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David Santos, Yuri

Published in:
Logic, Rationality, and Interaction - 7th International Workshop, LORI

DOI:
10.1007/978-3-662-60292-8_5

Consolidation of Belief in Two Logics of Evidence

Yuri David Santos(✉)

University of Groningen, Groningen, The Netherlands
y.david.santos@rug.nl

Abstract. Recently, several logics have emerged with the goal of modelling evidence in a more relaxed sense than that of justifications. Here, we explore two of these logics, one based on neighborhood models and the other being a four-valued modal logic. We establish grounds for comparing these logics, finding, for any model, a counterpart in the other logic which represents roughly the same evidential situation. Then we propose operations for *consolidation*, answering our central question: *What should the doxastic state of a rational agent be in a given evidential situation?* These operations map evidence models to Kripke models. We then compare the consolidations in the two logics, finding conditions under which they are isomorphic. By taking this dynamic perspective on belief formation we pave the way for, among other things, a study of the complexity, and an AGM-style analysis of rationality of these belief-forming processes.

Keywords: Evidence logics · Epistemic logic · Many-valued logic

1 Introduction

Epistemic and doxastic logics have been used for decades to model the knowledge and beliefs of agents [16,22]. Intelligent agents, especially in real-world settings, however, build up their beliefs from inputs that might be incomplete or even inconsistent. We think of these inputs as *evidence*, broadening of the concept of justification featured in justification logics [4–6,19,23]. Real agents normally have access to raw, imperfect data, which they process into a (preferably consistent) set of beliefs, which only then can be used to make sensible decisions and to act.

Like [12–15,20,24,30], the paper [29] presents a multi-agent *four-valued epistemic logic* (FVEL) to model evidence. But differently from those, it does not feature a belief modality. Our initial goal here is to add beliefs to that framework. It is of little use to model evidence and not derive any beliefs from it. In the spirit of [12], we assume that rational belief can be determined from evidence. However, we do not do that by extending FVEL models, similarly to the strategy in [12]. Instead, we extract a doxastic Kripke model representing the agents' beliefs from the FVEL model, which represents their evidence. With that, we not only accomplish the first goal of adding beliefs to the FVEL framework, but also introduce a dynamic perspective on forming beliefs from evidence. This new perspective, compared to the static one in [14], where evidence and belief

P. Blackburn et al. (Eds.): LORI 2019, LNCS 11813, pp. 57–70, 2019.
https://doi.org/10.1007/978-3-662-60292-8_5

coexist, is akin to public announcement logic [16,25,26] compared to epistemic logic: it adds a model-changing aspect. Rational beliefs, although pre-encoded in evidence, are not obtained for free, but require “computation”. This process of forming beliefs from evidence, which we call *consolidation*, is represented by transformations from evidence models to Kripke models. This idea generalises the static approach, because we can represent the “consolidation” of models where belief and evidence coexist as an automorphism from these models to themselves.

This paper is structured as follows. In Sect. 2 we introduce FVEL, a logic that models evidence but no beliefs. In Sect. 3, we present the main idea of this paper, the so-called *cautious consolidation*, a transformation from FVEL evidence models to doxastic Kripke models. We also discuss some of its properties. The remainder of the paper is concerned with comparing our work with another approach in the literature: the work started by Van Benthem and Pacuit [14] and extended together with Fernández-Duque [12,30]. Baltag et al. [8] also built upon those logics, offering more general topological semantics, but for the purpose of this paper the models of [14] will suffice. We cannot compare our consolidations with the ones from Van Benthem et al. if we cannot compare those evidence models in the first place, so that is what is done in Sect. 4. Then in Sect. 5 we finally compare the consolidations per se. We lay out our conclusions and ideas left for future work in Sect. 6. Proofs were omitted, but are available online[1].

2 A Multi-agent Logic of Evidence

Now we concisely describe the *four-valued epistemic logic* (FVEL, in short) [29], the logic of evidence to which we apply our idea of consolidations.

Definition 1. *[29] Let At be a countable set of atomic propositions and A a finite set of agents. A formula φ in the language $\mathscr{L}^n_{\Box\sim}$ is defined as follows:*

$$\varphi ::= p \mid {\sim}\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi$$

with $p \in At$ and $i \in A$. Let $(\varphi \vee \psi) \stackrel{\text{def}}{=} \neg(\neg\varphi \wedge \neg\psi)$.

The intended readings of literals such as p and $\neg p$ are *there is evidence for* p and *there is evidence against* p, respectively. We read ${\sim}$ as classical negation: ${\sim}\varphi$ means that *it is not the case that* φ. Formulas with the modal operator such as $\Box_i\varphi$ and $\Box_i\neg\varphi$, finally, have the intended meaning of *agent i knows that there is evidence for* φ and *agent i knows that there is evidence against* φ, respectively.

