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Existence of Bifurcating Quasipatterns in Steady Bénard–Rayleigh Convection

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Abstract

Extending the results obtained in the sixties for bifurcating periodic patterns, the existence of bifurcating quasipatterns in the steady Bénard–Rayleigh convection problem is proved. These are two-dimensional patterns, quasiperiodic in any horizontal direction, invariant under horizontal rotations of angle π/q. There is a small divisor problem for $q \geqq 4$.

Using the results of Berti–Bolle–Procesi in 2010, we adapt it to a Navier–Stokes system ruling the Bénard–Rayleigh convection problem. Our solution is approximated by the truncated power series which was formally obtained by Iooss in 2009, but which is divergent in general (Gevrey series). First, we formulate the problem in introducing a suitable parameter, able to move the spectrum of the linearized operator, as a whole, as for the Swift–Hohenberg PDE model. For using the Nash–Moser process, we are faced with the problem of inverting a linear operator which is the differential at a non zero point.

There are two new difficulties: (i) First, the extra dimension leading to a more complicated spectrum of the linear operator. This first difficulty leads to use specific projections for reducing the spectrum of the studied operator, which we want to invert, to a finite set very close to 0. (ii) The second difficulty is the fact that the linearization $L^{(N)}$ at a non-zero point leads to a non-selfadjoint operator, contrary to what occurs in previous works. This is more serious, and leads to use the spectrum of $L^{(N)}L^{(N)*}$ which depends mainly quadratically on the main parameter. A careful study of the "bad set" of parameters, with an assumption on the convexity of the eigenvalues of this operator, allows us to obtain a good estimate, as it is necessary for using the results of Berti et al. for solving "the range equation". We again use separation properties of the Fourier spectrum (see the Bourgain and Craig results) for obtaining an estimate in high Sobolev norms. It then remains to solve the one-dimensional "bifurcation equation.

For any $q \geqq 4$, and provided that a weak transversality conjecture is realized, we prove the existence of a bifurcating convective quasipattern of order $2q$, above the critical Rayleigh number.

1. Introduction

The Bénard–Rayleigh convection system is one of the most popular in hydrodynamic stability theory, and it was the subject of numerous papers and books, mostly in physics literature. The mathematical existence of steady convective patterns, as rolls or hexagonal cells, was first proved by YUDOVICH et al. in a series of papers in the sixties [21,25–27]. For other mathematical results on this problem, see RABINOWITZ [17], GÖRTLER et al. [9], KIRCHGÄSSNER et al. [14].

Here, about 50 years later, we are studying the same problem, but looking for a different type of steady convective pattern. Quasipatterns are two-dimensional patterns which have no translation symmetries and are quasiperiodic in any spatial direction (see Fig. 1). The spatial Fourier transforms of quasipatterns have discrete rotational order (most often, 8, 10 or 12-fold) and were first discovered in nonlinear pattern-forming systems in the Faraday wave experiment [6,8], in which a layer of fluid is subjected to vertical oscillation. Since their discovery, they have also been observed in shaken convection [18,23].

In many of these experiments, the domain is large compared to the size of the pattern, and the boundaries appear to have little effect. Furthermore, the pattern is usually formed in two directions (x_1 and x_2), while the third direction (z) plays little role. Mathematical models of the experiments are therefore often posed with two unbounded directions, and the basic symmetry of the problem is the Euclidean group of rotations, translations and reflections of the (x_1, x_2) plane. This is in particular the case for the studies made in the works [3,11,19,20].

Quasipatterns do not fit into any spatially periodic domain and have Fourier expansions with wavevectors that live on a *quasilattice* (defined below). At the onset

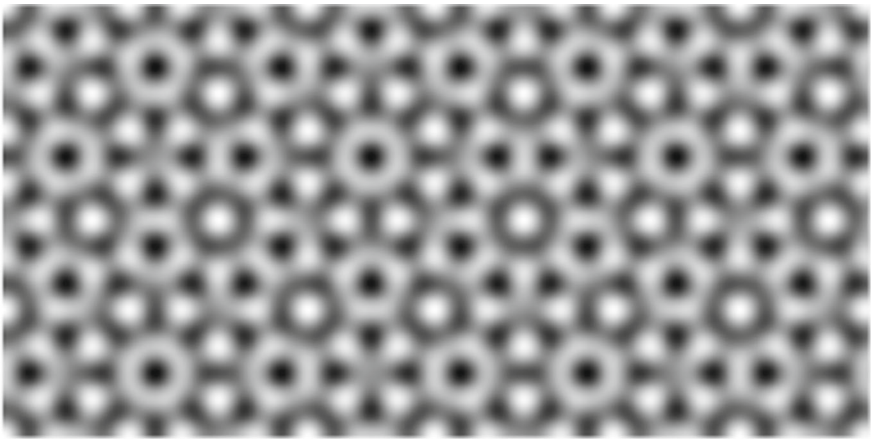

Fig. 1. Example eightfold quasipattern. This is an approximate solution of the Swift–Hohenberg equation, see [11].

of pattern formation, the primary modes have zero growth rate, and there are other modes on the quasilattice which have negative growth rates arbitrarily close to zero, and techniques (like Lyapunov–Schmidt reduction, or center manifold reduction) which are used for periodic patterns cannot be applied. These small growth rates appear as *small divisors*, as seen below.

This paper is in the spirit on the paper [3] dealing with the Swift–Hohenberg PDE. It is known that this PDE is a simple model of Bénard–Rayleigh convection for the bifurcation to a steady spatially periodic convective regime. In the present paper we solve the same problem but ruled by the full Boussinesq equations, which are usually taken for the study of Bénard–Rayleigh convection between two horizontal planes. This problem was studied in [10], where Gevrey estimates are given for the formal series solution of the problem. Summing this series by an incomplete Borel resummation, provides a solution of our problem *only up to an exponentially small term* (as the Rayleigh number tends towards its critical value).

In the present paper, we first define the functional setting in Sections 2, 3 and 4 for our unknown u. In Section 5 we formulate the problem in suitable form. In Section 6 we study in details the linearized operator, and the criticality conditions. This determines the critical value λ_0 of the bifurcation parameter λ, linked to the Rayleigh number by $\lambda = \mathcal{R}^{-1/2}$, and the critical wave number k_c. We then give the formal series for (u, λ) in powers of the amplitude ε of the bifurcating solution. We use the truncated series as the center of the neighborhood where one applies later the Nash–Moser process. Section 7 reformulates the problem for adapting it to the method used in [2,3] which exploits the fact that the parameter $\mu = \lambda_0 - \lambda$ appears in a way which moves the spectrum of the linearized operator, as a whole. This introduces finally parameters ε, μ', where μ' is a scaling of μ (see (63)). We are now faced with new difficulties: the problem is no longer in 2 dimensions, since we now have the vertical coordinate z introducing a dependency of Fourier coefficients in z. This leads to an infinite dimensional system, even when we truncate the Fourier modes at a finite number N (as in [3]). This needs the use of a new projection, complicating the operator to be inverted (see Section 7.4, Lemma 36 and Section 7.6).

The second new difficulty is that the linear operator which we have to invert in the Nash–Moser process is *no longer selfadjoint*. This serious complication is treated in Section 8. In particular, this requires the use of singular values of the truncated operator, instead of its eigenvalues as in [3]. The square of these singular values mainly behave quadratically in the parameter. We need an assumption on the convexity of these singular values for being able to bound suitably the "bad set" of parameters and obtain directly a good estimate for the inverse of the linearized operator in the basic space with small Sobolev norm (denoted $\mathcal{K}_{0,s_0}$). We then need to use separation properties of the eigenvalues $\lambda_0(|\mathbf{k}|^2)$ of the unperturbed operator, near λ_0, where the wave vectors $\mathbf{k}$ of the Fourier modes are restricted to $N_{\mathbf{k}} \leqq N$ ($N_{\mathbf{k}}$ is the $\mathbb{Z}^d$ norm in the quasilattice). This tool, introduced by Bourgain [4] and Craig [7], was already used on simpler systems in [1,3] and is necessary for obtaining good estimates in high Sobolev norms.

We show in Section 9 that we can adapt the method developed in [2] by Berti, Bolle, Procesi.

The existence of bifurcating convective quasipatterns is proved in Section 10. It results from the non empty intersection of a curve (H) defined by the bifurcation equation in the plane of parameters (ε, μ), and the complement of the "bad set" of parameters. This needs a transversality assumption depending on q.

We sum up our result in the following:

Theorem 1. *Let $q \geqq 4$ be an integer and let $d \leqq q$ be the dimension of the $\mathbb{Q}$-vector space spanned by the wave vectors $\mathbf{k}_j$, $j = 1, \ldots, 2q$ in $\mathbb{R}^2$ equally spaced on a circle centered at the origin (see the definition (4)). Assume that the neutral stability curve $\mathcal{R}(|\mathbf{k}|^2)$ leading to the critical Rayleigh number $\mathcal{R}_c = 1/\lambda_0^2$ for $|\mathbf{k}| = k_c$ has a unique minimum, such that $\mathcal{R}"(k_c^2) > 0$ (see Fig. 4 and Condition 32). We assume a convexity condition 47 and we assume that transversality Conjecture 58 is verified. Then, there exists $s_0 > d/2$, $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, there exists a 1-dimensional set $\overline{\Lambda}_\varepsilon$ centered on μ_4, with the following property: for any $\varepsilon < \varepsilon_0$, belonging to a set of asymptotically full measure as $\varepsilon \to 0$, there exists $\overline{\mu}_\varepsilon \in \overline{\Lambda}_\varepsilon$ such that the steady Bénard–Rayleigh system (35) admits a quasipattern solution $(u(\varepsilon), \lambda(\varepsilon))$, C^1 in the parameter ε, $u(\varepsilon) \in \mathcal{K}_{0,s_0}$ (see Definition 19), invariant under rotations of angle π/q, of the form*

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \mathcal{O}(\varepsilon^5),$$
$$\lambda = \lambda_0 - \mu_2\varepsilon^2 - \mu_3\varepsilon^3 - \varepsilon^4\overline{\mu}_\varepsilon$$

where $\mu_2 > 0$, $\overline{\mu}_\varepsilon = \mu_4 + O(\varepsilon)$. The quasiperiodic function u_1 spans the kernel of $\lambda_0 - \mathcal{A}$, and coefficients μ_j, u_j occurring in formulae above, are the ones defined in the truncated asymptotic expansion of the solution (see Section 6.3).

Remark 2. Condition 32 is "generic" and can be checked numerically, while Transversality Conjecture 58 depends on q. This one is hard to check but maybe weakened as indicated in Remarks 60 and 61. This is then probably valid for all q. Notice that for any $s_0' > s_0$, the result of the Theorem above is still valid, maybe for a smaller ε_0.

Remark 3. Hypothesis 47 is used for bounding the measure of the bad set of parameters. The quadratic dependence on $\tilde{\mu} = \mathcal{O}(\varepsilon^4)$ of the truncated selfadjoint linear operator, needs to control the convexity of its eigenvalues, while we have no means to provide a reasonable bound for their second derivative. This is an open question here.

Remark 4. The expression that we obtain for the bifurcating set, solution of (35), is under parametric form. The bifurcating set (u, λ) lies on a C^1 curve. At Fig. 2, we sketch the projection of this curve in the (ε, λ) plane.

2. The Bénard–Rayleigh Convection Problem

Consider a viscous fluid filling the region between two horizontal planes. Each planar boundary may be a rigid plane, or a "free" boundary. In addition, we assume that the lower and upper planes are at temperatures T_0 and T_1, respectively, with

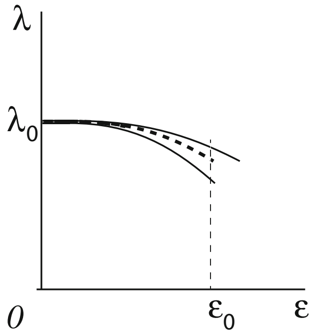

Fig. 2. Bifurcation curve. The set of "good" ε's is of asymptotically full measure.

$T_0 > T_1$. The difference of temperature between the two planes modifies the fluid density, tending to place the lighter fluid below the heavier one. The gravity then induces, through the Archimedian force, an instability of the "conduction regime" where the fluid is at rest, while the temperature depends linearly on the vertical coordinate z. This instability is prevented up to a certain level by viscosity ν, so that there is a critical value of the temperature difference below which nothing happens and above which a steady "convective regime" bifurcates.

The Navier–Stokes momentum equation needs to be completed with an equation for energy conservation. In the Boussinesq approximation, the dependency of the density ρ in function of the temperature T reads

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right),$$

where the (constant) volume expansion coefficient α, is taken into account in the momentum equation, only in the external volumic gravity force $-\rho g e_z$, introducing a coupling between the particles velocity and pressure (V, p), and T. We refer to [12, Vol. II] for a very complete discussion and bibliography on various geometries and boundary conditions in this problem.

Several different scalings are used in literature. We are only considering *steady solutions*, so we adopt here the formulation derived in [15] (after a scaling by $\mathcal{R}^{1/2}$ for V and by $\mathcal{R}$ for θ), which leads to the following system:

$$\begin{aligned} V \cdot \nabla V + \nabla p &= \mathcal{P}(\theta e_z + \mathcal{R}^{-1/2} \Delta V), \\ V \cdot \nabla \theta &= \mathcal{R}^{-1/2} \Delta \theta + V \cdot e_z, \\ \nabla \cdot V &= 0. \end{aligned} \tag{1}$$

Here $\mathcal{R}\theta$ is the deviation of the temperature from the conduction profile, which satisfies the boundary conditions, and $V = (V^{(H)}, v^{(z)})$, $V^{(H)} = (v_1, v_2)$, p, and θ are functions of $X = (\mathbf{x}, z)$, with $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ the horizontal coordinates and $z \in (0, 1)$ the vertical coordinate, e_z being the unitary ascendent vector. There are two dimensionless constant numbers in this problem: the Prandtl number $\mathcal{P}$ and the Rayleigh number $\mathcal{R}$ defined as

$$\mathcal{P} = \frac{\nu}{\kappa}, \quad \mathcal{R} = \frac{\alpha g d^3 (T_0 - T_1)}{\nu \kappa},$$

where d is the distance between the planes, κ is the thermal diffusivity. The system (1) is completed by the boundary conditions

$$v_z = \theta = 0, \quad z = 0, 1,$$

together with either a "rigid surface" condition

$$v_1 = v_2 = 0, \tag{2}$$

or a "free surface" condition (in fact no tangential stress condition)

$$\frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} = 0, \tag{3}$$

on the planes $z = 0$ or $z = 1$. Notice that we shall not consider here the case of free surface condition on both planes $z = 0$ and 1, since this case induces an additional (little) difficulty, which is exposed below.

Our next task is to formulate the problem ruled by the system (1) in a suitable function space, and find critical values of the parameters, for being able to use a method similar to the one in [3].

3. Quasilattices and Diophantine Bounds

Consider an integer $q \geqq 4$, where $2q$ is the *order of a quasipattern*, and define equally spaced wavevectors in $\mathbb{R}^2$

$$\mathbf{k}_j = k_c \left(\cos\left(\pi \frac{j-1}{q}\right), \sin\left(\pi \frac{j-1}{q}\right)\right) = R_{(j-1)\pi/q}\mathbf{k}_1, \quad j = 1, 2, \ldots, 2q, \tag{4}$$

where k_c is a positive number which is defined later, and R_θ is the rotation of angle θ around the vertical axis (see Fig. 3a). We define the *quasilattice* $\Gamma \subset \mathbb{R}^2$ to be the set of points spanned by integer combinations $\mathbf{k_m}$ of the form

$$\mathbf{k_m} = \sum_{j=1}^{2q} m_j \mathbf{k}_j, \quad \text{where} \quad \mathbf{m} = (m_1, m_2, \ldots, m_{2q}) \in \mathbb{N}^{2q}. \tag{5}$$

The set Γ is dense in $\mathbb{R}^2$. Since $\mathbf{k}_j$ and $-\mathbf{k}_j = \mathbf{k}_{j+q}$ belong to Γ, then $\mathbf{k_m}$ and $-\mathbf{k_m}$ are both in Γ. This, allows us to obtain real quantities of the form

$$\sum_{\mathbf{k}\in\Gamma} u_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2, \; u_\mathbf{k} \in \mathbb{C},$$

provided that

$$\overline{u_{-\mathbf{k}}} = u_\mathbf{k}.$$

We know (see [24]) that the $\mathbb{Q}-$ vector space spanned by $\{\mathbf{k}_j, \, j = 1, 2, \ldots, 2q\}$ has dimension $d = \varphi(2q) = 2(l_0 + 1)$ where φ is the Euler totient function, and $l_0 + 1$ is the order of the algebraic integer $\omega := 2\cos\pi/q$ ($l_0 = 1$ for $q = 4, 5, 6,$

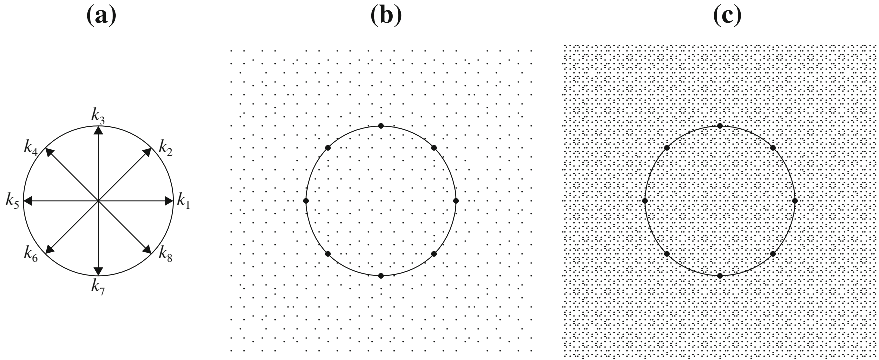

Fig. 3. Example of quasilattice with $2q = 8$, after [19]. **a** The 8 wavevectors with $|\mathbf{k}| = 1$ which form the basis of the quasilattice. **b**, **c** The truncated quasilattices Γ_9 and Γ_{27}. The small dots mark the positions of combinations of up to 9 or 27 of the 8 basis vectors on the unit circle.

$l_0 = 2$ for $q = 7\ldots$) with $2(l_0 + 1) \leqq q$. Let us define the subset of the d vectors $\{\mathbf{k}_j^*,\ j = 1, 2, \ldots, d\}$ of $\{\mathbf{k}_j,\ j = 1, 2, \ldots, 2q\}$ which forms a basis. Then

$$\mathbf{k}_j = \sum_{s=1}^{d} \alpha_{js}\mathbf{k}_s^*, \quad \alpha_{js} \in \mathbb{Q},$$

and any $\mathbf{k} \in \Gamma$ may be written in two different ways,

$$\mathbf{k} = \sum_{j=1}^{2q} m_j\mathbf{k}_j = \sum_{s=1}^{d} r_s\mathbf{k}_s^*, \quad m_j \in \mathbb{N},\ r_s \in \mathbb{Q},$$

where $r_s = \sum_{j=1}^{2q} m_j\alpha_{js}$.

Let us define $\alpha_{js} := \frac{n_{js}}{d_{js}}$ with irreducible fractions and

$$\mathfrak{d} = l.c.m_{\substack{j=1,\ldots 2q\\ s=1,\ldots d}}\{d_{js}\},\ \text{then } \mathfrak{d}\alpha_{js} = \beta_{js} \in \mathbb{Z}.$$

Remark 5. Notice that we have $\mathfrak{d} = 1$ for example for $q = 4, 5, 6, 7, 8, 9, 10, 11, 12$ where we can choose $\mathbf{k}_s^* = \mathbf{k}_s$, $s = 1, \ldots, d$ (see [3]).

Then $m_s^* := \mathfrak{d}r_s = \sum_{j=1}^{2q} m_j\beta_{js} \in \mathbb{Z}$ and

$$\mathbf{k} = \mathfrak{d}^{-1}\sum_{s=1}^{d} m_s^*\mathbf{k}_s^* =: \mathbf{k}(\mathbf{m}^*), \tag{6}$$

where $\mathbf{m}^* := (m_1^*, \ldots, m_d^*)$ and we define the following *norm in the lattice* Γ, identified with a subset of $\mathbb{Z}^d$:

$$\sum_{s=1}^{d} |m_s^*| =: N_{\mathbf{k}}.$$

Remark 6. If $\mathfrak{d} = 1$ we can identify Γ with $\mathbb{Z}^d$. If $\mathfrak{d} > 1$, for an arbitrary $\mathbf{m}^* \in \mathbb{Z}^d \backslash \{0\}$, we don't know a priori if there exists $\mathbf{k} \in \Gamma$ such that $\mathbf{k}(\mathbf{m}^*) = \mathbf{k}$.

Remark 7. Whenever solutions are computed numerically, it is necessary to use only a finite number of Fourier modes, so we define the *truncated quasilattice* Γ_N to be

$$\Gamma_N = \left\{ \mathbf{k} \in \Gamma : N_{\mathbf{k}} \leqq N \right\}. \tag{7}$$

Figure 3(b,c) shows the truncated quasilattices Γ_9 and Γ_{27} in the case $q = 4$.

In what follows we need a lower bound of quantities as

$$(k_c^2 - |\mathbf{k}|^2)^2, \mathbf{k} \in \Gamma,$$

which occur in the denominator of the inverse of the linear operator, when they are not 0. We show, in [3] (after a trivial scaling),

Lemma 8. *Assume $q \geqq 4$, then for any $\mathbf{k} \in \Gamma$ such that $|\mathbf{k}| \neq k_c$, i.e. $\mathbf{k} \neq \mathbf{k}_j$, $j = 1, \ldots, 2q$ the following estimate holds true:*

$$||\mathbf{k}|^2 - k_c^2| \geqq \frac{c}{(1 + N_{\mathbf{k}}^2)^{l_0}} \tag{8}$$

for a certain $c > 0$ only depending on q.

4. Function Spaces and Operators

We characterise the functions of interest by their Fourier coefficients on the quasilattice Γ generated by the $2q$ basic vectors $\mathbf{k}_j$:

$$u(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2.$$

Define now the (Sobolev) space of scalar functions

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} : ||u||_s^2 = \sum_{\mathbf{k} \in \Gamma} (1 + N_{\mathbf{k}}{}^2)^s |u_{\mathbf{k}}|^2 < \infty \right\}, \tag{9}$$

which becomes a Hilbert space with the scalar product

$$\langle w, v \rangle_s = \sum_{\mathbf{k} \in \Gamma} (1 + N_{\mathbf{k}}{}^2)^s w_{\mathbf{k}} \overline{v}_{\mathbf{k}}. \tag{10}$$

The two following Lemmas are classical results on Sobolev spaces:

Lemma 9. *Assume $q \geqq 4$, then for $s > d/2$, for any $u \in \mathcal{H}_s$ and any $v \in \mathcal{H}_0$, we have*

$$||uv||_0 \leqq c_s ||u||_s ||v||_0$$

for a certain constant $c_s > 0$.

Lemma 10. *(Moser–Nirenberg inequality) Assume* $q \geqq 4$, *and let* $s \geqq s_0 > d/2$ *and let* $u, v \in \mathcal{H}_s$. *Then,*

$$\|uv\|_s \leqq C(s, s_0)(\|u\|_s \|v\|_{s_0} + \|u\|_{s_0} \|v\|_s)$$

for some positive constant $C(s, s_0)$ *that depends only on* s *and* s_0. *For* $\ell \geqq 0$ *and* $s > \ell + d/2$, $\mathcal{H}_s$ *is continuously embedded into* $\mathcal{C}^\ell$.

In fact we need more complicate function spaces for our system (1). This is due to the necessity to control in terms of $|\mathbf{k}|$ (instead of $N_\mathbf{k}$) the gain of regularity provided by the inverse of the linear operator on the complementary space of its kernel (here, contrary to [3,11], the nonlinear term looses one derivative), hence the inverse of the linear operator is used to regain this loss (for large $|\mathbf{k}|$), while the loss due to the small divisor problem (for $|\mathbf{k}|$ close to k_c) is in terms of $N_\mathbf{k}$.

4.1. Projection $\mathfrak{P}$

First, as is the "rule" for Navier–Stokes systems, we define a projection operator $\mathfrak{P}$ on divergence free vector fields. Let us consider a vector field $V(\mathbf{x}, z)$ under the form

$$V(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} V_\mathbf{k}(z) e^{i\mathbf{k}\cdot\mathbf{x}},$$

which, for a fixed z belongs to $(\mathcal{H}_s)^3$. We would like to decompose V as follows:

$$V = W + \nabla\phi, \quad \nabla \cdot W = 0, \quad w^{(z)}|_{z=0,1} = 0.$$

Then we consider the system

$$\begin{aligned} W_\mathbf{k}^{(H)} + i\mathbf{k}\phi_\mathbf{k} &= V_\mathbf{k}^{(H)}, \\ w_\mathbf{k}^{(z)} + \frac{d\phi_\mathbf{k}}{dz} &= v_\mathbf{k}^{(z)}, \\ i\mathbf{k} \cdot W_\mathbf{k}^{(H)} + \frac{dw_\mathbf{k}^{(z)}}{dz} &= 0, \end{aligned} \tag{11}$$

where $V_\mathbf{k} = (V_\mathbf{k}^{(H)}, v_\mathbf{k}^{(z)})$, $V_\mathbf{k}^{(H)}$ and $v_\mathbf{k}^{(z)}$ are, respectively, the horizontal and vertical components of $V_\mathbf{k}$, and where we want to satisfy the boundary condition

$$w_\mathbf{k}^{(z)}|_{z=0,1} = 0 \tag{12}$$

for the unknown vector field $W_\mathbf{k} = (W_\mathbf{k}^{(H)}, w_\mathbf{k}^{(z)})$. We then obtain the following equation for $\phi_\mathbf{k}$:

$$\begin{aligned} \frac{\mathrm{d}^2\phi_\mathbf{k}}{\mathrm{d}z^2} - |\mathbf{k}|^2\phi_\mathbf{k} &= i\mathbf{k} \cdot V_\mathbf{k}^{(H)} + \frac{\mathrm{d}v_\mathbf{k}^{(z)}}{\mathrm{d}z}, \\ \frac{\mathrm{d}\phi_\mathbf{k}}{\mathrm{d}z}|_{z=0,1} &= v_\mathbf{k}^{(z)}|_{z=0,1}. \end{aligned} \tag{13}$$

For $\mathbf{k} \neq \mathbf{0}$, it is well known that, if $V_{\mathbf{k}}^{(H)} \in \{L^2(0,1)\}^2$, $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, then there is a unique solution $\phi_{\mathbf{k}} \in H^2(0,1)$ of this Neumann problem, which satisfies the estimates

$$|\mathbf{k}|^2 ||\phi_{\mathbf{k}}||^2 + \left\| \frac{\mathrm{d}\phi_{\mathbf{k}}}{\mathrm{d}z} \right\|^2 \leqq ||V_{\mathbf{k}}||^2, \tag{14}$$

and there exists a constant $c_1 > 0$ ($c_1 = 7$) such that

$$|\mathbf{k}|^4 ||\phi_{\mathbf{k}}||^2 + |\mathbf{k}|^2 \left\| \frac{\mathrm{d}\phi_{\mathbf{k}}}{\mathrm{d}z} \right\|^2 + \left\| \frac{\mathrm{d}^2\phi_{\mathbf{k}}}{\mathrm{d}z^2} \right\|^2 \leqq c_1 \left\{ \left\| \frac{\mathrm{d}v_{\mathbf{k}}^{(z)}}{\mathrm{d}z} \right\|^2 + |\mathbf{k}|^2 ||V_{\mathbf{k}}||^2 \right\}. \tag{15}$$

In the case when $\mathbf{k} = \mathbf{0}$, we have $w_{\mathbf{0}}^{(z)} = 0$, $W_{\mathbf{0}}^{(H)} = V_{\mathbf{0}}^{(H)}$, and $\frac{\mathrm{d}\phi_{\mathbf{0}}}{\mathrm{d}z} = v_{\mathbf{0}}^{(z)}$ defines $\phi_{\mathbf{0}}$ up to a constant. Hence, this remark, with (14) and (15) and the identity

$$\int_0^1 \left\{ i\mathbf{k}\phi_{\mathbf{k}} \cdot \overline{W_{\mathbf{k}}^{(H)}} + \frac{\mathrm{d}\phi_{\mathbf{k}}}{\mathrm{d}z} \overline{w_{\mathbf{k}}^{(z)}} \right\} \mathrm{d}z = 0, \tag{16}$$

lead to

$$||W_{\mathbf{k}}||_{L^2}^2 = \langle V_{\mathbf{k}}, W_{\mathbf{k}} \rangle_{L^2},$$

hence

$$||W_{\mathbf{k}}||_{L^2} \leqq ||V_{\mathbf{k}}||_{L^2},$$

$$|\mathbf{k}|^2 ||W_{\mathbf{k}}||_{L^2}^2 + \left\| \frac{\mathrm{d}W_{\mathbf{k}}}{\mathrm{d}z} \right\|_{L^2}^2 \leqq c^2 \left\{ |\mathbf{k}|^2 ||V_{\mathbf{k}}||_{L^2}^2 + \left\| \frac{\mathrm{d}V_{\mathbf{k}}}{\mathrm{d}z} \right\|_{L^2}^2 \right\} \tag{17}$$

for a constant c^2 independent of $\mathbf{k} \in \Gamma$.

Definition 11. The operator $\mathfrak{P}$ is the linear operator defined as

$$V = \sum_{\mathbf{k}\in\Gamma} V_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}} \overset{\mathfrak{P}}{\mapsto} W = \sum_{\mathbf{k}\in\Gamma} W_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $W_{\mathbf{k}}$ is solution of (11).

We notice that if V is divergence free and satisfies $v^{(z)}|_{z=0,1} = 0$ then $\mathfrak{P}$ acts as the identity. Hence *the operator* $\mathfrak{P}$ *is a projection*.

