of system trajectories provided that such trajectories come from sufficiently excited dynamics. While this result has been (more or less explicitly) used in the context of data-driven control [14], [15], [23], [24], [25], we feel that certain implications of the so-called Willems *et al.*'s fundamental lemma have not yet been fully captured.

In this paper, we show that Willems *et al.*'s fundamental lemma, can be used to obtain a data-based representation of the closed-loop system transition matrix, where the controller is itself parametrized through the sample trajectories (Theorem 1). Theorem 1 turns out to have surprisingly straightforward, yet profound, implications for control design. We discuss this fact in Section IV. The main point is that the parametrization provided in Theorem 1 can be naturally related to the classic Lyapunov stability inequalities. This makes it possible to cast the problem of designing a stabilizing controller in terms of a simple Linear Matrix Inequality (LMI) [26] (Theorem 2). In Theorem 3, the very same arguments are used to solve a Linear Quadratic Regulation problem through a convex optimization program. The main derivations are given for state feedback control. The case of output feedback control (Theorem 4) is discussed in Section V. Discussion and some research directions are given in Section VI. Throughout the paper we report several numerical examples to substantiate the analysis.

The proofs as well as further results such as the data-driven stabilization of *continuous-time systems*, the design of stabilizing controllers when *data are corrupted by noise*, the stabilization of unstable equilibria of *nonlinear systems* and the output feedback stabilization problem for MIMO systems, are omitted due to space limitations and can be found in [27].

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Notation. Given a signal $z : \mathbb{Z} \rightarrow \mathbb{R}^\sigma$, we denote by $z_{[k,k+T]}$, where $k \in \mathbb{Z}$, $T \in \mathbb{N}$, the restriction in vectorized form of the signal z to the interval $[k, k+T] \cap \mathbb{Z}$, namely

$$z_{[k,k+T]} = \begin{bmatrix} z(k) \\ \vdots \\ z(k+T) \end{bmatrix}.$$

When the signal is not restricted to an interval then it is simply denoted by its symbol, say z . To avoid notational burden, we use $z_{[k,k+T]}$ to denote also the sequence $\{z(k), \dots, z(k+T)\}$. For the same reason, we simply write $[k, k+T]$ to denote the discrete interval $[k, k+T] \cap \mathbb{Z}$.

We denote the Hankel matrix associated to z as

$$Z_{i,t,N} = \begin{bmatrix} z(i) & z(i+1) & \dots & z(i+N-1) \\ z(i+1) & z(i+2) & \dots & z(i+N) \\ \vdots & \vdots & \ddots & \vdots \\ z(i+t-1) & z(i+t) & \dots & z(i+t+N-2) \end{bmatrix}$$

where $i \in \mathbb{Z}$ and $t, N \in \mathbb{N}$. The first subscript denotes the time at which the first sample of the signal is taken, the second one the number of samples per each column, and the last one the number of signal samples per each row. Sometimes – but not always – if $t = 1$, noting that $Z_{i,t,N}$ has only one block row, then for compactness we simply write $Z_{i,N} = [z(i) \ z(i+1) \ \dots \ z(i+N-1)]$.

II. PRELIMINARIES AND THE WILLEMS *et al.*'S FUNDAMENTAL LEMMA

In this section, we revisit the main result in [22] and state a few auxiliary results inspired by subspace identification [2], which will be useful throughout the paper.

Consider $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ and a controllable and observable discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \quad (1a)$$

$$y(k) = Cx(k) + Du(k). \quad (1b)$$

Let $(u_{d,[0,T-1]}, y_{d,[0,T-1]})$ be the input-output data of the system collected during an experiment, and let

$$\begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} \quad (2)$$

be the corresponding Hankel matrix. We also introduce the matrix $X_{0,T-t+1} = [x_d(0) \ x_d(1) \ \dots \ x_d(T-t)]$ where $x_d(i)$ are the state samples produced by the system (1a) under the input $u_{d,[0,T-1]}$. For u_d , y_d , and x_d , we use the subscript d so as to emphasize that these are the sample data collected from the system during some experiment.