Definition 2. *[29] Given a set $A = \{1, 2, ..., n\}$ of agents, an FVEL model is a tuple $\mathscr{M} = (S, R, \mathscr{V})$, where $S \neq \emptyset$ is a set of states, $R = (R_1, R_2, ..., R_n)$ is an n-tuple of binary relations on S and $\mathscr{V} : At \times S \rightarrow \mathscr{P}(\{0, 1\})$ is a valuation*

[1] https://github.com/ydsantos/appendix_cons/blob/master/proofs.pdf.

function that assigns to each proposition at each state one of four truth values[2]. *With* $p \in At$, $s \in S$, $i \in A$ *and* $\varphi, \psi \in \mathscr{L}^n_{\Box\sim}$, *the relation* $\models$ *is defined as follows:*

$$\mathscr{M}, s \models p \text{ iff } 1 \in \mathscr{V}(p,s) \qquad \mathscr{M}, s \models \neg p \text{ iff } 0 \in \mathscr{V}(p,s)$$

$$\begin{array}{lll}
\mathscr{M}, s \models (\varphi \wedge \psi) & \textit{iff} & \mathscr{M}, s \models \varphi \textit{ and } \mathscr{M}, s \models \psi \\
\mathscr{M}, s \models \neg(\varphi \wedge \psi) & \textit{iff} & \mathscr{M}, s \models \neg\varphi \textit{ or } \mathscr{M}, s \models \neg\psi \\
\mathscr{M}, s \models \Box_i \varphi & \textit{iff} & \forall t \in S \textit{ s.t. } sR_i t : \mathscr{M}, t \models \varphi \\
\mathscr{M}, s \models \neg\Box_i \varphi & \textit{iff} & \exists t \in S \textit{ s.t. } sR_i t \textit{ and } \mathscr{M}, t \models \neg\varphi \\
\mathscr{M}, s \models {\sim}\varphi & \textit{iff} & \mathscr{M}, s \not\models \varphi
\end{array}$$

$$\mathscr{M}, s \models \neg{\sim}\varphi \text{ iff } \mathscr{M}, s \models \varphi \qquad \mathscr{M}, s \models \neg\neg\varphi \text{ iff } \mathscr{M}, s \models \varphi$$

Definition 3. *[29] The* extended valuation function $\overline{\mathscr{V}} : \mathscr{L}^n_{\Box\sim} \times S \to \mathscr{P}(\{0,1\})$ *is defined as follows:* $1 \in \overline{\mathscr{V}}(\varphi, s)$ *iff* $\mathscr{M}, s \models \varphi$; $0 \in \overline{\mathscr{V}}(\varphi, s)$ *iff* $\mathscr{M}, s \models \neg\varphi$.

Using Definition 3, we say that φ has value *both* at s, for example, iff $\overline{\mathscr{V}}(\varphi, s) = \{0,1\}$, which is the case when both $\mathscr{M}, s \models \varphi$ and $\mathscr{M}, s \models \neg\varphi$. Semantic conditions for negated and non-negated formulas are defined separately, due to the independence of positive and negative atoms. Based on this semantics, it will be handy to define formulas discriminating which of the four truth values a formula φ has:

Definition 4. *[29]* $\varphi^n \stackrel{\text{def}}{=} ({\sim}\varphi \wedge {\sim}\neg\varphi)$; $\varphi^f \stackrel{\text{def}}{=} {\sim}{\sim}({\sim}\varphi \wedge \neg\varphi)$; $\varphi^t \stackrel{\text{def}}{=} {\sim}{\sim}(\varphi \wedge {\sim}\neg\varphi)$; $\varphi^b \stackrel{\text{def}}{=} {\sim}{\sim}(\varphi \wedge \neg\varphi)$.

Now we can read $\Box_i \varphi^x$ as *Agent i knows that the status of evidence for* φ *is* x (where $x \in \{t, f, b, n\}$).

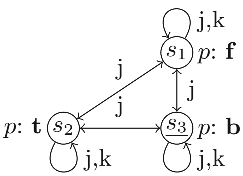

Fig. 1. Some evidence about p.

Example 1. John (j) knows that there are studies about health effects of coffee. However, he never read those articles, so he is sure that there is evidence *for* or *against* (or even *both for and against*) *coffee being beneficial for health* (p), but he does not know what the status of the evidence about p is, only that there is *some* information. Looking at Fig. 1, one can see that $\Box_j((p \wedge {\sim}\neg p) \vee (\neg p \wedge {\sim} p) \vee (p \wedge \neg p))$, which is equivalent to $\Box_j(p \vee \neg p)$, holds in the "actual" world (s_3).

[2] Abbrev.: $\{0\}$ is *false* or f, $\{1\}$ is *true* or t, $\{\}$ is *none* or n, and $\{0,1\}$ is *both* or b.

Kate (k), on the other hand, is a researcher on the effects of coffee on health, and for this reason she knows exactly what evidence is available (R_k has only reflexive arrows). Notice that $\mathscr{M}, s_3 \models \Box_k(p \wedge \neg p)$, that is, in the actual state, Kate knows that there is evidence both for and against the benefits of coffee. Moreover, John knows Kate and her job, so he also knows that she knows about p, whatever its status is: $\Box_j(\Box_k p^f \vee \Box_k p^t \vee \Box_k p^b)$. Likewise, Kate knows that John simply knows that there is some information about p: $\Box_k(\Box_j(p \vee \neg p) \wedge {\sim}\Box_j(p \wedge \neg p))$.

Thus, FVEL expresses two types of facts: whether there is evidence for and/or against propositions (in a public sense); and first and higher-order knowledge of agents about these evidential facts.

3 A Consolidation Operation

Now that we have seen how FVEL works, we want to be able to extract a Kripke model from an FVEL model, representing the beliefs obtained from the evidence in the latter, constituting a so-called *consolidation* operation.

