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Bauso, Dario; Tembine, Hamidou

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1 **GAME-THEORETIC LEARNING AND ALLOCATIONS IN ROBUST**
2 **DYNAMIC COALITIONAL GAMES***

3 M. SMYRNAKIS [†], D. BAUSO [‡], AND H. TEMBINE [§]

4 **Abstract.** The problem of allocation in coalitional games with noisy observations and dynamic
5 environments is considered. The evolution of the excess is modelled by a stochastic differential
6 inclusion involving both deterministic and stochastic uncertainties. The main contribution is a
7 set of linear matrix inequality conditions which guarantee that the distance of any solution of the
8 stochastic differential inclusions from a predefined target set is second-moment bounded. As a direct
9 consequence of the above result we derive stronger conditions still in the form of linear matrix
10 inequalities to hold in the entire state space, which guarantee second-moment boundedness. Another
11 consequence of the main result are conditions for convergence almost surely to the target set, when the
12 Brownian motion vanishes in proximity of the set. As further result we prove convergence conditions
13 to the target set of any solution to the stochastic differential equation if the stochastic disturbance
14 has bounded support. We illustrate the results on a simulated intelligent mobility scenario involving
15 a transport network.

16 **Key words.** Coalitional Games, Transferable Utility (TU), Second-moment boundedness, In-
17 telligent mobility network, Robust control.

18 **AMS subject classifications.** 68Q25, 68R10, 68U05

19 **1. Introduction.** The theory of coalitional games with transferable utility stud-
20 ies stable allocations for groups of agents who decide to cooperate (Osborne, 2004;
21 Shapley, 1953; Aumann et al., 1960; Schmeidler, 1969; Aumann, 1961; Luce and Raiffa,
22 1957; Maschler et al., 1979). Cooperation materializes in different forms such as shar-
23 ing facilities, sharing costs, placing joint bids. Coalitional games arise in many areas
24 such as: communication networks (Saad et al., 2009), smart grids (Saad et al., 2012),
25 reconfigurable robotics (Ramaekers et al., 2011), swarm robotics (Cheng et al., 2008),
26 multi-robot task allocation (Bayram et al., 2016).

27 A research area where coalitional games are an active topic is robust control
28 (Bauso and Timmer, 2012; Wada and Fujisaki, 2017; Fele et al., 2017). A widely used
29 approach to solve robust control problems, (Bauso, 2017; Garud, 2005; Bauso et al.
30, 2015), is approachability theorem (Blackwell, 1956). In Lehrer (2003), Blackwell's
31 approachability theorem was used in order to analyse an allocation process based
32 on coalitional games. Another technique which has been used in order to analyse
33 game-theoretic learning algorithms is stochastic approximation. In his seminal paper
34 Benaim et al. (2005) showed that stochastic approximation methods can be seen as
35 a continuous asymptotic version of approachability theorem. Based on this result in
36 this article stochastic approximation methods are used in order to analyse coalitional
37 games.

38 The results we provide collocate within the learning, control and optimisation
39 research areas. This research direction finds applications in various problems such as
40 wind energy (Opathella and Venkatesh, 2013; Bayens et al., 2013), and the inventory
41 control problem Bauso et al. (2008); Bauso et al. (2010).

42 In accordance with the classification provided in (Saad et al., 2009), this paper

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[†]Learning and Game Theory Laboratory, New York University Abu Dhabi
(m.smyrnakis@nyu.edu).

[‡]Department of Automatic Control and Systems Engineering (d.bauso@sheffield.ac.uk).

[§]Learning and Game Theory Laboratory, New York University Abu Dhabi (tembine@nyu.edu).

43 answers most of the questions arising in canonical coalitional games with transferable
 44 utility (TU) within the framework of robust stabilizability. The underlying idea is
 45 that the cooperative agents, now viewed as players, form a coalition which includes all
 46 players, namely the grand coalition and need to reach agreement on how to redistribute
 47 the reward deriving from forming such a grand coalition in a way that makes the grand
 48 coalition stable. Stability is generally linked to the possibility of allocating to each
 49 sub-coalition a quantity greater than the reward itself that the sub-coalition could
 50 guarantee for itself without coalizing with the rest of the players (players outside
 51 that sub-coalition). When this occurs, we say that no players or subsets of players
 52 gain from quitting the grand coalition. This corresponds to saying that the *excess*,
 53 namely the difference between the allocated rewards and the value of the coalition
 54 is non-negative. In the broad context of coalitional games, consider the possibility
 55 that the reward of a coalition is divided among the players of the coalitions. By value
 56 of the coalition we mean the reward produced by that coalition. The procedure to
 57 allocate the reward which needs to be agreed by the players, constitutes the so-called
 58 allocation rule. Under the assumption that the values of the coalitions are time-
 59 varying and uncertain, and the allocation process occurs continuously in time, the
 60 resulting game is called *robust coalitional game*. Such a game was first formulated by
 61 (Bauso and Timmer, 2009, 2012). The evolution of the excesses is also captured by a
 62 fluid flow system of the type discussed in (Bauso et al., 2010).

63 The contribution of this paper is three-fold. We first formulate the problem of
 64 allocation in TU games with noisy observations and dynamic environments. In the
 65 considered scenario the evolution of the excess is subjected to both deterministic
 66 and stochastic uncertainty. The resulting dynamics can be expressed in the form
 67 of a stochastic differential inclusion, involving also a Brownian motion. For this
 68 game, as main result we provide conditions which guarantee that the distance of
 69 any solution of the stochastic differential inclusion from a predefined target set is
 70 second-moment bounded. We show that these conditions can take the form of a
 71 linear matrix inequality to be verified in different regions of the state space (Boyd et
 72 al., 1994, Chapter 6). As direct consequence of the above result we derive stronger
 73 conditions still in the form of linear matrix inequalities to hold in the entire state
 74 space, which guarantee second-moment boundedness. Further to the above main
 75 result we provide conditions for convergence almost surely to the target set, when the
 76 influence of the Brownian motion vanishes with decreasing distance from the set. The
 77 resulting dynamics mimics a geometric Brownian motion. As further result we prove
 78 convergence conditions to the target set of any solution to the stochastic differential
 79 equation if the stochastic disturbance has bounded support.

80 The rest of the paper is organised as follows. Section 2 introduces preliminaries
 81 on coalitional games. Section 4 discusses the model and states the problem. Section 5
 82 links the model to saturated control and population game dynamics. Section 6 in-
 83 cludes the main results of the paper. Section 7 specializes the model to an intelligent
 84 mobility scenario. Section 8 contains numerical examples. Finally, Section 9 provides
 85 conclusions and future works.

86 **2. Preliminaries on TU games.** This section overviews coalitional games with
 87 transferable utility (TU). Let a set $N = \{1, \dots, n\}$ of players be given and a function
 88 $\eta : S \mapsto \mathbb{R}$ defined for each non-empty coalition $S \in \mathcal{S}$, where \mathcal{S} is the set of all
 89 possible non-empty coalitions, with cardinality $|\mathcal{S}| = 2^n - 1$. We denote by $\langle N, \eta \rangle$
 90 the TU game with players set N and characteristic function η , which quantifies the
 91 gain of coalition S .

Let us introduce some arbitrary mapping of \mathcal{S} into $M := \{1, \dots, q\}$ where $q = 2^n - 1$, is the number of non-empty coalitions, namely, the cardinality of \mathcal{S} . Denote a generic element of M by j . In other words, we can see j standing for the labelling of the j th element of \mathcal{S} , say S_j , according to some arbitrary but fixed ordering. Let the grand coalition be denoted by N . Furthermore, let η_j be the value of the characteristic function η associated with a non-empty coalition $S_j \in \mathcal{S}$.

Given a TU game, we wish first to investigate if the grand coalition is stable, i.e. if it is possible for the players to get better rewards by choosing a smaller coalition.

A partial answer to the above question lies in the concept of *imputation set*. The imputation set $I(\eta)$ is the set of allocations that are

- *efficient*, that is, the sum of the components of the allocation vector is equal to the value of the grand coalition, and
- *individually rational*, namely there is no individual which is benefited, increase his reward, by splitting from the grand coalition and playing alone.

More formally, the imputation set is a convex polyhedron defined as:

$$I_\eta = \left\{ \tilde{u} \in \mathbb{R}^n \mid \underbrace{\sum_{i \in N} \tilde{u}_i = \eta_N}_{\text{Efficiency}}, \underbrace{\tilde{u}_i \geq \eta_{S_i}, \forall S_i \in \mathcal{S}'}_{\text{individual rationality}} \right\},$$

where \tilde{u}_i is the reward allocated to player i , N here represents the grand coalition where all the players participate, \mathcal{S}' is the set of all coalitions which consist of a single player and η_{S_j} is the gain of coalition S_j .