Throughout the paper, having the rank condition

$$\text{rank} \begin{bmatrix} U_{0,t,T-t+1} \\ X_{0,T-t+1} \end{bmatrix} = n + tm \quad (3)$$

satisfied plays an important role. Even though it is in general difficult to check this condition when only input-output data are accessible, it is possible to have it satisfied when the input is persistently exciting of sufficient order. We first recall the notion of persistency of excitation.

Definition 1: [22, p. 327] The signal $z_{[0,T-1]} \in \mathbb{R}^\sigma$ is persistently exciting of order L if the matrix

$$Z_{0,L,T-L+1} = \begin{bmatrix} z(0) & z(1) & \dots & z(T-L) \\ z(1) & z(2) & \dots & z(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ z(L-1) & z(L) & \dots & z(T-1) \end{bmatrix}$$

has full rank σL . ■

For a signal z to be persistently exciting of order L , it must be sufficiently long, namely $T \geq (m+1)L - 1$. We can now recall two results by Willems *et al.* [22] which are key for the developments of the paper.

Lemma 1: [22, Corollary 2] Consider system (1a). If the input $u_{d,[0,T-1]}$ is persistently exciting of order $n+t$, then condition (3) holds. ■

Lemma 2: [22, Theorem 1] Consider system (1). Then the following holds:

- (i) If $u_{d,[0,T-1]}$ is persistently exciting of order $n+t$, then any t -long input/output trajectory of system (1) can be expressed as

$$\begin{bmatrix} u_{[0,t-1]} \\ y_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g$$

where $g \in \mathbb{R}^{T-t+1}$.

- (ii) Given a T -long input/output trajectory of system (1), any linear combination of the columns of the matrix in (2), that is

$$\begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g,$$

is a t -long input/output trajectory of (1). ■

Lemma 1 shows that, for a fixed t , if T is taken sufficiently large, then the rank condition (3) turns out to be satisfied. In turn, condition (3) makes it possible to represent any input/output trajectory of the system as a linear combination of previously collected input/output data. Lemma 2 has been originally proven in [22, Theorem 1] using the behavioral language, and it was later referred to as the *fundamental lemma* [28] to describe a linear system through a finite collection of its input/output data.

III. DATA-BASED SYSTEM REPRESENTATION

Theorem 1 below shows how one can parametrize feedback interconnections just by using data. This result will be key later on for deriving control design methods that skip any system identification step.

Consider a persistently exciting input sequence $u_{d,[0,T-1]}$ of order $t+n$ with $t=1$, that is such that $U_{0,1,T}$ is full-row rank. Notice that the only requirement on T is that $T \geq (n+1)m + n = (m+1)n + m$, which is necessary for the persistence of excitation condition to hold. By Lemma 1,

$$\text{rank} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} = n + m \quad (4)$$

From now on, we will directly refer to condition (4), bearing in mind that this condition requires persistently exciting

inputs of order $n + 1$. In this respect, we point out that condition (4) can always be tested if the state of the system is accessible. This is the standing assumption we will make in Section IV when considering state-feedback control design problems. Later in Section V we will provide a condition alternative to (4) for the case where only input/output data are accessible.

A. Data-based closed-loop representation

We now exploit Lemma 2 to derive a parametrization of system (1a) in closed-loop with a feedback law $u = Kx$.

Theorem 1: Let condition (4) be satisfied. Then system (1a) in closed-loop with a state feedback $u = Kx$ has the following equivalent representation $x(k+1) = X_{1,T}G_Kx(k)$ where G_K is a $T \times n$ matrix satisfying

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} G_K. \quad (5)$$

In particular $u(k) = U_{0,1,T}G_Kx(k)$. ■

B. From indirect to direct data-driven control

A crucial observation that emerges from Theorem 1 is that also the controller K can be parametrized through data via (5). Thus for design purposes one can regard G_K as a *decision variable*, and search for the matrix G_K that guarantees prescribed stability and performance specifications. In fact, as long as G_K satisfies the condition $X_{0,T}G_K = I_n$ in (5) we are ensured that $X_{1,T}G_K$ provides an equivalent representation of the closed-loop matrix $A + BK$ with feedback matrix $K = U_{0,1,T}G_K$. As shown in the next section, this corresponds to a design procedure that avoids any identification step.