3.1 Definitions

To define this operation we will need some essential notions:

Definition 5 *(Selection Function and Accepted Valuations). Let $Val = \{v : At \to \{0,1\}\}$ be the set of all* binary valuations*. Given an FVEL model $\mathscr{M} = (S, R, \mathscr{V})$ and the set of agents $A = \{1, 2, ..., n\}$, we define $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n)$, where $\mathcal{V}_i(s) \subseteq Val$ and $\mathcal{V}_i(s) \neq \emptyset$, for all $i \in A$ and $s \in S$. $\mathcal{V}$ is called a* (valuation) selection function *for $\mathscr{M}$, and $\mathcal{V}_i(s)$ is the set of binary valuations that agent i* accepts *at s. $U_s = \bigcup_{i \in A} \mathcal{V}_i(s)$ are the valuations accepted by* some *agent at s.*

Intuitively, the selection function $\mathcal{V}$ gives the set of valuations that each agent finds plausible at each state. The idea is that these plausible valuations will bear a strong connection to the evidence possessed, by means of constraints imposed on $\mathcal{V}$. In principle, however, $\mathcal{V}$ can be any function conforming to Definition 5.

We use s_v to denote the pair (s, v), where $s \in S$ and $v \in Val$. Now we define *cluster consolidations* (Definition 6). Ideally, the consolidation would generate one state for each state in $\mathscr{M}$, with the same valuation. If FVEL were two-valued, that would be possible, but since it is four-valued, we generate a cluster of states for each state s, with one state s_v for each valuation v accepted at s according to $\mathcal{V}$.

Definition 6 *(Cluster Consolidation). Let $\mathscr{M} = (S, R, \mathscr{V})$ be an FVEL model, $\mathcal{V}$ be a selection function for $\mathscr{M}$. The* cluster consolidation *of $\mathscr{M}$ (based on $\mathcal{V}$) is the Kripke model $\mathscr{M}! = (S', R', V)$, where: (i) $S' = \{s_v \mid s \in S, v \in U_s\}$; (ii) if $s_v, t_u \in S'$ then: $s_v R'_i t_u$ iff $sR_i t$ and $u \in \mathcal{V}_i(t)$; and (iii) $V(p, s_v) = v(p)$.*[3]

[3] Since the number of states in $\mathscr{M}!$ can be exponential in the number of elements of At, if At is countably infinite, S' may be uncountable (by Cantor's Theorem).

Definition 6 hopefully covers most reasonable consolidations, modulo some notion of equivalence. It covers a lot of unreasonable ones too. It does not reflect, however, any specific "consolidating policy": it only defines a technically convenient class of consolidations, due to their modular nature (each state generating a cluster of states) and the way they link accepted valuations and evidence.

Now we define a type of cluster consolidation reflecting an actual policy: *cautious consolidation*. It is based on the following consolidating principle: *If there is only positive evidence for a proposition, then the agent believes it; if there is only negative evidence, then the agent believes its negation; otherwise, the agent has no opinion about it.* Consider the set of functions $H = \{h : \mathscr{P}(\{0,1\}) \to \{-1,0,1\}\}$, mapping status of evidence to doxastic attitudes (1 standing for belief, 0 for disbelief and -1 for abstention of judgement). This principle, then, can be codified in a function h_1 such that $h_1(n) = h_1(b) = -1$, $h_1(t) = 1$, $h_1(f) = 0$.[4]

Definition 7[5] *(Compatibility). Let $h \in H$ and $Val^h_s = \{v \in Val \mid$ for all $p \in At$, if $h(\mathscr{V}(p,s)) \neq -1$ then $v(p) = h(\mathscr{V}(p,s))\}$ be the set of binary valuations* h-compatible *with $\mathscr{V}$ at s.*

Definition 8 *(Implementation). If $\mathcal{V}_i(s) = Val^h_s$ for all $s \in S$ and some $i \in A$, we say that $\mathcal{V}$* implements *h for agent i.*

Definition 9 *(h-consolidation). Let $h \in H$. $\mathscr{M}!$ is called an h-consolidation of $\mathscr{M}$ for agent i iff $\mathscr{M}!$ is the cluster consolidation of $\mathscr{M}$ based on $\mathcal{V}$, and $\mathcal{V}$ implements h for agent i.*

Let *cautious consolidation* be synonymous with h_1-consolidation. A consolidation is characterised in Definition 9 relative to an agent. This allows consolidations to implement different belief formation policies for each agent.

3.2 Examples

Figure 2 (left) shows a simple cautious consolidation, with one agent and one proposition with value *true*. The selection function is cautious, so the set of valuations accepted by the agent has to be h_1-compatible with $\mathscr{V}$ at s_1. This is the case for a valuation v only if $v(p) = 1$. Then, according to Definition 6, there is only one state in the consolidated model (s'_1), which conforms to v (that is, p holds) and has a reflexive arrow, because the original state s_1 has one as well. In Fig. 2 (right), the value *both* for p admits two h_1-compatible valuations: one in which p holds, and one in which p does not hold. Then, by Definition 6, two states

[4] Out of 81 functions in H, only h_1 and h_0 ($h_0(x) = -1$, $x \in \{t,f,b,n\}$) respect some permissive postulates. They are: if evidence is only positive (negative) then you should not disbelieve (believe); if only positive (negative) evidence is not enough to generate belief (disbelief), nothing is; $h(n) = h(b) = -1$, justified by the fact that $\varphi^b = (\neg\varphi)^b$ (similarly for n), so only abstention can avoid inconsistency; and $h(t) = 1$ iff $h(f) = 0$, justified by the fact that $\varphi^t = (\neg\varphi)^f$ and $\varphi^f = (\neg\varphi)^t$ in FVEL.

