Self-Triggered Stochastic MPC for Linear Systems With Disturbances

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Self-Triggered Stochastic MPC for Linear Systems With Disturbances

Zhongqi Sun, Vahab Rostampour, and Ming Cao

***Abstract*—In this letter, we present a self-triggering mechanism for stochastic model predictive control (SMPC) of discrete-time linear systems subject to probabilistic constraints, where the controller and the plant are connected by a shared communication network. The proposed triggering mechanism requires that only one control input is allowed to be transmitted through the network at each triggering instant which is then applied to the plant for several steps afterward. By doing so, communication is effectively reduced both in terms of frequency and total amount. We establish the theoretical result for recursive feasibility in the light of proper reformulation of constraints on the nominal system trajectories, and also provide stability analysis for the proposed self-triggered SMPC. A numerical example illustrates the efficiency of the proposed scheme in reducing the communication as well as ensuring meeting the probabilistic constraints.**

***Index Terms*—Self-triggered control, stochastic model predictive control (SMPC), linear systems.**

I. Introduction

Model predictive control (MPC) has gained considerable attention in the last decades due to its powerful ability to explicitly handle constrained systems [1]. In networked control systems (NCSs), the controllers and the sensors do not need to be implemented on dedicated platforms, but using shared communication networks [2]. Such flexibility has given rise to new challenges in control design, such as how to deal with limited communication bandwidths and energy sources. So it is of great interest to study event-based MPC for the NCSs under constraints in the form of the amount of communication per unit time or in total.

Some developments in event-based MPC are reported in the recent literature [3]–[11], where [3]–[6] focus on the event-triggering schemes and [7]–[11] fall into the self-triggering categories. For uncertain systems, most results take into account the co-design of tightening bounds and triggering conditions aiming at ensuring recursive feasibility under hard constraints. Such methods are often categorized into the robust MPC (RMPC) based approach (see [3], [5], [6], [9]), which, however, may suffer from conservatism problems due to the fact that they have to tolerate the worst-case realization of the system uncertainties [12].

Stochastic MPC (SMPC) aims at exploiting the stochastic nature of the uncertainty which allows constraint violations in prescribed probabilistic manner by reformulating the hard constraints into soft ones leading to lower costs (or better system performances). Interested readers are referred to [13]–[15] and also to [12] and [16] for an overview of SMPC. Note that the studies on event-based SMPC schemes are limited and, to the best of our knowledge, are reported only in the most recent works [17], [18], in which the robust self-triggering approach in [9] is extended to a stochastic setting. This triggering strategy might not necessarily relieve the communication load from the controller side, since a control sequence is required to be transmitted through the network at each triggering instant, while the load of transmitting m different data together at once is not less than transmitting one data a time for m times [11]. Along this line, we propose a self-triggered SMPC scheme for linear systems subject to probabilistic constraints, in which *only one* control input instance in the sequence at each triggering instant is allowed to be transmitted through the network. This may have the potential advantage in reducing the communication load. The aim of reducing the number of transmitting control input is also studied in [19] and [20] which mainly focus on deterministic systems. This letter will deal with this problem in a stochastic setting.

The main contributions of this letter are as follows: 1) a self-triggered SMPC scheme is proposed which reduces the size of the data to be transmitted over the network. This is beneficial for large scale systems on bandwidth constrained and shared networks; 2) a self-triggering optimization problem is reformulated to be computationally tractable while ensuring sub-optimality with a specific level of trade-off; and 3) recursive feasibility and stability are established while guaranteeing a-priori probability of satisfying the constraint.

Notation: $\mathbb{P}\{\xi\}$ denotes the probability of an event ξ, $\mathbb{E}\{\xi\}$ is the expectation and $\mathbb{E}_k\{\xi\} = \mathbb{E}\{\xi|x_k\}$ is the conditional expectation of the event ξ given a random event x_k. Let the triple $(\Delta, \mathfrak{B}(\Delta), \mathbb{P})$ be a probability space where Δ is a metric space associate with the Borel σ-algebra $\mathfrak{B}(\Delta)$, and $\mathbb{P}$ is the probability measure function.

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II. Problem Formulation

Consider the following stochastic linear system:

$$x_{k+1} = Ax_k + Bu_k + w_k,\ \ k \in \mathbb{N}, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, $u_k \in \mathbb{R}^{n_u}$ is the control input, w_k is a realization of an unknown stochastic process defined on some probability space $(\mathcal{W}, \mathfrak{B}(\mathcal{W}), \mathbb{P})$, n_x, n_u and n_w are all positive integers. In addition, the pair of system matrices (A, B) is stabilizable.

Assumption 1: The disturbances $w_k \in \mathcal{W}$ are independent and identically distributed (i.i.d) with zero mean[1] and bounded support $\|w_k\|_\infty \le \bar{w}$.