Before proceeding, we note that Theorem 1 already gives an identification-free method for *checking* whether a candidate controller K is stabilizing or not. In fact, given K , any solution G_K to (5) is such that $X_{1,T}G_K = A + BK$. Hence, one can compute the eigenvalues of $X_{1,T}G_K$ to check whether K is stabilizing or not. Notice that this method does not require to place K into feedback, in the same spirit of unfalsified control theory [6].

IV. EXAMPLES OF DIRECT DATA-DRIVEN DESIGN

In this section, we discuss how Theorem 1 can be used to get identification-free design algorithms. Although the problems considered hereafter are all of practical relevance, we would like to regard them as application *examples* of Theorem 1. In fact, we are confident that Theorem 1 can be used to approach other, more complex, design problems such as H_∞ control and quadratic stabilization [26].

A. Stabilizing state feedback

By Theorem 1, the closed-loop system under state-feedback $u = Kx$ is such that $A + BK = X_{1,T}G_K$ where G_K satisfies (5). One can therefore search for a matrix G_K such that $X_{1,T}G_K$ satisfies the classic Lyapunov stability condition. As the next result shows, it turns out that this

problem can be actually cast in terms of a simple Linear Matrix Inequality (LMI).

Theorem 2: Let condition (4) be satisfied. Then, any matrix Q satisfying

$$\begin{bmatrix} X_{0,T}Q & X_{1,T}Q \\ Q^\top X_{1,T}^\top & X_{0,T}Q \end{bmatrix} \succ 0 \quad (6)$$

is such that

$$K = U_{0,1,T}Q(X_{0,T}Q)^{-1} \quad (7)$$

is a stabilizing feedback gain for system (1a). Conversely, if K is a stabilizing state-feedback gain for system (1a) then it can be written as in (7), with Q solution of (6). ■

Note that in the formulation (6) the parametrization of the closed-loop matrix $A + BK$ is given by $X_{1,T}Q(X_{0,T}Q)^{-1}$, that is $G_K = Q(X_{0,T}Q)^{-1}$ which satisfies $X_{0,T}G_K = I_n$ corresponding to the second identity in (5). On the other hand, the constraint corresponding to the first identity in (5) is guaranteed by the choice $K = U_{0,1,T}Q(X_{0,T}Q)^{-1}$. This is the reason why (6) is representative of closed-loop stability even if no constraint like (5) appears in the formulation (6). We also stress that this result characterizes the whole set of stabilizing state-feedback gains in the sense that any stabilizing feedback gain K can be expressed as in (7) for some matrix Q satisfying (6).

Illustrative example. As an illustrative example, consider the discretized version of a batch reactor [29] using a sampling time of 0.1s (up to the third digit),

$$[A|B] = \begin{bmatrix} 1.178 & 0.001 & 0.511 & -0.403 & | & 0.004 & -0.087 \\ -0.051 & 0.661 & -0.011 & 0.061 & | & 0.467 & 0.001 \\ 0.076 & 0.335 & 0.560 & 0.382 & | & 0.213 & -0.235 \\ 0 & 0.335 & 0.089 & 0.849 & | & 0.213 & -0.016 \end{bmatrix}.$$

The system to be controlled is open-loop unstable. The control design procedure is implemented in MATLAB. We generate the data with random initial conditions and by applying to each input channel a random input sequence of length $T = 15$ by using the MATLAB command `rand`. To solve (6) we used CVX [30], obtaining

$$K = \begin{bmatrix} 0.7610 & -1.1363 & 1.6945 & -1.8123 \\ 3.5351 & 0.4827 & 3.3014 & -2.6215 \end{bmatrix},$$

which stabilizes the closed-loop dynamics in agreement with Theorem 1. ■

B. Linear quadratic regulation

Matrix (in)equalities similar to the one in (6) are recurrent in control design, with the major difference that in (6) only information collected from data appears, rather than the system matrices. Yet, these matrix inequalities can inspire the data-driven solution of other control problems. Important examples are quadratic regulation problems. Consider the discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q_x^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \end{aligned} \quad (8)$$

where ξ is an external input to the system, and where z is a performance signal of interest; $Q_x \succeq 0$, $R \succ 0$ are weighting matrices with (Q_x, A) observable. The objective is to design a state-feedback law $u = Kx$ which renders $A + BK$ stable and minimizes the H_2 norm of the transfer function $g : \xi \rightarrow z$ [31, Section 4],

$$\|g\|_2 := \left[\frac{1}{2\pi} \int_0^{2\pi} \text{trace} \left(g(e^{j\theta})^\top g(e^{j\theta}) \right) d\theta \right]^{\frac{1}{2}}. \quad (9)$$