A stronger solution concept than the imputation set is the *core*. Given any allocation in the core, the players do not benefit from not only quitting the grand coalition and playing alone, but also from creating any sub-coalition. In this sense the core strengthens the conditions valid for the imputation set. Thus the core is still a polyhedral set which is included in the imputation set.

DEFINITION 2.1. *The core of a game $\langle N, u \rangle$ is the set of allocations that satisfy i) efficiency, ii) individual rationality, and iii) super-additivity, i.e. stability with respect to sub-coalitions:*

$$C_\eta = \left\{ \tilde{u} \in I(\eta) \mid \underbrace{\sum_{i \in S_j} \tilde{u}_i \geq \eta_{S_j}, \forall S_j \in \mathcal{S}}_{\text{stability w.r.t. subcoalitions}} \right\}.$$

Even though the core is a fundamental concept in coalitional games, it is not necessary that the core will be a non-empty set. Two broad categories of coalitional games with non-empty core are: convex (Shapley, 1971) and balanced games (Bondareva, 1963; Shapley, 1967).

DEFINITION 2.2. *A coalitional game $\langle N, \eta \rangle$ is convex if the following inequality is satisfied.*

$$\eta_{S_i} + \eta_{S_j} \leq \eta_{S_i \cap S_j} + \eta_{S_i \cup S_j}, \forall S_i, S_j \subset N.$$

DEFINITION 2.3. *A coalitional game $\langle N, \eta \rangle$ is balanced if for any balanced map α we have:*

$$\sum_{j \in \mathcal{S}} \alpha_{S_j} \eta_{S_j} \leq \eta_N.$$

118 In order to overcome the problem of an empty core in (Shapley and Shubik, 1966)
 119 the notion of ϵ -core was introduced

DEFINITION 2.4. For a real number ϵ the ϵ -core is defined as:

$$C_\eta = \{\tilde{u} \in I(\eta) \mid \sum_{i \in S_j} \tilde{u}_i \geq \eta_{S_j} - \epsilon, \forall S_j \in \mathcal{S}\}.$$

120 In order to assess stability of the grand coalition, the core, both its value η_N , and
 121 the reward allocated to each player is needed. Therefore, there is a need to define
 122 an allocation mechanism of the coalition's rewards among the players. One of the
 123 most used allocation mechanisms is the Shapley value (Shapley, 1953, 1971). An
 124 additional reason for choosing Shapley's value is its connection with feedback control
 125 and uncertainty as it was shown in (Bauso and Timmer, 2012)

DEFINITION 2.5. The Shapley value of player i , given a coalitional game $\langle N, \eta \rangle$ is defined as:

$$\phi_i(\eta) = \sum_{S_j \subset N \setminus \{i\}} \frac{|S_j|!(|N| - |S_j| - 1)!}{|N|!} (\eta_{S_j \cup \{i\}} - \eta_{S_j}).$$

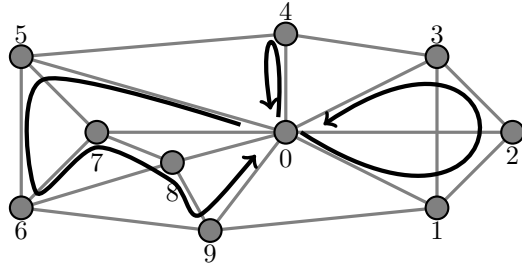
126 The Shapley value can be interpreted as the expected weighted contribution of
 127 player i when it joins the grand coalition in a random order.

128 **3. Motivating example.** Various applications of the TU games have been con-
 129 sidered in literature. Examples include Market games (Shapley and Shubik, 1969),
 130 public good games (Bodwin, 2017), the bankruptcy problem (Aumann and Maschler,
 131 1985) and inventory problems (Chinchuluun et al., 2008). Applications which com-
 132 bine TU games with optimisation and learning include micro-grid problems (Saad et
 133 al., 2013) and coordinated replenishment (Bauso and Timmer, 2009).

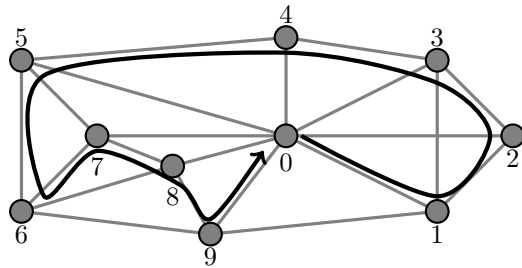
134 The case study which is considered in this article, the intelligent mobility network
 135 application, falls in the category of the inventory problems. Players should decide if
 136 it is more beneficial to create a coalition and share the cost of the inventory or it is
 137 better to bear the cost alone.

138 Intelligent mobility deals with the smart transport of items, goods or individuals
 139 from source to destination nodes using shared facilities like buses, trams, electric
 140 vehicles. Suppose that items are initially stored in the supply centre indexed by 0
 141 and need to be transported to different destination centres generically indexed by i ,
 142 $i = 1, \dots, n$. Destination centres are characterized by a time-varying demand which
 143 is independent identically distributed across time and centres.

144 Note here that the capacitate vehicle routing problem is usually solved in two
 145 parts. In the first one the assignment problem is solved, i.e. one makes decisions
 146 about the sites that should be visited. In the second part the optimal route is found
 147 through traveller salesman algorithms for example. In this article we focus on the
 148 first part, where the network topology is not playing a significant role. The manager
 149 of destination center i bids the quantity to be transported from the supply center
 150 and terminating in center i based on his forecast of the future demand. Managers
 151 can collaborate and place joint bids with the advantage of compensating potential
 152 fluctuation of the their demand. This can be represented using a graph and a cycle,
 153 namely, a closed path with source and destination in node zero, see for instance the
 154 three transport cycles originating from and terminating in 0 and touching destination
 155 centres $\{1, 2, 3\}$, $\{4\}$, and $\{5, \dots, 9\}$ in the network of Figure 1(a).



(a) Three transport cycles originating from and terminating in 0 and touching destination centers $\{1, 2, 3\}$, $\{4\}$, and $\{5, \dots, 9\}$.



(b) One transport cycle originating from and terminating in 0 and touching all destination centers.

FIG. 1. *Example of a distribution network*

156 When all managers act jointly, we say that they form a grand coalition. In such
 157 case a single cycle will touch all destination centres as described by the transport
 158 cycle originating from and terminating in 0 and touching all destination centres in
 159 Figure 1(b).

160 In stable environments, in cases where the cost function of players is deterministic,
 161 and it possible to obtain observations without noise the conventional analysis of TU
 162 games can be applied, i.e. results about the existence of the core, or the evaluation
 163 of nucleus or Shapley’s value.

In particular, consider the scenario where $N = \{1, \dots, n\}$ be the set of receiving centres. For each coalition $S \in \mathcal{S}$, let D_S be a random variable representing the aggregate demand faced by that coalition. Let us assume that D_S has continuous probability density function $f(D_S)$. In other words, the probability that the aggregate demand is between a and b is

$$\mathbb{P}(a \leq D_S \leq b) = \int_a^b f(D_S) dD_S.$$

The continuous cumulative distribution function (CDF) is $F(b)$, and represents the probability that the aggregate demand is less than or equal to b :

$$F(b) := \mathbb{P}(D_S \leq b) = \int_0^b f(D_S) dD_S.$$

164 Let Θ be the order quantity, p in \mathbb{R}_+ be the sale price, s in \mathbb{R}_+ be the penalty

165 price for shortage, when demand exceeds supply, and let h in \mathbb{R}_+ be the penalty price
166 for holding, when supply exceeds demand.