It is important to note that we do not require the distribution of the disturbances to be known explicitly, as will be explained later. We only need a finite number of realizations of the uncertainty w_k and it is sufficient to assume that they are i.i.d. Consider the following probabilistic constraint on the system trajectories in the form of

$$\mathbb{P}\{g^{\mathrm{T}} x_k \le h\} \ge 1 - \varepsilon,\ \ k \in \mathbb{N}_{\ge 1} \tag{2}$$

where $g^{\mathrm{T}} \in \mathbb{R}^{n_x}$, $h \in \mathbb{R}$ and $\varepsilon \in (0, 1)$ is the given level of constraint violation. The probability measure $\mathbb{P}$ is assigned for the uncertain state x_k, since it is a function of past values of disturbances as in (1). Note that in the case of several probabilistic constraints, one can treat them in a similar way using the worst-case reformulation [21]. Moreover, the control input is usually subject to hard constraints, which can be handled using the standard constraint tightening approach in the robust MPC framework [7]. We omit it here for brevity.

In a periodic time-triggered stochastic MPC setup, given the initial (current) state x_k and a fixed prediction horizon N, the predicted state trajectories can be obtained using

$$x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + w_{i+k}\ ,\ i \in \mathbb{N}_{[0,N-1]}, \tag{3}$$

with the initial condition $x_{0|k} = x_k$. The predicted controller can be designed to be

$$u_{i|k} = Ke_{i|k} + v_{i|k} \tag{4}$$

where $e_{i|k} = x_{i|k} - z_{i|k}$, $z_{i|k} = \mathbb{E}_k\{x_{i|k}\}$ is the nominal state with $z_{0|k} = x_k$, $v_{i|k} \in \mathbb{R}^{n_u}$ is the free decision vector, and K is designed such that $\Phi = A + BK$ is a Schur matrix. By minimizing a generic convex cost function of the predicted states and control inputs, one can obtain an optimal sequence:

$$\boldsymbol{v}_k^* = \{v_{0|k}^*, v_{1|k}^*, \ldots, v_{N-1|k}^*\}. \tag{5}$$

Following the traditional receding horizon principle, at each time step k, $u_k = v_{0|k}^*$ is applied to system (1), noting $e_{0|k} = 0$. In this setup, the state and control information is required to be transmitted through a network at each time instant, which may require more communication.

To address such a shortcoming, an intuitive way is to design a self-triggering mechanism such that the state is only measured and transmitted at triggering instants $k_j \in \mathbb{N}$, $j \in \mathbb{N}$, with $k_{j+1} = k_j + m_j$. Then, as an automated mechanism, the inter-execution time $m_j \in \mathbb{N}_{[1,N]}$ is determined by a self-triggering mechanism at k_j. At each triggering time instant, the predicted control sequence is updated as

$$\boldsymbol{v}_{k_j}^* = \{v_{0|k_j}^*, v_{1|k_j}^*, \ldots, v_{N-1|k_j}^*\} \tag{6}$$

and the first m_j control input instances in $\boldsymbol{v}_{k_j}^*$, i.e., $\{v_{0|k_j}^*, v_{1|k_j}^*, \ldots, v_{m_j-1|k_j}^*\}$ are transmitted to the actuator through the communication network (see [9], [17], [18]). However, this strategy might not really reduce the size of the data to be transmitted over the network from the controller side as we have discussed in Section I.

In this letter, we develop a self-triggered SMPC scheme such that it allows *only* one state measurement and one control input to be transmitted at each triggering instant to reduce the frequency and amount of the communication at both sides. To this end, the following sub-optimal sequence is obtained by maximizing the inter-execution time m_j^*:

$$\boldsymbol{v}_{k_j}^* = \{\underbrace{v_{0|k_j}^*, \ldots, v_{0|k_j}^*}_{m_j^*}, v_{m_j^*|k_j}^*, \ldots, v_{N-1|k_j}^*\} \tag{7}$$

and *only* one control input instance, i.e., $v_{0|k_j}^*$, is required to be sent to the actuator. During the interval $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$, the destabilizing error feedback, i.e., $Ke_{i|k}$, is not applicable to the system, and thus the actual control is given by

$$u_k = v_{0|k_j}^*,\ \ k \in \mathbb{N}_{[k_j,k_{j+1}-1]} \tag{8}$$

in an open-loop fashion. At k_{j+1}, the sensor will be waked up and the inter-execution time and the optimal control sequence will be updated according to the new measurement $x_{k_{j+1}}$.

III. Self-Triggered SMPC

In this section, we first formulate a prototype optimization problem for a *fixed* inter-execution time m in Section III-A and develop the probabilistic constraint handling strategy in Section III-B. The self-triggering optimization problem which provides a maximal inter-execution time m_j^* at each triggering instant k_j is presented in Section III-C.