This problem can be equivalently formulated as a convex program [32], [33]. In fact, as a natural counterpart of the continuous-time formulation in [32], the optimal controller K can be found by solving

$$\begin{aligned} & \min_{K,W,X} \text{trace}(Q_x W) + \text{trace}(X) \\ & \text{s.t.} \begin{cases} (A + BK)W(A + BK)^\top - W + I_n \preceq 0 \\ W \succeq I_n \\ X - R^{1/2}KWK^\top R^{1/2} \succeq 0 \end{cases} \end{aligned} \quad (10)$$

This can be cast as a convex optimization problem by means of suitable change of variables [32]. Based on this formulation, it is straightforward to derive a data-dependent formulation of this optimization problem.

Theorem 3: Let condition (4) be satisfied. The optimal H_2 state-feedback controller K for system (8) can be computed as $K = U_{0,1,T}Q(X_{0,T}Q)^{-1}$ where Q optimizes

$$\begin{aligned} & \min_{Q,X} \text{trace}(Q_x X_{0,T}Q) + \text{trace}(X) \\ & \text{s.t.} \begin{cases} \begin{bmatrix} X & R^{1/2}U_{0,1,T}Q \\ Q^\top U_{0,1,T}^\top R^{1/2} & X_{0,T}Q \end{bmatrix} \succeq 0 \\ \begin{bmatrix} X_{0,T}Q - I_n & X_{1,T}Q \\ Q^\top X_{1,T}^\top & X_{0,T}Q \end{bmatrix} \succeq 0 \end{cases} \end{aligned} \quad (11)$$

Illustrative example. We consider the batch reactor system of the previous subsection. As before, we generate the data with random initial conditions and by applying to each input channel a random input sequence of length $T = 15$ by using the MATLAB command `rand`. We let $Q_x = I_n$ and $R = I_m$. To solve (11) we used CVX, obtaining

$$K = \begin{bmatrix} 0.0639 & -0.7069 & -0.1572 & -0.6710 \\ 2.1481 & 0.0875 & 1.4899 & -0.9805 \end{bmatrix}$$

This controller coincides with the controller \bar{K} obtained with the MATLAB command `dare` which solves the classic DARE equation. In particular, $\|K - \bar{K}\| \approx 10^{-7}$. ■

V. INPUT-OUTPUT DATA: THE CASE OF SISO SYSTEMS

In Section IV-A, the measured data are the inputs and the full state, and the starting point is to express the trajectories of the system and the control gain in terms of the Hankel matrix of input-state data. This section considers the case where only input-output data of the system are accessible. The main derivations are given for single-input single-output (SISO) systems.

Consider a SISO systems as in (1) in left difference operator representation [34, Section 2.3.3],

$$\begin{aligned} y(k) + a_n y(k-1) + \dots + a_2 y(k-n+1) + a_1 y(k-n) \\ = b_n u(k-1) + \dots + b_2 u(k-n+1) + b_1 u(k-n) \end{aligned} \quad (12)$$

This representation allows us to reduce the output measurement case to the state measurement case with minor effort. Define the state vector

$$\begin{aligned} \chi(k) := \text{col}(y(k-n), y(k-n+1), \dots, y(k-1), \\ u(k-n), u(k-n+1), \dots, u(k-1)), \end{aligned} \quad (13)$$

from (12) we obtain the state space system (14) on the next page. Note that we turned our attention to a system of order $2n$, which is not minimal.