167 Introduce the stock variable $Z_S = \Theta - D_S$. Denote the indicator function by

$$168 \quad (1) \quad \mathbf{I}_{\mathbb{R}_+}(Z_S) = \begin{cases} 1 & \text{if } Z_S \in \mathbb{R}_+ \\ 0 & \text{otherwise.} \end{cases}$$

169 Then, the expected profit for the generic coalition $S \in \mathcal{S}$ under the order quantity
170 Θ is given by

$$171 \quad (2) \quad \langle \mathcal{P}_S(D_S, \Theta) \rangle = \mathbb{E} \left[p \min(\Theta, D_S) - c\Theta - [s\mathbf{I}_{\mathbb{R}_+}(Z_S) - h\mathbf{I}_{\mathbb{R}_+}(-Z_S)] |Z_S| \right].$$

172 In the above we express the expected profit as function of the expected shortage and
173 expected holding, which are given by

$$174 \quad (3) \quad \begin{aligned} \mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(-Z_S) |Z_S| \right] &= \int_{\Theta}^{\infty} f(D_S)(D_S - \Theta) dD_S, \\ \mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(Z_S) |Z_S| \right] &= \int_0^{\Theta} f(D_S)(\Theta - D_S) dD_S. \end{aligned}$$

175 We can then rewrite the expected profit as

$$176 \quad (4) \quad \begin{aligned} \langle \mathcal{P}_S(D_S, \Theta) \rangle &= \mathbb{E} [p \min(\Theta, D_S)] - c\Theta \\ &\quad - s\mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(-Z_S) |Z_S| \right] - h\mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(Z_S) |Z_S| \right]. \end{aligned}$$

177 The following relation between the expected shortage \mathbf{E}_s and the expected holding
178 \mathbf{E}_h holds:

$$179 \quad \begin{aligned} \mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(Z_S) |Z_S| \right] &= \int_0^{\Theta} f(D_S)Z_S dD_S \\ &= \int_0^{\infty} f(D_S)Z_S dD_S - \int_{\Theta}^{\infty} f(D_S)Z_S dD_S \\ &= \Theta - \langle D_S \rangle + \mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(-Z_S) |Z_S| \right], \end{aligned}$$

180 where $\langle y_s \rangle$ is the mean demand and is given by $\int_0^{\infty} f(D_S)D_S dD_S$. The problem faced
181 by the coalition is the one of maximizing the expected profit with respect to the order
182 quantity Θ , which is the decision variable:

$$183 \quad \max_{\Theta} \left\{ \mathbb{E} [p \min(\Theta, D_S)] - c\Theta - s\mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(-Z_S) |Z_S| \right] - h\mathbb{E} \left[\mathbf{I}_{\mathbb{R}_+}(Z_S) |Z_S| \right] \right\}.$$

184 Assuming concavity of $\langle \mathcal{P}_S(D_S, \Theta) \rangle$ the optimal order quantity Θ^* is obtained by
185 computing the derivative of $\langle \mathcal{P}_S(D_S, \Theta) \rangle$ with respect to Θ and taking it equal to
186 zero. To do this, after rearranging the first term $\mathbb{E} \min(\Theta, D_S)$ in the above equation
187 as below

$$188 \quad \begin{aligned} \mathbb{E} \min(\Theta, D_S) &= \int_0^{\Theta} D_S f(D_S) dD_S + \int_{\Theta}^{\infty} \Theta f(D_S) dD_S \\ &= \langle D_S \rangle - \int_{\Theta}^{\infty} D_S f(D_S) dD_S + \int_{\Theta}^{\infty} \Theta f(D_S) dD_S \end{aligned}$$

189 we can rewrite the expected profit as

$$190 \quad \begin{aligned} \langle \mathcal{P}_S(D_S, \Theta) \rangle &= p\langle D_S \rangle - c\Theta \\ &\quad - s\Theta \int_0^{\Theta} f(D_S) dD_S + s \int_0^{\Theta} D_S f(D_S) dD_S \\ &\quad + (p+h)\Theta \int_{\Theta}^{\infty} f(D_S) dD_S - (p+h) \int_{\Theta}^{\infty} D_S f(D_S) dD_S. \end{aligned}$$

191 Then for the derivative we have

$$\begin{aligned}
 & \frac{d}{dC} (\langle \mathcal{P}_S(D_S, \Theta) \rangle) \\
 &= -c - s \int_0^{\Theta} f(D_S) dD_S - s\Theta f(\Theta) + s\Theta f(\Theta) \\
 192 &+ (p+h) \int_{\Theta}^{\infty} f(D_S) dD_S - (p+h)\Theta f(\Theta) + (p+h)\Theta f(\Theta) \\
 &= -c - s \int_0^{\Theta} f(D_S) dD_S + (p+h) \int_{\Theta}^{\infty} f(D_S) dD_S \\
 &= -c - sF(\Theta) + (p+h)[1 - F(\Theta)],
 \end{aligned}$$

193 where F is the cumulative distribution function (CDF) of y . The optimal order
 194 quantity is given by:

$$195 \quad (5) \quad F(\Theta_S^*) = \frac{p+h-c}{p+h+s}.$$

196 Let F^{-1} be the inverse function of F then it holds

$$197 \quad (6) \quad \Theta_S^* = F^{-1}\left(\frac{p+h-c}{p+h+s}\right).$$

198 Then, the optimal expected profit is

$$\begin{aligned}
 & \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle = p\mu - c\Theta_S^* - s \int_0^{\Theta_S^*} (\Theta_S^* - D_S) f(D_S) dD_S \\
 & \quad - (p+h) \int_{\Theta_S^*}^{\infty} (D_S - \Theta_S^*) f(D_S) dD_S \\
 199 \quad (7) &= p\mu - c\Theta_S^* - s(\Theta_S^* - \mu + \mathbf{E}_h^*) - (p+h)\mathbf{E}_h^* \\
 &= p\mu - cF^{-1}\left(\frac{p+h-c}{p+h+s}\right) - s\left(F^{-1}\left(\frac{p+h-c}{p+h+s}\right)\right. \\
 & \quad \left. - \mu + \mathbf{E}_h^*\right) - (p+h)\mathbf{E}_h^*,
 \end{aligned}$$

200 where we denote by \mathbf{E}_h^* the expected surplus under the optimal order quantity Θ_S^* .

201 Consider a sequence of sampling intervals indexed by $k = 0, 1, \dots$. We build on
 202 the results for the optimal order quantity (6) and expected profit (7), which we have
 203 obtained above. We assume that the demand at interval k has a Normal distribution
 204 with mean $D_S(k-1)$ and variance σ^2 :

$$205 \quad (8) \quad D_S(k) - D_S(k-1) \sim \mathcal{N}(0, \sigma^2).$$

We can rewrite the optimal order quantity in terms of the number of standard deviations away from the mean:

$$\Theta_S^* = D_S(k-1) + k^* \sigma,$$

where k has standard Normal distribution. Denote by $\Phi(k)$ the CDF of a standard Normal distribution, from (5) we have

$$\Phi(k^*) = \frac{p+h-c}{p+h+s}.$$

To obtain (6) from (5), we introduced the inverse function F^{-1} . We follow the same procedure here and consider the inverse function Φ^{-1} of Φ . Then, for the optimal k^* it holds

$$k^* = \Phi^{-1}\left(\frac{p+h-c}{p+h+s}\right).$$

Denote the expected surplus of k as

$$G(k) = \int_k^\infty (D_S - k)f(D_S) dD_S.$$

206 Then, from (7) the optimal expected profit is

$$\begin{aligned} \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle &= p\mu - c(D_S(k-1) + k^*\sigma) \\ &\quad - s[k^*\sigma + \sigma G(k^*)] - (p+h)\sigma G(k^*) \\ 207 &= p\mu - cy_{k-1} - \underbrace{\sigma(c+s)k^*}_{<0} - \underbrace{\sigma(s+p+h)G(k^*)}_{<0}. \end{aligned}$$

208 Note that the expected profit decreases with the standard deviation σ , namely, the
209 volatility of the demand.

210 Coalition games that are subject to probabilistic demand/ characteristic function,
211 as in the aforementioned example, have been also studied in the context of stochastic
212 cooperative games (Suijs et al., 1997; Toriello and Nelson, 2017). In that context
213 conditions for a stable core were devised. Similarly the news agent problem (Muller
214 et al., 2002; Hartman and Dror, 2005; Slikker et al., 2005) is a coalition problem where
215 probabilistic utilities emerge. The literature concerning this problem also focuses on
216 conditions for non-empty core and fair allocations.

217 In the current article a different approach is adopted. The control of the stochastic
218 process in order to be bounded around the core is considered, instead of trying to
219 define suitable conditions for the core of the game to be non-empty. As a result a
220 formulation of TU games with dynamically changing characteristic function, which
221 allows its representation as a stochastic process is provided. A saturated controller is
222 used in order for the process to be bounded around the core. The proposed controller
223 resembles the ‘‘Best response’’ decision making process. Hence, stochastic differential
224 inclusions emerge from the control process. Therefore, analysis of a stochastic process
225 which can be occurred through the TU game formulation is provided, based on the
226 theory of stochastic differential inclusions Benaim et al. (2005).