[5] For this and coming definitions, keep in mind that whenever $\mathscr{V}$, S or $\mathcal{V}$ are mentioned, they are always relative to an underlying FVEL model $\mathscr{M} = (S,R,\mathscr{V})$.

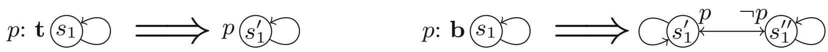

Fig. 2. Cautious consolidations on positive (left) and conflicting evidence (right).

must exist in the consolidation, and they should contain all possible arrows, because the original state has a reflexive arrow. The consolidation would be identical if p had value *none*: cautious consolidations do not distinguish between *none* and *both* (due to h_1). Figure 3 illustrates cautious consolidation applied to Example 1.

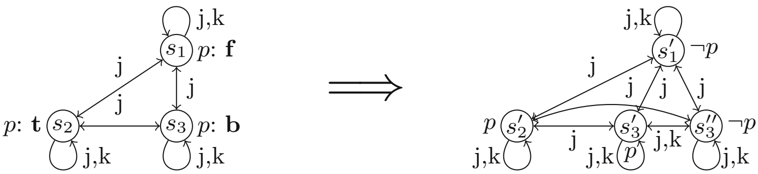

Fig. 3. Cautious consolidation of Example 1.

3.3 Properties

In this section we explore formal properties of the consolidations. Proposition 1 represents a desideratum for cluster consolidations: that they "respect" the function h upon which they are based. In a cautious consolidation, for example, we want that if an agent a knows that the status of evidence for p is t in state s, that is, $\mathscr{M}, s \models \Box_a p^t$, then in the corresponding state of $\mathscr{M}!$ a will believe p. Now if $\Box_a p^f$ holds, a will believe $\neg p$, and otherwise a will believe neither p nor $\neg p$. Proposition 1 generalises this result for any function $h \in H$, for any number of "stacked boxes", and for disjunctions of truth values of p. For example, with h_1, if $\Box_a(p^b \vee p^n)$ holds, then the agent will not form beliefs about p. Let $h^{-1}(y)$ be the preimage of y by h: $h^{-1}(y) = \{x \in \mathscr{P}(\{0,1\}) \mid h(x) = y\}$.

Proposition 1. *Given any FVEL model $\mathscr{M} = (S, R, \mathscr{V})$ and a function $h \in H$, consider an h-consolidation $\mathscr{M}! = (S', R', V)$ of $\mathscr{M}$ for agent i_0. For any such consolidation, for all $p \in At$ and $s \in S$: $\mathscr{M}, s \models \Box_{i_n} ... \Box_{i_0}(p^{x_1} \vee ... \vee p^{x_m}) \Rightarrow$*

$$\begin{cases} \mathscr{M}!, f(s) \models B_{i_n} ... B_{i_0} p & \textit{if } \{x_1, ..., x_m\} \subseteq h^{-1}(1) \\ \mathscr{M}!, f(s) \models B_{i_n} ... B_{i_0} \neg p & \textit{if } \{x_1, ..., x_m\} \subseteq h^{-1}(0) \\ \mathscr{M}!, f(s) \not\models B_{i_n} ... B_{i_0} p & \textit{if } \{x_1, ..., x_m\} \cap h^{-1}(1) = \emptyset \\ \mathscr{M}!, f(s) \not\models B_{i_n} ... B_{i_0} \neg p & \textit{if } \{x_1, ..., x_m\} \cap h^{-1}(0) = \emptyset \end{cases}$$

where for all $s \in S$, $f(s) = s_v$ for some $s_v \in S'$, and B_a is the belief modality associated with R'_a.

Function h is respected in a weak way, namely, only for atoms. Now consider the following translation function for formulas.

Definition 10. *Let* $\mathfrak{t} : \mathscr{L}^n_{\Box\sim} \to \mathscr{L}^n_B$ *be a function that translates FVEL formulas into a standard multimodal language with modal operators* B_a *for each* $a \in A$ *such that* $\sim$ *is replaced by* $\neg$, $\Box_a$ *is replaced by* B_a, *and the rest remains the same.*

The following result, as Proposition 1, establishes a correspondence between formulas in an FVEL model and in its consolidation. The result is limited to formulas with "classically-valued" atoms, but encompasses all formulas instead of only atoms.

Proposition 2. *Let* $\mathscr{M} = (S, R, \mathscr{V})$ *be an FVEL model and* $\mathscr{M}! = (S', R', V)$ *its cautious consolidation, and let* φ *be an FVEL formula such that for all atoms* p *occurring in* φ, $\mathscr{V}(p, s) \in \{\{0\}, \{1\}\}$ *for all* $s \in S$. *Then, for all* $s \in S$, $\mathscr{M}, s \models \varphi$ *iff* $\mathscr{M}!, s_v \models \mathfrak{t}(\varphi)$, *for any* $s_v \in S'$.