Consider now the matrix in (4) written for the system $\chi(k+1) = \mathcal{A}\chi(k) + \mathcal{B}u(k)$ in (14), with $T \geq 2n+1$. If this matrix is full-row rank, then the analysis in the previous sections can be repeated also for system (14). We observe that written for system (14), the matrix in question is

$$\begin{bmatrix} U_{0,1,T} \\ \hat{X}_{0,T} \end{bmatrix} = \begin{bmatrix} u_d(0) & u_d(1) & \dots & u_d(T-1) \\ \chi_d(0) & \chi_d(1) & \dots & \chi_d(T-1) \end{bmatrix}, \quad (15)$$

where $\chi_d(i+1) = \mathcal{A}\chi_d(i) + \mathcal{B}u_d(i)$ for $i \geq 0$ and where $\chi_d(0)$ is the initial condition in the experiment,

$$\begin{aligned} \chi_d(0) = \text{col}(y_d(-n), y_d(-n+1), \dots, y_d(-1), \\ u_d(-n), u_d(-n+1), \dots, u_d(-1)). \end{aligned}$$

We have the following result.

Lemma 3: The identity

$$\begin{bmatrix} U_{0,1,T} \\ \hat{X}_{0,T} \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ Y_{-n,n,T} \\ U_{-n,n,T} \end{bmatrix} \quad (16)$$

holds. Moreover, if $u_{d,[0,T-1]}$ is persistently exciting of order $2n+1$ then

$$\text{rank} \begin{bmatrix} U_{0,1,T} \\ \hat{X}_{0,T} \end{bmatrix} = 2n+1. \quad (17)$$

A. Direct data-driven design of output feedback controllers

Consider the left difference operator representation (12), its realization (14) and the input/state pair (u, χ) . We introduce a controller of the form

$$\begin{aligned} y^c(k) + c_n y^c(k-1) + \dots + c_1 y^c(k-n) \\ = d_n u^c(k-1) + \dots + d_1 u^c(k-n) \end{aligned} \quad (18)$$

Its state space representation, with state χ^c defined similar to (13), input u^c and output y^c , can be given analogously to (14). We omit it due to lack of space. In the closed-loop system, we enforce the following interconnection conditions relating the process and the controller $u^c(k) = y(k)$, $y^c(k) = u(k)$, $k \geq 0$. Note in particular the identity, for $k \geq n$,

$$\chi(k) = \begin{bmatrix} y_{[k-n,k-1]} \\ u_{[k-n,k-1]} \end{bmatrix} = \begin{bmatrix} u_{[k-n,k-1]}^c \\ y_{[k-n,k-1]}^c \end{bmatrix} = \begin{bmatrix} \mathbb{0}_{n \times n} & I_n \\ I_n & \mathbb{0}_{n \times n} \end{bmatrix} \chi^c(k).$$

$$\chi(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ -a_1 & -a_2 & -a_3 & \dots & -a_n & b_1 & b_2 & b_3 & \dots & b_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_A \chi(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_B u(k) \quad (14)$$

$$y(k) = \underbrace{[-a_1 \ -a_2 \ -a_3 \ \dots \ -a_n \ b_1 \ b_2 \ b_3 \ \dots \ b_n]}_C \chi(k)$$

Hence, for $k \geq n$, there is no loss of generality in considering the system

$$\chi(k+1) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ -a_1 & -a_2 & -a_3 & \dots & -a_n & b_1 & b_2 & b_3 & \dots & b_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\ d_1 & d_2 & d_3 & \dots & d_n & -c_1 & -c_2 & -c_3 & \dots & -c_n \end{bmatrix} \chi(k). \quad (19)$$

as the closed-loop system. In the following statement we say that (18) stabilizes system (12), meaning that the closed-loop system (19) is asymptotically stable.

Theorem 4: Let condition (17) be satisfied. Then:

- (i) The closed-loop system (19) has the equivalent representation $\chi(k+1) = \hat{X}_{1,T} G_{\mathcal{K}} \chi(k)$, where $G_{\mathcal{K}}$ is a $T \times 2n$ matrix such that

$$\begin{bmatrix} \mathcal{K} \\ I_{2n} \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ \hat{X}_{0,T} \end{bmatrix} G_{\mathcal{K}}, \quad (20)$$

and

$$\mathcal{K} := [d_1 \ \dots \ d_n \ -c_1 \ \dots \ -c_n] \quad (21)$$

is the vector of coefficients of the controller (18).