Remark 12. Notice that for $V_{\mathbf{k}} \in \{L^2(0,1)\}^3$ such that $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, $\mathbf{k} \in \Gamma$, (which is the case when V is divergence free), the boundary values $v_{\mathbf{k}}^{(z)}|_{z=0,1}$ have a meaning, then we still have $||W_{\mathbf{k}}||_{L^2} \leqq ||V_{\mathbf{k}}||_{L^2}$.

4.2. Function Spaces

Let us define function spaces for the 4-components vector field $U = (V, \theta)$ as follows:

$$\mathcal{H}_{r,s} = \left\{ U = (V,\theta)(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}}; \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s ||U_{\mathbf{k}}||_r^2 \right) < \infty \right\}, \tag{18}$$

where

$$||U_{\mathbf{k}}||_r^2 = \sum_{0\leqq l\leqq r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2.$$

Notice the following equivalence between (squared) norms in (18):

$$\sum_{0\leqq l\leqq r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2 \sim \sum_{0\leqq l\leqq r} (1+|\mathbf{k}|^2)^{(r-l)} ||\frac{d^l U_{\mathbf{k}}}{dz^l}||_{L^2}^2.$$

The space $\mathcal{H}_{r,s}$ has a natural Hilbertian structure. For example, for U, $U' \in \mathcal{H}_{0,s}$, the scalar product reads

$$\langle U, U'\rangle_{0,s} = \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s \int_0^1 U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}} dz \right),$$

where $U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}}$ is the usual hermitian scalar product in $\mathbb{C}^4$.

Now, denoting $\mathfrak{P}U = (\mathfrak{P}V, \theta)$, we have the following:

Proposition 13. *The projection $\mathfrak{P}$ is bounded in $\mathcal{H}_{r,s}$ for $r \geqq 1$, and bounded in the subspace $\mathcal{H}'_{0,s}$ of $\mathcal{H}_{0,s}$ such that $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, $\mathbf{k} \in \Gamma$. For any U, $U' \in \mathcal{H}_{1,s}$, or $\mathcal{H}'_{0,s}$, we have*

$$\langle U, \mathfrak{P}U'\rangle_{0,s} = \langle \mathfrak{P}U, \mathfrak{P}U'\rangle_{0,s}.$$

Remark 14. The above Proposition means that $(\mathbb{I}-\mathfrak{P})\mathcal{H}_{1,s}$ is orthogonal to $\mathfrak{P}\mathcal{H}_{1,s}$ with the scalar product of $\mathcal{H}_{0,s}$. In other words, $\mathfrak{P}$ is an orthogonal projection in $\mathcal{H}_{0,s}$ restricted to subspaces $\mathcal{H}_{1,s}$ and $\mathcal{H}'_{0,s}$. Moreover, for $U \in \mathcal{H}'_{0,s}$, then $\mathfrak{P}U \in \mathcal{H}_{0,s}$ is orthogonal to any $(\nabla\phi, 0) \in \mathcal{H}_{0,s}$, and $||\mathfrak{P}U||_{0,s} \leqq ||U||_{0,s}$ (see (16)).

Proof. The boundedness of $\mathfrak{P}$ in $\mathcal{H}_{1,s}$ results immediately from (17), and in $\mathcal{H}'_{0,s}$ from Remark 12. For the boundedness in $\mathcal{H}_{r,s}$ for $r > 1$, this follows easily after differentiating (11) and (13). Now assume U, $U' \in \mathcal{H}_{1,s}$ or $\mathcal{H}'_{0,s}$, and define $\mathfrak{P}U' = (V', \theta')$, then from the form of $V_{\mathbf{k}} - W_{\mathbf{k}} = (\nabla\phi, 0)_{\mathbf{k}}$ indicated in (11), we have (notice that V' satisfies de conditions required on W in (11))

$$\begin{aligned}\langle (\mathbb{I}-\mathfrak{P})U, \mathfrak{P}U'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s \int_0^1 \left(i\mathbf{k}\phi_{\mathbf{k}} \cdot \overline{V_{\mathbf{k}}^{\prime(H)}} + \frac{\mathrm{d}\phi_{\mathbf{k}}}{\mathrm{d}z} \overline{v_{\mathbf{k}}^{\prime(z)}} + 0 \right) \mathrm{d}z \right) \\ &= \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s \int_0^1 \phi_{\mathbf{k}} \left(i\mathbf{k} \cdot \overline{V_{\mathbf{k}}^{\prime(H)}} - \frac{d\overline{v_{\mathbf{k}}^{\prime(z)}}}{\mathrm{d}z} \right) dz \right) \\ &= 0.\end{aligned}$$

Now we need to extend the definition of the orthogonal projector $\mathfrak{P}$ in all $\mathcal{H}_{0,s}$. Let us consider the orthogonal projection $\mathfrak{P}_0$ in $\mathcal{H}_{0,s}$ on the orthogonal complement of the subspace

$$\mathcal{G}_{0,s} = \{U = (\nabla\phi, 0); \phi \in \mathcal{H}_{1,s}^{(1)}\} \subset \mathcal{H}_{0,s},$$

where we denote by an upper index $^{(1)}$ a space of scalar functions. Then, $\mathfrak{P}_0$ is an extension of $\mathfrak{P}$ obtained by density of $H^1(0,1)$ in $L^2(0,1)$ for all $v_{\mathbf{k}}^{(z)}$, $\mathbf{k} \in \Gamma$. This then results in

Lemma 15. *The projection $\mathfrak{P}$ is bounded in $\mathcal{H}_{r,s}$ for $r \geqq 0$. It is an orthogonal projection in $\mathcal{H}_{0,s}$, orthogonal to elements of $\mathcal{G}_{0,s}$.*

In what follows we need to use analogues of Lemma 10.

Lemma 16. *Let $u, v \in \mathcal{H}_{1,s}^{(1)}$ (scalar functions) with $s \geqq s_0 > d/2$. Then $uv \in \mathcal{H}_{1,s}^{(1)}$ and there exists $c(s, s_0) > 0$ such that*

$$||uv||_{1,s} \leqq c(s, s_0)(||u||_{1,s}||v||_{1,s_0} + ||u||_{1,s_0}||v||_{1,s}).$$

Lemma 17. *Let u, v be scalar functions respectively in $\mathcal{H}_{1,s}^{(1)}$ and $\mathcal{H}_{0,s}^{(1)}$ with $s \geqq s_0 > d/2$. Then $uv \in \mathcal{H}_{0,s}^{(1)}$ and there exists $c(s, s_0) > 0$ such that*

$$||uv||_{0,s} \leqq c(s, s_0)(||u||_{1,s}||v||_{0,s_0} + ||u||_{1,s_0}||v||_{0,s}).$$

Lemma 18. *Let u, v be scalar functions respectively in $\mathcal{H}_{1,s}^{(1)}$ and $\mathcal{H}_{0,0}^{(1)}$ with $s \geqq s_0 > d/2$. Then $uv \in \mathcal{H}_{0,0}^{(1)}$ and there exists $c(s) > 0$ such that*

$$||uv||_{0,0} \leqq c(s)||u||_{1,s}||v||_{0,0}.$$

Lemma 19. *Let u, v be scalar functions respectively in $\mathcal{H}_{1,0}^{(1)}$ and $\mathcal{H}_{0,s}^{(1)}$ with $s \geqq s_0 > d/2$. Then $uv \in \mathcal{H}_{0,0}^{(1)}$ and there exists $c(s) > 0$ such that*

$$||uv||_{0,0} \leqq c(s)||u||_{1,0}||v||_{0,s}.$$

The proofs of these Lemmas are made in "Appendix B".

5. Formulation of the Convection Problem

5.1. Operators L, A and B

Definition 20. We say that U satisfies *Condition* **b.c.** if one of the following boundary conditions are satisfied:

(i) $V^{(H)}|_{z=0,1} = 0$ (rigid-rigid),
(ii) $V^{(H)}|_{z=0} = \frac{dV^{(H)}}{dz}|_{z=1} = 0$ (rigid-free),
(iii) $\frac{dV^{(H)}}{dz}|_{z=0} = V^{(H)}|_{z=1} = 0$ (free-rigid).

Then, we define the following function spaces for r and s non-negative integers:

$$\mathcal{K}_{r,s} = \mathfrak{P}\mathcal{H}_{r,s} = \{U = (V, \theta) \in \mathcal{H}_{r,s};\ \nabla \cdot V = 0,\ v^{(z)}|_{z=0,1} = 0\},$$
$$D_s(L) = \mathcal{K}_{2,s} \cap \{U \text{ satisfies Condition } \textbf{b.c.,}\ \theta|_{z=0,1} = 0\}, \tag{19}$$

and we put on these subspaces the norms of $\mathcal{H}_{r,s}$ and $\mathcal{H}_{2,s}$. We notice that we do not consider the case of conditions $\frac{\mathrm{d}V^{(H)}}{\mathrm{d}z}|_{z=0,1} = 0$ (free–free) (see Remark 23 below).

Definition 21. For any $U \in D_s(L)$ operators L and A are defined by

$$LU = (\mathfrak{P}\Delta V, \Delta\theta),\quad U \in D_s(L)$$
$$AU = (\mathfrak{P}(\theta e_z), V \cdot e_z),\ U \in \mathcal{K}_{0,s},$$

and the quadratic operator B by

$$B(U, U) = \left(\frac{1}{\mathcal{P}}\mathfrak{P}(V \cdot \nabla V), V \cdot \nabla\theta\right), U \in \mathcal{K}_{1,s}.$$

It is clear that L maps continuously $D_s(L)$ to $\mathcal{K}_{0,s}$. For $s > d/2$ the quadratic operator B maps continuously $D_s(L)$ to $\mathcal{K}_{1,s}$ as this results easily from the fact that $H^1(0,1)$ is an algebra, as well as $\mathcal{H}_s$ for $s > d/2$ (see Lemma 16 and see "Appendix C" for the rest of the proof). This means that there exists $c(s, s_0)$ such that for any $U, U' \in D_s(L)$, and $s \geqq s_0 > d/2$ we have

$$||B(U, U')||_{1,s} \leqq c(s, s_0)(||U||_{2,s}||U'||_{2,s_0} + ||U||_{2,s_0}||U'||_{2,s}), \tag{20}$$

where we define the bilinear symmetric operator $(U, U') \mapsto B(U, U')$ as

$$2B(U, U') =: \left(\frac{1}{\mathcal{P}}\mathfrak{P}(V \cdot \nabla V' + V' \cdot \nabla V), V \cdot \nabla\theta' + V' \cdot \nabla\theta\right).$$

Moreover, we also have easily $B(U, U) \in \mathcal{K}_{0,s}$ for $U \in \mathcal{K}_{1,s}$, as this results from the fact that the product of a function in $H^1(0,1)$ with another in $L^2(0,1)$ lies in $L^2(0,1)$, then $V \cdot \nabla V \in \mathcal{H}_{0,s}$ (see "Appendix C") and for $U, U' \in \mathcal{K}_{1,s}$ and $s \geqq s_0 > d/2$ we have the estimate

$$||B(U, U')||_{0,s} \leqq c(s, s_0)(||U||_{1,s}||U'||_{1,s_0} + ||U||_{1,s_0}||U'||_{1,s}). \tag{21}$$

Now solving the system (1) reduces to solving the equation

$$(\lambda L + A)U - B(U, U) = 0,\quad U \in D_s(L), \tag{22}$$

where $\lambda =: \mathcal{R}^{-1/2}$.

Then, we show the following useful basic properties of operators L, A and B:

Lemma 22. *For any $s \geqq 0$, the unbounded operator L with domain $D_s(L)$ is selfadjoint, definite negative, in the space $\mathcal{K}_{0,s}$. Moreover, for $U \in D_s(L)$, there exits a scalar function $c(\lambda)$ such that*

$$\langle(\lambda L + A)U, U\rangle_{0,s} \leqq c(\lambda)||U||_{0,s}^2 \tag{23}$$

holds, with $c(\lambda) = 1 - 2\lambda < 0$ for $\mathcal{R} < 4$ (in the case of free–free boundary condition, which we exclude, $c(\lambda) = 0$.)

For $s > d/2$, and $U, U' \in \mathcal{K}_{1,s}$ and U, U' real, i.e. $U = \overline{U}$, $U' = \overline{U'}$ we have

$$\langle B(U, U), U\rangle_{0,0} = 0, \tag{24}$$

$$\langle 2B(U, U'), U\rangle_{0,0} = -\langle B(U, U), U'\rangle_{0,0}. \tag{25}$$

Proof. First we have, by using Lemma 15,

$$\begin{aligned}\langle(\lambda L + A)U, U'\rangle_{0,s} &= \langle(\mathfrak{P}(\lambda\Delta V + \theta e_z), \lambda\Delta\theta + V \cdot e_z), (V', \theta')\rangle_{0,s} \\ &= \lambda\langle(\Delta V, \Delta\theta), (V', \theta')\rangle_{0,s} + \langle(\theta e_z, V \cdot e_z), (V', \theta')\rangle_{0,s} \\ &= \lambda\langle\Delta V, V'\rangle_{0,s} + \lambda\langle\Delta\theta, \theta'\rangle_{0,s} + \langle\theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)}, \theta'\rangle_{0,s}.\end{aligned}$$

Then we observe that $\langle\theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)}, \theta'\rangle_{0,s}$ is symmetric in (U, U'). Moreover by integrating by parts, since $\theta_{\mathbf{k}}|_{z=0,1} = 0$,

$$\begin{aligned}\langle\Delta\theta, \theta'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \int_0^1 \left(\frac{d^2\theta_{\mathbf{k}}}{dz^2} - |k|^2\theta_{\mathbf{k}}\right)\overline{\theta'_{\mathbf{k}}}dz \\ &= -\sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \int_0^1 \left(\frac{d\theta_{\mathbf{k}}}{dz}\frac{d\overline{\theta'_{\mathbf{k}}}}{dz} + |k|^2\theta_{\mathbf{k}}\overline{\theta'_{\mathbf{k}}}\right)dz,\end{aligned}$$

which is symmetric in (U, U'). The same computation holds by using the boundary conditions satisfied by V for $U \in D_s(L)$, and shows that $\langle\Delta V, V'\rangle_{0,s}$ is symmetric in (U, U'). This proves that

$$\langle(\lambda L + A)U, U'\rangle_{0,s} = \langle U, (\lambda L + A)U'\rangle_{0,s},$$

i.e. the operators L and A are symmetric in $\mathcal{K}_{0,s}$.

The operator L is selfadjoint in $\mathcal{K}_{0,s}$ because it is easy to prove that L^{-1} is symmetric in $\mathcal{K}_{0,s}$, bounded from $\mathcal{K}_{0,s}$ into $D_s(L)$ (with norm of $\mathcal{K}_{2,s}$), see "Appendix A". The operator A is symmetric and bounded in $\mathcal{K}_{0,s}$. Hence by theorem 4.3 in [13] p.287, the sum $\lambda L + A$ with domain $D_s(L)$ is also selfadjoint in $\mathcal{K}_{0,s}$.

To prove the inequality (23), we come back to the computation above, valid for $U \in D_s(L)$:

$$\begin{aligned}\langle(\lambda L + A)U, U\rangle_{0,s} &= -\lambda\langle\nabla V, \nabla V\rangle_{0,s} - \lambda\langle\nabla\theta, \nabla\theta\rangle_{0,s} + 2\mathrm{Re}\langle\theta, v^{(z)}\rangle_{0,s} \\ &\leqq 2||V||_{0,s}||\theta||_{0,s} \leqq 2||U||_{0,s}^2.\end{aligned} \tag{26}$$

For all boundary conditions (see Definition 20) we have Poincaré inequalities: θ, $v^{(z)}$ and $V^{(H)}$ cancel at $z = 0$ or (and) $z = 1$, so, for example,

$$|v^{(z)}(z)|^2 = |\int_0^z Dv^{(z)}(s)\mathrm{d}s|^2 \leqq z\int_0^1 |Dv^{(z)}(s)|^2\mathrm{d}s,$$

and integrating on (0, 1) leads to the Poincaré estimates

$$||V||_{0,s} \leqq \frac{1}{\sqrt{2}}||\nabla V||_{0,s}, \quad ||\theta||_{0,s} \leqq \frac{1}{\sqrt{2}}||\nabla\theta||_{0,s}. \tag{27}$$

Hence this leads to

$$|2\mathrm{Re}\langle\theta, v^{(z)}\rangle_{0,s}| \leqq ||\nabla V||_{0,s}||\nabla\theta||_{0,s} \leqq 1/2||\nabla V||_{0,s}^2 + 1/2||\nabla\theta||_{0,s}^2$$

and

$$\langle(\lambda L + A)U, U\rangle_{0,s} \leqq (1/2-\lambda)[||\nabla V||_{0,s}^2 + ||\nabla\theta||_{0,s}^2] < 0 \text{ for } \lambda > 1/2, \text{ i.e. } \mathcal{R} < 4.$$

Hence for $\mathcal{R} < 4$ (i.e. $\lambda > 1/2$) we have

$$\langle(\lambda L + A)U, U\rangle_{0,s} \leqq -(2\lambda - 1)[||V||_{0,s}^2 + ||\theta||_{0,s}^2] = c(\lambda)||U||_{0,s}^2,$$

with

$$c(\lambda) = 1 - 2\lambda.$$

Remark 23. In the case of free–free Boundary conditions which we exclude here, we have not $||V^{(H)}||_{0,s} \leqq \frac{1}{\sqrt{2}}||\nabla V^{(H)}||_{0,s}$, hence we only have

$$\begin{aligned}\langle(\lambda L + A)U, U\rangle_{0,s} \leqq (1/2 - \lambda)[||\nabla v^{(z)}||_{0,s}^2 + ||\nabla\theta||_{0,s}^2] \\ - \lambda||\nabla V^{(H)}||_{0,s}^2 \leqq 0 \text{ for } \lambda \geqq 1/2.\end{aligned}$$

In such a case, 0 is an eigenvalue of $\lambda L + A$ corresponding to the eigenvector $U = (V^{(H)}, 0)$ where $V^{(H)} = Const.$

In the same way as above, for $U \in \mathcal{K}_{1,s}$ we have

$$\langle B(U, U), U\rangle_{0,s} = \frac{1}{\mathcal{P}}\langle V \cdot \nabla V, V\rangle_{0,s} + \langle V \cdot \nabla\theta, \theta\rangle_{0,s},$$

and by using $\overline{\theta_{\mathbf{p}+\mathbf{q}}} = \theta_{\mathbf{r}}$ when $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$, since θ is real, we have

$$\begin{aligned}\langle V \cdot \nabla\theta, \theta\rangle_{0,0} &= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{q} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}} + v_{\mathbf{p}}^{(z)}\frac{\mathrm{d}\theta_{\mathbf{q}}}{\mathrm{d}z}\right)\theta_{\mathbf{r}}\mathrm{d}z \\ &= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{r} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{\mathrm{d}\theta_{\mathbf{r}}}{\mathrm{d}z}\right)\theta_{\mathbf{q}}\mathrm{d}z \\ &= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((-i\mathbf{p} \cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{\mathrm{d}(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{\mathrm{d}z}\right)\mathrm{d}z \\ &= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left(\frac{\mathrm{d}v_{\mathbf{p}}^{(z)}}{\mathrm{d}z}\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{\mathrm{d}(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{\mathrm{d}z}\right)\mathrm{d}z = 0.\end{aligned}$$

In the same way, we have

$$\langle V\cdot\nabla V, V\rangle_{0,0} = \langle V\cdot\nabla V^{(H)}, V^{(H)}\rangle_{0,s} + \langle V\cdot\nabla v^{(z)}, v^{(z)}\rangle_{0,s}$$
$$= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma}\int_0^1 \frac{\mathrm{d}(v_{\mathbf{p}}^{(z)}V_{\mathbf{q}}\cdot V_{\mathbf{r}})}{\mathrm{d}z}\mathrm{d}z = 0,$$

which ends the proof of (24). Identity (25) is a consequence of (24); indeed let us consider the identity

$$\langle B(U+tU', U+tU'), U+tU'\rangle_{0,0} = 0\ ,$$

which holds for any $t\in\mathbb{R}$. It results that the coefficient of degree 1 in t of this polynomial is zero, which is exactly the property (25). □

5.2. New Formulation

For applying a method analogous to the one developed in [2,3], we need to control a parameter able to move all of the spectrum of the linearized operator. In the present problem, we are lucky enough to have λ in front of an invertible operator, allowing us to suitably reformulate the problem.

We know that the operator $-L$ is selfadjoint and positive, so we can define the selfadjoint positive operator $(-L)^{1/2}$ with dense domain (see [13] section V.11 p.281) as the inverse of

$$(-L)^{-1/2} = \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}(\zeta - L)^{-1}\mathrm{d}\zeta,$$

which is selfadjoint and bounded, with the following properties: first, for $U\in D_s(L)$ we have

$$(-L)^{1/2}(-L)^{1/2}U = -LU.$$

Let us define the Hilbert space, adapted to boundary conditions **b.c.** (see Definition 20),

$$\widetilde{\mathcal{K}_{1,s}} = \{U=(V,\theta)\in\mathcal{K}_{1,s};\ \theta = v^{(z)}|_{z=0,1} = 0,\ V^{(H)}|_{z=0} = 0,$$
$$\text{or (and) } V^{(H)}|_{z=1} = 0\}.$$

We can take in $\widetilde{\mathcal{K}_{1,s}}$ the norm

$$||U||_{\widetilde{1,s}} := \{||\nabla V||_{0,s}^2 + ||\nabla\theta||_{0,s}^2\}^{1/2}, \tag{28}$$

which is equivalent to the usual norm in $\mathcal{K}_{1,s}$, due to Poincaré inequalities (27). Then, because of the identity

$$\langle -LU, U\rangle_{0,s} = ||\nabla V||_{0,s}^2 + ||\nabla\theta||_{0,s}^2,$$

valid for any $U\in D_s(L)$, it is clear that the following identity holds:

$$||(-L)^{1/2}U||_{0,s} = ||U||_{\widetilde{1,s}}. \tag{29}$$

This can be extended to any $U \in D_s[(-L)^{1/2}]$ the domain of $(-L)^{1/2}$ acting in $\mathcal{K}_{0,s}$. This then shows that the domain $D_s[(-L)^{1/2}]$ (dense in $\mathcal{K}_{0,s}$) satisfies

$$D_s[(-L)^{1/2}] \subset \widetilde{\mathcal{K}_{1,s}} \tag{30}$$

with a continuous embedding.

Definition 24. We denote that

$$\mathcal{D}_{1/2,s} := D_s[(-L)^{1/2}].$$

This is an Hilbert subspace of $\mathcal{K}_{1,s}$, with the scalar product associated with the norm (28) in $\widetilde{\mathcal{K}_{1,s}}$.

Remark 25. In the sequel, the norm in $\mathcal{D}_{1/2,s}$ is denoted by $||\cdot||_{\widetilde{1,s}}$ or $||\cdot||_{1,s}$ or $||\cdot||_{\mathcal{D}_{1/2,s}}$ as well.

Now let us consider the following equation in $\mathcal{K}_{0,s}$:

$$\lambda u - \mathcal{A}u + \mathcal{B}(u,u) = 0, \tag{31}$$

where operators $\mathcal{A}$ and $\mathcal{B}$ are defined as

$$\begin{aligned}\mathcal{A} &:= (-L)^{-1/2} A (-L)^{-1/2},\\ \mathcal{B}(u,u) &:= (-L)^{-1/2} B((-L)^{-1/2}u, (-L)^{-1/2}u).\end{aligned}$$

Since the operator A is bounded in $\mathcal{K}_{0,s}$ this is also the case for $\mathcal{A}$. Now for the quadratic operator $\mathcal{B}$ we have

Lemma 26. *Assume $s > d/2$, then the quadratic operator $\mathcal{B}$ is bounded from $\mathcal{K}_{0,s}$ to $\mathcal{D}_{1/2,s} \hookrightarrow \widetilde{\mathcal{K}_{1,s}} \hookrightarrow \mathcal{K}_{0,s}$. Moreover for $u, u' \in \mathcal{K}_{0,s}$, with $s \geqq s_0 > d/2$ we have*

$$\begin{aligned}||\mathcal{B}(u,u')||_{0,s} &\leqq ||(-L)^{-1/2}||_{0,s} ||\mathcal{B}(u,u')||_{\widetilde{1,s}}\\ &\leqq c(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s}).\end{aligned} \tag{32}$$

Moreover, for $u \in \mathcal{K}_{0,s}$, with $s > d/2$, the linear operator $v \mapsto \mathcal{B}(u,v)$ is bounded in $\mathcal{K}_{0,0}$ with the estimate

$$||\mathcal{B}(u,v)||_{0,0} \leqq c||u||_{0,s}||v||_{0,0}. \tag{33}$$

Proof. Using (29) and (21) we obtain

$$\begin{aligned}||\mathcal{B}(u,u')||_{\widetilde{1,s}} &= ||B((-L)^{-1/2}u, (-L)^{-1/2}u')||_{0,s}\\ &\leqq c(s,s_0)(||(-L)^{-1/2}u||_{\widetilde{1,s}}||(-L)^{-1/2}u'||_{\widetilde{1,s_0}}\\ &\quad + ||(-L)^{-1/2}u||_{\widetilde{1,s_0}}||(-L)^{-1/2}u'||_{\widetilde{1,s}})\\ &\leqq c(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s})\end{aligned}$$

and

$$||\mathcal{B}(u,u')||_{0,s} \leqq ||(-L)^{-1/2}||_{0,s}|\mathcal{B}(u,u')||_{\widetilde{1,s}}$$
$$= c_1(s,s_0)(||u||_{0,s}||u'||_{0,s_0} + ||u||_{0,s_0}||u'||_{0,s}). \tag{34}$$

For finding estimate (33) we just need to prove that for $((-L)^{-1/2}u, (-L)^{-1/2}v) = (U,V) \in \mathcal{K}_{1,s} \times \mathcal{K}_{1,0}$ then $||B(U,V)||_{0,0} \leqq c'||U||_{1,s}||V||_{1,0}$. This is proved in "Appendix C". □

Then we have

Lemma 27. *Assuming $s > d/2$ and $\lambda > 0$, finding a solution $u \in \mathcal{K}_{0,s}$ of*

$$\lambda u - \mathcal{A}u + \mathcal{B}(u,u) = 0, \tag{35}$$

where the linear operator $\mathcal{A}$ is bounded and selfadjoint in $\mathcal{K}_{0,s}$, implies that $u \in \mathcal{D}_{1/2,s}$, and is equivalent to finding a solution $U = (-L)^{-1/2}u \in D_s(L)$ of

$$\lambda LU + AU - B(U,U) = 0. \tag{36}$$

Proof. Indeed, we notice that for $u \in \mathcal{K}_{0,s}$ solution of (31), then $(-L)^{-1/2}u \in \mathcal{D}_{1/2,s} \subset \mathcal{K}_{1,s}$, hence $B((-L)^{-1/2}u, (-L)^{-1/2}u) \in \mathcal{K}_{0,s}$ (see (21)) and finally $\mathcal{B}(u,u) \in \mathcal{D}_{1/2,s}$. It is also clear that $\mathcal{A}u \in \mathcal{D}_{1/2,s}$. For $\lambda \neq 0$ this last property and (31) show that $u \in \mathcal{D}_{1/2,s}$, and we can apply the operator $(-L)^{1/2}$ to (31). Then defining $U = (-L)^{-1/2}u$ gives U in $D_s(L)$ verifying (36). Conversely, the knowledge of a solution U of (36) gives a solution $u = (-L)^{1/2}U$ of (31). We may observe that the quadratic operator $\mathcal{B}$ is bounded in $\mathcal{K}_{0,s}$ (see (34)). Now due to the selfadjointness of operators A and $(-L)^{-1/2}$ in $\mathcal{K}_{0,s}$, the operator $\mathcal{A}$ *is also selfadjoint in* $\mathcal{K}_{0,s}$. □

Remark 28. We might think that it would be advantageous to work in $\mathcal{D}_{1/2,s}$ instead of $\mathcal{K}_{0,s}$. However for the method we are using in what follows, it is necessary that $\mathcal{A}$ be selfadjoint. If we consider this operator in $\mathcal{D}_{1/2,s}$, then it can be shown that this is not true for boundary conditions (ii) and (iii) in Definition 20.