Now let us check the preservation of frame properties under consolidations. Seriality, transitivity and Euclideanicity are preserved in general. Reflexivity and symmetry, however, are only preserved if there is a certain similarity among the selection functions $\mathcal{V}_i$. Notice that for all R'_i to be reflexive, all functions $\mathcal{V}_i$ have to be equal. The following propositions are all relative to an FVEL model $\mathscr{M} = (S, R, \mathscr{V})$ and a cluster consolidation $\mathscr{M}! = (S', R', V)$ of $\mathscr{M}$, where $R = (R_1, ..., R_n)$ and $R' = (R'_1, ..., R'_n)$.

Proposition 3. *If* R_i *is serial (transitive, Euclidean), then* R'_i *is serial (transitive, Euclidean).*

Proposition 4. *If* R_i *is reflexive, then* R'_i *is reflexive iff for all* $j \in A$ *and all* $s \in S$ *it holds that* $\mathcal{V}_j(s) \subseteq \mathcal{V}_i(s)$.

Proposition 5. *If* R_i *is symmetric, then* R'_i *is symmetric iff for all* $s, t \in S$ *such that* sR_itR_is *it holds that* $\mathcal{V}_j(s) \subseteq \mathcal{V}_i(s)$ *for all* $j \in A$.

In the case where all the agents consolidate in the same manner (for example, through cautious consolidation), reflexivity, symmetry, transitivity, seriality and Euclideanicity are all preserved. Since we want the consolidated model to be a *doxastic* model, it is desirable that its relation be Euclidean, serial and transitive (KD45 models). These results provide sufficient conditions for that.

3.4 A Unified Language for Evidence and Beliefs

A detailed study of an extension of the language and logic of FVEL with beliefs is beyond the scope of this paper, but we will suggest here how this can be done.

First, we have to recall that propositional formulas in FVEL are not about facts, but about evidence. For this reason, it is better to define belief over formulas of $\mathscr{L}_B$, the doxastic language of the consolidated model. We can define belief in FVEL model as follows:

$$\mathscr{M}, s \models B_a \mathfrak{t}(\varphi) \quad \text{iff} \quad \mathscr{M}!, s_v \models B_a \mathfrak{t}(\varphi)$$

where $\mathscr{M}! = (S', R', V)$ is the cautious consolidation of $\mathscr{M}$, and $s_v \in S'$.

In this language it is now possible to talk about formulas such as $\Box_a p^t \tilde{\leftrightarrow} B_a p$ or $\Box_a p^f \tilde{\leftrightarrow} B_a \neg p$, i.e., only positive (negative) evidence equals belief (disbelief), where $\varphi \tilde{\leftrightarrow} \psi \overset{\text{def}}{=} {\sim}(\varphi \wedge {\sim}\psi) \wedge {\sim}(\psi \wedge {\sim}\varphi)$. These formulas are valid, but if we employ another type of consolidation in the semantic definition above, they may not be.

Notice also that if $\mathscr{M}!$ is a $\mathsf{KD45}$ model, for example, the behaviour of this new B_a operator in FVEL will be governed by that logic. But since the consolidation is completely determined by the original FVEL model, it should be possible to define semantics for B_a in FVEL without mentioning $\mathscr{M}!$.

4 Equivalence Between Evidence Models

Now we introduce Van Benthem and Pacuit's (hereafter, B&P) models [14].

Definition 11. *[14] A* B&P model *is a tuple $M = (S, E, V)$ with $S \neq \emptyset$ a set of states, $E \subseteq S \times \mathscr{P}(S)$ an evidence relation, and $V : At \to \mathscr{P}(S)$ a valuation function. We write $E(w)$ for the set $\{X \mid wEX\}$. We impose two constraints on E: for all $w \in S$, $\emptyset \notin E(w)$ and $S \in E(w)$.*

In B&P models, propositional formulas are about facts (not evidence), as usual.

Definition 12. *[12] A* w-scenario *is a maximal $\mathcal{X} \subseteq E(w)$ such that for any finite $\mathcal{X}' \subseteq \mathcal{X}$, $\bigcap \mathcal{X}' \neq \emptyset$. Let $Sce_E(w)$ be the collection of w-scenarios of E.*

Definition 13. *[14] A standard bimodal language $\mathscr{L}_{\Box B}$ (with $\Box$ for evidence and B for belief) is interpreted over a B&P model $M = (S, E, V)$ in a standard way, except for B and $\Box$:*

$$M, w \models \Box\varphi \textit{ iff } \exists X \textit{ with } wEX \textit{ and } \forall v \in X : M, v \models \varphi$$

$$M, w \models B\varphi \textit{ iff } \forall \mathcal{X} \in Sce_E(w) \textit{ and } \forall v \in \bigcap \mathcal{X}, M, v \models \varphi$$

Formulas such as $\Box\varphi$ mean that *the agent has evidence for* φ. Notice that an agent can have evidence for φ and $\neg\varphi$ at the same time, or have no evidence about φ whatsoever. This makes the status of evidence (in any given state) four-valued, just as in FVEL. Note also that the conditions for the satisfaction of $B\varphi$ tell us how the consolidation in B&P logic is done: *You believe what is supported by all pieces of evidence in all maximal consistent subsets of your evidence* (w-scenarios).