A. Stochastic Optimization Problem

Consider the aforementioned self-triggering scheme such that the control input predictions are expressed in the following form for $i \in \mathbb{N}_{[0,N-1]}$:

$$u_{i|k_j} = v_{0|k_j},\quad i \in \mathbb{N}_{[0,m-1]} \tag{9a}$$

$$u_{i|k_j} = Ke_{i|k_j} + v_{i|k_j},\quad i \in \mathbb{N}_{[m,N-1]} \tag{9b}$$

and the states evolve according to

$$x_{i|k_j} = z_{i|k_j} + e_{i|k_j},\ e_{0|k_j} = 0\,,\quad i \in \mathbb{N}_{[0,N-1]} \tag{10a}$$

$$z_{i+1|k_j} = Az_{i|k_j} + Bv_{i|k_j},\quad i \in \mathbb{N}_{[0,N-1]} \tag{10b}$$

$$e_{i+1|k_j} = Ae_{i|k_j} + w_{i+k_j},\quad i \in \mathbb{N}_{[0,m-1]} \tag{10c}$$

$$e_{i+1|k_j} = \Phi e_{i|k_j} + w_{i+k_j},\quad i \in \mathbb{N}_{[m,N-1]}. \tag{10d}$$

It is important to highlight that we split the system dynamics into a stochastic part $e_{i|k}$ and a deterministic part $z_{i|k}$, and due to the fact that the nominal state is initialized using the

[1]The zero-mean assumption can be easily relaxed to the case of non-zero-mean processes, provided that they are generated according to a dynamic model fed by a zero-mean random variable.

measured one, i.e., $z_{0|k_j} = x_{k_j}$, the initial value of the stochastic term is zero, i.e., $e_{0|k_j} = 0$. The stochastic term $e_{i|k_j}$ evolves free of control during $i \in \mathbb{N}_{[0,m-1]}$ as shown in (10c), since the feedback is unavailable in the implementing phase of this time period. Define the vector of decision variables to be $\boldsymbol{v}_{k_j} = \{v_{0|k_j}, \ldots, v_{0|k_j}, v_{m_j|k_j}, \ldots, v_{N-1|k_j}\}$ and consider the following cost function for an arbitrary triggering time k_j:

$$\bar{J}(x_{k_j}, \boldsymbol{v}_{k_j}) = \mathbb{E}_{k_j}\left\{\sum_{i=0}^{N-1}\left(\|x_{i|k_j}\|_Q^2 + \|u_{i|k_j}\|_R^2\right) + \|x_{N|k_j}\|_P^2\right\}$$

where $Q \succ 0$, $R \succ 0$, and $P \succ 0$ is determined using the Lyapunov equation $\Phi^{\mathrm{T}}P\Phi + Q + K^{\mathrm{T}}RK = P$.

We are now in a position to formulate the finite-horizon stochastic optimization problem for each triggering time k_j:

$$\bar{\mathcal{P}}^{(m)}(x_{k_j}):\begin{cases} \min\limits_{\boldsymbol{v}_{k_j}} & \bar{J}(x_{k_j}, \boldsymbol{v}_{k_j}) \\ \text{s.t.} & u_{i|k_j} = v_{0|k_j},\ i \in \mathbb{N}_{[1,m-1]} \\ & u_{i|k_j} = v_{i|k_j} + Ke_{i|k_j},\ i \in \mathbb{N}_{[m,N-1]} \\ & x_{i+1|k_j} = Ax_{i|k_j} + Bu_{i|k_j} + w_{i+k_j} \\ & z_{i|k_j} \in \mathbb{Z}_i^{(m)},\ i \in \mathbb{N}_{[1,N-1]} \\ & z_{N|k_j} \in \mathbb{Z}_f^{(m)} \end{cases}$$

where $z_{i|k_j}$ and $e_{i|k_j}$ are defined under Eq. (4), and the sets $\mathbb{Z}_i^{(m)}$ for all $i \in \mathbb{N}_{[1,N-1]}$ and the terminal set $\mathbb{Z}_f^{(m)}$ are designed in Lemma 3 in order to guarantee the probabilistic constraint and achieve recursive feasibility. Consider now the following auxiliary optimization problem:

$$\mathcal{P}^{(m)}(x_{k_j}):\begin{cases} \min\limits_{\boldsymbol{v}_{k_j}} & J(x_{k_j}, \boldsymbol{v}_{k_j}) \\ \text{s.t.} & z_{0|k_j} = x_{k_j} \\ & v_{i|k_j} = v_{0|k_j},\ i \in \mathbb{N}_{[1,m-1]} \\ & z_{i+1|k_j} = Az_{i|k_j} + Bv_{i|k_j} \\ & z_{i|k_j} \in \mathbb{Z}_i^{(m)},\ i \in \mathbb{N}_{[1,N-1]} \\ & z_{N|k_j} \in \mathbb{Z}_f^{(m)} \end{cases}$$

where the cost function is only evaluated for the nominal state variable as follows:

$$J(x_{k_j}, \boldsymbol{v}_{k_j}) = \sum_{i=0}^{N-1}\left(\|z_{i|k_j}\|_Q^2 + \|v_{i|k_j}\|_R^2\right) + \|z_{N|k_j}\|_P^2.$$

The following lemma provides a connection between $\bar{\mathcal{P}}^{(m)}(x_{k_j})$ and $\mathcal{P}^{(m)}(x_{k_j})$.

Lemma 1: Given the initial state x_{k_j}, $\bar{\mathcal{P}}^{(m)}(x_{k_j})$ and $\mathcal{P}^{(m)}(x_{k_j})$ have a same feasible set. In addition, if they both are feasible, the minimizers of the two problems coincide.