5.3. Rotationnal Symmetry

The system (1), completed with the boundary conditions included in the definition of $D_s(L)$, is invariant under horizontal rotations of angle π/q. To make this precise, let us define the linear operator $\mathbf{R}_{\pi/q}$ by

$$\mathbf{R}_{\pi/q}U(\mathbf{x},z) = \left(R_{\pi/q}V(R_{-\pi/q}\mathbf{x},z), \theta(R_{-\pi/q}\mathbf{x},z)\right),$$

where R_ϕ is the horizontal rotation of angle ϕ. More precisely, by using the identity $\mathbf{k}\cdot R_{-\phi}\mathbf{x} = R_\phi\mathbf{k}\cdot\mathbf{x}$ we have

$$\mathbf{R}_{\pi/q}\sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}\in\Gamma}\left(R_{\pi/q}V_{\mathbf{k}}(z), \theta_{\mathbf{k}}(z)\right)e^{iR_{\pi/q}\mathbf{k}\cdot\mathbf{x}}. \tag{37}$$

Definition 29. We say that $U = (V, \theta)$ is invariant under $\mathbf{R}_{\pi/q}$ if

$$R_{\pi/q} V_{\mathbf{k}}(z) = V_{R_{\pi/q}\mathbf{k}}(z), \ \ \theta_{\mathbf{k}}(z) = \theta_{R_{\pi/q}\mathbf{k}}(z).$$

Then, we have the following:

Lemma 30. *The linear operators* $L, A, \mathcal{A}$ *and the quadratic operators* B *and* $\mathcal{B}$ *commute with* $\mathbf{R}_{\pi/q}$ *: for* $U \in D_s(L)$ *and* $u \in \mathcal{K}_{0,s}$

$$\mathbf{R}_{\pi/q}(\lambda L + A)U = (\lambda L + A)\mathbf{R}_{\pi/q}U, \ \mathbf{R}_{\pi/q}\mathcal{A}u = \mathcal{A}\mathbf{R}_{\pi/q}u$$
$$\mathbf{R}_{\pi/q}B(U, U) = B(\mathbf{R}_{\pi/q}U, \mathbf{R}_{\pi/q}U), \ \mathbf{R}_{\pi/q}\mathcal{B}(u, u) = \mathcal{B}(\mathbf{R}_{\pi/q}u, \mathbf{R}_{\pi/q}u). \tag{38}$$

Proof. This results from the commutation of the original system (1) under any horizontal rotations, and from the commutation property

$$\mathbf{R}_{\pi/q}\mathfrak{P} = \mathfrak{P}\mathbf{R}_{\pi/q},$$

which is easy to check from the construction of projection $\mathfrak{P}$. Moreover the operator L commutes with $\mathbf{R}_{\pi/q}$, hence this is also valid for $(-L)^{-1/2}$. □

6. Criticality for $\mathcal{A} - \lambda\mathbb{I}$ and Formal Bifurcation

6.1. Study of Criticality

Let us consider the linear system

$$(\mathcal{A} - \lambda)u = G \in \mathcal{K}_{0,s}, \tag{39}$$

where we look for $u \in \mathcal{K}_{0,s}$. This system is equivalent to looking for $U = (-L)^{-1/2}u \in D_{1/2,s}$ such that

$$(\lambda L + A)U = G' = (-L)^{1/2}G = (F, g) \in (D_{1/2,s})^*, \tag{40}$$

where $G' = (F, g)$ is given in $(D_{1/2,s})^*$ (see the definition and properties of this dual space at Section A.1 of "Appendix A").

Let us define the Fourier components

$$U_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)}, \theta_{\mathbf{k}}),$$
$$G'_{\mathbf{k}} = (F, g)_{\mathbf{k}} = (F_{\mathbf{k}}^{(H)}, f_{\mathbf{k}}^{(z)}, g_{\mathbf{k}}),$$

then for a fixed $\mathbf{k}$, the system has the form

$$(\lambda L_{\mathbf{k}} + A_{\mathbf{k}})U_{\mathbf{k}} = G'_{\mathbf{k}}, \tag{41}$$

which is *exactly the same as the one obtained in the periodic case*, described in details for example in Chapter II of [5] and solved in details by Yudovich [25]. Notice that for each $|\mathbf{k}|$ the linear operator $\lambda L_{\mathbf{k}} + A_{\mathbf{k}}$ is selfadjoint in the space $\{\mathcal{K}_{0,s}\}_{\mathbf{k}}$ spanned by the $\mathbf{k}$-th Fourier components of elements in $\mathcal{K}_{0,s}$, and operator $L_{\mathbf{k}}$ is studied in particular in "Appendix A".

Remark 31. Notice that in [5] and in [25] only the case of $\lambda > 0$ is considered, since $\lambda = 1/\sqrt{\mathcal{R}}$, by definition. The result of this is that we don't know anything a priori on the operator, for $\lambda \leqq 0$. However we may observe that the homogeneous system associated with (41) is invariant when changing λ into $-\lambda$ and θ into $-\theta$ (see also (1) in changing $\sqrt{\mathcal{R}}$ into its opposite). Consequently, the spectrum of $\mathcal{A}$ is symmetric with respect to 0. Moreover, $\lambda = 0$ is an eigenvalue with infinite multiplicity.

Then, it is known (see YUDOVICH [25]) that for a fixed $|\mathbf{k}|$ there is a denumerable sequence of $\mathcal{R}_j$ $(= 1/\lambda_j^2)$ such that the system (41) has a non-trivial solution for $(F, g)_{\mathbf{k}} = 0$, and there is a variational principle for finding $\mathcal{R}_0(|\mathbf{k}|^2) = \min \mathcal{R}_j$ (see VELTE [22]). It is also known mathematically (see YUDOVICH [25]) that the function $\mathcal{R}_0(|\mathbf{k}|^2)$ is analytic, tends towards ∞ as $|\mathbf{k}|^2 \to 0$ and as $|\mathbf{k}|^2 \to \infty$, and that there is a minimum $\mathcal{R}_c$ obtained for a critical value k_c^2. However, it is *only known numerically* (see [5]) that *this minimum is unique* and the kernel of $\lambda L_{\mathbf{k}} + A_{\mathbf{k}}$ for $\mathbf{k} = \mathbf{k}_1 = (k_c, 0)$ is one-dimensional ([25]).

We now define $\lambda_0 = 1/\sqrt{\mathcal{R}_c}$. The result is that the kernel of the linear operator $(\mathcal{A} - \lambda_0 \mathbb{I})$ is $2q$-dimensional, spanned by $\boldsymbol{\xi}_j = (-L)^{1/2} \boldsymbol{\xi}'_j$, with

$$\boldsymbol{\xi}'_j = \mathbf{R}_{\frac{\pi(j-1)}{q}} \left(\widehat{U}_{\mathbf{k}_1}(z) e^{i\mathbf{k}_1 \cdot \mathbf{x}} \right), \quad j = 1, 2, \ldots, 2q \tag{42}$$

in the kernel of $\lambda_0 L + A$, where

$$\widehat{U}_{\mathbf{k}_1} = (V_{\mathbf{k}_1}^{(H)}, v_{\mathbf{k}_1}^{(z)}, \theta_{\mathbf{k}_1})$$

is solution of the homogeneous system (41) for $\mathbf{k} = \mathbf{k}_1$, and with $G'_{\mathbf{k}} = 0$, and $\mathcal{R} = \mathcal{R}_c$.

We now need to estimate the inverse of the linear operator defined by the system (41) for $\mathcal{R} = \mathcal{R}_c$ and $|\mathbf{k}| \neq k_c$. We follow the now standard study of the resolvent operator for a Navier–Stokes type of system ([28]), but here, with a periodic frame, we deduce that there is a function $c(|\mathbf{k}|^2)$ bounded as $|\mathbf{k}| \to \infty$ and $|\mathbf{k}| \to 0$ such that (we notice that $||G'||_{(D_{1/2,s})^*} \leqq c||G||_{0,s}$ for a certain $c > 0$)

$$||U_{\mathbf{k}}||_1^2 = ||DU_{\mathbf{k}}||_{L^2}^2 + (1 + |\mathbf{k}|^2)||U_{\mathbf{k}}||_{L^2}^2 \leqq [c(|\mathbf{k}|^2)]^2 ||G_{\mathbf{k}}||_{L^2}^2. \tag{43}$$

For $|\mathbf{k}|$ near k_c, we know that $c(|\mathbf{k}|^2)$ diverges as $|\mathbf{k}|^2 \to k_c^2$. In fact let us consider the dispersion equation, obtained when we look for eigenvectors of the homogeneous system (41), which has constant coefficients (see [5]). Then, the modulus of the dispersion equation, which cancels for $|\mathbf{k}| = k_c$, is bounded from below by the inverse $c(|\mathbf{k}|^2)^{-1}$. This dispersion equation depends analytically on $|\mathbf{k}|^2$ ([25]) and we now need

Condition 32. *We assume that the second derivative* $\mathcal{R}_0''(|\mathbf{k}|^2) \neq 0$ *for* $|\mathbf{k}| = k_c$ *at* $\mathcal{R}_0(k_c^2) = \mathcal{R}_c$.

Notice that we give a formula for $\frac{d^2}{d|\mathbf{k}|^2}(1/\sqrt{\mathcal{R}_c})$ in "Appendix D". The dispersion relation cancels with a (only) double root for $|\mathbf{k}|^2 = k_c^2$. This means that we have, in fact,

$$c(|\mathbf{k}|^2) = \frac{c_1(|\mathbf{k}|^2)}{\left(|\mathbf{k}|^2 - k_c^2\right)^2} \quad \text{for } |\mathbf{k}| \neq k_c, \tag{44}$$

where c_1 is bounded for all bounded $|\mathbf{k}|^2$ and is $O(|\mathbf{k}|^4)$ as $|\mathbf{k}| \to \infty$.

For $|\mathbf{k}| = k_c$ and $\mathbf{k} \in \Gamma$, this implies that $\mathbf{k}$ belongs to the basis of the quasipattern. Then, following [5,25] and [26], the system (41) is solvable provided that the compatibility conditions

$$\langle G, \boldsymbol{\xi}_j \rangle_{0,0} = \langle G', \boldsymbol{\xi}'_j \rangle_{0,0} = \int_0^1 G'_{\mathbf{k}_j} \cdot \overline{\widehat{U}_{\mathbf{k}_j}} \mathrm{d}z = \int_0^1 (F_{\mathbf{k}_j} \cdot \overline{\widehat{V}_{\mathbf{k}_j}} + g_{\mathbf{k}_j} \cdot \overline{\widehat{\theta}_{\mathbf{k}_j}}) \mathrm{d}z = 0,$$
$$j = 1, \ldots, 2q$$

hold. The result of this is that

$$||u_{\mathbf{k}}||_0 = ||U_{\mathbf{k}}||_1 \leqq c(|\mathbf{k}|^2)||G_{\mathbf{k}}||_0. \tag{45}$$

Remark 33. The classical linear stability theory ([5,27]) says that

$$\langle (\lambda_0 L + A)U, U \rangle_{0,0} < 0 \text{ for all } U \in D_s(L) \text{ not in } \ker(\lambda_0 L + A), \tag{46}$$

i.e., using that $\mathcal{D}_{1/2,s}$ is dense in $\mathcal{K}_{0,s}$,

$$\langle (\mathcal{A} - \lambda_0)u, u \rangle_{0,0} < 0 \text{ for all } u \in \mathcal{K}_{0,s} \text{ not in } \ker(\mathcal{A} - \lambda_0). \tag{47}$$

We also know, from the discussion above, that for any fixed $|\mathbf{k}|$ we have a decreasing sequence of positive eigenvalues, and a sequence of symmetric ones, for the selfadjoint bounded operator $\mathcal{A}$:

$$\lambda_0(|\mathbf{k}|^2) > \lambda_1(|\mathbf{k}|^2) ... \geqq \lambda_n(|\mathbf{k}|^2) \geqq ... \geqq 0 ... \geqq -\lambda_n(|\mathbf{k}|^2)$$
$$\geqq ... - \lambda_1(|\mathbf{k}|^2) > -\lambda_0(|\mathbf{k}|^2),$$

(see Fig. 4 for positive eigenvalues) corresponding to eigenvectors, depending on $\mathbf{x}$ as $e^{i\mathbf{k}\cdot\mathbf{x}}$. The largest eigenvalue reaches a maximum λ_0 at k_c^2. Now the lattice Γ is well defined, thanks to (4). When $\mathbf{k}$ varies in Γ, the set of values for $|\mathbf{k}|$ is dense on the half positive line. As a result, the spectrum (closed in $\mathbb{R}$) of $\mathcal{A}$ is the closed interval $[-\lambda_0, \lambda_0]$. Moreover, as this will be useful later, we notice that for all $\mathbf{k} \in \Gamma$,

$$\lambda_0 - \lambda_j(|\mathbf{k}|^2) > \delta_0, \;\; \lambda_0(|\mathbf{k}|^2) - \lambda_j(|\mathbf{k}|^2) \geqq \delta_0(|k|) > 0, \quad j = 1, 2, \ldots, \infty, \tag{48}$$

with $\delta_0(|k|) > \delta_0 > 0$ for $|k|$ close to k_c.

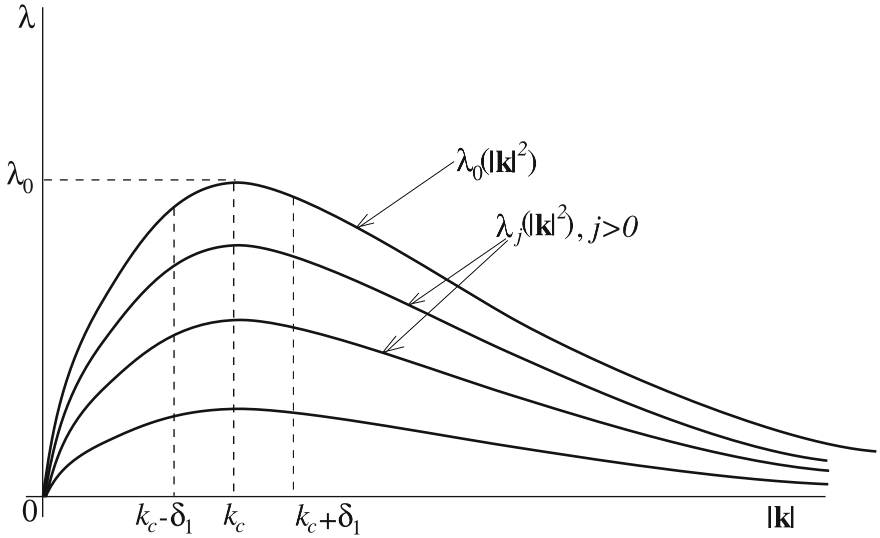

Fig. 4. Sketch of positive eigenvalues of $\mathcal{A}$ in function of $|\mathbf{k}|$, and definition of critical lambda λ_0. δ_1 is used to define the projection π_0 in section 7.4

6.2. Pseudo-Inverse of $\mathcal{A} - \lambda_0\mathbb{I}$

Let us define the orthogonal projection $\mathbf{P}_0$ on the kernel of $\mathcal{A} - \lambda_0\mathbb{I}$: for any $u \in \mathcal{K}_{0,s}$,

$$\mathbf{P}_0 u = \sum_{1 \leqq j \leqq Q} \gamma_j \boldsymbol{\xi}_j, \quad \gamma_j = \frac{\langle u, \boldsymbol{\xi}_j \rangle_{0,0}}{\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0}}, \tag{49}$$

where we notice that

$$\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0} = \langle \boldsymbol{\xi}_j, \boldsymbol{\xi}_j \rangle_{0,0}, \quad j = 2, \ldots, 2q.$$

We denote by $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ the projection on the complementary space (of codimension $2q$). Since the eigenvectors $\boldsymbol{\xi}_j$ belong to $\mathcal{K}_{0,s}$ for any s, the projection $\mathbf{Q}_0$ is bounded in $\mathcal{K}_{0,s}$ for any s. Notice that when u is invariant under $\mathbf{R}_{\pi/q}$, then $\gamma_j = \gamma_1$ for $j = 2, \ldots, 2q$.

Now, coming back to the linear system,

$$(\mathcal{A} - \lambda_0)u = G,$$

where $G \in \mathcal{K}_{0,s}$ satisfies the compatibility condition $\mathbf{P}_0 G = 0$, the above estimate (45), and the form (44) of $c(|\mathbf{k}|^2)$ show that there is a unique solution u satisfying $\mathbf{P}_0 u = 0$ and there exists a constant $c > 0$ such that

$$||u_{\mathbf{k}}||_0 \leqq c \left[\frac{(1 - \delta_{k_c}(|\mathbf{k}|))(1 + |\mathbf{k}|^2)^2}{\left(|\mathbf{k}|^2 - k_c^2\right)^2} + \delta_{k_c}(|\mathbf{k}|) \right] ||G_{\mathbf{k}}||_0,$$

where $\delta_{k_c}(|\mathbf{k}|) = 1$ if $|\mathbf{k}| = k_c$, and $= 0$ otherwise. By using the diophantine inequality (8), this leads to the following:

Lemma 34. *Assuming* $\lambda_0''(|\mathbf{k}|^2)|_{|\mathbf{k}|=k_c} \neq 0$ *(see (138)) for the second derivative of* λ_0 *at* $|\mathbf{k}| = k_c$*, then for any* $s \geqq 0$*, the linear operator* $(\mathcal{A} - \lambda_0)$ *has a bounded inverse from the subspace* $\mathbf{Q}_0\mathcal{K}_{0,s}$ *to the subspace* $\mathbf{Q}_0\mathcal{K}_{0,s-4l_0}$*. In other words, for any* $\delta_1 > 0$ *small enough, there exists* $c > 0$ *such that for* u *solution in* $\mathbf{Q}_0\mathcal{K}_{0,s-4l_0}$ *of* $(\mathcal{A} - \lambda_0)u = G \in \mathbf{Q}_0\mathcal{K}_{0,s}$*, the following estimate holds:*

$$||u_{\mathbf{k}}||_0 \leqq c(1 + N_{\mathbf{k}}^2)^{2l_0}||G_{\mathbf{k}}||_0, \ \textit{for} \ ||\mathbf{k}| - k_c| < \delta_1,$$
$$||u_{\mathbf{k}}||_0 \leqq \frac{c}{\delta_1^2}||G_{\mathbf{k}}||_0, \ \textit{for} \ ||\mathbf{k}| - k_c| \geqq \delta_1.$$

6.3. Formal Power Series for Bifurcating Solution

Let us rewrite the system (35) as

$$(\mathcal{A} - \lambda_0)u = -\mu u + \mathcal{B}(u, u), \tag{50}$$

where

$$\lambda_0 = \frac{1}{\sqrt{\mathcal{R}_c}}, \ \lambda = \lambda_0 - \mu.$$

We are looking for a solution of (50) in $\mathcal{K}_{0,s}$, $s > d/2$, which is invariant under $\mathbf{R}_{\pi/q}$ under the form of a formal expansion

$$u = \sum_{n\geqq 1} \varepsilon^n u_n, \tag{51}$$

$$\mu = \sum_{n\geqq 1} \varepsilon^n \mu_n, \tag{52}$$

where, in fact, $u_n \in \mathcal{D}_{1/2,s}$ (see Lemma 27). Identifying powers of ε at orders ε, ε^2, ε^3, leads to the system

$$(\mathcal{A} - \lambda_0)u_1 = 0, \tag{53}$$
$$(\mathcal{A} - \lambda_0)u_2 = -\mu_1 u_1 + \mathcal{B}(u_1, u_1) \tag{54}$$
$$(\mathcal{A} - \lambda_0)u_3 = -\mu_1 u_2 - \mu_2 u_1 + 2\mathcal{B}(u_1, u_2). \tag{55}$$

Equation (53) gives (here we choose the coefficient in front of the eigenvector, which determines the parameter ε)

$$u_1 = \sum_{1\leqq j\leqq 2q} \boldsymbol{\xi}_j, \tag{56}$$

which is invariant under $\mathbf{R}_{\pi/q}$, and we observe, thanks to property (24), still valid for $\mathcal{B}$, that

$$\langle \mathcal{B}(u_1, u_1), u_1 \rangle_{0,0} = 0,$$

and since

$$\mathbf{R}_{\pi/q}\mathcal{B}(u_1, u_1) = \mathcal{B}(u_1, u_1),$$

this means that (see the definition of projection $\mathbf{P}_0$ in (49))

$$\mathbf{P}_0\mathcal{B}(u_1, u_1) = 0,$$

hence Equation (54) is solvable with $\mu_1 = 0$, and since the Fourier series of $\mathcal{B}(u_1, u_1)$ is finite, we find a unique $u_2 \in \mathcal{D}_{1/2,s}$, orthogonal to u_1 in $\mathcal{K}_{0,0}$, such that

$$u_2 = \widetilde{(\mathcal{A} - \lambda_0)}^{-1}\mathcal{B}(u_1, u_1), \tag{57}$$

which is invariant under $\mathbf{R}_{\pi/q}$, and where $\widetilde{(\mathcal{A} - \lambda_0)}^{-1}$ is the pseudo-inverse of $(\mathcal{A} - \lambda_0)$ as defined by Lemma 34. Now, the compatibility condition for solving (55) gives

$$\langle \mu_2 u_1 - 2\mathcal{B}(u_1, u_2), u_1\rangle_{0,0} = 0. \tag{58}$$

Then we use the identity (25) to obtain

$$\begin{aligned}\langle 2\mathcal{B}(u_1, u_2), u_1\rangle_{0,0} &= -\langle \mathcal{B}(u_1, u_1), u_2\rangle_{0,0}\\ &= -\langle (\mathcal{A} - \lambda_0)u_2, u_2\rangle_{0,0} > 0.\end{aligned}$$

The result above, in the periodic case, was first obtained by Yudovich in [26]. The last inequality results from the fact that $\mathbf{P}_0 u_2 = 0$, and from the property (47). Then μ_2 is positive, determined by

$$\mu_2 = \frac{-\langle (\mathcal{A} - \lambda_0)u_2, u_2\rangle_{0,0}}{\langle u_1, u_1\rangle_{0,0}} > 0. \tag{59}$$

Now the unique solution u_3 of (55), orthogonal to u_1 in $\mathcal{K}_{0,0}$, takes the form

$$u_3 = 2\widetilde{(\mathcal{A} - \lambda_0)}^{-1}\mathbf{Q}_0\mathcal{B}(u_1, u_2), \tag{60}$$

and is invariant under $\mathbf{R}_{\pi/q}$ and again lies in $\mathcal{D}_{1/2,s}$ because of the finiteness of its Fourier series. Now, from

$$(\mathcal{A} - \lambda_0)u_4 = -\mu_2 u_2 - \mu_3 u_1 + 2\mathcal{B}(u_1, u_3) + \mathcal{B}(u_2, u_2),$$

we observe that $\mu_2 u_2$ is orthogonal to u_1. The factors $e^{i\mathbf{k}\cdot\mathbf{x}}$ in the expression for $2\mathcal{B}(u_1, u_3) + \mathcal{B}(u_2, u_2)$ are such that

$$\mathbf{k} = \sum_{1 \leqq j \leqq 2q} m_j\mathbf{k}_j, \text{ and } \sum m_j = 4,$$

so that the scalar product with u_1 may be different from 0, as this can be seen in the case when q is a multiple of 3. It results from the compatibility condition that

$$\mu_3 = \frac{\langle 2\mathcal{B}(u_1, u_3) + \mathcal{B}(u_2, u_2), u_1\rangle_{0,0}}{\langle u_1, u_1\rangle_{0,0}}, \tag{61}$$

and

$$u_4 = \widetilde{(\mathcal{A} - \lambda_0)}^{-1}\mathbf{Q}_0[-\mu_2 u_2 + 2\mathcal{B}(u_1, u_3) + \mathcal{B}(u_2, u_2)]$$

is invariant under $\mathbf{R}_{\pi/q}$ and still in $\mathcal{D}_{1/2,s}$. Going on the computation, we obtain, in particular, that

$$\mu_4 = \frac{\langle 2\mathcal{B}(u_1, u_4) + 2\mathcal{B}(u_2, u_3), u_1\rangle_{0,0}}{\langle u_1, u_1\rangle_{0,0}}. \tag{62}$$

We show in [10] that we can go on in computing the successive terms of the series which appear to be of *Gevrey type*. Making an incomplete Borel resummation of these series, invariant under $\mathbf{R}_{\pi/q}$, provides a solution of (50) up to an exponentially small term as ε tends towards 0. Our purpose now is to improve such a result in proving that there indeed exist quasipatterns solutions of (50).

7. Adapted Formulation and Splitting of the Space

7.1. Decomposition of u

In all what follows, we study functions u, v in $\mathcal{K}_{0,s}$, invariant under rotations $\mathbf{R}_{\pi/q}$. In this frame the kernel of the linear operator $(\mathcal{A} - \lambda_0)$ is one-dimensional. Let us define the new unknown function $\tilde{v}$ in rewriting the solution of (50) in $\mathcal{K}_{0,s}$, $s > d/2$ as

$$\begin{aligned} u &= u_\varepsilon + \varepsilon^4\widetilde{v}, \;\; \mu = \mu_\varepsilon + \varepsilon^3\mu', \\ u_\varepsilon &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4, \\ \mu_\varepsilon &= \varepsilon^2\mu_2 + \varepsilon^3\mu_3, \quad \widetilde{v} \in \{u_1\}^\perp \cap \mathcal{K}_{0,s}, \end{aligned} \tag{63}$$

where the coefficients $u_1, u_2, u_3, u_4, \mu_2, \mu_3$ are defined above, and we assume below that $\varepsilon > 0$ (the same proof applies for $\varepsilon < 0$). Then

$$(\mathcal{A} - \lambda_0)u_\varepsilon = -\mu_\varepsilon u_\varepsilon + \mathcal{B}(u_\varepsilon, u_\varepsilon) + \varepsilon^5 f_\varepsilon,$$

where f_ε is a known quasiperiodic function with a finite Fourier expansion with $N_\mathbf{k} \leqq 8$. Now, we have, by (50), that

$$(\mathcal{A} - \lambda_0)(u_\varepsilon + \varepsilon^4\widetilde{v}) + (\mu_\varepsilon + \varepsilon^3\mu')(u_\varepsilon + \varepsilon^4\widetilde{v}) - \mathcal{B}(u_\varepsilon + \varepsilon^4\widetilde{v}, u_\varepsilon + \varepsilon^4\widetilde{v}) = 0,$$

which becomes

$$\mathfrak{L}_\varepsilon v + \varepsilon^3\mu'\widetilde{v} + \mu'\varepsilon^{-1}u_\varepsilon + \varepsilon f_\varepsilon - \varepsilon^4\mathcal{B}(\widetilde{v}, \widetilde{v}) = 0, \tag{64}$$

with

$$\mathfrak{L}_\varepsilon\widetilde{v} = (\mathcal{A} - \lambda_0 + \mu_\varepsilon)\widetilde{v} - 2\mathcal{B}(u_\varepsilon, \widetilde{v}). \tag{65}$$

7.2. Decomposition of the System

Let us use the projection $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ on the orthogonal complement of u_1 in the subspace of $\mathcal{K}_{0,0}$ invariant under $\mathbf{R}_{\pi/q}$, defined at Section 6.2. We might notice that the formal computation made at Section 6.3 gives $\widetilde{v} = \varepsilon u_5 + \mathcal{O}(\varepsilon^2)$ in $\{u_1\}^{\perp}$, with $\mu' = \varepsilon\mu_4 + \mathcal{O}(\varepsilon^2)$.

Equation (64) decomposes into the *bifurcation equation*, using the projection $\mathbf{P}_0$ onto the kernel $\{u_1\}$ of $(\mathcal{A} - \lambda_0)$:

$$\mu' u_1 + \varepsilon \mathbf{P}_0 f_\varepsilon - 2\mathbf{P}_0\mathcal{B}(\widetilde{v}, u_\varepsilon) - \varepsilon^4 \mathbf{P}_0 \mathcal{B}(\widetilde{v}, \widetilde{v}) = 0, \tag{66}$$

with

$$\mathbf{P}_0 f_0 = -\mu_4 u_1,$$

and the *range equation* (projection onto $\{u_1\}^{\perp}$) :

$$\mathbf{Q}_0\mathfrak{L}_\varepsilon \widetilde{v} + \varepsilon^3 \mu' \widetilde{v} + \widetilde{g}(\varepsilon, \mu') - \varepsilon^4 \mathbf{Q}_0 \mathcal{B}(\widetilde{v}, \widetilde{v}) = 0, \tag{67}$$

where

$$\begin{aligned} \mathbf{Q}_0\mathfrak{L}_\varepsilon &= \mathbf{Q}_0(\mathcal{A} - \lambda_0 + \mu_\varepsilon) - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon, \cdot), \\ \widetilde{g}(\varepsilon, \mu') &:= \mu'\varepsilon(u_2 + \varepsilon u_3 + \varepsilon^2 u_4) + \varepsilon \mathbf{Q}_0 f_\varepsilon. \end{aligned}$$

7.3. Optimization of Variables

In what follows, we need to obtain a solution of (67) which is C^2- bounded in $\widetilde{\mu}$, so we need to have operators and functions in (67) with bounded first and second derivatives with respect to $\widetilde{\mu} = \varepsilon^3\mu'$. This is not the case for the term $\widetilde{g}(\varepsilon, \mu')$, so we need to slightly modify the definition of $\widetilde{v}$ in such a way that $\widetilde{g}(\varepsilon, \mu')$ has a more suitable form.

Let us define (see Lemma 26, using that $u_\varepsilon \in \mathcal{K}_{0,t}$ for all $t > 0$) the linear operator $\mathcal{S}_\varepsilon$ bounded by $c_s\varepsilon$ in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for any $s \geqq 0$, as

$$\begin{aligned} \mathbf{Q}_0\mathfrak{L}_\varepsilon &= \mathbf{Q}_0(\mathcal{A} - \lambda_0) + \mathcal{S}_\varepsilon, \\ \mathcal{S}_\varepsilon &:= \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon, \cdot). \end{aligned}$$

We notice that $\widetilde{g}(\varepsilon, \mu')$ has a finite Fourier expansion with $N_{\mathbf{k}} \leqq 8$ (because of f_ε). Hence $[\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\widetilde{g}(\varepsilon, \mu') \in \mathbf{Q}_0\mathcal{K}_{0,s}$ for any $s \geqq 0$. In the same way, we can define

$$\begin{aligned} h(\varepsilon, \mu') = &\left\{ \mathbb{I} + \sum_{n=1,2,3,4} (-1)^n \left([\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}(\mathcal{S}_\varepsilon + \widetilde{\mu}) \right)^n \right\} \\ &\times [\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\widetilde{g}(\varepsilon, \mu'), \end{aligned} \tag{68}$$

which is still well-defined in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for $s \geqq 0$, and h is analytic in its arguments (ε, μ'). Indeed, the operator $\left([\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}(\mathcal{S}_\varepsilon + \widetilde{\mu})\right)^n$ is bounded on a finite Fourier series in $e^{i\mathbf{k}\cdot\mathbf{x}}$ leading to a finite Fourier series with $N_{\mathbf{k}}$ increased by $4n$.

Finally $h(\varepsilon, \mu')$ has a finite Fourier expansion with wave vectors bounded by $N_{\mathbf{k}} = 16 + 8 = 24$.