Now we want to be able to compare consolidations of B&P models to consolidations of FVEL models. For this, first, we need a way of establishing that an FVEL model and a B&P model are "equivalent" with respect to how evidence is represented. It only makes sense to compare consolidations if they depart from (roughly) the same evidential situation.

The "logics of evidence" in B&P logic and FVEL differ, the former being non-normal (so, for example, $\Box\varphi \wedge \Box\psi$ does not imply $\Box(\varphi \wedge \psi)$ in B&P logic, while in FVEL it does), and the latter being *First Degree Entailment* (FDE)

[17,27][6]. Note, however, that this difference is more about how evidence is *manipulated* in these logics, than about how it is *represented*. For this reason, our equivalence in evidence is, fittingly, limited to literals.

Definition 14 *(ev-equivalence). Let* $M = (S, E, V)$ *be a B&P model and let* $\mathscr{M} = (S', R, \mathscr{V})$ *be an FVEL model. A relation* $\doteq \subseteq S \times S'$ *is an ev-equivalence between* M *and* $\mathscr{M}$ *iff:*

1. $\doteq$ *is a bijection;*
2. *If* $s \doteq s'$*, where* $s \in S$ *and* $s' \in S'$*, then, for all* $p \in At$*:* $M, s \models \Box p$ *iff* $\mathscr{M}, s' \models \Box p$*; and* $M, s \models \Box\neg p$ *iff* $\mathscr{M}, s' \models \Box\neg p$*.*

We write $M \doteq \mathscr{M}$ *if there exists an ev-equivalence between* M *and* $\mathscr{M}$*.* $M \doteq M'$*,* $\mathscr{M} \doteq M$ *and* $\mathscr{M} \doteq \mathscr{M}'$ *are defined analogously.*

Now our job is to find, for each B&P or FVEL model, a model of the other type which is ev-equivalent to it, that is, that *represents* the same evidence[7]. Since B&P models are single-agent, we assume from now on that all models are single-agent. Much of the conversions between models that follow will be about removing aspects of evidence that are not represented in the other type of model.

4.1 From B&P to FVEL Models

Consider the following *conversion* from B&P to FVEL models:

Definition 15. *Let* $M = (S, E, V)$ *be a B&P model. Define the FVEL model* $\mathsf{FV}(M) = (S, R, \mathscr{V})$*, where* $R = \{(s, s) \mid s \in S\}$ *and for all* $p \in At$ *and states* $s \in S$*:* $1 \in \mathscr{V}(p, s)$ *iff* $M, s \models \Box p$*; and* $0 \in \mathscr{V}(p, s)$ *iff* $M, s \models \Box\neg p$*.*

We cannot expect a complete correspondence between M and $\mathsf{FV}(M)$ in terms of satisfaction of formulas (in the vein of Proposition 11), for while propositional formulas in B&P models represent facts and $\Box$ formulas represent the agent's evidence, in FVEL propositional formulas represent generally available evidence, while $\Box$ formulas represent agents' knowledge of such evidence. This public/personal distinction for evidence in FVEL would be superfluous in B&P models, since they are not multi-agent. Nevertheless, we have the following correspondence:

Proposition 6. *For any B&P model* $M = (S, E, V)$ *and its FVEL counterpart* $\mathsf{FV}(M)$*, for all states* $s \in S$ *and all literals* $l \in \{p, \neg p\}$*, with* $p \in At$*, we have:*

$$M, s \models \Box l \ \textit{ iff } \ \mathsf{FV}(M), s \models l \ \textit{ iff } \ \mathsf{FV}(M), s \models \Box l$$

Corollary 1. *For any B&P model* M*,* $M \doteq \mathsf{FV}(M)$*.*

[6] In other words: if there is evidence for Σ and $\Sigma \vdash_{\mathsf{FDE}} \varphi$, then there is evidence for φ.

[7] I opted for Definition 14 instead of an equivalence between $\Box p$ in B&P and p in FVEL models, because even though we do restrict FVEL models to the single-agent case, these models are still multi-agent in nature. So, while $\mathscr{M}, s \models p$ indicates that *there is evidence for* p (at s), it is only when $\mathscr{M}, s \models \Box_a p$ holds that we should think that an agent a *has (knowledge of)* this evidence. On the other hand, in single-agent B&P models there is no semantic difference between *there is evidence for* p and *the agent has evidence for* p.

4.2 From FVEL to B&P Models

This direction is less straightforward than the conversion discussed above. Again we run into the problem of representing a four-valued model as a two-valued one.

Definition 16. *Let $\mathscr{M} = (S, R, \mathscr{V})$ be an FVEL model. We build a B&P model* $\mathsf{BP}(\mathscr{M}) = (S', E, V)$ *where* $S' = \{s_v \mid s \in S \text{ and } v \in Val_s^{h_1}\}$ *and* $s_v \in V(p)$ *iff* $v(p) = 1$. *Let* $C(s) = \{t_v \in S' \mid sRt\}$. *E is defined as follows:* $E(s_v) = \{S'\} \cup \{X_p \subseteq C(s) \mid X_p \neq \emptyset, p \in At; t_u \in X_p \text{ iff } \mathscr{M}, s \models \Box p \text{ and } t_u \in V(p)\} \cup \{X_{\neg p} \subseteq C(s) \mid X_{\neg p} \neq \emptyset, p \in At; t_u \in X_{\neg p} \text{ iff } \mathscr{M}, s \models \Box\neg p \text{ and } t_u \notin V(p)\}$.