Proof: First, we compare their cost functions by substituting the predicted control sequence and the state sequence as in (9) and (10), respectively, to the cost function $\bar{J}(x_{k_j}, \boldsymbol{v}_{k_j})$ as follows:

$$\bar{J}(x_{k_j}, \boldsymbol{v}_{k_j}) = \mathbb{E}_{k_j}\left\{\sum_{i=0}^{m-1}\left(\|z_{i|k_j} + e_{i|k_j}\|_Q^2 + \|v_{i|k_j}\|_R^2\right) + \sum_{i=m}^{N-1}\left(\|z_{i|k_j} + e_{i|k_j}\|_Q^2 + \|Ke_{i|k_j} + v_{i|k_j}\|_R^2\right) + \|z_{N|k_j} + e_{N|k_j}\|_P^2\right\}$$

where $e_{i|k_j} = \sum_{s=0}^{i-1} A^s w_{s+k_j}$ for $i \in \mathbb{N}_{[1,m]}$ and $e_{i|k_j} = \Phi^{i-m}\sum_{\ell=0}^{m-1} A^\ell w_{\ell+k_j} + \sum_{\ell=0}^{i-m-1}\Phi^\ell w_{m+\ell|k_j}$ for $i \in \mathbb{N}_{[m+1,N]}$. Since it is assumed that $\mathbb{E}_{k_j}\{e_{i|k_j}\} = 0$, $\forall i \in \mathbb{N}_{[0,N]}$, one can obtain

$$\bar{J}(x_{k_j}, \boldsymbol{v}_{k_j}) = \sum_{i=0}^{N-1}\left(\|z_{i|k_j}\|_Q^2 + \|v_{i|k_j}\|_R^2\right) + \|z_{N|k_j}\|_P^2 + \mathrm{c}$$

where $\mathrm{c} = \mathbb{E}_{k_j}\{\sum_{i=0}^{m-1}\|e_{i|k_j}\|_Q^2 + \sum_{i=m}^{N-1}\|e_{i|k_j}\|_{Q+K^{\mathrm{T}}PK}^2 + \|e_{N|k_j}\|_P^2\}$ is a constant term and can be neglected in the optimization. Note that in the last statement only the nominal states and inputs are involved. This highlights the fact that only the evolution of the nominal state in $\bar{\mathcal{P}}^{(m)}(x_{k_j})$ needs to be considered. It is straightforward to check that $\bar{\mathcal{P}}^{(m)}(x_{k_j})$ and $\mathcal{P}^{(m)}(x_{k_j})$ have a same feasible set by simply comparing their constraints, which are exactly the same. Moreover the optimizer of $\bar{\mathcal{P}}^{(m)}(x_{k_j})$ is a solution of $\mathcal{P}^{(m)}(z_{k_j})$ and vice versa. ■

B. Constraints Handling

To handle the probabilistic constraint (2), we now present an approach to convert it to a computationally solvable and equivalent deterministic constraint, while ensuring the probability of constraint fulfillment.

Lemma 2: Given the inter-execution time $m \in \mathbb{N}_{[1,N]}$ for an arbitrary triggering instant k_j, the predicted state satisfies the probabilistic constraint, i.e., $\mathbb{P}\{g^{\mathrm{T}}x_{i|k_j} \leq h\} \geq 1-\varepsilon$, if and only if its expectations $z_{i|k_j}$, $i \in \mathbb{N}_{[1,N]}$, conditioning on $z_{0|k_j} = x_{k_j}$ satisfy

$$g^{\mathrm{T}}z_{i|k_j} \leq \eta_i^{(m)}, \qquad i \in \mathbb{N}_{[1,N]}, \tag{11}$$

where $\eta_i^{(m)}$ is defined by

$$\eta_i^{(m)} = \max_{\eta} \quad \eta$$
$$\text{s.t.}\begin{cases} \mathbb{P}\left\{g^{\mathrm{T}}\sum\limits_{s=0}^{i-1}A^s w_{s+k_j} \leq h-\eta\right\} \geq 1-\varepsilon,\ i \in \mathbb{N}_{[1,m]} \\ \mathbb{P}\left\{g^{\mathrm{T}}\Phi^{i-m}\sum\limits_{\ell=0}^{m-1}A^\ell w_{\ell+k_j} + g^{\mathrm{T}}\sum\limits_{\ell=0}^{i-m-1}\Phi^\ell w_{m+\ell|k_j} \leq h-\eta\right\} \geq 1-\varepsilon,\ i \in \mathbb{N}_{[m+1,N]}. \end{cases}$$

Proof: Following Eqs. (10c) and (10d), one can derive the error caused by the uncertainty on the predicted state as $e_{i|k_j} = \sum_{s=0}^{i-1}A^s w_{s+k_j}$ for an arbitrary triggering instant k_j, for $i \in \mathbb{N}_{[1,m]}$, and $e_{i|k_j} = \Phi^{i-m}\sum_{\ell=0}^{m-1}A^\ell w_{\ell+k_j} + \sum_{\ell=0}^{i-m-1}\Phi^\ell w_{m+\ell|k_j}$ for $i \in \mathbb{N}_{[m+1,N]}$. Taking into consideration the fact that $x_{i|k_j} = z_{i|k_j} + e_{i|k_j}$, it is straightforward to observe the equivalence relation between the probabilistic constraint (2) and the aforesaid assertion. ■