227 Since the cost function is not constant throughout the game any more and in each
228 time step of the decision making process a fluctuated version of the cost function is
229 available because either of changes in the environment or noisy observations. This
230 analysis focuses on the control of the outcome of the stochastic process either to be
231 in the core or bounded in the ϵ -core based on the volatility of the perturbations.

232 **4. Model and problem statement.** This section is separated into two parts.
233 The first contains the description of the dynamic TU model and provides an illustrative
234 example of a 3-player game. The second part contains the representation of the
235 dynamic TU game as a stochastic process and a proposed control strategy which
236 allows an a solution bounded in the ϵ -core of the dynamic TU-game. The distance ϵ
237 from the core depends on the volatility of the stochastic process.

238 **4.1. TU Games with noisy observations.** A *dynamic TU game* is described
239 by $\langle N, \eta(t) \rangle$, where $\eta(t)$ is a time-varying characteristic function representing the
240 values of different coalitions. In real life applications there are many uncontrollable
241 processes which introduce uncertainty either on the rewards of the coalitional games
242 or the observations of the other players’ decisions. In the intelligent mobility network
243 problem, of the previous section, managers can have an estimate of the ordering
244 capacities of the other managers. This estimate can be of the form of a probability
245 distribution which changes over time. Therefore, the uncertainty can be modelled as
246 a stochastic process.

247 It possible to represent a dynamic TU game in Matrix form. In addition, fol-
 248 lowing the dynamic programming paradigm, all the constraints which arise from
 249 the definition of the core can be represented as inequalities. In particular, let $B_{\mathcal{H}}$
 250 be a $((q - 1) \times n)$ -matrix whose rows are the characteristic vectors $y^{S_j} \in \mathbb{R}^n$ of
 251 each coalition other than the grand coalition, i.e., $S_j \in \mathcal{S}, S_j \neq N$. In other words
 252 $B_{\mathcal{H}} = \{(y^{S_j})^T\}_{S_j \in \mathcal{S}, S_j \neq N}$.

253 The characteristic vectors are in turn binary vectors representing the participation
 254 or not of a player i in the coalition S_j , whereby $y_i^{S_j} = 1$ if $i \in S_j$ and $y_i^{S_j} = 0$ if $i \notin S_j$.
 255 For any allocation in the *core* of the game $C(\eta(t))$ we have:

256 (9)
$$\tilde{u}(t) \in C(\eta(t)) \Leftrightarrow B_{\mathcal{H}}\tilde{u}(t) \geq \eta(t),$$

258 where the inequality is to be interpreted component-wise, and for the grand coalition
 259 it is satisfied with equality due to the efficiency condition of the core, i.e., $\sum_{i=1}^n \tilde{u}_i(t) =$
 260 $\eta_{N(t)}$, where $\eta_{N(t)}$ denotes the q_{th} component of $\eta(t)$ and is equal to the grand coalition
 261 value.

262 Let

263 (10)
$$B = \begin{bmatrix} B_{\mathcal{H}} & -I \\ \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \in \{-1, 0, 1\}^{q \times n + (q-1)}.$$

264 Inequality (9) can be rewritten as an equality by using an augmented allocation
 265 vector given by $u := \begin{bmatrix} \tilde{u} \\ s \end{bmatrix} \in \mathbb{R}^{n+q-1}$ where s is a vector of $q - 1$ non-negative surplus
 266 variables. Then, we have

267 (11)
$$Bu(t) = \eta(t).$$

For a 3-player coalitional game equation (11) takes the form

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_B \underbrace{\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix}}_u = \underbrace{\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \end{bmatrix}}_{\eta}.$$

268 *Remark* Note here that in general TU coalitional games, as well as the formulation
 269 which is proposed in this article, suffer from the curse of dimensionality. In particular,
 270 the dimensionality of B will exponentially increase with the number of players and
 271 possible actions. In that case a distributed solution as the one in (Nedich and Bauso
 272 , 2013) can be used in order to cluster the problem to smaller sub-problems which are
 273 feasible to be solved.

274 **4.2. TU games as a stochastic process.** Let us assume that the perturba-
 275 tions of the characteristic function are bounded in an ellipsoid. Let $w(t)$ denote the
 276 perturbed observation of the players at time t , $w_0(t)$ being the time-varying charac-
 277 teristic function and $\tilde{w}(t)$ the perturbation term, such as a bias in the estimator of the

278 characteristic function $w_0(t)$. In the case of an additive perturbation term the drift
 279 from $w_0(t)$ can be expressed as $w(t) = [w_0(t) + \tilde{w}(t)]$. The analysis of the dynamic
 280 TU games which follows in the rest of this article is based on the assumption that the
 281 perturbations are bounded in an ellipsoid, i.e $w(t)$ can be written as:

$$282 \quad (12) \quad w(t) \in \mathcal{W} = \{w \in \mathbb{R}^q : w^T R w \leq 1\}.$$

283 The changes in the characteristic function as they are realised by the players can be
 284 written then as

$$285 \quad (13) \quad d\eta(t) = w(t)dt - \Sigma d\mathcal{B}(t), \quad \text{in } \mathbb{R}^q,$$

286 where $\Sigma d\mathcal{B}(t)$ is a random noise with zero mean and $\Sigma = \text{diag}((\Sigma_{ii})_{i=1,\dots,q}) \in \mathbb{R}^{q \times q}$
 287 for given scalars Σ_{ii} , all full column rank, and $\mathcal{B}(t) \in \mathbb{R}^q$ is a q -dimensional Brownian
 288 motion, which is independent across its components, independent of the initial state
 289 η_0 , and independent across time.

Instead of studying the evolution of the characteristic function in order to solve a
 TU game the surpluses s_j can be studied. Note that the difference between the allo-
 cated value and the coalitional S_j , corresponds to surplus variable s_j and is described
 as,

$$s_j(t) = \sum_{i \in \mathcal{S}_j} \tilde{u}_i(t) - \eta_j(t).$$

290 A positive value for $s_j(t)$ can be interpreted as a debit for the coalition, whereas
 291 a negative value can be interpreted as a credit. The main insight is that *if all the*
 292 *surpluses are non-negative, then the total allocation to any coalition exceeds the value*
 293 *of the coalition itself and the allocation vector lies in the core.* Also, note that there
 294 are only $q - 1$ surplus variables because coalition N has no surplus ($\sum_{i \in N} \tilde{u}_i - \eta_q = 0$)
 295 due to the efficiency condition of the core.

296 Let $x(t) \in \mathbb{R}^q$, denote the cumulative excess which is obtained as follows. In
 297 essence, every component of vector $Bu(t)$ is the total reward given to the members
 298 of a coalition at time t , and the drift from this reward, $w(t)$, is subtracted. Then, a
 299 positive $x(t)$ means positive cumulative excess.

300 Let us denote the controller in linear state feedback form as:

$$301 \quad (14) \quad u(x) = K(x, t)x,$$

302 where $K(x, t) \in \mathbf{co}\{K^{(i)}\}_{i \in I}$.

303 Then the problem of stabilising the core can be cast as a problem of solving the
 304 following stochastic differential inclusion:

$$305 \quad (15) \quad dx(t) \in F(x)dt + \Sigma d\mathcal{B}(t).$$

306 Also,

$$307 \quad (16) \quad F(x) := \{\xi \in \mathbb{R}^q \mid \xi = (BK(x, t) - I)x - w, \\ K(x, t) \in \mathbf{co}\{K^{(i)}\}_{i \in I}, w \in \mathcal{W}\},$$

308 for assigned polytopic sets $\mathbf{co}\{K^{(i)}\}_{i \in I}$, and ellipsoidal set \mathcal{W} , and where $\mathcal{B}(t)$ is a
 309 Brownian motion weighted by a matrix Σ and B defined as in (10).

310 The stability, well-posedness and existence of solution to (15), when saturated
 311 linear controllers are used has been studied in [Hu et al. \(2006\)](#); [Cai et al. \(2009\)](#); [Hu](#)
 312 [et al. \(2005\)](#); [Jokic et al. \(2008\)](#); [Grammatico et al. \(2014\)](#).

313 For any symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, define the function $V(x) =$
 314 $x^T P x$ and the ellipsoidal target set $\Pi = \{x \in \mathbb{R}^n : V(x) \leq 1\}$. We are interested in
 315 studying convergence of the solutions of (15) to the target set.

316 **5. Examples.** The stochastic differential inclusion (15) arises in the case of sat-
 317 urated controls, and in the case of two-population games. We discuss next these three
 318 examples.