Definition 16 creates clusters of states for each original state in $\mathscr{M}$ (similarly to the technique for cluster consolidations). Then, all clusters accessible from a state s_v are grouped together and "filtered" to form the "pieces of evidence" in $E(s_v)$, one for each literal that is known to be evidence in the corresponding state of the FVEL model. E.g. if in a state s only evidence for the literal $\neg p$ is known (that is, $\mathscr{M}, s \models \Box\neg p$), then $E(s_v)$ will be $\{S', X_{\neg p}\}$, where $X_{\neg p}$ is a piece of evidence made up of all states accessible from s_v where $\neg p$ holds. See Fig. 4.

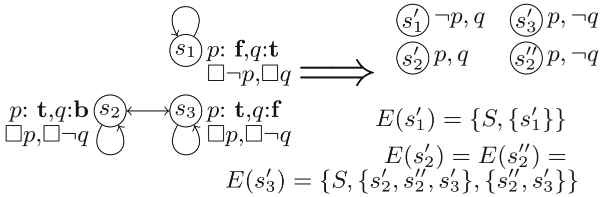

Fig. 4. An example of BP being applied to an FVEL model.

Proposition 7. *Let $\mathscr{M} = (S, R, \mathscr{V})$ be a serial FVEL model with* $\mathsf{BP}(\mathscr{M}) = (S', E, V)$. *Then, for all* $s \in S$, *all* v *such that* $s_v \in S'$ *and all* $l \in \{p, \neg p\}$, *with* $p \in At$: $\mathscr{M}, s \models \Box l$ *iff* $\mathsf{BP}(\mathscr{M}), s_v \models \Box l$.

Corollary 2. *For all serial FVEL models* $\mathscr{M}$, $\mathsf{BP}(\mathscr{M}) \doteq \mathscr{M}$.

4.3 Evaluating the Conversions

Our conversions are satisfactory enough to produce ev-equivalent models, but unfortunately the following proposition can be easily verified:

Proposition 8. *Let M be a B&P model and $\mathscr{M}$ be an FVEL model. Then, neither* $\mathsf{BP}(\mathsf{FV}(M)) \cong M$ *nor* $\mathsf{FV}(\mathsf{BP}(\mathscr{M})) \cong \mathscr{M}$ *are guaranteed to hold; where* $M \cong M'$ *denote that* M *is isomorphic to* M', *and similarly for* $\mathscr{M} \cong \mathscr{M}'$.

One reason why the above do not hold in general is simple: $\mathsf{BP}(\mathscr{M})$ has more states than $\mathscr{M}$ if the latter has any state where some atom has value b or n.

Definition 17. *Let $M = (S, E, V)$ be a B&P model. We define the following conditions on M:*

- ***Consistent Evidence (CONS)*** $\forall s \in S \forall X, Y \in E(s)$: *if* $\forall x \in X, M, x \models l$ *then* $\exists y \in Y, M, y \models l$, *for all literals* $l \in \{p, \neg p\}$, $p \in At$;
- ***Complete Evidence (COMP)*** $\forall s \in S \forall p \in At \exists X \in E(s)$ *s.t.* $\forall x \in X, M, x \models p$ *or* $\forall x \in X, M, x \models \neg p$;
- ***Good Evidence (GOOD)*** $s \in V(p)$ *iff* $\exists X \in E(s)$ *s.t.* $\forall x \in X, M, x \models p$
- ***Simple Evidence (SIMP)*** $\forall s \in S, E(s) = \{\{s\}, S\}$.

Proposition 9. *SIMP entails CONS, COMP and GOOD. CONS and COMP are sufficient and necessary for the preservation of S. CONS, COMP and GOOD are sufficient (but GOOD is not necessary) for preservation of V. SIMP is sufficient and necessary for preservation of E.*

Corollary 3. $\mathsf{BP}(\mathsf{FV}(M)) \cong M$ *iff SIMP holds.*

Definition 18. *Let $\mathscr{M} = (S, R, \mathscr{V})$ be an FVEL model. We define the following conditions on $\mathscr{M}$:*

- ***Classicality (CLAS)*** $\forall p \in At, \forall s \in S : \mathscr{V}(p, s) \in \{t, f\}$;
- ***Knowledge of Evidence (KNOW)*** $\mathscr{M}, s \models p$ *iff* $\mathscr{M}, s \models \Box p$; $\mathscr{M}, s \models \neg p$ *iff* $\mathscr{M}, s \models \Box \neg p$;
- ***Only-Reflexivity (REFL)*** $R = \{(s, s) \mid s \in S\}$

Proposition 10. *REFL entails KNOW. CLAS is necessary and sufficient for preservation of S. CLAS and KNOW are sufficient (but KNOW is not necessary) for preservation of $\mathscr{V}$. CLAS and REFL are the necessary and sufficient conditions for preservation of R.*

Corollary 4. $\mathsf{FV}(\mathsf{BP}(\mathscr{M})) \cong \mathscr{M}$ *iff CLAS and REFL hold.*

The desired correspondences only hold under fairly strong conditions. These conditions are not arbitrary restrictions, but idealising conditions[8]. This means that B&P and FVEL models have perfectly (ev-)equivalent counterparts under idealised scenarios, where evidence is factive, always present, complete and consistent, and where agents have perfect knowledge of what evidence is available. This correspondence breaks when we deviate from these assumptions to cover situations of imperfect evidence and imperfect knowledge. Now we can compare the two consolidations.