Lemma 2 is adapted to represent the probabilistic constraint (2) under the self-triggering mechanism in an equivalent manner. Since we have assumed that the probability density is not known explicitly, we introduce a sampling technique to determine $\eta_i^{(m)}$ in Lemma 2. This problem can be efficiently solved respecting a given confidence $1-\beta$ by drawing a sufficiently large number of N_s samples from $\mathcal{W}$, which can be obtained using off-line sampling of the uncertainties [22] or from a sample generator model built using historical data [23]. The detailed method to determine $\eta_i^{(m)}$ is the same as [15, Proposition 5] and is omitted here for space limitation.

In the following lemma, we design the sets $\mathbb{Z}_i^{(m)}$, $i \in \mathbb{N}_{[1,N-1]}$, and $\mathbb{Z}_f^{(m)}$ to guarantee the existence of at least one feasible solution at each triggering instant.

Lemma 3: Given the inter-execution time $m \in \mathbb{N}_{[1,N]}$ for an arbitrary triggering instant k_j, and $\mathbb{Z}_i^{(m)}$, $i \in \mathbb{N}_{[1,N-1]}$, $\mathbb{Z}_f^{(m)}$ are designed as follows:

$$\mathbb{Z}_i^{(m)} = \{z \in \mathbb{R}^{n_x} | g^{\mathrm{T}} z \leq \beta_i^{(m)}, \quad i \in \mathbb{N}_{[1,N-1]}\} \tag{12}$$

$$\mathbb{Z}_f^{(m)} = \{z \in \mathbb{R}^{n_x} | g^{\mathrm{T}} \Phi^{\ell} z \leq \beta_{N+\ell}^{(m)}, \quad \ell \in \mathbb{N}_{\geq 1}\} \tag{13}$$

where

$$\beta_i^{(m)} = \begin{cases} \eta_i^{(m)}, & i \in \mathbb{N}_{[1,m]} \\ \eta_1^{(m)} - b_i^{(m)} - \sum_{\ell=m+2}^{i} d_\ell^{(m)}, & i \in \mathbb{N}_{\geq m+1} \end{cases} \tag{14}$$

with $b_i^{(m)} = \max_{w_s \in \mathcal{W}} g^{\mathrm{T}} \Phi^{i-m} \sum_{s=0}^{m-1} A^s w_s$ and $d_\ell^{(m)} = \max_{w_\ell \in \mathcal{W}} g^{\mathrm{T}} \Phi^{\ell-m-1} w$. Then there exists at least one feasible solution satisfying the state constraint for $m = 1$ at k_{j+1}, provided $\mathcal{P}^{(m)}(x_{k_j})$ is feasible.

Proof: Suppose at k_j an optimal solution is found denoted by $\boldsymbol{v}_{k_j}^* = \{v_{0|k_j}^*, \ldots, v_{0|k_j}^*, v_{m_j|k_j}^*, \ldots, v_{N-1|k_j}^*\}$ and its corresponding nominal states sequence is given by $\boldsymbol{z}_{k_j}^* = \{z_{1|k_j}^*, z_{2|k_j}^*, \ldots, z_{N|k_j}^*\}$, and the control instance $v_{0|k_j}^*$ is applied to system (1) for the next m steps. At the next triggering instant k_{j+1}, the error can be determined by $x_{k_{j+1}} - z_{k_{j+1}|k_j}^* = \sum_{s=0}^{m-1} A^s w_{k_j+s}$ and the nominal system is initialized by $z_{0|k_{j+1}} = x_{k_{j+1}}$. Then a candidate solution $\tilde{\boldsymbol{v}}_{k_{j+1}}$ to $\mathcal{P}^{(1)}(x_{k_{j+1}})$ is defined by

$$\tilde{v}_{i|k_{j+1}} = \begin{cases} v_{m+i|k_j}^* + K\Phi^i \sum_{s=0}^{m-1} A^s w_{k_j+s}, & i \in \mathbb{N}_{[0,N-m-1]} \\ K\tilde{z}_{i|k_{j+1}}, & i \in \mathbb{N}_{[N-m,N-1]}. \end{cases}$$

We now examine the feasibility of state trajectories corresponding to the above possibly feasible solution. First consider the interval $i \in \mathbb{N}_{[1,N-m]}$, the state trajectory is

$$\tilde{z}_{i|k_{j+1}} = z_{m+i|k_j}^* + \Phi^i \sum_{s=0}^{m-1} A^s w_{k_j+s}\,.$$

By considering the constraint satisfaction of $z_{m+i|k_j}^*$, one has

$$g^{\mathrm{T}} \tilde{z}_{i|k_{j+1}} \leq \eta_1^{(m)} - \sum_{\ell=m+2}^{i} d_\ell^{(m)} = \beta_i^{(1)}$$

where $\beta_i^{(1)}$ is defined in accordance with (14). This implies $\tilde{z}_{i|k_{j+1}} \in \mathbb{Z}_i^{(1)}$, $i \in \mathbb{N}_{[0,N-m-1]}$.