319 **5.1. Example 1: saturated controls.** Assume that controls are bounded
 320 within polytopes

321 (17)
$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^{(q-1)+n} : u^- \leq u \leq u^+\},$$

322 where u^+, u^- are assigned vectors. Note that we can assume the characteristic func-
 323 tion centred at zero as in (12) as we can always center the hypercube of $u(t)$ around
 324 any desired value.

325 In addition, for any matrix $K \in \mathbb{R}^{n+(q-1) \times q}$, define as saturated linear state
 326 feedback control any policy

327 (18)
$$u = -\text{sat}\{Kx\} = \begin{cases} -Kx & \text{if } Kx \in \mathcal{U} \\ u(x) \in \partial\mathcal{U} & \text{otherwise,} \end{cases}$$

328 where $\partial\mathcal{U}$ indicates the frontier of set \mathcal{U} .

329 In the above, the $\text{sat}\{\cdot\}$ operator has to be interpreted component-wise, namely

330 (19)
$$u_i = \text{sat}_{[u_i^-, u_i^+]} \{-K_{i\bullet}x\},$$

where $K_{i\bullet}$ denotes the i th row of K and where, for any given scalar a and b

$$\text{sat}_{[a,b]} \{\zeta\} = \begin{cases} b, & \text{if } \zeta > b, \\ \zeta, & \text{if } a \leq \zeta \leq b, \\ a, & \text{if } \zeta < a. \end{cases}$$

331 Henceforth we omit the indices of the sat function.

332 Under the control $u = \text{sat}\{-Kx\}$, the closed-loop dynamics mimics the differen-
 333 tial inclusion (15) as follows

334
$$dx \in \{(-x + B\text{sat}\{-Kx\} - w)dt + \Sigma d\mathcal{B}(t), w \in \mathcal{W}\}.$$

335 **5.2. Example 2: distribution network.** Consider a distribution network
 336 problem where there is a demand for a specific commodity and the reward for sup-
 337 plying it is suitably described by our control law. When the demands are based on a
 338 diffusion process, their evolution can be written as:

339 (20)
$$\dot{d} = w(t) - \sum d\mathcal{B}(t).$$

Then (13) can be written with respect to \dot{d} as:

$$d\eta(t) = [w_0(t) + \dot{d}(t) + \Sigma d\mathcal{B}(t)]dt - \Sigma d\mathcal{B}(t).$$

340 The excess then can be written as

341 (21)
$$dx(t) = (-x(t) + B_{\mathcal{H}}u(t))dt - d\eta(t),$$

342 where u is the control vector as defined in (18).

$u^{(j)} \setminus w^{(k)}$	$w^{(1)}$	\dots	$w^{(\bar{q})}$
$u^{(1)}$	$Bu^{(1)} - w^{(1)}$	\dots	$Bu^{(1)} - w^{(\bar{q})}$
\vdots	\vdots		\vdots
$u^{(\bar{p})}$	$Bu^{(\bar{p})} - w^{(1)}$	\dots	$Bu^{(\bar{p})} - w^{(\bar{q})}$

TABLE 1
The possible vector payoffs.

343 **5.3. Example 3: approachability.** Equation (15) is in the same spirit as in
 344 Hart and Mas-Colell's paper (Hart and Mas-Colell, 2003) on continuous-time ap-
 345 proachability.

In particular (15), can be obtained when a 2-player repeated game with vector payoffs as displayed in Table 1, is considered. Let $A_1 = \{u^{(1)}, \dots, u^{(\bar{p})}\}$ and $A_2 = \{w^{(1)}, \dots, w^{(\bar{q})}\}$ be the actions sets of player 1 and 2. Denote $a_1 = [a_{11}, \dots, a_{1\bar{p}}]^T$ and $a_2 = [a_{21}, \dots, a_{2\bar{q}}]^T$ the mixed strategies of player 1 and 2, respectively. Introduce the mixed extension mapping $\Delta(A_1) \times \Delta(A_2) \rightarrow \mathcal{U} \times \mathcal{W}$, such that $(a_1, a_2) \mapsto (u, w)$ where

$$u = \sum_{j=1}^{\bar{p}} a_{1j} u^{(j)}, \quad w = \sum_{k=1}^{\bar{q}} a_{2k} w^{(k)}.$$

346 Consider the time-average expected (over opponent's play) payoff defined as

$$347 \quad \Gamma(s) = \frac{1}{s} \int_0^s (Bu - w) d\tau \in \mathbb{R}^q.$$

If we rescale the time window using $s = e^t$, take $x(t) = \Gamma(e^t)$ and differentiate with respect to t , we obtain the differential equation (15). Note that, after rescaling the time window, we have

$$x(0) = \int_0^1 (Bu - w) d\tau \in \mathbb{R}^q.$$

348 Adopting a ‘‘population-game dynamics’’ perspective, the state $x(t) \in \mathbb{R}^q$ repre-
 349 sents the current average payoff over the population.

350 **6. Main results.** In this section it is shown that the second moment of the
 351 deviations from the core, $x(t)$, is bounded, when a saturated linear feedback controller
 352 is used. This is achieved by the use of polytopic techniques (Mayne, 2003). Polytopic
 353 constraints are widely used in order to model problems related to robust control
 354 problems when the transition matrix of the process is state-dependent, i.e. $\dot{x} = A(x)x$.
 355 In addition, because no further constraints have been imposed on (15), the proposed
 356 methodology can be used to control dynamic TU games when (15) describes the
 357 dynamics of the game.

358 Our idea is to rewrite the above dynamics in the following polytopic form

$$359 \quad (22) \quad dx \in \{(BK(x, t) - I)x(t) - w(t)dt + \Sigma d\mathcal{B}(t), w \in \mathcal{W}\},$$

360 where the time varying matrices $K(x, t)$ are expressed as convex combinations of
 361 $|I|$ matrices $K^{(i)}$, $i \in I$. More precisely the expressions for $K(x, t)$ are

$$362 \quad (23) \quad K(x, t) = \sum_{i \in I} \tilde{\sigma}_i(x, t) K^{(i)}, \quad \sum_{i \in I} \tilde{\sigma}_i(x, t) = 1.$$

The control policy is then

$$u = Kx = \left(\sum_{i \in I} \tilde{\sigma}_i(x, t) K^{(i)} \right) x, \quad \sum_{i \in I} \tilde{\sigma}_i(x, t) = 1.$$

In the case of saturated controls the procedure to derive the weights in the above control policy are discussed in (Gomes da Silva, 2001).

THEOREM 6.1. *The distance of any solution of the stochastic differential inclusion (15) from the target set Π is second-moment bounded if for all $x \in X_j, j \in I$*

$$(24) \quad x^T \left[Q(\Psi^{(i)})^T + \Psi^{(i)} Q + \alpha Q + \frac{1}{\beta} R^{-1} \right] x \leq 0,$$

where $\Psi^{(i)} = [BK^{(i)} - I]$ and X_j is any subspace where $K^{(i)}$ is in the support S_j of K , i.e., the control is

$$u = Kx = \left(\sum_{i \in S_j} \tilde{\sigma}_i(x, t) K^{(i)} \right) x, \quad \sum_{i \in S_j} \tilde{\sigma}_i(x, t) = 1.$$

Proof. The analysis is then performed within the framework of stochastic stability theory (Loparo and Feng, 1996). To this end, consider the infinitesimal generator

$$(25) \quad \mathcal{L}[\cdot] = \lim_{dt \rightarrow 0} \frac{\frac{1}{2} \mathbb{E} \sum_{i \in I} dx^T \nabla_{xx}^2 [\cdot] dx + \mathbb{E} dx^T \nabla_x [\cdot]}{dt},$$

and the Lyapunov function $V(x) = x^T P x$. The stochastic derivative of $V(x)$ is obtained by applying (25) to $V(x)$, which yields

$$\begin{aligned} \mathcal{L}V(x(t)) &= \lim_{dt \rightarrow 0} \frac{\mathbb{E}V(x(t+dt)) - V(x(t))}{dt} \\ &= \lim_{dt \rightarrow 0} \frac{\frac{1}{2} \mathbb{E} \sum_{i \in I} dx^T \nabla_{xx}^2 [V(x)] dx + \mathbb{E} dx^T \nabla_x [V(x)]}{dt} \\ &= \frac{1}{2} \sum_{i \in I} \Sigma_{ii}^2(x) (\nabla_{xx}^2 [V(x)])_{ii} + [BK(\cdot)x - x - w]^T \cdot \\ &\quad \cdot \nabla_x [V(x)] + \nabla_x [V(x)]^T [BK(\cdot)x - x - w]. \end{aligned}$$