We can now check that

$$(\mathbf{Q}_0\mathfrak{L}_\varepsilon + \widetilde{\mu})h(\varepsilon, \mu') = \widetilde{g}(\varepsilon, \mu') + \Big((\mathcal{S}_\varepsilon + \widetilde{\mu})[\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\Big)^5 \widetilde{g}(\varepsilon, \mu').$$

We do not use Neumann series for inverting $(\mathbf{Q}_0\mathfrak{L}_\varepsilon + \widetilde{\mu})$ because of the small divisor difficulty. We notice that $\varepsilon^2\widetilde{g}(\varepsilon, \mu') = \widetilde{\mu}(u_2 + \varepsilon u_3 + \varepsilon^2 u_4) + \varepsilon^3\mathbf{Q}_0 f_\varepsilon$, hence $\varepsilon^2 h(\varepsilon, \mu') := \widetilde{h}(\varepsilon, \widetilde{\mu})$ is analytic in $(\varepsilon, \widetilde{\mu})$ with

$$||\widetilde{h}(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s(\varepsilon^3 + |\widetilde{\mu}|).$$

Now we define the new v as

$$v = \widetilde{v} + h(\varepsilon, \mu'), \tag{69}$$

so that (67) becomes

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}} v + g(\varepsilon, \widetilde{\mu}) - \varepsilon^4\mathbf{Q}_0\mathcal{B}(v, v) = 0, \tag{70}$$

with

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}} = \mathbf{Q}_0\mathfrak{L}_\varepsilon + \widetilde{\mu} + 2\varepsilon^2\mathbf{Q}_0\mathcal{B}(\widetilde{h}(\varepsilon, \widetilde{\mu}), \cdot), \tag{71}$$

$$g(\varepsilon, \widetilde{\mu}) = -\Big((\mathcal{S}_\varepsilon + \widetilde{\mu})[\mathbf{Q}_0(\mathcal{A} - \lambda_0)]^{-1}\Big)^5 \widetilde{g}(\varepsilon, \mu') - \mathbf{Q}_0\mathcal{B}(\widetilde{h}(\varepsilon, \widetilde{\mu}), \widetilde{h}(\varepsilon, \widetilde{\mu})).$$

We notice that the first term on the right hand side of $g(\varepsilon, \widetilde{\mu})$ is now C^4- bounded in $\widetilde{\mu}$ since, up to order $\widetilde{\mu}^4$ it is analytic, and the non analyticity only occurs at orders $\varepsilon^2\mu'\widetilde{\mu}^4$ and $\varepsilon\mu'\widetilde{\mu}^5$. Since we restrict to $\widetilde{\mu} \in [-\varepsilon, \varepsilon]$ the values for $\widetilde{\mu}$, we finally obtain in (70) the required properties for all terms, with

$$||g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s\varepsilon^2, \; ||\partial_{\varepsilon,\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s\varepsilon^2, \tag{72}$$

$$||\partial^2_{\widetilde{\mu}^2} g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s, \; ||\partial^2_{\varepsilon^2} g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s\varepsilon^2, \; ||\partial^2_{\varepsilon\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq c_s\varepsilon^2.$$

Let us define the linearized operator

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu},V} := \mathfrak{L}_{\varepsilon,\widetilde{\mu}} - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V, \cdot)$$

for $V \in \mathcal{K}_{0,s}$ for $s > d/2$. Then, we need a careful study of this linearized operator for applying the result of [2].

Lemma 35. *The operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ is analytic in its arguments for $(\varepsilon, \widetilde{\mu}, V) \in (0, \varepsilon_0) \times [-\varepsilon, \varepsilon] \times \mathbf{Q}_0\mathcal{K}_{0,s}$, $s \geqq s_0 > d/2$; it is acting in $\mathbf{Q}_0\mathcal{K}_{0,t}$ for $t \in [0, s]$ (see the result of Lemma 26), with*

$$\begin{aligned}\mathfrak{L}_{\varepsilon,\widetilde{\mu},V} \; &: = \mathbf{Q}_0(\mathcal{A} - \lambda_0) + \widetilde{\mu} + \mathcal{R}_{\varepsilon,\widetilde{\mu}} - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V, \cdot)\\ \mathcal{R}_{\varepsilon,\widetilde{\mu}} &= \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}(\varepsilon, \widetilde{\mu}), \cdot),\end{aligned} \tag{73}$$

and, for $||V||_{0,s_0} \leqq 1$, we have the estimates

$$\begin{aligned}||\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} &\leqq c_s\varepsilon||v||_{0,s},\\ ||\partial_{\widetilde{\mu}}\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} + ||\partial^2_{\widetilde{\mu}^2}\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} + ||\partial^2_{\varepsilon\widetilde{\mu}}\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} &\leqq c_s\varepsilon^2||v||_{0,s},\\ ||\partial_\varepsilon\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} + ||\partial^2_{\varepsilon^2}\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_{0,s} &\leqq c_s||v||_{0,s},\\ ||2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V, v)||_{0,s} &\leqq c_s\varepsilon^4(||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}).\end{aligned}$$

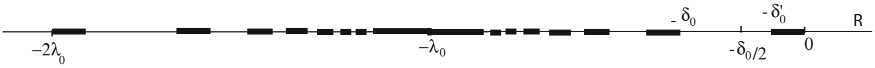

Fig. 5. Spectrum of $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$

7.4. First Splitting of the Space (Operator π_0)

We are interested in the inversion of the operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ in a certain subspace. The first difficulty comes from the infinite dimension of the system, despite of the use of a projection Π_N suppressing the Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ such that $N_{\mathbf{k}} > N$. Thus, we now use the property described in (48) for the spectrum of the operator $\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0$, which is selfadjoint in $\mathcal{K}_{0,s}$:

$$\begin{aligned}\lambda_0 - \lambda_0(|\mathbf{k}|^2) &\geqq 0,\\ \lambda_0 - \lambda_j(|\mathbf{k}|^2) &> \delta_0 > 0, \quad j = 1, 2, \ldots\\ \lambda_0(|\mathbf{k}|^2) &\to 0 \text{ as } |\mathbf{k}| \to 0 \text{ or } \infty.\end{aligned}$$

Let us consider $\delta_1 > 0$, defined at Lemma 34. Then for $\mathbf{k} \in \Gamma$, the inequality $||\mathbf{k}| - k_c| > \delta_1$ implies that $\lambda_0 - \lambda_0(|\mathbf{k}|^2) > \delta_0'$ $(= \mathcal{O}(\delta_1^2)$ (recall that $\lambda_0(|\mathbf{k}|^2)$ is analytic in $|\mathbf{k}|^2$ with a maximum λ_0 in k_c) and we choose δ_1 small enough to have $\delta_0' < \delta_0/2$. We now define the projection π_0, orthogonal in $\mathbf{Q}_0\mathcal{K}_{0,s}$, for any $s \geqq 0$, which consists in eliminating the Fourier modes $\mathbf{k} \in \Gamma$ such that $||\mathbf{k}| - k_c| > \delta_1$. We give at Fig. 5 a sketch of the spectrum of the selfadjoint operator $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$. We notice that the selfadjoint operator

$$(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)$$

has an inverse bounded by $1/\delta_0'$, since its eigenvalues (dense in the spectrum) are in absolute value larger than δ_0'.

Then, for $|\widetilde{\mu}| \leqq \varepsilon$ and $||V||_{0,s_0} \leqq 1$, $s_0 > d/2$, the operator

$$(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)$$

is a perturbation of order ε of $(\mathbb{I}-\pi_0)\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0(\mathbb{I}-\pi_0)$ (see (73)). For ε_0 small enough, we have, for $s \in [0, s_0]$, that

$$||\widetilde{\mu} + \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon, \cdot) - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V, \cdot)||_{0,s} \leqq c\varepsilon, \tag{74}$$

hence, for ε_0 small enough, and $\delta_0' > 2c\varepsilon$, the operator $(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)$ has an inverse bounded by $2/\delta_0'$ in $(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}$. Notice that a true estimate of the inverse in $\mathbf{Q}_0\mathcal{K}_{0,s}$ for $s > s_0$ would need a bound for $||V||_{0,s}$, which we do not have, except for $s = s_0$. Let us now show that the inversion of $\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}$ reduces to the inversion of a small perturbation $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ of $\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ for $d/2 < s_0$.

Indeed, let us consider the linear system

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}v = f \in \mathbf{Q}_0\mathcal{K}_{0,s_0}. \tag{75}$$

This leads to

$$\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(v_0+v_1)=\pi_0 f,$$
$$(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(v_0+v_1)=(\mathbb{I}-\pi_0)f,$$

where

$$v_0=\pi_0 v,\quad v_1=(\mathbb{I}-\pi_0)v.$$

Solving first with respect to v_1 gives

$$v_1=\mathfrak{Q}^{(1,1)}(\mathbb{I}-\pi_0)f+\mathfrak{Q}^{(1,0)}v_0, \tag{76}$$

with bounded operators $\mathfrak{Q}^{(1,1)}$ and $\mathfrak{Q}^{(1,0)}$ defined by

$$\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}=:[(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)]^{-1}\in\mathcal{L}((\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}), \tag{77}$$
$$\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V}=:-\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\in\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0},(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0}). \tag{78}$$

Then the system satisfied by v_0 becomes

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}v_0=\pi_0 f+\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)f, \tag{79}$$

with

$$\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}=:-\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}\in\mathcal{L}((\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s_0},\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}) \tag{80}$$
$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}:=\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}[\mathbb{I}+\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V}]\pi_0\in\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}). \tag{81}$$

We show in the next subsection, for $V\in\mathbf{Q}_0\mathcal{K}_{0,s}$ such that $||V||_{0,s_0}<1$ and δ'_0 well chosen, that there exists $c(s)>0$ with the following tame estimates, valid for $d/2<s_0\leqq s\leqq\overline{s}$ and $0\leqq\varepsilon\leqq\varepsilon_1(\overline{s})$:

$$||\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s}\leqq\frac{c(s)}{\delta'_0}\{||v||_{0,s}+\varepsilon^4||V||_{0,s}||v||_{0,s_0}\}\ \forall v\in(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s},$$
$$||\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s}\leqq\frac{c(s)}{\delta'_0}\varepsilon\{||v||_{0,s}+\varepsilon^4||V||_{0,s}||v||_{0,s_0}\}\ \forall v\in\pi_0\mathbf{Q}_0\mathcal{K}_{0,s},$$
$$||\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}v||_{0,s}\leqq\frac{c(s)}{\delta'_0}\varepsilon\{||v||_{0,s}+\varepsilon^4||V||_{0,s}||v||_{0,s_0}\}\ \forall v\in(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}. \tag{82}$$

7.5. Structure of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$

We need to study the structure of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ defined by (81). This is summed up in

Lemma 36. *For s such that $\overline{s}\geqq s\geqq s_0>d/2$, there exists $\varepsilon_0>0$ such that for $0<\varepsilon\leqq\varepsilon_1(\overline{s})\leqq\varepsilon_0$, $|\widetilde{\mu}|\leqq\varepsilon_0$, and $V\in\mathbf{Q}_0\mathcal{K}_{0,s}$, with $||V||_{0,s_0}\leqq 1$, we have*

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}=\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0+\widetilde{\mu}+\mathfrak{B}_\varepsilon+\varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}+\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}, \tag{83}$$

with

$$\mathfrak{B}_\varepsilon=-2\pi_0\mathbf{Q}_0\mathcal{B}(u_\varepsilon,\cdot)\mathbf{Q}_0\pi_0+\mathcal{O}(\varepsilon^2),$$

$\mathfrak{B}_\varepsilon$, $\mathfrak{C}_{\varepsilon,\widetilde{\mu}}$ and $\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}$ depend analytically on their arguments, with $\mathfrak{R}_{\varepsilon,\widetilde{\mu},0} = 0$ and a constant $c(s)$ such that for any $v \in \pi_0\mathbf{Q}_0\mathcal{K}_{0,s}$

$$\begin{aligned}
||\mathfrak{B}_\varepsilon v||_{0,s} &\leqq c\varepsilon||v||_{0,s},\\
||\mathfrak{C}_{\varepsilon,\widetilde{\mu}}v||_{0,s} + ||\partial_{\widetilde{\mu}}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}v||_{0,s} &\leqq c||v||_{0,s},\\
||\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leqq c\varepsilon^4\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\},\\
||\partial_{\widetilde{\mu}}\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leqq c\varepsilon^4\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\},\\
||\partial_\varepsilon\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}v||_{0,s} &\leqq c\varepsilon^3\{||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}\}.
\end{aligned} \tag{84}$$

Proof. We examine first $\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}$ which is the inverse of $(\mathbb{I} - \pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)$. Thanks to (73), we can write

$$\begin{aligned}
(\mathbb{I} - \pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0) = {}& (\mathbb{I} - \pi_0)\mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0(\mathbb{I} - \pi_0) + \widetilde{\mu}\mathbb{I}d\\
&+ (\mathbb{I} - \pi_0)\mathcal{P}(\varepsilon, \widetilde{\mu}, V)(\mathbb{I} - \pi_0),
\end{aligned} \tag{85}$$

where $\mathbb{I}d$ is the identity in the subspace $(\mathbb{I} - \pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$ and

$$\mathcal{P}(\varepsilon, \widetilde{\mu}, V) =: \mu_\varepsilon - 2\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h}(\varepsilon, \widetilde{\mu}), \cdot) - 2\varepsilon^4\mathbf{Q}_0\mathcal{B}(V, \cdot).$$

Now, for $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$, the operator $\mathcal{P}(\varepsilon, \widetilde{\mu}, V)$ takes values in $\mathcal{L}((\mathbb{I} - \pi_0)\mathbf{Q}_0\mathcal{K}_{0,s})$ for $d/2 < s_0 \leqq s \leqq \overline{s}$, and satisfies for $\varepsilon \in [0, \varepsilon_0]$, ε_0 small enough and $||V||_{0,s_0} \leqq 1$, $|\widetilde{\mu}| \leqq \varepsilon$,

$$||(\mathbb{I} - \pi_0)\mathcal{P}(\varepsilon, \widetilde{\mu}, V)(\mathbb{I} - \pi_0)v||_{0,s} \leqq c\{\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}\}.$$

Let us define the operator

$$\mathfrak{S} =: [(\mathbb{I} - \pi_0)\mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0(\mathbb{I} - \pi_0)]^{-1}(\mathbb{I} - \pi_0)\{\widetilde{\mu} + \mathcal{P}(\varepsilon, \widetilde{\mu}, V)(\mathbb{I} - \pi_0)\},$$

then

$$[(\mathbb{I} - \pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)]^{-1} = (\mathbb{I} + \mathfrak{S})^{-1}[(\mathbb{I} - \pi_0)\mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0(\mathbb{I} - \pi_0)]^{-1},$$

then we need to invert $(\mathbb{I} + \mathfrak{S})$ in checking a tame estimate.

For $\varepsilon < \varepsilon_1(\overline{s}) \leqq \varepsilon_0$, $|\widetilde{\mu}| \leqq \varepsilon$ and $||V||_{0,s_0} \leqq 1$, there exists a constant $c > 0$ such that for any $v \in (\mathbb{I} - \pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$

$$||\mathfrak{S}v||_{0,s} \leqq \frac{c}{\delta_0'}[\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}],$$

and for ε_0 small enough such that $(\varepsilon + \varepsilon^4) \leqq 2\varepsilon$, we have for any $p \in \mathbb{N}$

$$||\mathfrak{S}^p v||_{0,s} \leqq \frac{c}{\delta_0'}\left(\frac{2c\varepsilon_0}{\delta_0'}\right)^{p-1}[(|\widetilde{\mu}| + \varepsilon)||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}],$$

hence for any $v \in (\mathbb{I} - \pi_0)\mathbf{Q}_0\mathcal{K}_{0,s}$

$$||(\mathbb{I} + \mathfrak{S})^{-1}v||_{0,s} \leqq ||v||_{0,s} + \frac{c}{\delta_0'}\left(1 - \frac{2c\varepsilon_0}{\delta_0'}\right)^{-1}[\varepsilon||v||_{0,s} + \varepsilon^4||V||_{0,s}||v||_{0,s_0}].$$

It results that, for $\delta_0' > 4c\varepsilon_0$, $[(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)]^{-1} = \mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}$ is analytic in its arguments and satisfies, for ε_0 small enough, the estimate

$$||\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V} v||_{0,s} \leqq c'/\delta_0'(||v||_{0,s} + \varepsilon^4 ||V||_{0,s}||v||_{0,s_0}) \ \ \forall v \in (\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,s},$$

which is the first part of (82). Coming back to (81) with (78) and (80), we observe that

$$(\mathbb{I}-\pi_0)\mathbf{Q}_0[(\mathcal{A}-\lambda_0)+\mu_\varepsilon+\widetilde{\mu}]\mathbf{Q}_0\pi_0 = \pi_0\mathbf{Q}_0[(\mathcal{A}-\lambda_0)+\mu_\varepsilon+\widetilde{\mu}]\mathbf{Q}_0(\mathbb{I}-\pi_0) = 0,$$

because the coefficients of the linear operator are independent of $\mathbf{x}$. Then

$$\begin{aligned}
&(\mathbb{I}-\pi_0)\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0 \\
&\quad = -2(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h},\cdot)\mathbf{Q}_0\pi_0 - 2\varepsilon^4(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{B}(V,\cdot)\pi_0,\\
&\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I}-\pi_0)\\
&\quad = -2\pi_0\mathbf{Q}_0\mathcal{B}(u_\varepsilon - \varepsilon^2\widetilde{h},\cdot)\mathbf{Q}_0(\mathbb{I}-\pi_0) - 2\varepsilon^4\pi_0\mathbf{Q}_0\mathcal{B}(V,\cdot)(\mathbb{I}-\pi_0),
\end{aligned}$$

both operators being of order ε (with the tame estimates), and depending analytically on their arguments. It finally results from (78) and (80) that the rest of estimate (82) holds. Finally

$$\pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V} = \mathfrak{C}^{(1)}_\varepsilon + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}, \tag{86}$$

with $\mathfrak{C}^{(1)}_\varepsilon$, $\mathfrak{C}_{\varepsilon,\widetilde{\mu}}$ and $\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}$ analytic in their arguments, taking values in $\mathcal{L}(\pi_0\mathbf{Q}_0\mathcal{K}_{0,s})$ for $s_0 \leqq s \leq \overline{s}$, and a careful examination of (77), (78), (81), (73) leads $\forall v \in \pi_0\mathbf{Q}_0\mathcal{K}_{0,s}$, to

$$\begin{aligned}
\mathfrak{C}^{(1)}_\varepsilon &= \mathcal{O}(\varepsilon^2),\\
||\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V} v||_{0,s} &\leqq c\varepsilon^5(||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}),\\
||\partial_{\varepsilon,\widetilde{\mu}}\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V} v||_{0,s} &\leqq c\varepsilon^4(||v||_{0,s} + ||V||_{0,s}||v||_{0,s_0}).
\end{aligned}$$

Finally, from (81), we can write

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} = \pi_0\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\pi_0 + \mathfrak{C}^{(1)}_\varepsilon + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}, \tag{87}$$

where $\mathfrak{R}'_{\varepsilon,\widetilde{\mu},V}$ is at least linear in V. This leads to (83), (84) and to the result of the Lemma. □

Remark 37. We may observe that the spectrum of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ results from a perturbation of order ε of the spectrum of the selfadjoint operator $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$, the spectrum of which is the closure of the set of eigenvalues $\lambda_j(|\mathbf{k}|^2)-\lambda_0$, $j = 0, 1, \ldots$, $\mathbf{k} \in \Gamma$, with

$$\begin{aligned}
&-\delta_0' \leqq \lambda_0(|\mathbf{k}|^2) - \lambda_0 < 0, \text{ and } \pm\lambda_j(|\mathbf{k}|^2) - \lambda_0 < -\delta_0, \ \ j = 1, 2, \ldots\\
&\text{and } -\lambda_0(|\mathbf{k}|^2) - \lambda_0 < -\delta_0 \quad \text{where } \mathbf{k} \in \Gamma \text{ with } 0 < ||\mathbf{k}| - k_c| \leqq \delta_1.
\end{aligned}$$

It results that, for ε small enough, the spectrum of $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ in $\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ has a gap in its real part, between $-3\delta_0/4$ and $-\delta_0/2$. Hence the eigenvalues which might be close to 0, are those coming from $\lambda_0(|\mathbf{k}|^2) - \lambda_0$ uniquely, and this allows us to come back to a situation analogue to the one in [3], except for the selfadjointness of the operator which is not true here, starting at order ε.

Remark 38. We notice that the restriction on δ_1 leads to a restriction on $\delta_0' = \mathcal{O}(\delta_1^2)$. The restriction on δ_0' made in the proof of Lemma above is independent of ε_0, for ε_0 small enough.

Remark 39. The operator $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ depends analytically on $(\varepsilon, \widetilde{\mu}, V)$, therefore, we can give its expression for $\varepsilon = 0$. From Lemma 36 we have

$$\mathfrak{L}'_{0,\widetilde{\mu},V} = \pi_0 \mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0\pi_0 + \widetilde{\mu}\mathbb{I}. \tag{88}$$

Coming back to the linear Equation (75), we finally have

Lemma 40. *For* $s_0 > d/2$, $\overline{s} > s_0$, $0 < \varepsilon \leqq \varepsilon_1(\overline{s}) \leqq \varepsilon_0$, $|\widetilde{\mu}| \leqq \varepsilon_0$, $V \in \mathbf{Q}_0\mathcal{K}_{0,s}$ *such that* $||V||_{0,s_0} \leqq 1$, *and* $s \in [s_0, \overline{s}]$, *assume that there exists* $C(s) > 0$ *such that*

$$||\mathfrak{L}'^{-1}_{\varepsilon,\widetilde{\mu},V} f_0||_{0,s} \leqq C(s)[||f_0||_{0,s} + ||V||_{0,s}||f_0||_{0,s_0}], \text{ for any } f_0 = \pi_0 f, \\ \text{with } f \in \mathbf{Q}_0\mathcal{K}_{0,s}.$$

Then, for $s \in [s_0, \overline{s}]$, ε_0 *small enough and* $f \in \mathbf{Q}_0\mathcal{K}_{0,s}$,

$$||\mathfrak{L}^{-1}_{\varepsilon,\widetilde{\mu},V} f||_{0,s} \leqq C'(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \tag{89}$$

where $C'(s) = 3Cs) + c(s)/\delta_0'$.

Proof. We start with (79) and the estimate for $\mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}$ in (82). We obtain, for ε small enough

$$\begin{aligned} ||v_0||_{0,s} &\leqq C(s)[||\pi_0 f + \mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)f||_{0,s} \\ &\quad + ||V||_{0,s}||\pi_0 f + \mathfrak{Q}^{(0,1)}_{\varepsilon,\widetilde{\mu},V}(\mathbb{I} - \pi_0)f||_{0,s_0}] \\ &\leqq 2C(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}]. \end{aligned}$$

Now, using (76) with (82), we obtain, successively,

$$\begin{aligned} ||\mathfrak{Q}^{(1,0)}_{\varepsilon,\widetilde{\mu},V} v_0||_{0,s} &\leqq 2\varepsilon\frac{c(s)}{\delta_0'}C(s)[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \\ ||v_1||_{0,s} &\leqq \frac{2c(s)}{\delta_0'}[||f||_{0,s} + ||V||_{0,s}||f||_{0,s_0}], \end{aligned}$$

and $v_0 + v_1$ is $\mathfrak{L}^{-1}_{\varepsilon,\widetilde{\mu},V} f$ for which (89) holds in the norm $\mathbf{Q}_0\mathcal{K}_{0,s}$. □

7.6. Projection Π_N

We define the projection Π_N as the suppression of Fourier modes with $\mathbf{k} \in \Gamma$ such that $N_{\mathbf{k}} > N$. The range of this projection is then

$$E_N := \Pi_N \pi_0 \mathbf{Q}_0 \mathcal{K}_{0,s},$$

which is in fact independent of s (however its norm depends on s), and where we do not forget that coefficients are functions of $z \in [0, 1]$, here in L^2. A difference with the spaces E_N occurring in [2,3] (for example), is that our E_N is *infinite dimensional*. However *the spectrum of the linear operator* $\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N$ *is discrete* since, for a given V, it is a perturbation of the operator $\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$, where the number of Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ is finite (number $\mathcal{N}$ bounded by bN^d, d being defined in Section 3 and b independent of N), and that for any fixed $|\mathbf{k}|$, the spectrum of $\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$ is discrete, only composed with eigenvalues of finite multiplicities. Notice also that

$$\Pi_N\pi_0 = \pi_0\Pi_N,$$

and that Lemma 40 is still valid, when restricted to E_N.

8. Estimates of the Inverse of $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)$

8.1. Estimate of $(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}$ *in* $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$ *for small N*

Lemma 41. *Let* $s_0 > d/2$, $V \in \mathcal{K}_{0,s_0}$ *satisfies* $||V||_{0,s_0} \leqq 1$, *and assume* $(\varepsilon,\widetilde{\mu}) \in [0,\varepsilon_0]\times[-\varepsilon,\varepsilon]$. *Then for* $N \leqq M_\varepsilon$, *where* M_ε *is defined by* (92)*, we have the following estimates:*

$$||(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}v||_{0,s_0} \leqq 2c(1+N^2)^{2l_0}||v||_{0,s_0}, \text{ for } v\in\Pi_N\pi_0 Q_0\mathcal{K}_{0,s_0}, \tag{90}$$

$$||(\Pi_N\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}v||_{0,s_0} \leqq 2cc'(1+N^2)^{2l_0}||v||_{0,s_0} \text{ for } v\in\Pi_N Q_0\mathcal{K}_{0,s_0}. \tag{91}$$

Proof. We use Lemma 36 and

$$\begin{aligned}\Pi_N\pi_0\mathbf{Q}_0\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\mathbf{Q}_0\pi_0\Pi_N &= \Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N + \widetilde{\mu} + \\ &\quad + \Pi_N\pi_0\mathbf{Q}_0[\mathfrak{B}_\varepsilon + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}_{\varepsilon,\widetilde{\mu},V}]\mathbf{Q}_0\pi_0\Pi_N,\end{aligned}$$

with the estimates (for $|\widetilde{\mu}| \leqq \varepsilon$)

$$||\Pi_N\pi_0\mathbf{Q}_0[\widetilde{\mu} + \mathfrak{B}_\varepsilon + \varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}_{\varepsilon,\widetilde{\mu},V}]\mathbf{Q}_0\pi_0\Pi_N||_{0,s_0} \leqq c_1\varepsilon.$$

Now, by construction, and from Lemma 34, we have

$$||(\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N)^{-1}||_{0,s_0} \leqq c(1+N^2)^{2l_0}.$$

Then, if we have

$$cc_1\varepsilon(1+N^2)^{2l_0} \leqq 1/2,$$

we can use Neumann series to invert the operator $(\Pi_N\mathfrak{L}'_{\varepsilon,\tilde{\mu},V}\Pi_N)$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{0,s_0}$, and obtain (90) provided that

$$N \leqq M_\varepsilon = \left[\frac{c_2}{\varepsilon^{1/4l_0}}\right] \leqq \left(\frac{1}{(2cc_1\varepsilon)^{1/2l_0}} - 1\right)^{1/2}, \tag{92}$$

where the brackets $[\cdot]$ mean the integer part of. The result for $(\Pi_N\mathfrak{L}_{\varepsilon,\widetilde{\mu},V}\Pi_N)^{-1}$ comes from Lemma 40. □

8.2. Good Set of $\widetilde{\mu}$

Let us define for $M > 0$, $s_0 > d/2$ the set

$$\mathcal{U}_M^{(N)} := \{u \in C^2([0, \varepsilon_1[\times[-\varepsilon, \varepsilon], E_N);\ u(0, \widetilde{\mu}) = 0, \\ ||u||_{0,s_0} \leqq 1,\ ||\partial_{\varepsilon,\widetilde{\mu}} u||_{0,s_0} \leqq M,\ ||\partial^2_{\varepsilon,\widetilde{\mu}} u||_{0,s_0} \leqq M\}. \tag{93}$$

We do not forget that Lemma 36 says that operator $\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}$ is analytic in $(\varepsilon, \widetilde{\mu}, V)$.

Now, for $V \in \mathcal{U}_M^{(N)}$ we need to study the *inverse* of $\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N$, when it exists, in function of $\widetilde{\mu}$ for ε fixed. As an operator in $\mathcal{L}(E_N)$ with the norm induced by $\mathcal{L}(\mathcal{K}_{0,s_0})$, its eigenvalues result from a small perturbation of the selfadjoint operator $\Pi_N \pi_0 \mathbf{Q}_0(\mathcal{A} - \lambda_0)\pi_0 \Pi_N \mathbf{Q}_0$ which has a discrete set of eigenvalues (notice that since we do not impose a bound on $||V||_{0,s}$, the perturbation might not be small for $s > s_0$). Since we are only interested in the eigenvalues very close to 0, the eigenvalues which interest us are the ones which perturb the *(negative) eigenvalues* $\lambda_0(|\mathbf{k}|^2) - \lambda_0$ *close to* 0, obtained for $|\mathbf{k}|$ near k_c.