- (ii) Any matrix \mathcal{Q} satisfying

$$\begin{bmatrix} \hat{X}_{0,T} \mathcal{Q} & \hat{X}_{1,T} \mathcal{Q} \\ \mathcal{Q}^\top \hat{X}_{1,T}^\top & \hat{X}_{0,T} \mathcal{Q} \end{bmatrix} \succ 0, \quad (22)$$

is such that controller (18) with coefficients given by

$$\mathcal{K} = U_{0,1,T} \mathcal{Q} (\hat{X}_{0,T} \mathcal{Q})^{-1} \quad (23)$$

stabilizes system (12). Conversely, any controller (18) that stabilizes system (12) must have coefficients \mathcal{K} given by (23), with \mathcal{Q} a solution of (22). ■

As for the case of full-state measurements, the result above unveils the identity $\mathcal{A} + \mathcal{B}\mathcal{K} = \hat{X}_{1,T} G_{\mathcal{K}}$, which gives a method to check whether a controller (18) with coefficients

in (20) is stabilizing for system (12). As a final point, we note that given a solution \mathcal{K} as in (23) the resulting entries ordered as in (21) lead to the following state-space realization of order n for the controller

$$\xi(k+1) = \begin{bmatrix} -c_n & 1 & 0 & \dots & 0 \\ -c_{n-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -c_1 & 0 & 0 & \dots & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_2 \\ d_1 \end{bmatrix} y(k) \quad (24)$$

$$u(k) = [1 \ 0 \ 0 \ \dots \ 0 \ 0] \xi(k).$$

Illustrative example. Consider a system [35] made up by two carts. The two carts are mechanically coupled by a spring with unknown stiffness $\gamma \in [0.25, 1.5]$. The aim is to control the position of one cart by applying a force to the other cart. The system state-space description is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -\gamma & 0 & \gamma & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \gamma & 0 & -\gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array} \right]. \quad (25)$$

Assume that $\gamma = 1$ (unknown). The system is controllable and observable. All the open-loop eigenvalues are on the imaginary axis. The input-output discretized version using a sampling time of 1s is as in (12) with coefficients (up to the fourth digit)

$$\begin{aligned} [a_1 \ a_2 \ a_3 \ a_4] &= [1 \ -2.311 \ 2.623 \ -2.311] \\ [b_1 \ b_2 \ b_3 \ b_4] &= [0.039 \ 0.383 \ 0.383 \ 0.039]. \end{aligned}$$

We design a controller following Theorem 4. We generate the data with random initial conditions and by applying a random input sequence of length $T = 9$. To solve (22) we used CVX, obtaining from (23)

$$\mathcal{K} = \begin{bmatrix} 1.1837 & -1.5214 & 1.3408 & -1.4770 \\ 0.0005 & -0.5035 & -0.9589 & -0.9620 \end{bmatrix},$$

which stabilizes the closed-loop dynamics in agreement with Theorem 4. In particular, a minimal state-space representation (A_c, B_c, D_c, D_c) of this controller is given by (see (24))

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \left[\begin{array}{cccc|c} -0.9620 & 1 & 0 & 0 & -1.4770 \\ -0.9589 & 0 & 1 & 0 & 1.3408 \\ -0.5035 & 0 & 0 & 1 & -1.5214 \\ 0.0005 & 0 & 0 & 0 & 1.1837 \\ \hline & 1 & 0 & 0 & 0 \end{array} \right]. \quad \blacksquare$$

VI. CONCLUSIONS

We have shown the existence of a parametrization of feedback control systems that allows us to reduce the stabilization problem to an equivalent data-dependent linear matrix inequality. Since LMIs are ubiquitous in systems and control we expect that our approach will lead to data-driven solutions to many other control problems, such as H_∞ control and quadratic stabilization [26]. As first examples, we have considered the LQR problem and the case of output feedback controllers. The important extension to the case when data are corrupted by noise is tackled in [27].

A great leap forward will come from extending the methods of this paper to systems where identification is hard, such as nonlinear systems. Our results in [27, Section V.B] show that this approach is concretely promising for nonlinear systems, but we have only scratched the surface of this research area. Recent results have reignited the interest of the community on system identification for nonlinear systems, interestingly pointing out the importance of the concept of persistently exciting signals [36]. We are confident that our approach will also play a fundamental role in developing a systematic methodology for the data-driven design of control laws for nonlinear systems.

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