Using $\nabla_{xx}^2 [V(x)] = P$ and $\nabla_x [V(x)] = Px$ the above can be rewritten as follows, for all $x \notin \Pi$, and $w \in \mathcal{W}$

$$(26) \quad \begin{aligned} \mathcal{L}V(x) &= [-x + BK(x, t)x - w]^T P x \\ &+ x^T P [-x + BK(x, t)x - w] + \sum_{i=1}^q \Sigma_{ii}^2(x) P_{ii} \\ &= x^T [BK(x, t) - I]^T P x + x^T P [BK(x, t) - I] x \\ &- w^T P x - x^T P w + \sum_{i=1}^q \Sigma_{ii}^2 P_{ii} < 0. \end{aligned}$$

Let $\bar{\Pi} = \mathbb{R}^q \setminus \Pi$. From the S -procedure, we know that for all $x \in \bar{\Pi}$, and $w \in \mathcal{W}$ condition (26) holds if there exist $\alpha, \beta \geq 0$, such that for all $(x, w) \in \bar{\Pi} \times \mathcal{W}$

$$(27) \quad \begin{aligned} \mathcal{L}V(x) &= x^T [BK(x, t) - I]^T P x \\ &+ x^T P [BK(x, t) - I] x \\ &- w^T P x - x^T P w + \sum_{i=1}^q \Sigma_{ii}^2 P_{ii} \\ &\leq \alpha(1 - V(x)) + \beta(\|w\|_R^2 - 1) \leq 0. \end{aligned}$$

The last inequality is obtained from observing that

$$\bar{\Pi} \times \mathcal{W} := \{(\xi, \omega) : 1 - V(\xi) \leq 0, \|\omega\|_R^2 - 1 \leq 0\}.$$

383 Let $\Psi(x, t) = [BK(x, t) - I]$, inequality (27) can be rewritten as

$$384 \quad \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \Psi(x, t)^T P + P\Psi(x, t) + \alpha P & -P \\ -P & -\beta R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ -\alpha + \beta + \sum_{i=1}^q \Sigma_{ii}^2 P_{ii} \leq 0.$$

385 Trivially it must hold $\beta \leq \alpha$. Assume without loss of generality that $\beta = \alpha -$
386 $\sum_{i=1}^q \Sigma_{ii}^2 P_{ii}$.¹ Recall that α and β can be chosen arbitrarily. After pre and post-
387 multiplying by $Q = P^{-1}$, the above condition becomes

$$388 \quad \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} Q\Psi(x, t)^T + \Psi(x, t)Q + \alpha Q & -I \\ -I & -\beta R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0.$$

389 Now, as the state never leaves the region $S(\psi^\theta)$, i.e., $x(t) \in S(\psi^\theta)$, we can always
390 express $A(x(t))$ as a convex combination of the A_j s as in (23).

391 By convexity, the above condition is true if it holds, for all $j = 1, \dots, 2^n$,

$$392 \quad (28) \quad \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} Q(\Psi^{(j)})^T + \Psi^{(j)}Q + \alpha Q & -I \\ -I & -\beta R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0,$$

393 where $\Psi^{(j)} = [BK^{(j)} - I]$. Using the Shur complement condition (28) is implied
394 by (24).

395 Based on the above stated theorem we can infer that the solution of a dynamic TU
396 game when (15) is used will lie in the ϵ -core. This is because even if the disturbance
397 in 13 is a q -dimensional unbounded Brownian motion, the dynamics of the process
398 are bounded in the second moment.

399 Stronger conditions are established in the following corollary.

400 **COROLLARY 6.2.** *The distance of any solution of the stochastic differential inclu-*
401 *sion (15) from the target set Π is second-moment bounded, if there exists a scalar*
402 *$\alpha \geq 0$ such that, for all $K^{(i)}, i \in I$*

$$403 \quad (29) \quad Q[BK^{(i)} - I]^T + [BK^{(i)} - I]Q + \alpha Q + \frac{1}{\beta}R^{-1} < 0.$$

404 *Proof.* Straightforward from observing that (29) implies (24).

405 Note that conditions (24) simply impose that each one of the conditions (29) (for
406 fixed j) holds only in a specific region of the state space and not over the entire \mathbb{R}^n .
407 In this sense, condition (24) is weaker than (29).

408 Let $d(x, \Pi)$ be the distance of any given $x \in \mathbb{R}^q$ from the target set Π . Consider
409 a modified stochastic differential inclusion

$$410 \quad (30) \quad dx(t) \in F(x)dt + \Sigma(x)d\mathcal{B}(t),$$

411 where $\Sigma(x)$ is the weight of the random noise which is now upper bounded by the
412 distance of x from the target set, i.e., $\Sigma(x) \leq d(x, \Pi)$. We are in a position to
413 establish the next result relating to the case where the variance of the stochastic
414 process vanishes the closer the trajectory is to the target set.

¹ P_{ii} is not known a priori so we need to implement a guess method

415 COROLLARY 6.3. Let $\Sigma(x) \leq d(x, \Pi)$ and let $\Psi^{(i)} = [BK^{(i)} - I]$. Any solution of
 416 the stochastic differential inclusion (30) converges to the target set Π almost surely if
 417 for all $x \in X_i$, $i \in I$

$$418 \quad (31) \quad x^T \left[Q(\Psi^{(i)})^T + \Psi^{(i)}Q + \alpha Q + \frac{1}{\beta}R^{-1} \right] x \leq 0.$$

419 *Proof.* The underlying idea is that for all $x \notin \Pi$, and $w \in \mathcal{W}$

$$420 \quad (32) \quad \begin{aligned} & \lim_{x \rightarrow \Pi} \mathcal{L}(V(x)) \\ &= \lim_{x \rightarrow \Pi} \left\{ [-x + BK(x, t)x - w]^T P x \right. \\ & \quad \left. + x^T P [-x + BK(x, t)x - w] + \sum_{i=1}^q \Sigma_{ii}^2(x) P_{ii} \right\} \\ &= x^T [BK(x, t) - I]^T P x + x^T P [BK(x, t) - I] x \\ & \quad - w^T P x - x^T P w < 0. \end{aligned}$$

421 We then look for $\alpha, \beta \geq 0$, such that for all $(x, w) \in \bar{\Pi} \times \mathcal{W}$

$$422 \quad (33) \quad \begin{aligned} \mathcal{L}V(x) &= x^T [BK(x, t) - I]^T P x \\ & \quad + x^T P [BK(x, t) - I] x \\ & \quad - w^T P x - x^T P w \\ & \leq \alpha(1 - V(x)) + \beta(\|w\|_R^2 - 1) \leq 0, \end{aligned}$$

423 which is equivalent to setting $\beta \leq \alpha$ and solving

$$424 \quad \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \Psi(x, t)^T P + P\Psi(x, t) + \alpha P & -P \\ -P & -\beta R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ -\alpha + \beta \leq 0.$$

425 After pre and post-multiplying by $Q = P^{-1}$, and using convexity, the above condition
 426 leads to (28), and this concludes the proof.

427 Let $\underline{\mathcal{B}}(t)$ be a zero-mean random noise such that $\int d\underline{\mathcal{B}}(t)$ has bounded support.
 428 For instance, think of $\int d\underline{\mathcal{B}}(t)$ as a truncated Gaussian noise with bounded support
 429 in the interval $[-\bar{\kappa}\sigma, \bar{\kappa}\sigma]$ for a positive scalar $\bar{\kappa}$. The counterpart of (15) is then

$$430 \quad (34) \quad dx(t) \in F(x)dt + \Sigma d\underline{\mathcal{B}}(t).$$

Assume $\underline{\mathcal{B}}(t) \in [-\Sigma, \Sigma]$ and let $\tilde{W} := \{\omega : \omega = w + \tilde{\sigma}, w \in \mathcal{W}, \tilde{\sigma} \in [-\Sigma, \Sigma]\}$. Also, let \tilde{R} be such that

$$\tilde{W} \subseteq \bar{W} := \{\omega : \|\omega\|_R^2 - 1 \leq 0\}.$$

431 We are in a position to state the following main result.

432 THEOREM 6.4. Any solution of the stochastic differential inclusion (15) converges
 433 to the target set Π if for all for all $K^{(i)}$, $i \in I$

$$434 \quad (35) \quad \left[Q(\Psi^{(i)})^T + \Psi^{(i)}Q + \alpha Q + \frac{1}{\beta}\tilde{R}^{-1} \right] \leq 0.$$