For $s = s_0$, let us introduce the projection Π' commuting with $\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N$, which is associated with this group of eigenvalues close to 0 (separated from the rest of the spectrum at a distance at least $\delta_0/4$). We then apply the results (such as [13] Theorem 6.17 p.178) on bounded operators with a separation of the spectrum in two bounded parts. We then obtain that the spectrum of the operator

$$(\mathbb{I} - \Pi')\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N (\mathbb{I} - \Pi')$$

lies at a distance at least $3\delta_0/4$ from 0, hence its inverse is bounded by a constant C. We can then proceed exactly as with the projection π_0 at Section 7.4 and prove the following:

Lemma 42. *For* $s_0 > d/2$, $0 < \varepsilon \leqq \varepsilon_0$, $|\widetilde{\mu}| \leqq \varepsilon$, $V \in \mathbf{Q}_0\mathcal{K}_{0,s_0}$ *such that* $||V||_{0,s_0} \leqq 1$, *there exists* $c'' > 0$ *such that*

$$||(\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N)^{-1}||_{0,s_0} \leqq c''||(\Pi' \Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N \Pi')^{-1}||_{0,s_0}\ .$$

We are in the same finite-dimensional space as in [3]. The definition of the good set of $\widetilde{\mu}$ is only linked with the finite set of eigenvalues perturbing $\lambda_0(|\mathbf{k}|^2) - \lambda_0$ for $\mathbf{k} \in \Gamma$, $||\mathbf{k}| - k_c| \leqq \delta_1$, and located in the strip

$$-3\delta_0/4 < \mathrm{Re}(\cdot) < \delta_0/4$$

for ε small enough. However, we cannot use directly the method of [3], since *the operator* $\Pi' \Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N \Pi'$ *is not selfadjoint.*

From Lemma 36 we have

$$\Pi' \Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N \Pi' = \Pi' \Pi_N \pi_0 \mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0 \pi_0 \Pi_N \Pi' + \widetilde{\mu}\mathbb{I}d + \\ + \Pi' \Pi_N \mathfrak{B}_\varepsilon \Pi_N \Pi' + \varepsilon^2 \widetilde{\mu} \Pi' \Pi_N \mathfrak{C}_{\varepsilon,\widetilde{\mu}} \Pi_N \Pi' \\ + \Pi' \Pi_N \mathfrak{R}_{\varepsilon,\widetilde{\mu},V} \Pi_N \Pi',$$

where $\mathbb{I}d$ is the identity in $\Pi' E_N$. The new property is that the negative selfadjoint operator $\Pi'\Pi_N\pi_0\mathbf{Q}_0(A-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N\Pi'$ satisfies

$$||\Pi'\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N\Pi'||_{0,s_0} \leqq \delta_0', \tag{94}$$

which is the size of its spectrum even in absence of Π_N (the norm in $\mathcal{L}(E_N)$ is the norm induced by $\mathcal{L}(\mathcal{K}_{0,s_0})$).

In the sequel of this subsection and the next one, we simplify the notations in defining

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} =: \Pi'\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi', \tag{95}$$

which is analytic in $(\varepsilon,\widetilde{\mu})$ when $V=0$. Then we define

$$V(\varepsilon,\widetilde{\mu}) = V_0(\varepsilon) + V_1(\varepsilon,\widetilde{\mu}),$$

where V_0, V_1 are C^2 in their arguments, and V_1 satisfies (see properties required in $\mathcal{U}_M^{(N)}$)

$$||V_1(\varepsilon,\widetilde{\mu})||_{0,s_0} \leqq M|\widetilde{\mu}|,\ ||\partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu})||_{0,s_0} \leqq M,\ ||\partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu}_2) - \partial_{\widetilde{\mu}}V_1(\varepsilon,\widetilde{\mu}_1)|| \leqq M|\widetilde{\mu}_2-\widetilde{\mu}_1|.$$

Then we also decompose $\mathfrak{R}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}$ as

$$\mathfrak{R}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} = \varepsilon^4\mathfrak{R}^{(0)}_{\varepsilon} + \varepsilon^4\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}},$$

where $\mathfrak{R}^{(0)}_{\varepsilon}$, $\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}$ are C^2 in their arguments, and $\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}$ satisfies

$$||\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}v||_{0,s_0} \leqq M|\widetilde{\mu}|||v||_{0,s_0},\ ||\partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}}v||_{0,s_0} \leqq M||v||_{0,s_0}$$
$$||(\partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}_2} - \partial_{\widetilde{\mu}}\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}_1})v||_{0,s_0} \leqq M|\widetilde{\mu}_2-\widetilde{\mu}_1|||v||_{0,s_0}.$$

Then,

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} = (\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \widetilde{\mu}\mathbb{I}d + \varepsilon^2\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}},$$

with

$$(\widetilde{\mathcal{A}-\lambda_0})_N =: \Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$$

$$\mathfrak{B}'^{(N)}_{\varepsilon} = \Pi_N(\mathfrak{B}_{\varepsilon} + \varepsilon^4\mathfrak{R}^{(0)}_{\varepsilon})\Pi_N,$$
$$\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}} = \Pi_N(\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \varepsilon^2\mathfrak{R}^{(1)}_{\varepsilon,\widetilde{\mu}})\Pi_N,$$

Let us now consider the selfadjoint operator $\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}}$, which may now be written as

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}} = \widetilde{\mu}^2\mathbb{I}d + \widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} + \widetilde{\mathfrak{B}}^{(N)}_{\varepsilon}, \tag{96}$$

where (we simplify in omitting below the writting of Π')

$$\widetilde{\mathfrak{B}}^{(N)}_{\varepsilon} = (\widetilde{\mathcal{A}-\lambda_0})^2_N + \mathfrak{B}'^{(N)}_{\varepsilon}(\widetilde{\mathcal{A}-\lambda_0})_N + (\widetilde{\mathcal{A}-\lambda_0})_N\mathfrak{B}'^{(N)*}_{\varepsilon} + \mathfrak{B}'^{(N)}_{\varepsilon}\mathfrak{B}'^{(N)*}_{\varepsilon},$$
$$\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}} = \widetilde{\mu}[2(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \mathfrak{B}'^{(N)*}_{\varepsilon}] + \varepsilon^2[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)}_{\varepsilon} + \widetilde{\mu}]\mathfrak{C}'^{(N)*}_{\varepsilon,\widetilde{\mu}} + $$
$$+ \varepsilon^2\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}}[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}'^{(N)*}_{\varepsilon} + \widetilde{\mu}] + \varepsilon^4\mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}}\mathfrak{C}'^{(N)*}_{\varepsilon,\widetilde{\mu}},$$

where the adjoint is taken with the scalar product in E_N induced by the scalar product in $\mathcal{K}_{0,s_0}$. Operators $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$ and $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$ are C^1 in their arguments. Moreover there exists $c > 0$ such that

$$\begin{aligned} ||\widetilde{\mathfrak{B}}_{\varepsilon_2}^{(N)} - \widetilde{\mathfrak{B}}_{\varepsilon_1}^{(N)}||_{0s_0} &\leqq c(\delta_0' + \varepsilon)|\varepsilon_2 - \varepsilon_1|, \quad \widetilde{\mathfrak{C}}_{\varepsilon,0}^{(N)} = 0 \\ ||\widetilde{\mathfrak{C}}_{\varepsilon_2,\widetilde{\mu}_2}^{(N)} - \widetilde{\mathfrak{C}}_{\varepsilon_1,\widetilde{\mu}_1}^{(N)}||_{0,s_0} &\leqq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + |\widetilde{\mu}_2 - \widetilde{\mu}_1|), \\ ||\partial_{\widetilde{\mu}}\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}_2}^{(N)} - \partial_{\widetilde{\mu}}\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}_1}^{(N)}||_{0,s_0} &\leqq c\varepsilon^2|\widetilde{\mu}_2 - \widetilde{\mu}_1|. \end{aligned} \tag{97}$$

Let us now define

Definition 43. For $V \in \mathcal{U}_M^{(N)}$ and $\tau, \gamma > 0$ (to be determined later), the "good" set of $\widetilde{\mu}$ is the set

$$G_{\varepsilon,\gamma}^{(N)}(V) := \Big\{ \widetilde{\mu} \in [-\varepsilon, \varepsilon]; \ ||(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi')^{-1}v||_{0,s_0} \leqq \frac{N^\tau}{\gamma}||v||_{0,s_0}, \quad \text{for any } v \in \Pi' E_N \Big\},$$

where $||\cdot||_{0,s}$ means the norm in $\mathcal{L}(E_N)$ induced by $\mathcal{L}(\mathcal{K}_{0,s})$.

Saying that $\widetilde{\mu}$ is "good", i.e. $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V)$, implies that the positive selfadjoint operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ has all its eigenvalues larger than $(\frac{\gamma}{N^\tau})^2$. It is now possible to give a bound for the measure of the bad set for $\widetilde{\mu}$.

8.3. *Bad Set of* $\widetilde{\mu}$

By definition, the bad set of $\widetilde{\mu}$ is the complement of the good set. Hence, for $V \in \mathcal{U}_M^{(N)}$,

$$B_{\varepsilon,\gamma}^{(N)}(V) := \left\{ \begin{array}{c} \widetilde{\mu} \in [-\varepsilon, \varepsilon]; \ \exists v \in \Pi' E_N \text{ such that} \\ ||(\Pi'\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V}\Pi_N\Pi')^{-1}v||_{0,s_0} > \frac{N^\tau}{\gamma}||v||_{0,s_0}. \end{array} \right\}.$$

Now we prove the following:

Lemma 44. *Assume that $N > M_\varepsilon$, $d/2 < s_0, \tau > d + 12l_0$, $(\varepsilon, \widetilde{\mu}) \in (0, \varepsilon_1] \times [-\varepsilon, \varepsilon]$, and $V \in \mathcal{U}_M^{(N)}$. Moreover assume that Condition 47 holds, then there exists a constant $C > 0$, such that the measure of $B_{\varepsilon,\gamma}^{(N)}(V)$ is bounded by*

$$\frac{C\gamma}{N^{\tau-d}}.$$

The following proof only considers eigenvalues close to 0, i.e., we use, without mentioning it, the projection Π' which eliminates the infinite dimensional subspace corresponding to "large" eigenvalues.

Let us prove the following:

Lemma 45. *For ε small enough, $\widetilde{\mu} \in [-\varepsilon, \varepsilon], s_0 > d/2$, the eigenvalues of $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)} \mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ take the form*

$$\sigma_j(\varepsilon, \widetilde{\mu}) = \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu}), \tag{98}$$

where $f_j(\varepsilon, \widetilde{\mu})$ is Lipschitz in $(\varepsilon, \widetilde{\mu})$ with

$$|f_j(\varepsilon_2, \widetilde{\mu}_2) - f_j(\varepsilon_1, \widetilde{\mu}_1)| \leqq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + |\widetilde{\mu}_2 - \widetilde{\mu}_1|. \tag{99}$$

Moreover, for ε fixed, $f_j(\varepsilon, \widetilde{\mu})$ is C^2 with respect to $\widetilde{\mu}$.

Proof. We use the Lidskii theorem (see [13] theorem 6.10 p.126) for comparing the eigenvalues f_j of operators $\widetilde{\mathfrak{C}}_{\varepsilon_2,\widetilde{\mu}_2}^{(N)} + \widetilde{\mathfrak{B}}_{\varepsilon_2}^{(N)}$ and $\widetilde{\mathfrak{C}}_{\varepsilon_1,\widetilde{\mu}_1}^{(N)} + \widetilde{\mathfrak{B}}_{\varepsilon_1}^{(N)}$, and the estimate (97), which directly leads to (99). Then, it remains to add $\widetilde{\mu}^2$ for obtaing the eigenvalues σ_j of $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)} \mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$. The property that $f_j(\varepsilon, \widetilde{\mu})$ is C^2 with respect to $\widetilde{\mu}$ results from the selfadjointness and from [13]; see p.115 and the proof of theorem 6.8 p.122 applied on the reduced operator (using the eigenprojection associated with a group of eigenvalues which split for $\widetilde{\mu}$ close to $\widetilde{\mu}_0$).

Remark 46. Let us consider eigenvalues $\widetilde{\mu} g_j(\varepsilon, \widetilde{\mu})$ of the selfadjoint operator $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$ which we write as

$$\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)} = \widetilde{\mu} \widetilde{\mathfrak{C}}_{\varepsilon}^{(1)} + \widetilde{\mu} \widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(2)},$$

where $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(2)}$ is C^1 in $\widetilde{\mu}$ and

$$\widetilde{\mathfrak{C}}_{\varepsilon,0}^{(2)} = 0, \ \partial_{\widetilde{\mu}} \widetilde{\mathfrak{C}}_{\varepsilon,0}^{(2)} = 0, \ ||\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(2)}||_{0,s_0} \leqq c\varepsilon^2 |\widetilde{\mu}|. \tag{100}$$

By the Lidskii theorem we know that

$$\widetilde{\mu} g_j(\varepsilon, \widetilde{\mu}) = \widetilde{\mu} g_j^{(1)}(\varepsilon) + \widetilde{\mu} g_j^{(2)}(\varepsilon, \widetilde{\mu}),$$

with

$$g_j^{(1)}(\varepsilon) \text{ eigenvalue of } \widetilde{\mathfrak{C}}_{\varepsilon}^{(1)},$$

and $\{g_1^{(2)}(\varepsilon, \widetilde{\mu}), \ldots, g_{\mathcal{N}}^{(2)}(\varepsilon, \widetilde{\mu})\}$ belongs to the convex hull of the vectors obtained from $\{\gamma_1, \ldots \gamma_{\mathcal{N}}\}$ by all possible permutations, where γ_j 's are the eigenvalues of $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(2)}$ in E_N. Then, because of (100), we obtain

$$|g_j^{(2)}(\varepsilon, \widetilde{\mu})| \leqq c\varepsilon^2 |\widetilde{\mu}|.$$

Applying again the Lidskii theorem, in considering the eigenvalues $f_j(\varepsilon, \widetilde{\mu})$ of the selfadjoint operator $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)} + \widetilde{\mathfrak{B}}_{\varepsilon}^{(N)}$, this leads to

$$f_j(\varepsilon, \widetilde{\mu}) = s_\varepsilon + \widetilde{\mu} f_j^{(1)}(\varepsilon, \widetilde{\mu}),$$

where $\widetilde{\mu} f_j^{(1)}(\varepsilon, \widetilde{\mu})$ belongs to the convex hull of the vectors obtained from $\{\widetilde{\mu} g_1(\varepsilon, \widetilde{\mu}), \ldots \widetilde{\mu} g_{\mathcal{N}}(\varepsilon, \widetilde{\mu})\}$ by all possible permutations, where $\widetilde{\mu} g_j(\varepsilon, \widetilde{\mu})$'s are the eigenvalues of $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$ in E_N, and we cannot decompose $f_j^{(1)}(\varepsilon, \widetilde{\mu})$ as $f_j^{(1)}(\varepsilon, 0) +$

$f_j^{(2)}(\varepsilon,\widetilde{\mu})$, with $f_j^{(2)}(\varepsilon,\widetilde{\mu})$ Lipschitz in $\widetilde{\mu}$. For being able to claim such a decomposition, we need to control the Lipschitz constant with respect to $\widetilde{\mu}$ of the second derivative with respect to $\widetilde{\mu}$, in 0 of $f_j(\varepsilon,\widetilde{\mu})$. It is shown for example in [13] that such information uses a bound for the pseudo-inverse of $\widetilde{\mathfrak{B}}_\varepsilon^{(N)} - s_\varepsilon$, which is of the size of the inverse of the distance of s_ε from the spectrum of $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$. This distance is unfortunately very small, of order N^{-4l_0}.

Let us now try another way. For a given ε, let us consider an eigenvalue s_ε of $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$, and define the associated orthogonal eigenprojection $\mathbf{P}_\varepsilon$. Then, because $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$ is selfadjoint, we have

$$\mathbf{P}_\varepsilon(\widetilde{\mathfrak{B}}_\varepsilon^{(N)} - s_\varepsilon) = 0.$$

The operator $\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$ acts as a perturbation, and let us consider f_j which belongs to the s_ε-group of eigenvalues, resulting from the perturbation of s_ε, and denote by $\mathbf{P}_{\varepsilon,\widetilde{\mu}}$ the orthogonal eigenprojection associated with the s_ε-group of eigenvalues. Then, by definition, there is an eigenvector $\zeta_j(\varepsilon,\widetilde{\mu})$ satisfying

$$\{\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)} + \widetilde{\mathfrak{B}}_\varepsilon^{(N)} - f_j(\varepsilon,\widetilde{\mu})\}\zeta_j(\varepsilon,\widetilde{\mu}) = 0,$$

which is equivalent to

$$\mathbf{P}_\varepsilon\{\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)} + s_\varepsilon - f_j(\varepsilon,\widetilde{\mu})\}\zeta_j(\varepsilon,\widetilde{\mu}) = 0.$$

We have $\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu}) \in \mathbf{P}_\varepsilon E_N$, and also, since $\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}}$ is one to one from $\mathbf{P}_{\varepsilon,\widetilde{\mu}}E_N$ onto $\mathbf{P}_\varepsilon E_N$,

$$\zeta_j(\varepsilon,\widetilde{\mu}) = (\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})^{-1}\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu}),$$

which means that $\mathbf{P}_\varepsilon\zeta_j(\varepsilon,\widetilde{\mu})$ is an eigenvector belonging to the eigenvalue $f_j(\varepsilon,\widetilde{\mu}) - s_\varepsilon$ for the operator $\mathbf{P}_\varepsilon\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}(\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})^{-1}\mathbf{P}_\varepsilon$ acting in the subspace $\mathbf{P}_\varepsilon E_N$. We just need to decompose into a part which is linear in $\widetilde{\mu}$ plus a rest of order $\widetilde{\mu}^2$. Then, the problem is that we have no nice bound for the derivative $\partial_{\widetilde{\mu}}(\mathbf{P}_\varepsilon\mathbf{P}_{\varepsilon,\widetilde{\mu}})$ because there again occurs (see [13] p.77 formula (2.14)) the pseudo-inverse of $\widetilde{\mathfrak{B}}_\varepsilon^{(N)} - s_\varepsilon$, only bounded by the inverse of the (very small) distance of s_ε from the rest of spectrum of $\widetilde{\mathfrak{B}}_\varepsilon^{(N)}$.

Proof of Lemma 44. Assume that $\widetilde{\mu} \in B_{\varepsilon,\gamma}^{(N)}(V)$, then it results that the norm of $(\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*})^{-1}$ is $> (\frac{N^\tau}{\gamma})^2$ and that there exists j such that

$$0 \leqq \sigma_j(\varepsilon,\widetilde{\mu}) < \eta^2 =: \left(\frac{\gamma}{N^\tau}\right)^2. \tag{101}$$

We need to measure the set (depending on ε) of $\widetilde{\mu}$ such that

$$0 \leqq \widetilde{\mu}^2 + f_j(\varepsilon,\widetilde{\mu}) < \eta^2.$$

Let us consider the function of $\widetilde{\mu}$

$$\phi_\varepsilon(\widetilde{\mu}) =: \widetilde{\mu}^2 + f_j(\varepsilon,\widetilde{\mu}),$$

defined for $|\widetilde{\mu}| \leqq \varepsilon$. Thanks to (99), we then have

$$\widetilde{\mu}^2 - c(\delta_0' + \varepsilon)|\widetilde{\mu}| + f_j(\varepsilon, 0) \leqq \phi_\varepsilon(\widetilde{\mu}) \leqq \widetilde{\mu}^2 + c(\delta_0' + \varepsilon)|\widetilde{\mu}| + f_j(\varepsilon, 0), \quad (102)$$

which means that the graph of $\widetilde{\mu} \mapsto \phi_\varepsilon(\widetilde{\mu})$ is situated between two close parabolas. This implies that the roots $\widetilde{\mu}$ of $\phi_\varepsilon(\widetilde{\mu}) = \eta^2$ are bounded, when they exist. The maximal and minimal roots are denoted $\widetilde{\mu}^{\pm}$. Thus we have

$$\widetilde{\mu}^{+2} + f_j(\varepsilon, \widetilde{\mu}^{+}) = \eta^2,$$

with the same equation for $\widetilde{\mu}^{-}$. In the case when these roots do not exist, the bad set is empty for the eigenvalue $\sigma_j(\varepsilon, \widetilde{\mu})$.

In all cases, we have (positive operator)

$$\phi_\varepsilon(\widetilde{\mu}) \geqq 0 \text{ for } \widetilde{\mu} \in [\widetilde{\mu}^{-}, \widetilde{\mu}^{+}],$$

and the function has at least a minimum in $\widetilde{\mu}_m$ such that

$$\widetilde{\mu}^{-} < \widetilde{\mu}_m < \widetilde{\mu}^{+}, \ 0 \leqq \phi_\varepsilon(\widetilde{\mu}_m) < \eta^2.$$

Then this leads to

$$\widetilde{\mu}^{+2} - \widetilde{\mu}_m^2 + f_j(\varepsilon, \widetilde{\mu}^{+}) - f_j(\varepsilon, \widetilde{\mu}_m) < \eta^2,$$

and applying (99), we obtain

$$\widetilde{\mu}^{+2} - \widetilde{\mu}_m^2 - c(\delta_0' + \varepsilon)(\widetilde{\mu}^{+} - \widetilde{\mu}_m) < \eta^2,$$

hence,

$$(\widetilde{\mu}^{+} - \frac{c}{2}(\delta_0' + \varepsilon))^2 - \left(\widetilde{\mu}_m - \frac{c}{2}\left(\delta_0' + \varepsilon\right)\right)^2 < \eta^2.$$

If $\widetilde{\mu}_m - \frac{c}{2}(\delta_0' + \varepsilon)$ and $\widetilde{\mu}^{+} - \frac{c}{2}(\delta_0' + \varepsilon)$ have the same sign, we use now the property that $0 < a^2 - b^2 < \eta^2$ leads to $|a - b| < \eta$, when a and b have the same sign. This allows us to conclude that, in such a case

$$\widetilde{\mu}^{+} - \widetilde{\mu}_m < \eta.$$

In the same way, if $\widetilde{\mu}_m + \frac{c}{2}(\delta_0' + \varepsilon)$ and $\widetilde{\mu}^{-} + \frac{c}{2}(\delta_0' + \varepsilon)$ have the same sign,

$$(\widetilde{\mu}^{-} + \frac{c}{2}(\delta_0' + \varepsilon))^2 - \left(\widetilde{\mu}_m + \frac{c}{2}\left(\delta_0' + \varepsilon\right)\right)^2 < \eta^2$$

gives

$$\widetilde{\mu}_m - \widetilde{\mu}^{-} < \eta,$$

and finally the bad interval would be bounded by 2η.

Since we are unable to prove the suitable property for $f_j(\varepsilon, \widetilde{\mu})$, we need the following:

Condition 47. *Functions $f_j(\varepsilon, \widetilde{\mu})$ defined in (98) have their derivative with respect to $\widetilde{\mu}$ which are Lipschitz: for $\widetilde{\mu} \in [-\varepsilon, \varepsilon]$, there exists $0 < k < 2$ with*

$$|\partial_{\widetilde{\mu}} f_j(\varepsilon, \widetilde{\mu}_2) - \partial_{\widetilde{\mu}} f_j(\varepsilon, \widetilde{\mu}_1)| \leqq k|\widetilde{\mu}_2 - \widetilde{\mu}_1|. \quad (103)$$

We may observe that this assumption takes into account of a loss of boundedness from the estimate (97) for the operator $\widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$, since the Lipschitz constant for the derivative is $k < 2$ in place of $c\varepsilon^2$. However, this is a true assumption, with no proof at this time.

Now, in using Hypothesis (103), we claim that the function $\widetilde{\mu} \mapsto \phi_\varepsilon(\widetilde{\mu})$ is convex, i.e., that

$$\partial_{\widetilde{\mu}}\phi_\varepsilon(\widetilde{\mu}) = 2\widetilde{\mu} + \partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu})$$

is an increasing function of $\widetilde{\mu}$, cancelling in $\widetilde{\mu} = \widetilde{\mu}_m$. This property, combined with the property (102), leads to a unique minimum in $\widetilde{\mu}_m$, and to a measure of bad $\widetilde{\mu}$ in the (worse) case given when the graph of ϕ_ε is tangent to the axis. We have

$$\begin{aligned}\phi_\varepsilon(\widetilde{\mu}) - \phi_\varepsilon(\widetilde{\mu}_m) &= \int_{\widetilde{\mu}_m}^{\widetilde{\mu}} (2\widetilde{\mu} + \partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}))\mathrm{d}\widetilde{\mu} \\ &= \int_{\widetilde{\mu}_m}^{\widetilde{\mu}} (2(\widetilde{\mu} - \widetilde{\mu}_m) + \partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}) - \partial_{\widetilde{\mu}} f_j(\varepsilon,\widetilde{\mu}_m))\mathrm{d}\widetilde{\mu} \\ &\geqq \frac{(2-k)}{2}(\widetilde{\mu} - \widetilde{\mu}_m)^2.\end{aligned}$$

Since $\phi_\varepsilon(\widetilde{\mu}^{\pm}) = \eta^2$, we obtain

$$\widetilde{\mu}^+ - \widetilde{\mu}^- \leqq \frac{2\eta}{\sqrt{(1-k/2)}}.$$

Summing up for all eigenvalues, using that the dimension $\mathcal{N}$ of E_N is bounded by bN^d, the measure of the set of bad $\widetilde{\mu}$ is bounded by

$$\frac{2b\gamma}{\sqrt{(1-k/2)}N^{\tau-d}}. \tag{104}$$

□

Remark 48. We give precise results in Section 10 on the structure of the bad set in the plane $(\varepsilon, \tilde{\mu})$. It is shown that the curves $\widetilde{\mu}^-(\varepsilon), \widetilde{\mu}^+(\varepsilon)$ are H ölder continuous functions of ε with exponent 1/2.

The estimate of Lemma 44 is then proved with $C = \frac{2b}{\sqrt{(1-k/2)}}$. Finally let us observe that this measure is small with respect to the length $2\varepsilon^3$ of the interval for $\widetilde{\mu} = \varepsilon^3\mu'$, provided that

$$\varepsilon^3 N^{\tau-d} \geqq \varepsilon^3 M_\varepsilon^{12l_0} N^{\tau-d-12l_0} \geqq c_2' N^{\tau-d-12l_0}$$

is large enough. This is the case as soon as $\tau > d + 12l_0$. □

Then we have

Proposition 49. *Let $d = 2(l_0+1)$ be the dimension of the $\mathbb{Q}$- vector space spanned by the wave vectors k_j, $j = 1,\ldots,2q$, and $\tau > d + 2 + 24l_0$. Let N be $\geqq 1$. Assume moreover that $0 < \gamma \leqq \widetilde{\gamma} = \frac{c'}{c2^{2l_0+1}}$, (where c is the constant occurring in (90)) and $(\epsilon, \widetilde{\mu}, V) \in [0,\epsilon_1] \times [-\varepsilon,\varepsilon] \times \mathcal{U}_M^{(N)}$ with $\widetilde{\mu} \in G^{(N)}_{\varepsilon,\gamma}(V)$, ϵ_1 small* *enough. For* $s_0 > \frac{d}{2}$*, there exists* $c' > 0$ *independent of* N *and* γ*, such that for any* $v \in \Pi'\pi_0 E_N$*, we have*

$$||(\Pi'\Pi_N \mathcal{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi')^{-1}v||_{0,s_0} \leqq c'\frac{N^\tau}{\gamma}||v||_{0,s_0}, \tag{105}$$

and the same estimate holds for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ *for* $v \in E_N$*.*

Proof. If $N \geqq 1$, then $2c\gamma \leqq c'/2^{2l_0} \leqq \frac{c'N^\tau}{(1+N^2)^{2l_0}}$, i.e.

$$2c(1+N^2)^{2l_0} \leqq c'\frac{N^\tau}{\gamma}.$$

Then the estimate for $(\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}})^{-1}v$ follows for $N \leqq M_\varepsilon$ from (90). For $N > M_\varepsilon$ by definition of the good set of $\widetilde{\mu}$, the estimate on $(\Pi'\Pi_N \mathcal{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N\Pi')^{-1}v$ follows. For $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ the estimate follows from Lemma 42.

Remark 50. The choice to take $\tau > d + 2 + 24l_0$ will be explained later (see Lemma 55). With such a choice, we have $\frac{1}{N^{\tau-d-2}} \leqq \frac{1}{M_\varepsilon^{24l_0}} \leqq c\varepsilon^6$.

Definition 51. For $V \in \mathcal{U}_M^{(N)}$ and $\tau, \gamma > 0$, we define the set of good $\tilde{\mu}$ for all $K \leqq N$, as

$$\mathcal{G}^{(N)}_{\varepsilon,\gamma}(V) = \cap_{K\leqq N} G^{(K)}_{\varepsilon,\gamma}(V),$$

where we notice that $G^{(K)}_{\varepsilon,\gamma}(V) = [-\varepsilon, \varepsilon]$ for $K < M_\varepsilon$, thanks to Lemma 41.

Our aim is now to obtain an estimate for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$ in $\mathcal{K}_{0,s}$ for $s > s_0$. We may observe that it is not possible to obtain directly such an estimate in $\mathcal{K}_{0,s}$ for $s > s_0$, because the norm $||V||_{0,s}$ would appear in the estimates for f_j in the eigenvalues σ_j, and this is far to be controlled.