As for the next interval $i \in \mathbb{N}_{[N-m+1,N-1]}$, the feasible solution is given by $\tilde{v}_{i|k_{j+1}} = K\tilde{z}_{i|k_{j+1}}$. The corresponding states under this control sequence is

$$\tilde{z}_{i|k_{j+1}} = \Phi^{i-N+m} \left(z_{N|k_j}^* + \Phi^{N-m} \sum_{s=0}^{m-1} A^s w_{k_j+s} \right)$$

where $z_{N|k_j}^*$ should satisfy the terminal constraint. We now substitute it to evaluate its feasibility as follows:

$$g^{\mathrm{T}} \tilde{z}_{i|k_{j+1}} \leq \eta_1^{(1)} - b_i^{(1)} - \sum_{\ell=3}^{i} d_\ell^{(1)} = \beta_i^{(1)}\,.$$

This yields $\tilde{z}_{i|k_{j+1}} \in \mathbb{Z}_i^{(1)}$ for $i \in \mathbb{N}_{[N-m+1,N-1]}$. Similarly, one can check the terminal constraint $g^{\mathrm{T}} \Phi^{\ell} \tilde{z}_{N|k_{j+1}} \leq \beta_{N+\ell}^{(1)}$.

Note that following Lemma 2 the probabilistic constraints are ensured probabilistically for $i \in \mathbb{N}_{[1,m]}$, and we have $\beta_i^{(m)} \leq \eta_i^{(m)}$ which implies that the probabilistic constraint is also guaranteed for $i \in \mathbb{N}_{[m+1,N]}$. ■

Remark 1: For feasibility, we employ constraint tightening approach using a mixture of the worst-case recursive and stochastic trajectory prediction for constraint tightening inspired by [13] and [17]. This approach is the extension of the method for robust MPC [24], which utilizes worst-case prediction. Therefore, the stochastic MPC is slightly less conservative than the robust MPC.

C. Resulting Self-Triggered SMPC Scheme

We are now ready to present the self-triggered SMPC strategy using the optimization problem $\mathcal{P}^{(m)}(x_{k_j})$. At each triggering instant k_j, the goal is to determine the next triggering instant k_{j+1} by maximizing the inter-execution time m_j and find a sub-optimal sequence $\boldsymbol{v}_{k_j}^*$.

With a slight abuse of notation,[2] let $J^{(m)}(x_{k_j}, \boldsymbol{v}_{k_j}^*)$ be the optimal cost of $\mathcal{P}^{(m)}(x_{k_j})$ and $\bar{m}$ the upper bound of the open-loop phase. The self-triggering optimization problem is formulated by

$$\mathcal{SP}(x_{k_j}) \colon \begin{cases} \max\limits_{m \in \mathbb{N}_{[1,\bar{m}]}} \quad m \\ \quad \text{s.t. } J^{(m)}(x_{k_j}, \boldsymbol{v}_{k_j}^*) \leq J^{(1)}(x_{k_j}, \boldsymbol{v}_{k_j}^*) + \alpha \rho_{k_{j-1}} \end{cases}$$

where $\rho_{k_{j-1}} = (\|x_{k_{j-1}}\|_Q^2 + \|u_{k_{j-1}}\|_R^2)$, and $\alpha \in \mathbb{R}_{[0,1)}$ is tuned to reach a level of trade-off between the optimality and the amount of communication. Letting m_j^* with $m_0^* = 1$ be the maximum inter-execution time at k_j, the sequence $\boldsymbol{v}_{k_j}^*$ is then given by

$$\boldsymbol{v}_{k_j}^* = \arg\min\ J^{(m_j^*)}(x_{k_j}, \boldsymbol{v}_{k_j}).$$

The self-triggered SMPC scheme is summarized in Algorithm 1. Its main properties are stated in Theorems 1 and 2.

Theorem 1 (Recursive Feasibility): If $\mathcal{SP}(x_{k_j})$ is feasible at the initial time instant, then the proposed self-triggered SMPC in Algorithm 1 for system (1) is recursively feasible.

Proof: Suppose a solution to $\mathcal{SP}(x_{k_j})$ is found at k_j and $v_{0|k_j}^*$ is applied to the system for m_j^* steps. At k_{j+1}, $\tilde{m}_{j+1} = 1$ and the sequence presented in the proof of Lemma 3 can be verified to be a feasible solution. By induction, recursive feasibility of the self-triggered optimization problem is ensured. ■

[2]Note that the optimizer $\boldsymbol{v}_{k_j}^*$ to $J^{(m)}(x_{k_j}, \boldsymbol{v}_{k_j}^*)$ might be different from that to $J^{(n)}(x_{k_j}, \boldsymbol{v}_{k_j}^*)$ for $m, n \in \mathbb{N}_{[1,\bar{m}]}$ and $m \neq n$.