435 *Proof.* For all $x \notin \Pi$,

$$436 \quad (36) \quad \begin{aligned} \dot{V}(x) &\in \left\{ [-x + BK(x, t)x - w \pm \Sigma]^T P x \right. \\ & \quad \left. + x^T P [-x + BK(x, t)x - w \pm \Sigma], w \in \mathcal{W} \right\} \\ &= \left\{ x^T [BK(x, t) - I]^T P x + x^T P [BK(x, t) - I] x \right. \\ & \quad \left. - (w \pm \Sigma)^T P x - x^T P (w \pm \Sigma), w \in \mathcal{W} \right\} < 0. \end{aligned}$$

437 Recall that $\tilde{W} := \{\omega : \omega = w + \tilde{\sigma}, w \in \mathcal{W}, \tilde{\sigma} \in [-\Sigma, \Sigma]\}$. From the above we have
 438 that for all $x \notin \Pi$ it must hold

$$439 \quad (37) \quad \begin{aligned} \dot{V}(x) &\leq \max_{\omega \in \tilde{W}} \\ &\left\{ x^T [BK(x, t) - I]^T Px + x^T P [BK(x, t) - I] x \right. \\ &\quad \left. - \omega^T Px - x^T P \omega \right\} < 0. \end{aligned}$$

440 For all $x \in \bar{\Pi}$, and $\omega \in \tilde{W}$ the above condition holds if there exist $\alpha, \beta \geq 0$, such that
 441 for all $(x, w) \in \bar{\Pi} \times \mathcal{W}$

$$442 \quad (38) \quad \begin{aligned} \dot{V}(x) &= x^T [BK(x, t) - I]^T Px \\ &\quad + x^T P [BK(x, t) - I] x \\ &\quad - \omega^T Px - x^T P \omega \\ &\leq \alpha(1 - V(x)) + \beta(\|w\|_R^2 - 1) \leq 0. \end{aligned}$$

From the definition of \tilde{R} it holds

$$\tilde{W} \subseteq \bar{W} := \{\omega : \|\omega\|_R^2 - 1 \leq 0\}.$$

For all (x, w) in

$$\bar{\Pi} \times \bar{W} := \{(\xi, \omega) : 1 - V(\xi) \leq 0, \|\omega\|_R^2 - 1 \leq 0\},$$

443 condition (38) can be rewritten as

$$444 \quad (39) \quad \begin{bmatrix} x \\ \omega \end{bmatrix}^T \begin{bmatrix} Q(\Psi^{(i)})^T + \Psi^{(i)}Q + \alpha Q & -I \\ -I & -\beta\tilde{R} \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \leq 0.$$

445 and this concludes our proof.

446 **7. Intelligent Mobility Network.** In this section the stability analysis of the
 447 case study of the intelligent mobility network of Section 3 is presented.

448 Initially the deterministic version of dynamics (15) is decomposed as

$$449 \quad (40) \quad \begin{aligned} dx(t) &\in \{(-x(t) + Bu(t) - \tilde{w}(t))dt \\ &\quad + \Sigma d\mathcal{B}(t), \tilde{w}(t) \in \tilde{W}\}, \end{aligned}$$

450 where $\tilde{w}(t)$ is an uncertain but bounded deviation from the expected profit, given by

$$451 \quad (41) \quad \begin{aligned} \tilde{w}(t) &= [\mathcal{P}_S(y, \Theta_S^*) - \mathbb{E}\mathcal{P}_S(y, \Theta_S^*)]_{S \in \mathcal{S}} \\ &\in W^{(2)} := \{w \in \mathbb{R}^m \mid \underline{\delta} \leq w \leq \bar{\delta}\}. \end{aligned}$$

452 In the above expression $\bar{\delta}$ and $\underline{\delta}$ are upper and lower bounds respectively, and are
 453 obtained as

$$454 \quad (42) \quad \bar{\delta} := \mathcal{P}_S(\bar{D}_S, \Theta_S^*) - \mathbb{E}\mathcal{P}_S(y, \Theta_S^*),$$

$$455 \quad (43) \quad \underline{\delta} := \mathcal{P}_S(\underline{D}_S, \Theta_S^*) - \mathbb{E}\mathcal{P}_S(y, \Theta_S^*).$$

456 Before we calculate $\bar{\delta}^j$ and $\underline{\delta}^j$, note that to derive (40), we simply write the real
 457 profit as combination of expected profit $w_0(t)$ and deviation from the expected profit
 458 $\tilde{w}(t)$, namely $w(t) = w_0(t) + \tilde{w}(t)$. The expected profit is a priori known and given
 459 by $w_0(t) = [\langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle]_{S \in \mathcal{S}}$. We can then design a first control input $u_0(t)$ based

460 on the Shapley allocation to compensate the optimal expected profit. To do this, let
 461 $u_0(t)$ be obtained from the following equation:

$$462 \quad (44) \quad Bu_0(t) = w_0(t) = [\mathbb{E}_S J(y, \Theta_S^*)]_{S \in \mathcal{S}}.$$

463 To obtain an expression for $\bar{\delta}^j$ let us maximize the profit of the corresponding
 464 coalition S with respect to y , namely

$$465 \quad \begin{aligned} \bar{D}_S &:= \arg \max_{D_S} \mathcal{P}_S(D_S, \Theta_S^*) \\ &= \arg \max_{D_S} \{p\mu - c\Theta_S^* - s \max(0, \Theta_S^* - D_S) \\ &\quad - (p + h) \max(0, D_S - \Theta_S^*)\} = \Theta_S^*. \end{aligned}$$

Then, the maximal profit for coalition S is

$$\max_y \mathcal{P}_S(y, \Theta_S^*) = \mathcal{P}_S(\bar{D}_S, \Theta_S^*) = \mathcal{P}_S(\Theta_S^*, \Theta_S^*) = p\mu - c\Theta_S^*.$$

Substituting the above in (42), we have

$$\bar{\delta}^j := p\mu - c\Theta_S^* - \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle.$$

466 Similarly, to obtain $\underline{\delta}^j$ used in (43), let us minimize the profit of the corresponding
 467 coalition S with respect to y , namely

$$468 \quad \begin{aligned} \underline{D}_S &:= \arg \min_{D_S} \mathcal{P}_S(D_S, \Theta_S^*) \\ &= \arg \min_{D_S} \{p\mu - c\Theta_S^* - s \max(0, \Theta_S^* - D_S) \\ &\quad - (p + h) \max(0, D_S - \Theta_S^*)\} = 0. \end{aligned}$$

The above means that the minimal profit is obtained when the power output is zero, which leads to

$$\min_y \mathcal{P}_S(y, \Theta_S^*) = \mathcal{P}_S(\underline{D}_S, \Theta_S^*) = \mathcal{P}_S(0, \Theta_S^*) = p\mu - (s + c)\Theta_S^*.$$

Substituting the above in (43), we have

$$\underline{\delta}^j := p\mu - (s + c)\Theta_S^* - \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle.$$

469 We can conclude that

$$470 \quad \begin{aligned} \tilde{w}(t) \in \tilde{W} &:= \{w \in \mathbb{R}^m \mid \\ &[p\mu - (s + c)\Theta_S^* - \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle]_{S \in \mathcal{S}} \leq w \\ &\leq [p\mu - c\Theta_S^* - \langle \mathcal{P}_S(D_S, \Theta_S^*) \rangle]_{S \in \mathcal{S}}\}. \end{aligned}$$

As last step we define the parametrized ellipsoid

$$\Pi_k = \{\omega \in \mathbb{R}^m : k^2 \omega^T \Phi \omega \leq 1\},$$

where Φ is a matrix in $\mathbb{R}^{m \times m}$ and consider the problem of finding the smallest ellipsoid Π_k which contains $\mathcal{W}^{(2)}$:

$$k^* = \max_k \{k \mid \Pi_k \supset \mathcal{W}^{(2)}\}.$$

471 The dynamic model we obtain is then

$$472 \quad dx(t) \in \{(-x(t) + Bu(t) - \omega)dt + \Sigma dB(t), \omega \in \Pi_{k^*}\},$$

473 which is of the same form as in (15).

474 **8. Simulations.** An application of the multi-inventory coalitional model, which
 475 was described in the previous section, can be found in the electricity trade market.
 476 Consider the case of n electricity producers which should meet the electricity demands
 477 of a central distributor. The expected profit of a generic coalition is described by (2)
 478 under the following two assumptions (Baeyens et al., 2013):

- 479 • The structure of the network does not affect the prices and the demand of
 480 electricity.
- 481 • The electricity market system comprises of a single ex-ante forward penalty
 482 and a single ex-post imbalance penalty for variations from the contracted
 483 values.