8.4. Separation Properties (H1) and (H2)

The eigenvalues close to 0 of the unperturbed operator $\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A} - \lambda_0)\mathbf{Q}_0\pi_0\Pi_N$ are the negative numbers $\lambda_0(|\mathbf{k}|^2) - \lambda_0$ where $|\mathbf{k}| \neq k_c$, and $1 \leqq N_\mathbf{k} \leqq N$. Let $\rho > 0$. We need to have good separation properties of the singular set

$$S_{(N)} = \left\{\mathbf{k} \in \Gamma;\, \lambda_0 - \lambda_0(|\mathbf{k}|^2) < \rho,\ 1 \leqq N_\mathbf{k} \leqq N\right\}, \tag{106}$$

which contains the $\mathbf{k}$'s corresponding to the small denominators, whereas the regular set is

$$R_{(N)} := \left\{\mathbf{k} \in \Gamma;\, \lambda_0 - \lambda_0(|\mathbf{k}|^2) \geqq \rho,\ 1 \leqq N_\mathbf{k} \leqq N\right\}. \tag{107}$$

We have a bijection between $S_{(N)}$ and $S(N) := \{x \in \Gamma(N);\, \lambda_0 - \lambda_0(|\mathbf{k}(x)|^2) < \rho\}$ where $\mathbf{k}(x)$ is defined in (6) and

$$\Gamma(N) := \{x \in \mathbb{Z}^d;\ 0\leqq|x| \leqq N,\ \mathbf{k}(x) \in \Gamma\}.$$

We use the fact that for $||\mathbf{k}| - k_c| \leqq \delta_1$, there exist c_1 and $c_2 > 0$ such that

$$c_1(|\mathbf{k}|^2 - k_c^2)^2 \leqq \lambda_0 - \lambda_0(|\mathbf{k}|^2) \leqq c_2(|\mathbf{k}|^2 - k_c^2)^2, \qquad (108)$$

and (8) holds. Then, as in [3], we use the results of BOURGAIN in [4], CRAIG in [7], and [1], so that we obtain

Proposition 52. *There exists $\rho_0 > 0$ independent of N such that if $\rho \in]0, \rho_0]$ then there exists a decomposition of $S(N) = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ into a union of disjoint clusters Ω_α satisfying:*

- $(H1)$, *for all $\alpha \in \mathcal{A}$, $M_\alpha \leqq 2m_\alpha$ where $M_\alpha = \max_{x\in\Omega_\alpha} |x|$ and $m_\alpha = \min_{x\in\Omega_\alpha} |x|$;*
- $(H2)$, *there exists $\delta = \delta(d) \in]0, 1[$ independent of N such that if $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$ then*

$$\operatorname{dist}(\Omega_\alpha, \Omega_\beta) := \min_{x\in\Omega_\alpha, y\in\Omega_\beta} |x - y| \geqq \frac{(M_\alpha + M_\beta)^\delta}{2}.$$

8.5. Estimate of $(\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N)^{-1}$ *in* $\Pi_N \pi_0 \mathbf{Q}_0 \mathcal{K}_{0,s}$

We use the proof of [1] (see pages 628 to 636). In fact, we need the selfadjointness in $\Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$ (i.e. E_N with the adapted scalar product) of the operator

$$D_N =: \Pi_N \pi_0 \mathbf{Q}_0 (\mathcal{A} - \lambda_0) \mathbf{Q}_0 \pi_0 \Pi_N,$$

diagonal (see "Appendix E") with respect to Fourier components in $\Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$, for which we know all eigenvalues. Moreover, we have

$$\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} \Pi_N = D_N + \varepsilon T(\varepsilon, \widetilde{\mu}, V),$$

where the second part εT is a bounded operator (not diagonal) of order ε having the properties of a multiplication operator, as it is needed in [1] (see Lemma 3.9 in [1]); see the proof in "Appendix E".

Lemma 53. *Let $A, B \subset S(N) \cup R(N)$, and let $s_0 > d/2$. Then for any $s \geqq s_0 > d/2$ there exists $C(s) > 0$ such that the following estimate holds for any $V \in Q_0 \mathcal{K}_{0,s}$ such that $||V||_{0,s_0} \leqq 1$, and $h \in \Pi_N \pi_0 Q_0 \mathcal{K}_{0,0}$*

$$||T_B^A h||_{0,0} \leqq \frac{C(s)\varepsilon(1 + \varepsilon^3 ||V||_{0,s})||h||_{0,0}}{(1 + d(A, B))^{s-d/2}},$$

where $d(A, B)$ is the distance in $\mathbb{Z}^d$ between A and B, and T_B^A is the operator T acting in E_N restricted to elements with Fourier spectrum with $\{\mathbf{k}(x); x \in A\}$, the action being projected on elements with the Fourier spectrum such that $\{\mathbf{k}(x); x \in B\}$.

This property, and the estimate (105) used for any $K \leqq N$ (replaces the use of eigenvalues of $\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N$ as it is done in [1]), are the basic ingredients for the proof of the following:

Proposition 54. *Let* $d = 2(l_0+1)$ *be the dimension of the* $\mathbb{Q}$*- vector space spanned by the wave vectors* k_j, $j = 1, \ldots, 2q$, *and* $\tau > d + 2 + 24l_0$ *as in Lemma 49. Assume moreover that* $0 < \gamma \leqq \widetilde{\gamma} = \frac{c'}{c2^{2l_0+1}}$, *and* $(\epsilon, \widetilde{\mu}, V) \in [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times \mathcal{U}_M^{(N)}$, *with* $\widetilde{\mu} \in \mathcal{G}_{\epsilon,\gamma}^{(N)}(V)$, ϵ_1 *small enough. There exists* $s_0(d, \delta, \tau) > \frac{d}{2}$ *where* δ *is the number introduced in separation property (H2), and let* $\overline{s} > s_0$. *There exists* $m(d, \delta, \tau)$ *such that for all* $s \in [s_0, \overline{s}]$ *there exists* $K(s) > 0$ *such that for any* $h \in \Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$, *we have*

$$||(\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N)^{-1} h||_{0,s} \leqq K(s) \frac{N^m}{\gamma} (||h||_{0,s} + ||V(\epsilon, \widetilde{\mu})||_{0,s} ||h||_{0,s_0}), \tag{109}$$

and the same estimate holds for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})} \Pi_N)^{-1}$.

9. Resolution of the Range Equation

In this section we use [2] for finding $v = V(\epsilon, \widetilde{\mu})$ in $\mathcal{U}_M^{(N)}$, defined for $(\epsilon, \widetilde{\mu})$ in $[0, \epsilon_1] \times [-\varepsilon, \varepsilon]$, bounded by $O(\epsilon)$, of class C^2 in its arguments, solution of $\mathcal{F}(\epsilon, \widetilde{\mu}, v) = 0$ (see (110) below) in a suitably large subset of $(0, \epsilon_1) \times [-\varepsilon, \varepsilon]$.

All operators (linear and non linear) satisfy good tame estimates in the scale of Sobolev spaces $\Pi_N \pi_0 Q_0 \mathcal{K}_{0,s}$ $s > d/2$ and the projection Π_N plays the role of a smoothing operator (see [3]):

$$||\Pi_N u||_{0,s+r} \leqq (1 + N^2)^{r/2} ||u||_{0,s}, \quad \forall u \in \mathcal{K}_{0,s},$$
$$||(\mathbb{I} - \Pi_N)u||_{0,s} \leqq (1 + N^2)^{-r/2} ||u||_{0,r+s}, \quad \forall u \in \mathcal{K}_{0,s+r}.$$

Indeed, we have the good functional setting and the good "tame" properties of the map (see Lemmas 26, 35, 54):

$$\begin{aligned} \mathcal{F}(\epsilon, \widetilde{\mu}, v) \; = \; : \mathfrak{L}_{\varepsilon,\widetilde{\mu}} v + g(\varepsilon, \widetilde{\mu}) - \varepsilon^4 \mathbf{Q}_0 \mathcal{B}(v, v) \\ (\epsilon, \widetilde{\mu}, v) \mapsto \mathcal{F}(\epsilon, \widetilde{\mu}, v) : [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times Q_0 \mathcal{K}_{0,s} \to Q_0 \mathcal{K}_{0,s} \text{ for } s \geqq s_0 > d/2, \end{aligned} \tag{110}$$

with (see (67))

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}} = \mathbf{Q}_0(\mathcal{A} - \lambda_0 + \widetilde{\mu} + \mu_\varepsilon) - 2\mathbf{Q}_0 \mathcal{B}(u_\varepsilon - \varepsilon^2 \widetilde{h}(\varepsilon, \widetilde{\mu}), \cdot),$$
$$\mathcal{F}(0, 0, 0) = 0 \text{ (for } \varepsilon = 0, \text{ we have } \widetilde{\mu} = 0).$$

The mapping $\mathcal{F}$ appears to be C^3 with the following estimates for $v \in Q_0 \mathcal{K}_{0,s}$, $s \in [s_0, \overline{s}]$, $s_0 > d/2$, and $||v||_{0,s_0} \leqq 1$:

$$||\mathfrak{L}_{\varepsilon,\widetilde{\mu}} v||_{0,s} \leqq C(s) ||v||_{0,s},$$
$$||\varepsilon^4 \mathbf{Q}_0 \mathcal{B}(v, v')||_{0,s} \leqq \varepsilon^4 C(s) [||v||_{0,s} ||v'||_{0,s_0} + ||v'||_{0,s}],$$
$$||g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq \varepsilon^2 C(s),$$
$$||\partial_{\varepsilon,\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_{0,s} + ||\partial^2_{\varepsilon^2} g(\varepsilon, \widetilde{\mu})||_{0,s} + ||\partial^2_{\varepsilon\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_{0,s} + ||\partial^2_{\widetilde{\mu}^2} g(\varepsilon, \widetilde{\mu})||_{0,s} \leqq C(s),$$
$$||\partial_\varepsilon \mathfrak{L}_{\varepsilon,\widetilde{\mu}} v||_{0,s} \leqq C(s) ||v||_{0,s}.$$

We may notice that

$$D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u]=\mathfrak{L}_{\varepsilon,\widetilde{\mu}}u-2\varepsilon^4\mathbf{Q}_0\mathcal{B}(v,u),$$
$$D_v^2\mathcal{F}(\epsilon,\widetilde{\mu},v)[v_1,v_2]=-2\varepsilon^4\mathbf{Q}_0\mathcal{B}(v_1,v_2),$$
$$D_v^3\mathcal{F}(\epsilon,\mu',v)=0,$$

hence

$$||\partial_\varepsilon D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u]||_{0,s}\leqq C(s)[||u||_{0,s}+\varepsilon^3||v||_{0,s}||u||_{0,s_0}],$$
$$||\partial_{\widetilde{\mu}}D_v\mathcal{F}(\epsilon,\widetilde{\mu},v)[u]||_{0,s}\leqq C(s)||u||_{0,s}.$$

Moreover, Lemma 54 says that for any $(\epsilon,\widetilde{\mu},V)\in[0,\epsilon_1]\times[-\varepsilon,\varepsilon]\times U_M^{(N)}$, $V\in\mathcal{K}_{0,s}$ with $\widetilde{\mu}\in\mathcal{G}_{\varepsilon,\gamma}^{(N)}(V)$

$$||(\Pi_N D_v\mathcal{F}(\epsilon,\widetilde{\mu},V(\epsilon,\widetilde{\mu}))\Pi_N)^{-1}v||_{0,s}\leqq K(s)\frac{N^m}{\gamma}(||v||_{0,s}+||V(\epsilon,\widetilde{\mu})||_{0,s}||v||_{0,s_0}),$$

so that assumptions (F1), (F2), (F3), (F4) and on the invertibility of the linearized operator, made in [2] are satisfied. We also satisfy additionnal properties $(F2)^+$, $(F4)^+$ required in "Appendix F" on higher order derivatives, useful for getting a solution V which is C^2 in $(\varepsilon,\widetilde{\mu})$. Moreover the required property (L) in [2] needs to be satisfied, which brings us to

Lemma 55. *Choose $N_2\geqq N_1\geqq M_\varepsilon$, and $V_1\in\mathcal{U}_M^{(N_1)}$, $V_2\in\mathcal{U}_M^{(N_2)}$. For $\varepsilon\in(0,\varepsilon_1)$, consider the set of $\widetilde{\mu}$ which are "good" for V_1, but "bad" for V_2 :*

$$\widetilde{\mu}\in\left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c\cap\mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1),$$

where the apex c denotes the complementary in $[-\varepsilon,\varepsilon]$. Assume that $||V_2-V_1||_{0,s_0}\leqq N_1^{-\sigma}$, with $\sigma>2d-6+32l_0$, and $\tau>d+2+24l_0$, then for ε_1 small enough, in particular for $\varepsilon_1\leqq\gamma^{4l_0}$:

$$\text{meas}\left\{\left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c\cap\mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1)\right\}\cap[-\varepsilon,\varepsilon]\leqq C_1\gamma\frac{\varepsilon^6}{N_1}.$$

Proof.

$$\left(\mathcal{G}_{\varepsilon,\gamma}^{(N_2)}(V_2)\right)^c\cap\mathcal{G}_{\varepsilon,\gamma}^{(N_1)}(V_1)=\left(\cup_{M_\varepsilon\leqq K\leqq N_2}B_{\varepsilon,\gamma}^{(K)}(V_2)\right)\cap\left(\cap_{M_\varepsilon\leqq K\leqq N_1}G_{\varepsilon,\gamma}^{(K)}(V_1)\right)$$
$$\subset\left(\cup_{M_\varepsilon\leqq K\leqq N_1}B_{\varepsilon,\gamma}^{(K)}(V_2)\cap G_{\varepsilon,\gamma}^{(K)}(V_1)\right)\cup\left(\cup_{N_1\leqq K\leqq N_2}B_{\varepsilon,\gamma}^{(K)}(V_2)\right).$$

Moreover, according to Lemmas 35 and 36 and a careful study of the form of operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ in (96), we have for $K\leqq N_1$ that

$$||\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_2)*}-\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(K,V_1)*}||_{0,s_0}\leqq c\varepsilon^4||V_2-V_1||_{0,s_0}\leqq\frac{c\varepsilon^4}{N_1^\sigma}.$$

Let us assume that $\widetilde{\mu} \in B^{(K)}_{\varepsilon,\gamma}(V_2) \cap G^{(K)}_{\varepsilon,\gamma}(V_1)$, then there is at least one eigenvalue (> 0) of $\mathfrak{L}^{(K,V_2)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(K,V_2)*}_{\varepsilon,\widetilde{\mu}}$ which is $< (\frac{\gamma}{K^\tau})^2$. Then, by the Lidskii theorem (see [13] p.126), the selfadjoint operator $\mathfrak{L}^{(K,V_1)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(K,V_1)*}_{\varepsilon,\widetilde{\mu}}$ has an eigenvalue $< (\frac{\gamma}{K^\tau})^2 + \frac{c\varepsilon^4}{N_1^\sigma}$. Since $\widetilde{\mu} \in G^{(K)}_{\varepsilon,\gamma}(V_1)$, this eigenvalue is $> (\frac{\gamma}{K^\tau})^2$. Hence, the bad $\widetilde{\mu}$ correspond to an interval

$$\left[\left(\frac{\gamma}{K^\tau}\right)^2, \left(\frac{\gamma}{K^\tau}\right)^2 + \frac{c\varepsilon^4}{N_1^\sigma}\right],$$

containing the above eigenvalue of $\mathfrak{L}^{(K,V_1)}_{\varepsilon,\widetilde{\mu}}\mathfrak{L}^{(K,V_1)*}_{\varepsilon,\widetilde{\mu}}$. The same proof as the one made for Lemma 44, shows that the measure of corresponding bad set of $\widetilde{\mu}$ is bounded by

$$\frac{2}{\sqrt{(1-k/2)}}\sqrt{\frac{c\varepsilon^4}{N_1^\sigma}}.$$

Hence,

$$\begin{aligned}
\text{meas}\left(\cup_{M_\varepsilon \leqq K \leqq N_1} B^{(K)}_{\varepsilon,\gamma}(V_2) \cap G^{(K)}_{\varepsilon,\gamma}(V_1)\right) &\leqq \frac{2}{\sqrt{(1-k/2)}}\sqrt{\frac{c\varepsilon^4}{N_1^\sigma}} \sum_{M_\varepsilon \leqq K \leqq N_1} bK^d \\
&\leqq \frac{2b\sqrt{c}}{\sqrt{(1-k/2)}} \frac{\varepsilon^2}{N_1^{\sigma/2-d-1}} \\
&\leqq \frac{2b\sqrt{c}}{\sqrt{(1-k/2)}} \frac{\varepsilon^2}{N_1} \frac{1}{M_\varepsilon^{\sigma/2-d-2}} \\
&\leqq \frac{c'\gamma}{N_1}\varepsilon^{2+\frac{\sigma/2-d-3}{4l_0}} \leqq \frac{c'\gamma\varepsilon^6}{N_1}.
\end{aligned}$$

Now

$$\begin{aligned}
\text{meas}\left(\cup_{N_1 \leqq K \leqq N_2} B^{(K)}_{\varepsilon,\gamma}(V_2)\right) &\leqq \sum_{N_1 \leqq K \leqq N_2} \frac{C\gamma}{K^{\tau-d}} \leqq \frac{C\gamma(\tau-d-1)}{N_1^{\tau-d-1}} \\
&\leqq \frac{c''\gamma}{N_1}\varepsilon^{\frac{\tau-d-2}{4l_0}} \leqq \frac{c''\gamma\varepsilon^6}{N_1}.
\end{aligned}$$

Finally

$$\text{meas}\left(\mathcal{G}^{(N_2)}_{\varepsilon,\gamma}(V_2)\right)^c \cap \mathcal{G}^{(N_1)}_{\varepsilon,\gamma}(V_1) \leqq \frac{(c'+c'')\gamma\varepsilon^6}{N_1},$$

which is the result of the Lemma. □

We may then apply a simple adaptation of theorem 3 of Berti--Bolle--Procesi [2] to solve equation $\mathcal{F}(\epsilon, \widetilde{\mu}, V) = 0$, and find the solution V which is C^2 in the parameters $(\varepsilon, \widetilde{\mu})$, and such that $V \in \mathcal{U}^{(N)}_M$. Let γ, m, s_0 be as in Proposition 54. Moreover, let $\overline{s} > s_0 + 4(m+1) + 8m = s_0 + 4 + 12m$.

From proposition 54, it follows that if $(\epsilon, \widetilde{\mu}, V) \in [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times \mathcal{U}_M^{(N)}$, $V \in \mathcal{K}_{0,s}$ and $\widetilde{\mu} \in G_{\epsilon,\gamma}^{(N)}(V)$ then $(\epsilon, \widetilde{\mu}, V) \in J_{\gamma,m}^{(N)}$ (as defined in (4) of [2], that is (109) holds for $s \in [s_0, \overline{s}]$.

In [2] [theorem 3] one considers $N \geqq N_0 = N_0(\gamma)$ with $N_0(\gamma)$ sufficiently large and $0 < \epsilon \leqq \epsilon_2(\gamma)$ with $\epsilon_2(\gamma)$ sufficiently small. We may choose $N_0 = N_0(\gamma) = M_{\epsilon_2(\gamma)}$ with a suitable $\epsilon_2(\gamma)$ and we consider, in that which follows, $0 < \epsilon \leqq \epsilon_2(\gamma)$.

Theorem 56. *Let s_0 and $\tilde{\gamma}$ be as in Proposition 54. Then for all $0 < \gamma < \tilde{\gamma}$ there exist $\epsilon_2(\gamma) \in [0, \epsilon_0]$ and a C^2-map $V : (0, \epsilon_2(\gamma)) \times [-\varepsilon, \varepsilon] \to \Pi_N \pi_0 Q_0 \mathcal{K}_{0,s_0}$, such that $V(0,0) = 0$, $||\partial_{\widetilde{\mu}} V||_{0,s_0} \leqq M$, and if $\epsilon \in (0, \epsilon_2(\gamma))$, $\widetilde{\mu} \in ([-\varepsilon, \varepsilon] \backslash C_{\epsilon,\gamma})$, the function $V(\epsilon, \widetilde{\mu})$ is solution of $\mathcal{F}(\epsilon, \widetilde{\mu}, V) = 0$ (110). Here $C_{\epsilon,\gamma}$ is a subset of $[-\varepsilon, \varepsilon]$ which is a Hölder continuous function of ε, and has Lebesgue-measure less than $C\gamma\epsilon^6$ for some constant $C > 0$ independent of ϵ and γ.*

The proof is the same as in [3], except for Hölder continuity which is proved in the next section. In fact $C_{\epsilon,\gamma}$ is a union of intervals $I_\varepsilon^{(N_n)}$ (see definition 57, with $N_n = (N_0(\gamma))^{2^n}$, so that each end of each interval is a fonction of ε which is Hölder continuous in ε with exponent 1/2.

10. Resolution of the Bifurcation Equation

Let V be the function obtained in Theorem 56. It is C^2 in $(\varepsilon, \widetilde{\mu})$. Replacing $V(\varepsilon, \widetilde{\mu})$ in the bifurcation equation (66), and replacing $\tilde{\mu}$ by $\varepsilon^3\mu'$, we can solve with respect to μ' and find a function $\tilde{h}(\varepsilon)$ which is C^1 in (ε), such that

$$\mu' = \varepsilon\mu_4 + \varepsilon\tilde{h}(\varepsilon), \quad (H), \ \tilde{h}(\varepsilon) = \mathcal{O}(\varepsilon) \tag{111}$$

for $\varepsilon \in (0, \varepsilon_2(\gamma))$, provided that ε_2 is small enough, and $\mu' \in [-1, 1]$.

For obtaining solutions valid for our system, *the condition $\mu_4 \neq 0$ is not required* (see (62) for μ_4). Indeed, in case $\mu_4 = 0$, the curve (H) in the (ε, μ') plane is just more flat near $\varepsilon = 0$. This coefficient μ_4 has not been computed yet, but it can be computed in principle, depending a priori on q only.

Let us show that in the plane (ε, μ') the bad set is located in "bad strips". Then we shall need a transversality condition to ensure that these bad strips intersect transversally the "curve" (H), such that any point of this curve, which does not belong to bad strips, indeed gives an eligible solution of our problem.

10.1. Transversality Condition for "Bad Strips"

In the plane (ε, μ'), the bad strips are bounded by the curves given by the solutions $\widetilde{\mu}^{\pm}(\varepsilon)$ (where $\widetilde{\mu} = \varepsilon^3\mu'$) of

$$\sigma_j(\varepsilon, \widetilde{\mu}) = \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu}) = \eta^2,$$

where $\eta = \gamma/N^\tau$, not forgetting that σ_j depends on N.

Definition 57. For N and V fixed, a set of "bad strips" is defined by

$$BS_N(V) = \{(\varepsilon, \mu') \in [0, \varepsilon_2(\gamma)] \times [-1, 1]; \quad \varepsilon^3\mu' \in I_\varepsilon^{(N)}\},$$

where $I_\varepsilon^{(N)}$ is one of the intervals $(\tilde{\mu}_j^-(\varepsilon), \tilde{\mu}_j^+(\varepsilon))$, or with one of the bounds replaced by ε^3 (right bound), or by $-\varepsilon^3$ (left bound), as defined at Section 8.3.

Let us show that the limiting curves $\tilde{\mu}_j^-(\varepsilon), \tilde{\mu}_j^+(\varepsilon)$ are Hölder continuous with exponent 1/2. We have for $\varepsilon_2 > \varepsilon_1$, along a limiting curve, that

$$\sigma_j(\varepsilon_2, \widetilde{\mu}_2) - \sigma_j(\varepsilon_1, \widetilde{\mu}_1) = 0, \ \widetilde{\mu}_j = \widetilde{\mu}(\varepsilon_j),$$
$$\sigma_j(\varepsilon_2, \widetilde{\mu}_2) - \sigma_j(\varepsilon_1, \widetilde{\mu}_1) = \widetilde{\mu}_2^2 - \widetilde{\mu}_1^2 + f_j(\varepsilon_2, \widetilde{\mu}_2) - f_j(\varepsilon_1, \widetilde{\mu}_1),$$

and thanks to (99)), assuming $\widetilde{\mu_2} > \widetilde{\mu_1}$,

$$\widetilde{\mu}_2^2 - \widetilde{\mu}_1^2 \leqq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + \widetilde{\mu}_2 - \widetilde{\mu}_1),$$

hence

$$\left[\widetilde{\mu}_2 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 - \left[\widetilde{\mu}_1 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 \leqq c'(\delta_0' + \varepsilon)(\varepsilon_2 - \varepsilon_1),$$

and since the two quantities in brackets have the same sign when $|\tilde{\mu}_2 - \tilde{\mu}_1|$ is small enough, if

$$\left[\widetilde{\mu}_2 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 - \left[\widetilde{\mu}_1 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 > 0,$$

we may use the argument that when $0 < a^2 - b^2 < \eta^2$, with $ab \geqq 0$, then $|a - b| \leqq |\eta|$, which leads to

$$\widetilde{\mu}_2 - \widetilde{\mu}_1 \leqq \sqrt{c'(\delta_0' + \varepsilon)(\varepsilon_2 - \varepsilon_1)},$$

which is the Hölder continuity. If, on the contrary,

$$\left[\widetilde{\mu}_2 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 - \left[\widetilde{\mu}_1 - \frac{c}{2}(\delta_0' + \varepsilon)\right]^2 < 0,$$

we need to use Condition 47, as in Section 8.3. For $|\widetilde{\mu}_2 - \widetilde{\mu}_1|$ small enough, we may assume that either $\widetilde{\mu}_m(\varepsilon_1) < \widetilde{\mu}_1 < \widetilde{\mu}_2$ (upper limit curve), or $\widetilde{\mu}_1 < \widetilde{\mu}_2 < \widetilde{\mu}_m(\varepsilon_2)$ (lower limit curve). In the first case, we obtain

$$\begin{aligned}\sigma_j(\varepsilon_1, \widetilde{\mu}_2) - \sigma_j(\varepsilon_1, \widetilde{\mu}_1) &\geqq (1 - k/2)[(\widetilde{\mu}_2 - \widetilde{\mu}_m(\varepsilon_1))^2 - (\widetilde{\mu}_1 - \widetilde{\mu}_m(\varepsilon_1))^2] \\ &\geqq (1 - k/2)(\widetilde{\mu}_2 - \widetilde{\mu}_1)^2.\end{aligned}$$

In the second case, we obtain

$$\begin{aligned}|\sigma_j(\varepsilon_2, \widetilde{\mu}_2) - \sigma_j(\varepsilon_2, \widetilde{\mu}_1)| &\geqq (1 - k/2)[(\widetilde{\mu}_m(\varepsilon_2) - \widetilde{\mu}_1)^2 - (\widetilde{\mu}_m(\varepsilon_2) - \widetilde{\mu}_2)^2] \\ &\geqq (1 - k/2)|\widetilde{\mu}_2 - \widetilde{\mu}_1|^2.\end{aligned}$$

On the other hand, we have

$$|\sigma_j(\varepsilon_1,\widetilde{\mu}_2)-\sigma_j(\varepsilon_1,\widetilde{\mu}_1)| = |\sigma_j(\varepsilon_1,\widetilde{\mu}_2)-\sigma_j(\varepsilon_2,\widetilde{\mu}_2)| \leqq c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|$$
$$|\sigma_j(\varepsilon_2,\widetilde{\mu}_2)-\sigma_j(\varepsilon_2,\widetilde{\mu}_1)| = |\sigma_j(\varepsilon_2,\widetilde{\mu}_1)-\sigma_j(\varepsilon_1,\widetilde{\mu}_1)| \leqq c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|.$$

Hence, in all cases

$$|\widetilde{\mu}_2-\widetilde{\mu}_1|^2 \leqq (1-k/2)^{-1}c'(\delta_0'+\varepsilon)|\varepsilon_2-\varepsilon_1|,$$

which is Hölder continuity. □

In the case when $\widetilde{\mu}$ *is not exceptional, i.e. if the eigenvalue* σ_j *is not multiple,* the slope of the tangent to the curves $\tilde{\mu}_j^-(\varepsilon)$, $\tilde{\mu}_j^+(\varepsilon)$ is

$$t(\varepsilon) = -\frac{\partial_\varepsilon\sigma_j(\varepsilon,\widetilde{\mu}_j^+)}{\partial_{\widetilde{\mu}}\sigma_j(\varepsilon,\widetilde{\mu}_j^+)}, \tag{112}$$

given here for $\widetilde{\mu}_j^+(\varepsilon)$ (analogous formulae holding for the other curve). Now in a more precise way, for $(\varepsilon,\tilde{\mu})$ not exceptional, and taking into account the form (96), we obtain by standard arguments for simple eigenvalues that

$$\partial_{\widetilde{\mu}}\sigma_j(\varepsilon,\widetilde{\mu}^+) = 2\langle(\mathcal{A}-\lambda_0)\zeta_j(\varepsilon,\widetilde{\mu}^+),\zeta_j(\varepsilon,\tilde{\mu}^+)\rangle + 2\tilde{\mu}^+ + \mathcal{O}(\varepsilon) = \mathcal{O}(\delta_0'+\varepsilon),$$
$$\partial_\varepsilon\sigma_j(\varepsilon,\widetilde{\mu}^+) = -4\langle\mathcal{B}(u_1,(\mathcal{A}-\lambda_0)\zeta_j(\varepsilon,\widetilde{\mu}^+),\zeta_j(\varepsilon,\widetilde{\mu}^+))\rangle + \mathcal{O}(\varepsilon) = \mathcal{O}(\delta_0'+\varepsilon),$$

where $\zeta_j(\varepsilon,\widetilde{\mu}^+)$ is the eigenvector with norm 1 belonging to the eigenvalue $\sigma_j(\varepsilon,\widetilde{\mu}^+)$ of the operator $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$. Even though the operator $(\mathcal{A}-\lambda_0)$ is definite negative in the subspace where ζ_j lives, we may notice that $(\mathcal{A}-\lambda_0)\zeta_j(\varepsilon,\widetilde{\mu}^+)$ may be very small, so the term $\mathcal{O}(\varepsilon)$ above might be the dominant order in $\partial_{\widetilde{\mu}}\sigma_j$ and $\partial_\varepsilon\sigma_j$. It is then difficult to be more precise for any transversallity condition of the strips $BS_N(V)$ with respect to the curve (H) defined by (111).

Now, let us consider for (N,ε) fixed, the bad set of $\widetilde{\mu}$ which we know is of a measure bounded by $c_3\gamma\varepsilon^6/N$ (see Proposition 49 and Lemma 55). In case of the intersection of a bad strip with (H), we need to measure the corresponding set of "bad ε''. The proof of Theorem 56 via a Nash–Moser process considers a sequence $N_n = (N_0(\gamma))^{2^n}$ and successive approximations V_n of the solution V. For estimating the intersections of the bad strips with the curve (H) we are led to make a transversality conjecture.