5 Comparing Consolidations

In [12], a method for obtaining a relation from B&P models is provided:

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[8] S is added in SIMP and in the evidence sets generated by BP just to comply with the last condition of Definition 11. If we remove it from both places, Proposition 9 still holds.

Definition 19. *[12] Given a B&P model* $M = (S, E, V)$*, define* $B_E \subseteq S \times S$ *by* $sB_E t$ *if* $t \in \bigcap \mathcal{X}$ *for some* $\mathcal{X} \in Sce_E(s)$*.*

Consider a monomodal language $\mathscr{L}_B$ with B as its modality.

Proposition 11. *Let* $M = (S, E, V)$ *be a B&P model and* $M! = (S, B_E, V)$ *its relational counterpart. Then, for all* $\varphi \in \mathscr{L}_B$ *and* $s \in S$*:* $M, s \models \varphi$ *iff* $M!, s \models \varphi$*.*

This effectively proves that $M!$ is the consolidation for M found "implicitly" in [12]. Now given two models M (B&P) and $\mathscr{M}$ (FVEL) such that $M \mathrel{\mathring{=}} \mathscr{M}$, how does $M!$ compare to $\mathscr{M}!$ ($\mathscr{M}$'s cautious consolidation)?

Definition 20. *Given* $M \mathrel{\mathring{=}} \mathscr{M}$ *under bijection* f*, we say that* V *matches* $\mathscr{V}$ *iff: for all* $p \in At$ *and all* $s' \in S'$*,* $\mathscr{V}(p, s') \in \{t, f\}$*; and* $s \in V(p)$ *iff* $\mathscr{V}(p, f(s)) = t$*.*

Proposition 12. *Let* $M \mathrel{\mathring{=}} \mathscr{M}$ *under bijection* f*.* $M! \cong \mathscr{M}!$ *iff:* V *matches* $\mathscr{V}$*, and* $f(s)Rf(t)$ *iff* $t \in \bigcap \mathcal{X}$ *for some* $\mathcal{X} \in Sce_E(s)$*.*

So the conditions for consolidations of ev-equivalent B&P and FVEL models to be isomorphic are rather strong: they must have matching valuations and $\mathscr{M}$'s relation has to mirror B_E.

6 Conclusion

We introduced *consolidation* as the process of forming beliefs from a given evidential state, formally represented by transformations from evidential (FVEL and B&P) models into doxastic Kripke models. We established the grounds for comparison between these different models, and then found the conditions under which their consolidations are isomorphic. Future work can use bisimilarity instead of isomorphism, and extend this methodology to other evidence logics. Would it be possible to define belief without resorting to two-valued Kripke models? Certainly, as all information used in the consolidation is already in the initial evidential models. The rationale here is that, since Kripke models are standard and widely-accepted formal representations of belief, we should be able to represent the beliefs that implicitly exist in evidential models using this tool. We also wanted to highlight the *process* of transforming evidence into beliefs.

The dynamic perspective on consolidations allows us to study, for example, the complexity of these operations, which is important if we are concerned with real agents forming beliefs from imperfect data. It is clear that consolidations of FVEL models tend to be much larger than those of B&P models, but, on the other hand, might be much easier to compute, given that B&P consolidations rely on the hard-to-compute concept of maximally consistent sets. FVEL models can also deal with multiple agents, and accept a function from status of evidence to doxastic attitude as a parameter (in this case, function $h_1 \in H$), allowing for some flexibility in consolidation policies. It would also be interesting to see if a consolidation like B&P's, where maximal consistent evidence sets are taken into account, would be possible in the context of FVEL. Is the converse possible: to apply the idea of H functions in B&P models?

A future extension of this work taking computational costs of consolidations into account would be in line with other work that tries to fight "logical omniscience" or to model realistic resource-bounded agents [1–3,7,18]. Other *aspects* of evidence can also be considered, such as the *amount* of evidence for or against a certain proposition, the *reliability* of a source or a piece of evidence, etc.

Agents form different beliefs in ev-equivalent situations when departing from an FVEL or a B&P model. Part of this is explained by the fact that these logics do not represent exactly the same class of evidence situations. But clearly the consolidation policies also differ. Is one better than the other? At first glance, both seem to be reasonable, but more investigation could be done in this direction.

Moreover, how are changes in an FVEL (or other) evidence model reflected in its consolidation? *Evidence dynamics* for B&P logic are explored in [14], in line with other dynamic logics of knowledge update and belief revision [9–11,16, 21,26,28,31].

Acknowledgement. Thanks to Rineke Verbrugge and Barteld Kooi, and also to Malvin Gattinger, Stipe Pandžić, Davide Grossi, Thiago Dias Simão, the anonymous reviewers and the RUG MAS group, for very useful suggestions. Research supported by Ammodo KNAW project "Rational Dynamics and Reasoning".

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