484 The dynamic demand of such system can be defined as the diffusion process of
 485 (20) and the excess is defined as in (21). In the simulations of this section a saturated
 486 controller of the form of (18) is used here $K = kB^{-1}$ and $k = \frac{2}{3}$. In our simulations
 487 we consider the case of four players/energy producers that should decide if they will
 488 be part of a coalition and share the costs and profits from energy production. The
 489 initial demand was set to $[0.1693 \ 0.2019 \ 0.1304 \ 0.0562]^T$. The drift parameter
 490 w was bounded in $w^T R w \leq 1$ and R was set to be the identity matrix. Figures 2-4
 491 depict the evolution of the excess, the variance of the excess and the Shapley value
 492 respectively.

493 As it is evident from Figure 2 the excess is always non-negative for all the coalitions
 494 which is an indication of a non-empty core. In addition the excess is grouped according
 495 to the number of the coalition's members. In particular, the excess for the coalitions
 496 with one member have greater excess than the coalitions with two members and
 497 the coalitions with two members have greater excess than the coalitions with three
 498 members. The grand coalition has excess near to zero.

499 Figure 3 depicts the variance of the excess of all possible coalitions. As it can be
 500 seen from Figure 3 the variances of all coalitions converge to a constant value smaller
 501 than one.

502 Figure 4 depicts the Shapley's value for all players over time. Since the excess
 503 value is always positive we can conclude that the core is non-empty.

504 **9. Conclusion.** The problem of controlling the allocations in dynamic TU games
 505 is considered. Stochastic differential inclusions are used to model the uncertainty of
 506 dynamic TU games, which can be occurred either as a result of a dynamic environ-
 507 ment or noisy observations. A model is proposed, which extends the results of Bauso
 508 et al. (2010) that allows allocation to be controlled by taking into account the de-
 509 terministic and stochastic uncertainty which exists in the evolution of the excess of
 510 a coalition. In particular based on linear matrix inequality conditions it is shown
 511 that the stochastic differential inclusion solutions are second-moment bounded. An
 512 intelligent mobility scenario is used to show the applicability of the proposed method-
 513 ology. Additionally simulations in a distribution network are employed which support
 514 the theoretical results, by showing stability of the core and bounded variance of the
 515 coalitions' excesses.

516 Future work could include a distributed version of the proposed model. This will
 517 increase the efficiency of the proposed methodology's applicability in scenarios which
 518 include thousand of players. In addition the performance of the proposed methodology
 519 and limitation which may arise from the usage of real distribution network's data in
 520 the simulations will be considered.

521 References.

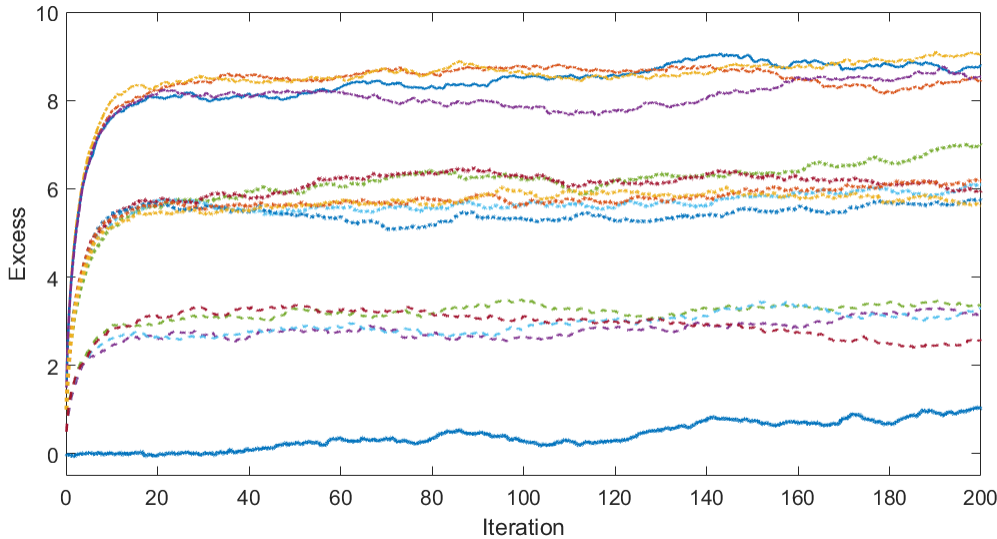


FIG. 2. Evolution of excess. The combined dotted and dashed lines depict the coalitions with a single member, the dotted lines depict the coalitions with two members, the dashed lines depict the coalitions with three members and the solid line depicts the grand coalition.

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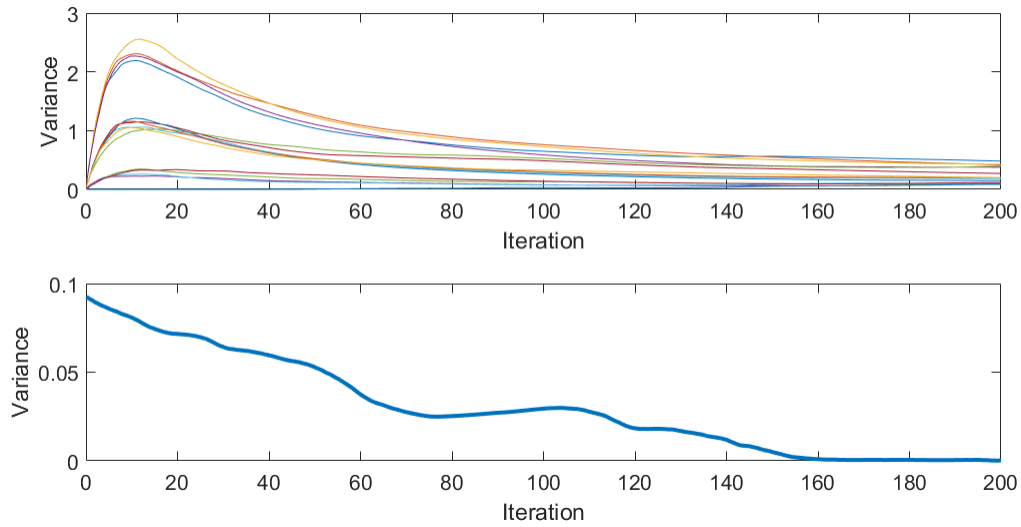


FIG. 3. Variance of the excess for each coalition. The top plot depicts the variance of all coalitions. The bottom panel depicts the variance of the grand coalition.

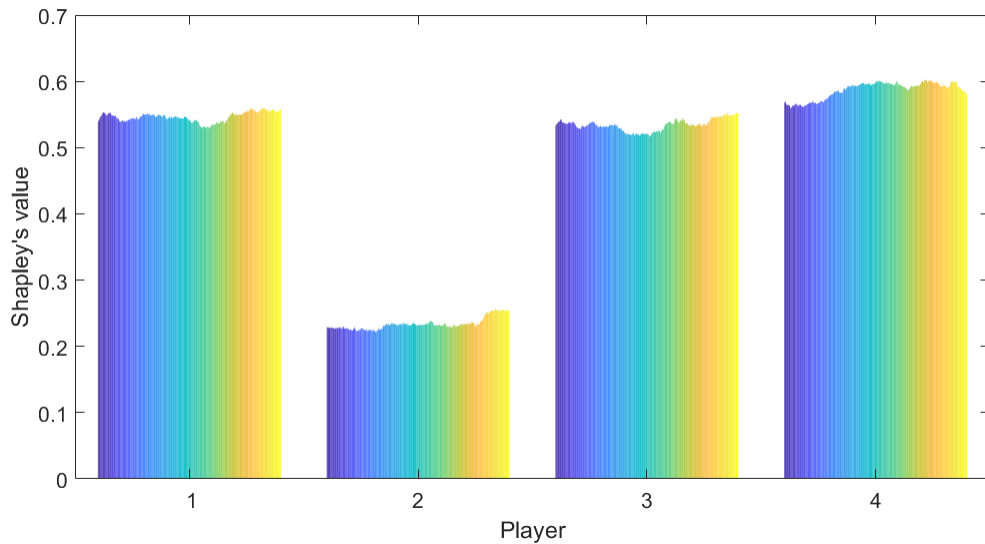


FIG. 4. Evolution of Shapley's value for the four players.

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