Algorithm 1 Self-Triggered SMPC Scheme

1: Initialize $\mathcal{P}^{(m)}(x_{k_j})$ with $z_{0|k_j} = x_{k_j}$;
2: Obtain the inter-execution time m_j^* and the optimal sequence $\mathbf{v}_{k_j}^*$ by solving $\mathcal{SP}(x_{k_j})$;
3: Send only the first entry of $\mathbf{v}_{k_j}^*$, i.e., $v_{0|k_j}^*$, to the actuator and apply this control action to the system for m_j^* steps;
4: At k_{j+1}, wake up the sensor to transmit a new measurement $x_{k_{j+1}}$ and go to step 1.

Theorem 2 (Stability Property): The closed-loop system under Algorithm 1 satisfies the following mean-square convergence performance

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\Big\{\|x_{k_j}\|_Q^2 + \|u_{k_j}\|_R^2\Big\} \le \frac{1}{1-\alpha}\gamma \tag{15}$$

where $\gamma = \mathbb{E}\{\sum_{s=0}^{\bar{m}-1} \|A^s w_s\|_P^2\}$.

Proof: Let $V(x_{k_j}) = J^{(m_j^*)}(x_{k_j}, \mathbf{v}_{k_j}^*)$ be the stochastic Lyapunov function and $\tilde{J}^{(1)}(x_{k_j}, \tilde{\mathbf{v}}_{k_j})$ the value function associated with the potential feasible solution presented in the proof of Lemma 3. By solving the self-triggering problem $\mathcal{SP}(x_{k_j})$, it holds for the closed-loop system that

$$\begin{aligned}
&\mathbb{E}_{k_j}\Big\{V(x_{k_{j+1}})\Big\} - V(x_{k_j})\\
&= \mathbb{E}_{k_j}\Big\{J^{(m_{j+1})}(x_{k_{j+1}}, \mathbf{v}_{k_{j+1}}^*)\Big\} - J^{(m_j^*)}(x_{k_j}, \mathbf{v}_{k_j}^*)\\
&\le \mathbb{E}_{k_j}\Big\{\tilde{J}^{(1)}(x_{k_{j+1}}, \tilde{\mathbf{v}}_{k_{j+1}})\Big\} + \alpha(\|x_{k_j}\|_Q^2 + \|u_{k_j}\|_R^2)\\
&\quad - J^{(m_j^*)}(x_{k_j}, \mathbf{v}_{k_j}^*).
\end{aligned} \tag{16}$$

First consider the first term $\mathbb{E}_{k_j}\{\tilde{J}^{(1)}(x_{k_{j+1}}, \tilde{\mathbf{v}}_{k_{j+1}})\}$, which can be rewritten as

$$\begin{aligned}
&\mathbb{E}_{k_j}\Big\{\tilde{J}^{(1)}(x_{k_{j+1}}, \tilde{\mathbf{v}}_{k_{j+1}})\Big\} = \sum_{i=m_j^*}^{N-1}\Big(\|z_{i|k_j}^*\|_Q^2 + \|v_{i|k_j}^*\|_R^2\Big)\\
&+ \sum_{i=N}^{N+m_j^*-1} \|\Phi^{i-N} z_{N|k_j}^*\|_{Q+K^\mathrm{T}RK}^2 + \|\Phi^{m_j^*} z_{N|k_j}^*\|_P^2\\
&+ \mathbb{E}_{k_j}\left\{\sum_{i=0}^{N-1} \|\Phi^i e_{k_{j+1}|k_j}\|_{Q+K^\mathrm{T}RK}^2 + \|\Phi^N e_{k_{j+1}|k_j}\|_P^2\right\}.
\end{aligned} \tag{17}$$

From the fact $\Phi^\mathrm{T} P\Phi + Q + K^\mathrm{T} RK = P$, one has the relations $\|\Phi^i e_{k_{j+1}|k_j}\|_{Q+K^\mathrm{T}RK}^2 + \|\Phi^{i+1} e_{k_{j+1}|k_j}\|_P^2 = \|\Phi^i e_{k_{j+1}|k_j}\|_P^2$ for $i \in \mathbb{N}_{[0,N-1]}$, which leads to $\mathbb{E}_{k_j}\{\sum_{i=0}^{N-1} \|\Phi^i e_{k_{j+1}|k_j}\|_{Q+K^\mathrm{T}RK}^2 + \|\Phi^N e_{k_{j+1}|k_j}\|_P^2\} = \mathbb{E}_{k_j}\{\|e_{k_{j+1}|k_j}\|_P^2\}$. Substituting this result and (17) into (16), summing up and taking expectation on both sides for $j \in \mathbb{N}_{[0,n-1]}$ result in

$$\begin{aligned}
&(1-\alpha)\sum_{j=0}^{n-1}\mathbb{E}\Big(\|x_{k_j}\|_Q^2 + \|u_{k_j}\|_R^2\Big) - n\mathbb{E}\left\{\sum_{s=0}^{\bar{m}-1}\|A^s w_s\|_P^2\right\}\\
&\le \mathbb{E}\{V(x_0)\} - \mathbb{E}\{V(x_{k_n})\}.
\end{aligned} \tag{18}$$