Conjecture 58. *Let* $\widetilde{\mu}^{\pm(N_n)}(\varepsilon)$ *be any one of the limiting curves of the bad strips of* $BS_{N_n}(V_{n-1})$, $n\in\mathbb{N}$. *Then we assume that for any of these curves there exists* $c>0$ *independently of* N_n, *such that for* $h\in\mathbb{R}$ *in a neighborhood of* 0, *the following inequality holds:*

$$|\widetilde{\mu}(\varepsilon+h)-\widetilde{\mu}(\varepsilon)| \geqq c\varepsilon^2|h|.$$

Remark 59. This is indeed a *very weak assumption* for the slopes defined by (112), since this means that the slopes $t(\varepsilon)$ have a lower bound $|t(\varepsilon)|>c\varepsilon^2$. This insures transversality with the bifurcation curve (H) , the slope of which is $\mathcal{O}(\varepsilon^3)$. However we have no means to check its validity. Moreover, if, unluckily, a curve $\widetilde{\mu}(\varepsilon)$

belonging to one of the bad strips of $BS_{N_n}(V_{n-1})$ intersects (H) at an exceptional point $(\varepsilon, \widetilde{\mu}(\varepsilon))$, where an eigenvalue σ_j is multiple, then we cannot a priori define the "slope" of the corresponding limiting curve of the bad strip. This is why we took the above formulation for the Transversality conjecture even though we might just eliminate the corresponding exceptional values of ε (we have no bound for their measure).

Remark 60. In taking μ_ε in (63) at a higher order than ε^3, we should find $\widetilde{\mu}$ of higher order than ε^4 which flattens the slope of the bifurcation curve (H). Then we could *weaken the transversality condition* and replace ε^2 by a an order in ε larger than 2, which still guarantees the transversality with (H).

Let us denote by $\delta\widetilde{\mu}$ the measure of the bad $\widetilde{\mu}$, and by $\delta\varepsilon$ the corresponding measure for bad ε. Then we have (see the right side of Fig. 6)

$$\delta\varepsilon < \frac{\delta\widetilde{\mu}}{|t|} < \frac{\delta\widetilde{\mu}}{c\varepsilon^2}.$$

Let us define for ε fixed the set $B_\varepsilon S_N(V)$, which is the section of $BS_N(V)$ for some ε. In summing the measure of the bad set for ε after all iterations, we obtain a measure of the bad set for ε, bounded by the measure of $C_{\varepsilon,\gamma} = \cup_{n\geqq 1} B_\varepsilon S_{N_n}(V_{n-1})$ divided by $c\varepsilon^2$, i.e. a bad set bounded by $C\gamma\varepsilon^4$ (see Theorem 56). The complementary subset in $(0, \varepsilon_3)$ constitutes the good set of ε, which is of asymptotic full measure, since $\frac{\varepsilon - C\gamma\varepsilon^4}{\varepsilon} \to 1$ as $\varepsilon \to 0$.

Remark 61. In the case when we need to weaken the transversality condition 58, as indicated in the Remark above, we can also increase the order (here ε^6) for the size of bad $\widetilde{\mu}$ in Theorem 56, just in increasing τ in Proposition 54, so that we can keep an order of smallness ε^4 for the bad ε's.

Remark 62. If we consider $\widetilde{\mu}$ in an interval independent of ε, we can look at the situation for $\varepsilon = 0$, as in Remark 39. We see that the eigenvalues $\sigma_j(0, \widetilde{\mu})$ have the form

$$\sigma_j(0, \widetilde{\mu}) = (\widetilde{\mu} + \lambda_0(|\mathbf{k}|^2) - \lambda_0)^2, \quad N_{\mathbf{k}} \leqq N.$$

This leads to bad intervals for $\widetilde{\mu}$ of the form

$$[\lambda_0 - \lambda_0(|\mathbf{k}|^2) - \frac{\gamma}{N^\tau}, \lambda_0 - \lambda_0(|\mathbf{k}|^2) + \frac{\gamma}{N^\tau}], \quad \text{with } \mathbf{k} \text{ such that } N_{\mathbf{k}} \leqq N. \tag{113}$$

We notice that $\lambda_0 - \lambda_0(|\mathbf{k}|^2) \sim c(|\mathbf{k}|^2 - k_c^2)$ with $c \neq 0$ because of Assumption 32. Hence

$$\lambda_0 - \lambda_0(|\mathbf{k}|^2) > \frac{c'}{N^{4l_0}},$$

which gives intervals (113) "far" from 0 for τ large enough (which is one of our assumptions in Proposition 54).

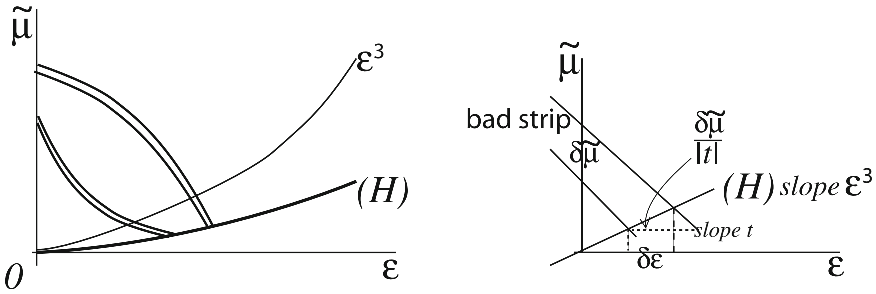

Fig. 6. Sketch of the bad set in the plane $(\varepsilon, \tilde{\mu})$. (H) is the "curve" given by (111) approximated by $\varepsilon^4\mu_4$ ($\mu_4 > 0$ is assumed here). The drawing on the right side explains the bound for the measure of $\delta\varepsilon$.

10.2. Final Result

If the Transversality Conjecture 58 is verified, then there is a good set for ε, with asymptotic full measure as $\varepsilon \to 0$, such that there exists a couple $(\varepsilon, \widetilde{\mu}(\epsilon))$ on the curve (H) which lies in the good set (see Fig. 6). Then this gives the existence of a solution $(\varepsilon, \mu'(\varepsilon))$ of (111), as ε tends towards 0.

Now we observe that we can write $\mu' = \varepsilon\overline{\mu}$, with $\overline{\mu}$ centered in μ_4. This defines the good 1-dimensional set $\overline{\Lambda}_\varepsilon$ of all good $\overline{\mu}_\varepsilon$.

Finally with (63), we obtain a solution of (35) under the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \varepsilon^4 (V(\varepsilon, \varepsilon^4\overline{\mu}_\varepsilon) - h(\varepsilon, \varepsilon\overline{\mu}_\varepsilon))$$
$$\lambda = \lambda_0 - \mu_2\varepsilon^2 - \mu_3\varepsilon^3 - \varepsilon^4\overline{\mu}_\varepsilon.$$

This ends the proof of Theorem 1, with little adaptation of notations. Notice that the solution $(\lambda, u)(\varepsilon)$ is C^1, restricted to the "good" values of ε. (see Fig. 2).

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A. Inverse of L

In this appendix we compute and estimate the inverse of the operator L. By construction, solving the equation

$$LU = G = (F, g) \in \mathcal{K}_{0,s}, \text{ with } U \in D_s(L), \tag{114}$$

means that we have

$$\Delta V - \nabla q = F, \ \nabla \cdot V = 0,$$
$$\Delta\theta = g,$$

i.e.

$$(D^2 - k^2)v_{\mathbf{k}}^{(z)} - Dq_{\mathbf{k}} = f_{\mathbf{k}}^{(z)}, \tag{115}$$

$$(D^2 - k^2)V_{\mathbf{k}}^{(H)} - i\mathbf{k}q_{\mathbf{k}} = F_{\mathbf{k}}^{(H)}, \tag{116}$$

$$(D^2 - k^2)\theta_{\mathbf{k}} = g_{\mathbf{k}}, \tag{117}$$

$$Dv_{\mathbf{k}}^{(z)} + i\mathbf{k} \cdot V_{\mathbf{k}}^{(H)} = 0, \tag{118}$$

where $k = |k|$, and with the boundary conditions

$$\theta = v_{\mathbf{k}}^{(z)} = 0 \text{ in } z = 0, 1,$$

$$V_{\mathbf{k}}^{(H)}|_{z=0,1} = 0, \text{ or } V_{\mathbf{k}}^{(H)}|_{z=0} = DV_{\mathbf{k}}^{(H)}|_{z=1} = 0, \text{ or } V_{\mathbf{k}}^{(H)}|_{z=1} = DV_{\mathbf{k}}^{(H)}|_{z=0} = 0.$$

The above system is a classical one (Stokes operator and Laplace operator), already obtained in the periodic case. The only thing to check concerns the estimates with respect to $k \in \Gamma$. The scalar product of (115) with $v_{\mathbf{k}}^{(z)}$ plus the scalar product of (116) with $V_{\mathbf{k}}^{(H)}$ and integration by parts, taking into account $Dv_{\mathbf{k}}^{(z)} + ik \cdot V_{\mathbf{k}}^{(H)} = 0$ and the boundary values, leads to

$$||DV_{\mathbf{k}}||_0^2 + k^2||V_{\mathbf{k}}||_0^2 = -\int_0^1 F_{\mathbf{k}} \cdot \overline{V}_{\mathbf{k}} \mathrm{d}z \leqq ||F_{\mathbf{k}}||_0 ||V_{\mathbf{k}}||_0 \leqq \frac{k^2}{2}||V_{\mathbf{k}}||_0^2 + \frac{1}{2k^2}||F_{\mathbf{k}}||_0^2. \tag{119}$$

Moreover, thanks to the boundary conditions, we have also the Poincaré estimate (see (27))

$$||V_{\mathbf{k}}||_0 \leqq \frac{1}{\sqrt{2}}||DV_{\mathbf{k}}||_0.$$

It results that there exists $c > 0$ such that for any $k \in \Gamma$ we have

$$(1 + k^2)||DV_{\mathbf{k}}||_0^2 + (1 + k^2)^2||V_{\mathbf{k}}||_0^2 \leqq c||F_{\mathbf{k}}||_0^2. \tag{120}$$

The same is valid for θ:

$$(1 + k^2)||D\theta_{\mathbf{k}}||_0^2 + (1 + k^2)^2||\theta_{\mathbf{k}}||_0^2 \leqq c||g_{\mathbf{k}}||_0^2. \tag{121}$$

Now (117) leads to

$$||D^2\theta_{\mathbf{k}}||_0 \leqq k^2||\theta_{\mathbf{k}}||_0 + ||g_{\mathbf{k}}||_0 \leqq c'||g_{\mathbf{k}}||_0,$$

hence, using (121),

$$||\theta_{\mathbf{k}}||_2 \leqq c_1||g_{\mathbf{k}}||_0. \tag{122}$$

Let us show that the same type of estimate holds for $V_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)})$. We observe that the divergence free condition on F leads to

$$Df_{\mathbf{k}}^{(z)} + i\mathbf{k} \cdot F_{\mathbf{k}}^{(H)} = 0,$$

which implies

$$(D^2 - k^2)q_{\mathbf{k}} = 0, \tag{123}$$

$$(D^2 - k^2)^2 v_{\mathbf{k}}^{(z)} = (D^2 - k^2)f_{\mathbf{k}}^{(z)}, \tag{124}$$

with boundary conditions on $v_{\mathbf{k}}^{(z)}$ as $v_{\mathbf{k}}^{(z)}|_{z=0,1} = 0$, $Dv_{\mathbf{k}}^{(z)}|_{z=0,1} = 0$ or $Dv_{\mathbf{k}}^{(z)}|_{z=0} = 0$, $D^2v_{\mathbf{k}}^{(z)}|_{z=1} = 0$, or $Dv_{\mathbf{k}}^{(z)}|_{z=1} = 0$, $D^2v_{\mathbf{k}}^{(z)}|_{z=0} = 0$. Now taking the scalar product of (124) with $v_{\mathbf{k}}^{(z)}$ in $L^2(0,1)$, and integrations by parts, leads to

$$\begin{aligned} &||D^2v_{\mathbf{k}}^{(z)}||^2 + 2k^2||Dv_{\mathbf{k}}^{(z)}||^2 + k^4||v_{\mathbf{k}}^{(z)}||^2 \\ &= \int_0^1 i\mathbf{k}\cdot F_{\mathbf{k}}^{(H)} D\overline{v}_{\mathbf{k}}^{(z)}\mathrm{d}z - k^2\int_0^1 f_{\mathbf{k}}^{(z)}\overline{v}_{\mathbf{k}}^{(z)}\mathrm{d}z \\ &\leqq k||Dv_{\mathbf{k}}^{(z)}||_0||F_{\mathbf{k}}^{(H)}||_0 + k^2||v_{\mathbf{k}}^{(z)}||_0||f_{\mathbf{k}}^{(z)}||_0. \end{aligned}$$

Taking into account of (120), we immediately obtain

$$||v_{\mathbf{k}}^{(z)}||_2 \leqq c_2||F_{\mathbf{k}}||_0. \tag{125}$$

Now, in using (115) we can say that

$$||Dq_{\mathbf{k}}|| \leqq c_3||F_{\mathbf{k}}||_0, \tag{126}$$

where c_3 is independent of $k \in \Gamma$. Now (123) gives

$$q_{\mathbf{k}} = \alpha_{\mathbf{k}}e^{kz} + \beta_{\mathbf{k}}e^{-kz},$$

and

$$Dq_{\mathbf{k}} = k\alpha_{\mathbf{k}}e^{kz} - k\beta_{\mathbf{k}}e^{-kz}$$

should satisfy (126). It is easy to check that this implies that

$$k\alpha_{\mathbf{k}}^2e^{2k} + k\beta_{\mathbf{k}}^2 - 4k^2\alpha_{\mathbf{k}}\beta_{\mathbf{k}}$$

is bounded by $2c_3^2||F_{\mathbf{k}}||_0^2$ for large k. Now, since

$$|4k^2\alpha_{\mathbf{k}}\beta_{\mathbf{k}}| \leqq 8k^3\alpha_{\mathbf{k}}^2 + \frac{k}{2}\beta_{\mathbf{k}}^2,$$

and since, for large k, $8k^3 << ke^{2k}$, we conclude that for large k the quantity $k\alpha_{\mathbf{k}}^2e^{2k} + k\beta_{\mathbf{k}}^2$ is bounded by $c_4||F_{\mathbf{k}}||_0^2$. Now computing $||kq_{\mathbf{k}}||^2$, we see the same behavior in $k\alpha_{\mathbf{k}}^2e^{2k} + k\beta_{\mathbf{k}}^2$ for large k. It follows that we have

$$||kq_{\mathbf{k}}|| \leqq c_5||F_{\mathbf{k}}||_0,$$

and (116) allows us to conclude that

$$||V_{\mathbf{k}}^{(H)}||_2 \leqq c_6||F_{\mathbf{k}}||_0.$$

Collecting all the above estimates gives, for a certain constant $c > 0$,

$$||U_{\mathbf{k}}||_2 \leqq c||G_{\mathbf{k}}||_0,$$

which is the desired estimate for L^{-1} now bounded from $\mathcal{K}_{0,s}$ to $D_s(L) \subset \mathcal{K}_{2,s}$.

A.1. Extension of the Inverse of L

Let us consider now the same Equation (114) but with a less regular right hand side. Now we take $G \in (\mathcal{D}_{1/2,s})^*$ which is the dual of $\mathcal{D}_{1/2,s}$ defined in (24). This means that for any $V \in \mathcal{D}_{1/2,s}$, we have the following bound for the semi-linear form $\langle G, V\rangle_{0,s}$:

$$|\langle G, V\rangle_{0,s}| \leqq ||G||_{(\mathcal{D}_{1/2,s})^*}||V||_{\widetilde{1,s}}.$$

We are now looking for $U \in \mathcal{D}_{1/2,s}$ defined by a variational formulation (also classical for the Stokes linear operator, as well as for the Laplace operator (see [28]), both written in Fourier components)

$$\langle U, V\rangle_{\widetilde{1,s}} = -\langle G, V\rangle_{0,s} \text{ for any } V \in \mathcal{D}_{1/2,s},$$

where the definition of $\langle U, V\rangle_{\widetilde{1,s}}$ comes from (28). For the type of discussion which follows, we may also refer to [16] p. 223–224, adapted to each Fourier component here.

It is easy to check, in looking at the first equality in (119) and its analogue for $\theta_{\mathbf{k}}$, that

$$\langle LU, V\rangle_{0,s} = \langle G, V\rangle_{0,s} \text{ for any } V \in \mathcal{D}_{1/2,s}$$

holds, where the brackets are dual products. This proves that the unique solution $U \in \mathcal{D}_{1/2,s}$, hence by definition, $(-L)^{1/2}U \in \mathcal{K}_{0,s}$ and

$$||U||_{\widetilde{1,s}} \leqq ||G||_{(\mathcal{D}_{1/2,s})^*}, \tag{127}$$

which means that the operator L which is bounded from $\mathcal{D}_{1/2,s}$ to $(\mathcal{D}_{1/2,s})^*$ has its inverse bounded from $(\mathcal{D}_{1/2,s})^*$ to $\mathcal{D}_{1/2,s}$.

B. Proof of Lemmas 16

Let u be a scalar function in $\mathcal{H}^{(1)}_{1,s}$, which means that

$$u(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

with

$$\sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s||u_{\mathbf{k}}||_1^2 < \infty, \quad ||u_{\mathbf{k}}||_1^2 = \int_0^1 \left(|Du_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|u_{\mathbf{k}}|^2\right) \mathrm{d}z.$$

Assume now that u and v are scalar functions in $\mathcal{H}^{(1)}_{1,s}$, then

$$||uv||^2_{\mathcal{H}_{1,s}} = \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \left(|D(uv)_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right) \mathrm{d}z,$$

and using $(a + b)^2 \leqq 2a^2 + 2b^2$,

$$\leqq \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \left(2|(vDu)_{\mathbf{k}}|^2 + 2|(uDv)_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right) \mathrm{d}z.$$

From Lemma 10 we have that

$$\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s|(uDv)_{\mathbf{k}}(z)|^2$$
$$\leqq 2C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}(z)|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}|Dv_{\mathbf{m}}(z)|^2\right)+$$
$$+2C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}|u_{\mathbf{l}}(z)|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s|Dv_{\mathbf{m}}(z)|^2\right),$$

and the analogue holds for vDu.
Now, introduce u' and $\widetilde{u}$ defined by

$$\widetilde{u}_{\mathbf{k}}=|u_{\mathbf{k}}| \quad u'_{\mathbf{k}}=\sqrt{1+\mathbf{k}^2}\,\widetilde{u}_{\mathbf{k}},$$

then, in using $(1+|\mathbf{l}+\mathbf{m}|^2)\leqq 2((1+|\mathbf{l}|^2)+2(1+|\mathbf{m}|^2)$,

$$\sqrt{1+\mathbf{k}^2}|(uv)_{\mathbf{k}}|\leqq\sqrt{1+\mathbf{k}^2}\sum_{\mathbf{k}=\mathbf{l}+\mathbf{m}}\frac{u'_{\mathbf{l}}v'_{\mathbf{m}}}{\sqrt{1+\mathbf{l}^2}\sqrt{1+\mathbf{m}^2}}$$
$$\leqq\sqrt{2}\sum_{\mathbf{k}=\mathbf{l}+\mathbf{m}}\widetilde{u}_{\mathbf{l}}v'_{\mathbf{m}}+u'_{\mathbf{l}}\widetilde{v}_{\mathbf{m}}=\sqrt{2}[(\widetilde{u}v')_{\mathbf{k}}+(u'\widetilde{v})_{\mathbf{k}}].$$

Hence

$$(1+\mathbf{k}^2)|(uv)_{\mathbf{k}}|^2\leqq 4(|(\widetilde{u}v')_{\mathbf{k}}|^2+|(u'\widetilde{v})_{\mathbf{k}}|^2),$$

and using Lemma 10 again we obtain

$$\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s(1+|\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2$$
$$\leqq 8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}|v_{\mathbf{m}}|^2\right)+$$
$$+8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^s|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^{s_0}(1+|\mathbf{m}|^2)|v_{\mathbf{m}}|^2\right)+$$
$$+8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s|v_{\mathbf{m}}|^2\right)+$$
$$+8C(s,s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1+N_{\mathbf{l}}^2)^{s_0}|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1+N_{\mathbf{m}}^2)^s(1+|\mathbf{m}|^2)|v_{\mathbf{m}}|^2\right).$$

Now we can use

$$\int_0^1|Du_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2\mathrm{d}z\leqq c||u_{\mathbf{l}}||_{H^1}^2||v_{\mathbf{m}}||_{H_1}^2$$
$$\int_0^1(1+|\mathbf{l}|^2)|u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2\mathrm{d}z\leqq c(1+|\mathbf{l}|^2)||u_{\mathbf{l}}||_{L^2}^2||v_{\mathbf{m}}||_{H_1}^2$$

and the similar symmetric estimates to show that there is a constant $c^2(s, s_0) = 10cC^2(s, s_0)$ such that finally

$$||uv||^2_{1,s} \leqq c^2(s, s_0)(||u||^2_{1,s}||v||^2_{1,s_0} + ||u||^2_{1,s_0}||v||^2_{1,s}),$$

Lemma 16 is proved.
Assume now that u and v are scalar functions, respectively in $\mathcal{H}^{(1)}_{1,s}$ and $\mathcal{H}^{(1)}_{0,s}$ with $s \geqq s_0 > d/2$. Then

$$||uv||^2_{0,s} = \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N^2_{\mathbf{k}})^s|(uv)_{\mathbf{k}}|^2 \mathrm{d}z$$

which gives, by Lemma 10

$$\begin{aligned}\sum_{\mathbf{k}\in\Gamma}(1 + N^2_{\mathbf{k}})^s|(uv)_{\mathbf{k}}|^2 \leqq 2C(s, s_0)^2 &\left(\sum_{\mathbf{l}\in\Gamma}(1 + N^2_{\mathbf{l}})^s|u_{\mathbf{l}}|^2\right)\\ &\times\left(\sum_{\mathbf{m}\in\Gamma}(1 + N^2_{\mathbf{m}})^{s_0}|v_{\mathbf{m}}|^2\right) +\\ &+ 2C(s, s_0)^2\left(\sum_{\mathbf{l}\in\Gamma}(1 + N^2_{\mathbf{l}})^{s_0}|u_{\mathbf{l}}|^2\right)\\ &\times\left(\sum_{\mathbf{m}\in\Gamma}(1 + N^2_{\mathbf{m}})^s|v_{\mathbf{m}}|^2\right).\end{aligned}$$

Now we use

$$\int_0^1 |u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2 \mathrm{d}z \leqq c||u_{\mathbf{l}}||^2_{H^1}||v_{\mathbf{m}}||^2_{L^2},$$

which leads to

$$||uv||^2_{0,s} \leqq 2cC(s, s_0)^2(||u||^2_{1,s}||v||^2_{0,s_0} + ||u||^2_{1,s_0}||v||^2_{0,s})$$

which gives Lemma 17.
Now by Lemma 9 we have for all $z \in (0, 1)$ the two inequalities

$$\sum_{\mathbf{k}\in\Gamma}|(uv)_{\mathbf{k}}|^2 \leqq 2c_s\left(\sum_{\mathbf{l}\in\Gamma}(1 + N^2_{\mathbf{l}})^s|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}|v_{\mathbf{m}}|^2\right), \tag{128}$$

and

$$\sum_{\mathbf{k}\in\Gamma}|(uv)_{\mathbf{k}}|^2 \leqq 2c_s\left(\sum_{\mathbf{l}\in\Gamma}|u_{\mathbf{l}}|^2\right)\left(\sum_{\mathbf{m}\in\Gamma}(1 + N^2_{\mathbf{m}})^s|v_{\mathbf{m}}|^2\right). \tag{129}$$

We also have, for some $c > 0$, that

$$\int_0^1 |u_{\mathbf{l}}|^2|v_{\mathbf{m}}|^2 \mathrm{d}z \leqq c \min\{||u_{\mathbf{l}}||^2_{H^1}||v_{\mathbf{m}}||^2_{L^2}, ||u_{\mathbf{l}}||^2_{L^2}||v_{\mathbf{m}}||^2_{H^1}\}.$$

Then summing (128) on (0, 1) and using the last inequality leads to Lemma 18, while summing (129) and using the last inequality leads to Lemma 19.

C. Proofs of the Bounds for the Quadratic Term

For $U \in \mathcal{K}_{2,s}$, we have all components of ∇V and $\nabla\theta$ which are in $\mathcal{H}^{(1)}_{1,s}$. Moreover, for $U \in \mathcal{K}_{2,s}$ and $U' \in \mathcal{K}_{2,s}$ Lemma 16 says that the components of

$$V \cdot \nabla V', \quad V \cdot \nabla\theta', \quad V' \cdot \nabla V, \quad V' \cdot \nabla\theta$$

satisfy estimates given by this Lemma in $\mathcal{H}_{1,s}$. The projection $\mathfrak{P}$ does not change the estimates, hence

$$||B(U, U')||_{1,s} \leqq c(s, s_0)(||U||_{2,s}||U'||_{2,s_0} + ||U||_{2,s_0}||U'||_{2,s}),$$

which is (20).
For proving (21) we have $U \in \mathcal{K}_{1,s}$, hence components of ∇V and $\nabla\theta \in \mathcal{H}^{(1)}_{0,s}$ and Lemma 17 show that the components of $V \cdot \nabla V'$ and $V \cdot \nabla\theta'$ lie in $\mathcal{H}^{(1)}_{0,s}$. To obtain $B(U, U')$ we just need to apply the projection $\mathfrak{P}$ to $V \cdot \nabla V'$ and to $V' \cdot \nabla V$. Then estimate (21) results immediately from estimate of Lemma 17.
For proving (33) we need to prove that for $(U, V) \in \mathcal{K}_{1,s} \times \mathcal{K}_{1,0}$,

$$||B(U, V)||_{0,0} \leqq c'||U||_{1,s}||V||_{1,0}.$$

Indeed, components of ∇U and ∇V belong to $\mathcal{H}_{0,s}$ and $\mathcal{H}_{0,0}$, respectively, and we need to consider products of functions of the forms $\mathcal{H}_{0,s} \times \mathcal{H}_{1,0}$ and $\mathcal{H}_{1,s} \times \mathcal{H}_{0,0}$. Then Lemmas 18 and 19, and projecting by $\mathfrak{P}$ (as above), allow us to prove that $B(U, V) \in \mathcal{K}_{0,0}$ with the required estimate (33).

D. Study of the Nondegeneracy Condition Leading to (44)

Let us come back to the homogeneous system associated with (41), which gives for every fixed $\mathbf{k} \in \Gamma$ the discrete set of eigenvalues $\lambda_j(|\mathbf{k}|)$, $j = 0, 1, 2, \ldots$ (below, for the sake of simplicity, we omit to consider λ_0 as a function of $|k|^2$). Below, we only consider $\mathbf{k}$ in $\mathbb{R}^+$ since we know that only its modulus matters. We are interested in the concavity of the graph of $\lambda_0(k)$ in the neighborhood of $k = k_c > 0$, where $\frac{\mathrm{d}\lambda_0}{\mathrm{d}k}(k_c) = 0$.
By construction, we have

$$\begin{aligned}
\lambda_0(D^2 - k^2)v^{(z)}_{\mathbf{k}} + \theta_{\mathbf{k}} - Dq_{\mathbf{k}} &= 0,\\
\lambda_0(D^2 - k^2)V^{(H)}_{\mathbf{k}} - ik\mathbf{e}_1 q_{\mathbf{k}} &= 0,\\
\lambda_0(D^2 - k^2)\theta_{\mathbf{k}} + v^{(z)}_{\mathbf{k}} &= 0,\\
Dv^{(z)}_{\mathbf{k}} + ik\mathbf{e}_1 \cdot V^{(H)}_{\mathbf{k}} &= 0,
\end{aligned} \tag{130}$$

where $D = \mathrm{d}/\mathrm{d}z$, $\mathbf{e}_1$ is the unit vector along the positive x axis, and where

$$v^{(z)}_{\mathbf{k}}|_{z=0,1} = \theta_{\mathbf{k}}|_{z=0,1} = 0,$$

and either

$$V^{(H)}_{\mathbf{k}}|_{z=0,1} = 0, \quad \text{or } V^{(H)}_{\mathbf{k}}|_{z=0} = DV^{(H)}_{\mathbf{k}}|_{z=1} = 0, \text{ or } V^{(H)}_{\mathbf{k}}|_{z=1} = DV^{(H)}_{\mathbf{k}}|_{z=0} = 0.$$

For $k=k_c>0$ the eigenvalue $\lambda_0(k)$ reaches $\lambda_0>0$ where $\frac{d\lambda_0}{dk}(k_c)=0$, as this results from the analyticity of the function $\lambda_0(k)$ with $\lambda_0\to 0$ as $k\to 0$ and as $k\to\infty$ (see [25,26]). Our purpose is to compute $\frac{d^2\lambda_0}{dk^2}(k_c)$. We need $\frac{d^2\lambda_0}{dk^2}(k_c)\neq 0$ for establishing (44) since the denominator in (44) corresponds, up to a factor, to $\lambda(k)-\lambda_0$ in a neighborhood of k_c (notice that the function $\lambda(k)$ is even in k). In fact it is only known numerically that there is only one maximum and that the graph is concave at this point, so we intend to just give a formula for $\lambda_0''=\frac{d^2\lambda_0}{dk^2}(k_c)$.
More precisely, let us differentiate (130) with respect to k as follows:

$$\begin{aligned}
\lambda_0'(D^2-k^2)v_{\mathbf{k}}^{(z)}-2\lambda_0 k v_{\mathbf{k}}^{(z)}+\lambda_0(D^2-k^2)v_{\mathbf{k}}'^{(z)}+\theta_{\mathbf{k}}'-Dq_{\mathbf{k}}'&=0,\\
\lambda_0'(D^2-k^2)V_{\mathbf{k}}^{(H)}-2\lambda_0 k V_{\mathbf{k}}^{(H)}-i\mathbf{e}_1 q_{\mathbf{k}}+\lambda_0(D^2-k^2)V_{\mathbf{k}}'^{(H)}-ik\mathbf{e}_1 q_{\mathbf{k}}'&=0,\\
\lambda_0'(D^2-k^2)\theta_{\mathbf{k}}-2\lambda_0 k\theta_{\mathbf{k}}+\lambda_0(D^2-k^2)\theta_{\mathbf{k}}'+v_{\mathbf{k}}'^{(z)}&=0,\\
Dv_{\mathbf{k}}'^{(z)}+ik\mathbf{e}_1\cdot V_{\mathbf{k}}'^{(H)}+i\mathbf{e}_1\cdot V_{\mathbf{k}}^{(H)}&=0,
\end{aligned}\tag{131}$$

which, for $k=k_c$, gives

$$\begin{aligned}
-2\lambda_0 k_c v_{\mathbf{k}}^{(z)}+\lambda_0(D^2-k_c^2)v_{\mathbf{k}}'^{(z)}+\theta_{\mathbf{k}}'-Dq_{\mathbf{k}}'&=0,\\
-2\lambda_0 k_c V_{\mathbf{k}}^{(H)}-i\mathbf{e}_1 q_{\mathbf{k}}+\lambda_0(D^2-k_c^2)V_{\mathbf{k}}'^{(H)}-ik_c\mathbf{e}_1 q_{\mathbf{k}}'&=0,\\
-2\lambda_0 k_c\theta_{\mathbf{k}}+\lambda_0(D^2-k_c^2)\theta_{\mathbf{k}}'+v_{\mathbf{k}}'^{(z)}&=0,\\
Dv_{\mathbf{k}}'^{(z)}+ik_c\mathbf{e}_1\cdot V_{\mathbf{k}}'^{(H)}+i\mathbf{e}_1\cdot V_{\mathbf{k}}^{(H)}&=0,
\end{aligned}\tag{132}$$

with the same boundary conditions for $(V_{\mathbf{k}}'^{(H)}, v_{\mathbf{k}}'^{(z)}, \theta_{\mathbf{k}}')$ as for the eigenvector $U_{\mathbf{k}}=(V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)}, \theta_{\mathbf{k}})$. Before going further we need to determine the derivative with respect to k of the eigenvector $U_{\mathbf{k}}$ in $k=k_c$. We observe that the last equation in (132) is not exactly as in (130), so we need to make a little change of notation for being able to use the pseudo-inverse of $\lambda_0 L_{\mathbf{k}_c}+A_{\mathbf{k}_c}$ in $\mathbf{k}_c=k_c\mathbf{e}_1$.
Let us define

$$\widetilde{U}_{\mathbf{k}}'=(\widetilde{V}_{\mathbf{k}}'^{(H)}, v_{\mathbf{k}}'^{(z)}, \theta_{\mathbf{k}}'),\ \text{with}\ \ \widetilde{V}_{\mathbf{k}}'^{(H)}=V_{\mathbf{k}}'^{(H)}+\frac{1}{k_c}V_{\mathbf{k}}^{(H)},$$

then (132) becomes

$$\begin{aligned}
\lambda_0(D^2-k_c^2)v_{\mathbf{k}}'^{(z)}+\theta_{\mathbf{k}}'-Dq_{\mathbf{k}}'&=2\lambda_0 k_c v_{\mathbf{k}}^{(z)},\\
\lambda_0(D^2-k_c^2)\widetilde{V}_{\mathbf{k}}'^{(H)}-ik_c\mathbf{e}_1 q_{\mathbf{k}}'&=2\lambda_0 k_c V_{\mathbf{k}}^{(H)}+\frac{2\lambda_0}{k_c}(D^2-k_c^2)V_{\mathbf{k}}^{(H)},\\
\lambda_0(D^2-k_c^2)\theta_{\mathbf{k}}'+v_{\mathbf{k}}'^{(z)}&=2\lambda_0 k_c\theta_{\mathbf{k}},\\
Dv_{\mathbf{k}}'^{(z)}+ik_c\mathbf{e}_1\cdot\widetilde{V}_{\mathbf{k}}'^{(H)}&=0.
\end{aligned}\tag{133}$$

The system (133) holds because of the property $\lambda_0'=0$, which implies that the compatibility condition is realized for the right hand side (cancelling the scalar product of the 3 first lines resp. with $(v_{\mathbf{k}}^{(z)}, V_{\mathbf{k}}^{(H)}, \theta_{\mathbf{k}})$), so we have

$$2\lambda_0 k_c||U_{\mathbf{k}}||_0^2+\frac{2\lambda_0}{k_c}\int_0^1(D^2-k_c^2)V_{\mathbf{k}}^{(H)}\cdot\overline{V_{\mathbf{k}}^{(H)}}\mathrm{d}z=0,$$

i.e., after integrating by parts

$$k_c^2(||v_{\mathbf{k}}^{(z)}||_0^2 + ||\theta_{\mathbf{k}}||_0^2) - ||DV_{\mathbf{k}}^{(H)}||_0^2 = 0. \tag{134}$$

Notice that for $\mathbf{k} = k\mathbf{e}_1$, the functions $v_{\mathbf{k}}^{(z)}$, $\theta_{\mathbf{k}}$, $v_{\mathbf{k}}^{\prime(z)}$, $\theta_{\mathbf{k}}^{\prime}$ are real valued, while $V_{\mathbf{k}}^{(H)}$ and $V_{\mathbf{k}}^{\prime(H)}$ are purely imaginary.

Remark 63. We can also give a formula for any $\mathbf{k}$ in using (131):

$$\lambda_0'(k)||U_{\mathbf{k}}||_{\widetilde{1}}^2 = \frac{2\lambda_0}{k}\left[||DV_{\mathbf{k}}^{(H)}||_0^2 - k^2(||v_{\mathbf{k}}^{(z)}||_0^2 + ||\theta_{\mathbf{k}}||_0^2)\right], \tag{135}$$

where

$$||U_{\mathbf{k}}||_{\widetilde{1}}^2 = ||DU_{\mathbf{k}}||_0^2 + |\mathbf{k}|^2||U_{\mathbf{k}}||^2, \tag{136}$$

which corresponds to the norm of the $\mathbf{k}$-component in the definition (28) of norm $||\cdot||_{\widetilde{1,s}}$.

From (133) we can now write

$$\widetilde{U}_{\mathbf{k}}' = (\widetilde{\lambda_0 L_{\mathbf{k}_c} + A_{\mathbf{k}_c}})^{-1}\left[2\lambda_0 k_c U_{\mathbf{k}} + \mathfrak{P}_k\left(\frac{2\lambda_0}{k_c}(D^2 - k_c^2)V_{\mathbf{k}}^{(H)}, 0, 0\right)\right],$$

where $(\widetilde{\lambda_0 L_{\mathbf{k}_c} + A_{\mathbf{k}_c}})^{-1}$ is the pseudo-inverse of $(\lambda_0 L_{\mathbf{k}_c} + A_{\mathbf{k}_c})$ taking values in the orthogonal of its kernel (selfadjoint operator) and $\mathfrak{P}_k$ is the k-component of the projection $\mathfrak{P}$ defined in Section 4.1. Hence

$$\begin{aligned} U_{\mathbf{k}}' = {} & (\widetilde{\lambda_0 L_{\mathbf{k}_c} + A_{\mathbf{k}_c}})^{-1}\left[2\lambda_0 k_c U_{\mathbf{k}} + \mathfrak{P}_k\left(\frac{2\lambda_0}{k_c}(D^2 - k_c^2)V_{\mathbf{k}}^{(H)}, 0, 0\right)^t\right] \\ & - \left(\frac{1}{k_c}V_{\mathbf{k}}^{(H)}, 0, 0\right)^t. \end{aligned} \tag{137}$$

Differentiating (135) with respect to k in $k = k_c$ then gives

$$\lambda_0''||U_{\mathbf{k}}||_{\widetilde{1}}^2 = 2\lambda_0\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{k}||DV_{\mathbf{k}}^{(H)}||_0^2 - k(||v_{\mathbf{k}}^{(z)}||_0^2 + ||\theta_{\mathbf{k}}||_0^2)\right)|_{k=k_c}, \tag{138}$$

which is the desired formula, where all terms are now known.

E. Proof of Lemma 53

We refer extensively to [1], pages 628–636, here adapted to an operator in an infinite-dimension space (since we do not consider the projection Π').
The operator $(\mathcal{A} - \lambda_0)$ is diagonal (all $\mathbf{k}$-th Fourier components are uncoupled for operators $\Delta, \mathfrak{P}, L, A$) as well as for orthogonal projections π_0 and Π_N. The projection $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ is also diagonal, since it just modifies each Fourier component $e^{i\mathbf{k}_j\cdot\mathbf{x}}$, $j = 1, 2, \ldots, 2q$. Moreover, notice that $\mathbf{k}_j$ belongs to the singular set $S_{(N)}$ for any N since $|\mathbf{k}_j| = k_c$. However 0 is not an eigenvalue because of

the z dependency of coefficients of $e^{i\mathbf{k}_j\cdot\mathbf{x}}$, the corresponding eigenvalues being $\lambda_j(k_c^2)-\lambda_0<-\delta_0<0$, $j=1,2,\ldots$.
Eigenvalues of $D_N=\Pi_N\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0\Pi_N$ are $\lambda_j(|\mathbf{k}|^2)-\lambda_0$, $j=0,1,\ldots$ with $||\mathbf{k}|-k_c|\leqq\delta_1$, and $N_{\mathbf{k}}\leqq N$, the eigenvalues close to 0 corresponding to $j=0$, with the estimate (108) (notice that the operator $\mathbf{Q}_0$ eliminates the eigenvalue 0). Then, the required estimates on $(D_N)^{-1}$ restricted to the subspace corresponding to parts of $\Omega_N=R_{(N)}+S_{(N)}$ are valid. For example, since we have for $\mathbf{k}\in R_{(N)}$, $\lambda_0-\lambda_0(|\mathbf{k}|^2)\geqq\rho$, and since the operator is self adjoint in $\mathcal{K}_{0,s}$,

$$||D_Rh||_{0,s}\geqq\rho||h||_{0,s}\text{ for any }h\in E_N,$$

where D_R is the operator D_N restricted to Fourier modes with $\mathbf{k}\in R_{(N)}$.
Let us now show the "multiplication property" of operator εT, where Lemma 36 gives, for $(\varepsilon,\widetilde{\mu},V)\in[0,\varepsilon_1]\times[-\varepsilon,\varepsilon]\times\mathbf{Q}_0\mathcal{K}_{0,s}$, $||V||_{0,s_0}\leqq 1$,

$$\varepsilon T(\varepsilon,\widetilde{\mu},V)=:\Pi_N(\widetilde{\mu}+\mathfrak{B}_\varepsilon+\varepsilon^2\widetilde{\mu}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}+\mathfrak{R}_{\varepsilon,\widetilde{\mu},V})\Pi_N,\tag{139}$$

with estimates (84).
First, for $U\in\mathcal{K}_{1,s}$, $s\geqq s_0>d/2$ and $H\in\mathcal{K}_{1,0}$, we see with the definition 21 of $B(U,H)$ that for $U=(V,\theta)$ and $H=(V_H,\theta_H)$ there are functions occurring in components of

$$V\cdot\nabla V_H,\ V_H\cdot\nabla V,\ V\cdot\nabla\theta_H,\ V_H\cdot\nabla\theta,$$

each one denoted by T_1H that lies in $\mathcal{H}_{0,0}$ (see Lemmas 18, 19), satisfying a bound such that, for $A,B\subset\Omega(N)$ (see definition of T_B^A at Lemma 53),

$$||[T_1]_B^AH||_{0,0}\leqq c(s)\frac{||U||_{1,s}}{(1+d(A,B))^{s-d/2}}||H||_{1,0},$$

as is obtained by the same proof as Lemma 3.9 in [1]. We observe that the projection $\mathfrak{P}$ is diagonal in Fourier components, so that the above estimate stays valid for $B(U,H)$ in $\mathcal{K}_{0,0}$. Now the operator $(-L)^{-1/2}$ is also diagonal, and bounded from $\mathcal{K}_{0,s}$ to $\mathcal{K}_{1,s}$ for all $s\geqq 0$. It then results from the definition of $\mathcal{B}$ that we have the following generalization of (26) for any $V\in\mathcal{K}_{0,s}$, $s\geqq s_0>d/2$ and $h\in\mathcal{K}_{0,0}$:

$$||[\mathcal{B}(V,\cdot)]_B^Ah||_{0,0}\leqq c(s)\frac{||V||_{0,s}}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.\tag{140}$$

We then look at the operator appearing in (73):

$$\widetilde{\mu}+\mu_\varepsilon-2\mathbf{Q}_0\mathcal{B}(u_\varepsilon,\cdot)-2\varepsilon^4\mathbf{Q}_0\mathcal{B}_1(V,\cdot).$$

The operator $\mathbf{Q}_0$ is diagonal, hence the above estimate (140) leads to a bound in $\mathcal{K}_{0,0}$ as

$$c(s)\frac{(\varepsilon+\varepsilon^4||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.\tag{141}$$

Now we need to track the estimate for the transformed operator after the splitting by π_0 (see Section 7.5). For its computation we need first to look at operator $\mathfrak{Q}_{\varepsilon,\widetilde{\mu},V}^{(1,1)}$ acting in $(\mathbb{I}-\pi_0)\mathbf{Q}_0\mathcal{K}_{0,0}$. It is obtained via a Neumann series of powers of operators satisfying estimates as (141), provided that $||V||_{0,s_0} \leqq 1$, and proofs of Lemmas 3.10, 3.11 of [1] apply analogously, leading to

$$||[\mathfrak{Q}^{(1,1)}_{\varepsilon,\widetilde{\mu},V}]^A_B h||_{0,0} \leqq c(s)\frac{(1+\varepsilon^4||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.$$

The composition of two operators satisfying the estimates, as above, also satisfies the same estimate, with modified constants, so that finally, for (139) and for any $V \in \mathcal{K}_{0,s}$, $||V||_{0,s_0} \leqq 1$, $s \geqq s_0 > d/2$ and $h \in \mathcal{K}_{0,0}$,

$$||\varepsilon T(\varepsilon,\widetilde{\mu},V)]^A_B h||_{0,0} \leqq c(s)\frac{\varepsilon(1+\varepsilon^3||V||_{0,s})}{(1+d(A,B))^{s-d/2}}||h||_{0,0}.$$

F. A C^2 Property for the Nash–Moser Theorem in [2]

The starting point is the Nash–Moser theorem 3 in BERTI--BOLLE--PROCESI [2]. We want to extend this theorem from the C^1-case to the C^2-case. We assume the conditions of that theorem with $\nu = 0$, and moreover that $F(\epsilon,\lambda,u)$ is C^3 in (ϵ,λ,u) on $[0,\epsilon_0) \times \Lambda \times X_{s_0}$ and that the following conditions are fulfilled for $z := (\epsilon,\lambda) \in [0,\epsilon_0) \times \Lambda$ and $u \in X_s$, $s \in [s_0, S)$, with $||u||_{s_0} \leqq 1$:

$(F2)^+$ $\quad ||\partial^2_\lambda F(z,u)||_s \leqq C(s)(||u||_s + 1)$

$(F3)^+$ $\quad ||D^3_u F(z,u)[v_1,v_2,v_3]||_s \leqq C(s)(||u||_s||v_1||_{s_0}||v_2||_{s_0}||v_3||_{s_0}$
$+||v_1||_s||v_2||_{s_0}||v_3||_{s_0} + ||v_2||_s||v_1||_{s_0}||v_3||_{s_0}|| + ||v_3||_s||v_1||_{s_0}||v_2||_{s_0})$

$(F4)^+$ $\quad ||\partial^2_\lambda D_u F(z,u)[v]||_s \leqq C(s)(||u||_s||v||_{s_0} + ||v||_s)$,
$||\partial_\lambda D^2_u F(z,u)[v_1,v_2]||_s \leqq C(s)(||u||_s||v_1||_{s_0}||v_2||_{s_0}$
$+ ||v_1||_s||v_2||_{s_0} + ||v_2||_s||v_1||_{s_0})$.

Then Theorem 1 of [2] holds with $\nu = 0$ and $\partial^2_\lambda u$ exists and belongs to $C([0,\epsilon_2) \times \Lambda, X_{s_0})$. To prove this we show that *the sequence* $(\partial^2_\lambda u_n)_{n\geqq 0}$ *converges in* $C([0,\epsilon_2) \times \Lambda, X_{s_0})$*, where* u_n *is as in* [2]. *Moreover, given* $\eta \in (0,1)$*, we may choose* $N_0(\gamma)$ *large enough such that for* $\partial^2_\lambda u_n : [0,\epsilon_2) \times \Lambda \to E_{n+1}$*, the properties* $(Pj)_n$*,* $j = 1,2,3,4$ *are supplemented by*

$$(P1)^+_n \quad 1 + ||\partial^2_\lambda u_n||_{s_0} \leqq C(\gamma)N_0^\sigma, \tag{142}$$

$$(P2)^+_n \quad ||\partial^2_\lambda (u_{n+1} - u_n)||_{s_0} \leqq N_{n+1}^{-1+\eta}, \tag{143}$$

$$(P4)^+_n \quad B''_n = 1 + ||\partial^2_\lambda u_n||_{\bar{s}} \leqq 2N_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{144}$$

Finally in $(P4)_n$ *we have* $B_n \leqq 2N_{n+1}^{\mu+\eta}$, $B'_n \leqq 2N_{n+1}^{\sigma/4+\mu+2\eta}$.
We denote formula numbers from [2] in what follows by adding a zero in front of that number. Thus (041) corresponds to (41) in [2]. First we remark that corresponding to (034) and (038) we also have, for $z \in \mathcal{N}(A_{n+1}, 2\gamma N_{n+1}^{-\sigma/2})$, that

$$||\tilde{h}_{n+1}||_{\bar{s}} \leqq N_{n+1}^{2\mu+2\eta}, \tag{145}$$

$$||\partial_z \tilde{h}_{n+1}||_{s_0} \leqq N_{n+1}^{-3\sigma/4-1+2\eta}, \tag{146}$$

$$||\partial_z \tilde{h}_{n+1}||_{\bar{s}} \leqq N_{n+1}^{\sigma/2+2\mu+3\eta}, \tag{147}$$

and $||h_{n+1}||_{\bar{s}} \leqq N_{n+1}^{2\mu+2\eta}$ with a proof quite similar to that in [2]. Similarly, it follows from this that [2, Theorem 1] holds in case $\nu = 0$.
To prove the C^2 property in λ we will follow the induction process in [2]. First functions $\tilde{u}_0$ and $\tilde{h}_n$ are constructed. Then $u_0 := \psi_0 \tilde{u}_0, h_n := \psi_n \tilde{h}_n, u_{n+1} := u_n + h_{n+1}$, where the cut-off function ψ_n is defined in (050), but now with the extra property that it is C^2 and

$$|\partial_z \psi_n| \leqq C\gamma^{-1} N_n^{\sigma/2}, |\partial_z^2 \psi_n| \leqq C^2 \gamma^{-2} N_n^{\sigma}. \tag{148}$$

From the implicit function theorem it follows that $\tilde{h}_n$ is C^2 in λ and then the same follows for h_n and u_n.
Next we have to estimate the norms of these functions in order to show that the sequence $\partial_\lambda^2 u_n \in C([0, \epsilon_2) \times \Lambda, E_n)$ converges in $C([0, \epsilon_2) \times \Lambda, X_{s_0})$.
By (032) we have $\Pi_{n+1} F(z, u) = 0$ if $u = u_n + \tilde{h}_{n+1} =: u_n^+$ and $z \in \mathcal{N}(A_{n+1}, 2\gamma N_n^{-\sigma/2})$. This also holds for $n = -1$ with $u_{-1} = 0, u_{-1}^+ = \tilde{u_0} = \tilde{h_0}$.
Applying ∂_λ^2 to this equation leads to

$$L_{n+1}^+ \partial_\lambda^2 \tilde{h}_{n+1} + M_{n+1} = 0,$$

where $L_{n+1}^+(z) := \Pi_{n+1} D_u F(z, u_n^+)$ which is invertible by [2, Lemma 2.3] and

$$\begin{aligned} M_{n+1} := \Pi_{n+1}[&\partial_\lambda^2(F(z, u_n^+)) + 2\partial_\lambda D_u(F(z, u_n^+)[\partial_\lambda u_n^+] \\ &+ D_u^2(F(z, u_n^+))[\partial_\lambda u_n^+, \partial_\lambda u_n^+] + D_u(F(z, u_n^+))[\partial_\lambda^2 u_n] \end{aligned}$$

for z as above.
First let $n = -1$. Then $||M_0||_s$ may be estimated using $(F2)^+$, $(F3)$ and $F(4)$. Thus

$$\begin{aligned} ||M_0||_s \leqq C(s)[&||\tilde{u_0}||_s(1 + 2||\partial_\lambda \tilde{u_0}||_{s_0} + ||\partial_\lambda \tilde{u_0}||_{s_0}^2) \\ &+ 2||\partial_\lambda \tilde{u_0}||_s(1 + ||\partial_\lambda \tilde{u_0}||_{s_0}) + 1]. \end{aligned}$$

From [2, p. 385] we have

$$\begin{aligned} ||\tilde{u_0}||_{s_0} \leqq \rho_0 = C_0 \gamma^{-1} N_0^{\mu} \epsilon, ||\partial_\lambda \tilde{u_0}||_{s_0} \leqq K\gamma^{-1} N_0^{\mu}, \\ ||\tilde{u_0}||_{\bar{s}} \leqq K(\gamma) N_0^{\mu} \epsilon, ||\partial_\lambda \tilde{u_0}||_{\bar{s}} \leqq K(\gamma) N_0^{\mu}. \end{aligned}$$

Then we get

$$||M_0||_s \leqq C_1(\gamma) N_0^{2\mu}$$

for both $s = s_0$ and $s = \bar{s}$. Then we apply (015) and (016) to $\partial_\lambda^2 \tilde{u}_0 = -(L_0^+)^{-1} M_0$ and obtain $||\partial_\lambda^2 \tilde{u_0}||_s \leqq C(\gamma) N_0^{3\mu}$ for both values of s.
From $u_0 := \psi_0 \tilde{u}_0$ and (148) we deduce $(P1)_0$ and $(P4)_0$ for $\eta > 0$ and N_0 sufficiently large.

For $n \geqq 0$ we write $M_{n+1} = \Pi_{n+1} \sum_{j=0}^{6} A_j$ with

$$\begin{aligned}
A_0 &= \partial_\lambda^2 F(z, u_n) + 2\partial_\lambda D_u F(z, u_n)[\partial_\lambda u_n] + D_u^2 F(z, u_n)[\partial_\lambda u_n, \partial_\lambda u_n] \\
&\quad + D_u F(z, u_n)[\partial_\lambda^2 u_n] \\
A_1 &= \partial_\lambda^2 (F(z, u_n^+) - F(z, u_n)) = \int_0^1 \partial_\lambda^2 D_u(F(z, u_n + \theta \tilde{h}_{n+1}))\mathrm{d}\theta[\tilde{h}_{n+1}] \\
A_2 &= 2\partial_\lambda D_u (F(z, u_n^+) - F(z, u_n))[\partial_\lambda u_n^+] \\
&= 2\int_0^1 \partial_\lambda D_u^2 (F(z, u_n + \theta \tilde{h}_{n+1}))\mathrm{d}\theta[\tilde{h}_{n+1}, \partial_\lambda u_n^+] \\
A_3 &= 2\partial_\lambda D_u F(z, u_n))[\partial_\lambda \tilde{h}_{n+1}] \\
A_4 &= D_u^2 (F(z, u_n^+) - F(z, u_n))[\partial_\lambda u_n^+, \partial_\lambda u_n^+] \\
&= \int_0^1 D_u^3 (F(z, u_n + \theta \tilde{h}_{n+1})) d\theta[\tilde{h}_{n+1}, \partial_\lambda u_n^+, \partial_\lambda u_n^+] \\
A_5 &= D_u^2 F(z, u_n)([\partial_\lambda u_n^+, \partial_\lambda u_n^+] - [\partial_\lambda u_n, \partial_\lambda u_n]), \\
A_6 &= D_u (F(z, u_n^+) - F(z, u_n))[\partial_\lambda^2 u_n] \\
&= \int_0^1 D_u^2 (F(z, u_n + \theta \tilde{h}_{n+1}))\mathrm{d}\theta[\tilde{h}_{n+1}, \partial_\lambda^2 u_n].
\end{aligned}$$

Similarly as to [2], using $(S1), (F4), (F4)^+, (F3)^+$ and the estimates for $||\tilde{h}_{n+1}||_s, ||\partial_z \tilde{h}_{n+1}||_s, ||u_n||_s$, we obtain that there are constants $C_1(s, \gamma)$ independent of n such that

$$||\Pi_{n+1}(A_1 + A_2 + A_4)||_{s_0} \leqq C_1(s_0, \gamma) N_{n+1}^{-\sigma-1} \tag{149}$$

$$||\Pi_{n+1}(A_1 + A_2 + A_4)||_{\overline{s}} \leqq C_1(\overline{s}, \gamma) N_{n+1}^{2(\mu+\eta)}. \tag{150}$$

Furthermore, using (F4) it follows that there exist positive constants K independent of n, which may be different in different places such that

$$||\Pi_{n+1} A_3||_{s_0} \leqq K N_{n+1}^{-3\sigma/4-1+2\eta}, \; ||\Pi_{n+1} A_3||_{\overline{s}} \leqq K N_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{151}$$

In A_5 we may replace $[\partial_\lambda u_n^+, \partial_\lambda u_n^+] - [\partial_\lambda u_n, \partial_\lambda u_n]$ by $[\partial_\lambda \tilde{h}_{n+1}, \partial_\lambda (2u_n + \tilde{h}_{n+1})]$ and then with (F3), (142), (145), (146) and $(P4)_n$ we obtain

$$||\Pi_{n+1} A_5||_{s_0} \leqq K N_{n+1}^{-3\sigma/4-1+2\eta}, \tag{152}$$

$$||\Pi_{n+1} A_5||_{\overline{s}} \leqq K N_{n+1}^{\sigma/2+2\mu+3\eta}. \tag{153}$$

Also, using (144), we have

$$||\Pi_{n+1} A_6||_{s_0} \leqq K N_{n+1}^{-\sigma-1}, \tag{154}$$

$$||\Pi_{n+1} A_6||_{\overline{s}} \leqq K N_{n+1}^{2(\mu+\eta)}. \tag{155}$$

Finally, using $(F2)^+, (F3), (F4), (F6), (P4)_n$ and $(P4)_n^+$, we get

$$||\Pi_{n+1} A_0||_{\overline{s}} \leqq K N_{n+1}^{\sigma/2+2\mu+3\eta}.$$

With [2, $(S2)$] it follows as in (047) that

$$||\Pi_{n+1}A_0||_{s_0} = ||\Pi_{n+1}(I - \Pi_n)A_0||_{s_0} \leqq KN_n^{-\bar{s}+s_0}||\Pi_{n+1}A_0||_{\bar{s}} \leqq K'N_{n+1}^{-\sigma/2-2+3\eta}.$$

Combining the estimates for A_j, $j = 0, \ldots 6$ it follows that

$$||M_{n+1}||_{s_0} \leqq KN_{n+1}^{-\sigma/2-2+3\eta}$$

and

$$||M_{n+1}||_{\bar{s}} \leqq KN_{n+1}^{\sigma/2+2\mu+3\eta}.$$

From $(P4)_n^+$ and [2, Lemma 2.3] we obtain

$$||\partial_\lambda^2\tilde{h}_{n+1}||_{s_0} \leqq KN_{n+1}^{-\sigma/2+\mu-2+3\eta}, \; ||\partial_\lambda^2\tilde{h}_{n+1}||_{\bar{s}} \leqq KN_{n+1}^{\sigma/2+3(\mu+\eta)}. \tag{156}$$

With $h_{n+1} = \psi_{n+1}\tilde{h}_{n+1}$ and (148) it follows that

$$||\partial_\lambda^2 h_{n+1}||_s \leqq ||\partial_\lambda^2\tilde{h}_{n+1}||_s + 2|\partial_\lambda\psi_{n+1}|||\partial_\lambda\tilde{h}_{n+1}||_s + |\partial_\lambda^2\psi_{n+1}|||\tilde{h}_{n+1}||_s, \tag{157}$$

and from the corresponding estimates for $\tilde{h}_{n+1}$ in (145), (146), (147) and (156), we get that

$$||\partial_\lambda^2 h_{n+1}||_{s_0} \leqq N_{n+1}^{-1+\eta}, \; ||\partial_\lambda^2 h_{n+1}||_{\bar{s}} \leqq N_{n+1}^{\sigma+2\mu+4\eta}. \tag{158}$$

From this and $u_{n+1} = u_n + h_{n+1}$ we deduce $(P2)_{n+1}^+$ and $(P1)_{n+1}^+$. Furthermore, with $(P4)_n^+$ it follows that

$$\begin{aligned} B''_{n+1} &\leqq B''_n + ||\partial_\lambda^2 h_{n+1}||_{\bar{s}} \leqq 2N_{n+1}^{\sigma/2+2\mu+3\eta} + N_{n+1}^{\sigma+2\mu+4\eta} \leqq 2N_{n+1}^{\sigma+2\mu+4\eta} \\ &= 2N_{n+2}^{\sigma/2+\mu+2\eta}, \end{aligned}$$

and so $(P4)_{n+1}$ holds and the induction step is proven. Finally this implies, as in [2, section 2.4], the statement on the convergence of the maps $\partial_\lambda^2 u_n$ in $C([0, \epsilon_2) \times \Lambda, X_{s_0})$ to $\partial_\lambda^2 u$.

G. Compliance with ethical standards

There are no conflicts of interest in this paper, and it follows all the rules required in the instructions for authors in the paragraphs "Ethical responsibilities of authors" and "Compliance with ethical standards".

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