Since the right hand of (18) is bounded, it is straightforward that (15) holds. This completes the proof. ∎

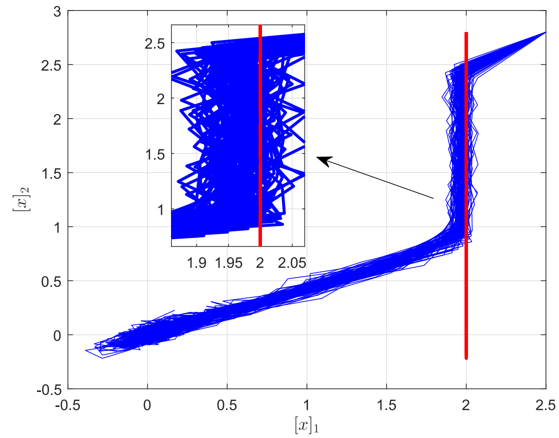

Fig. 1. State trajectories by SMPC.

Remark 2: Since this letter and [17] use different triggering mechanisms, prediction control laws and horizon settings, the analyses in the stability results are different. Moreover, the convergence result in this letter depends on the parameter α which shows the trade-off between the convergence performance and the amount of communication.

IV. Numerical Study

Consider the following stochastic linear system studied in [15], [17]:

$$x_{k+1} = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u_k + w_k.$$

Let the system state be subject to a probabilistic constraint $\mathbb{P}\{[\,1\quad 0\,]x_k \le 2\} > 1 - 0.2$, and assume that the disturbances w_k are truncated Gaussian random variables with zero mean and covariance $\Sigma = 0.004^2 I_2$ truncated at $\|w\|^2 \le 0.02$. In the cost function, the weight matrices are set to be $Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$, $R = 1$ and $P = \begin{bmatrix} 1.5 & -2.87 \\ -2.87 & 26.9 \end{bmatrix}$. The disturbance attenuation in the predictions is chosen as the unconstrained LQR with $K = [\,0.28\quad 0.49\,]$. The prediction horizon is $N = 10$ and the simulation time is set to be $N_{\text{sim}} = 20$ steps. In the self-triggering optimization problem, we set the maximum open-loop phase as $\bar{m} = 3$ and the sub-optimal factor $\alpha = 0.02$.

To show the efficiency of the proposed scheme, we compare the results with a periodic time-triggered SMPC. For brevity, we will use SMPC to stand for the periodic time-triggered SMPC and STSMPC to represent the proposed self-triggered SMPC. Fig. 1 and Fig. 2 present 100 realizations of the closed-loop trajectories obtained via SMPC and STSMPC, respectively. The a-postoriori probability of constraint violation in the first 6 steps is 19.5% by SMPC, and 7.2% by STSMPC. It is also worth mentioning that the convergence performance deteriorates when we use the self-triggered SMPC, which is attributed to the self-triggering mechanism. To further study the convergence performance, we compute

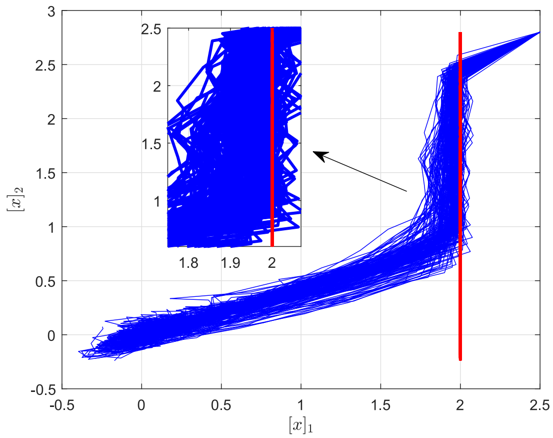

Fig. 2. State trajectories by STSMPC.

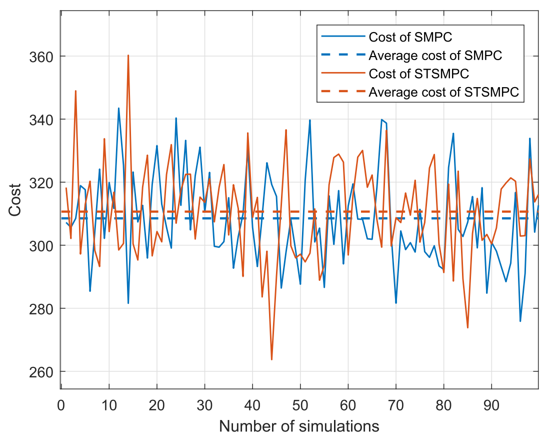

Fig. 3. Comparison of the measured costs by SMPC and STSMPC.

the cost measure in each simulation, i.e.,

$$J_{\text{meas}} = \sum_{i=0}^{N_{\text{sim}}} \left(\|x_{i|k_j}\|_Q^2 + \|u_{i|k_j}\|_R^2 \right) \tag{19}$$

and its average of 100 Monte Carlo simulations. The comparison result in Fig. 3 shows that the measured cost of STSMPC is slightly higher than that of SMPC. This slightly worse performance is justified by the reduction of communication, which coincides with the theoretical analysis. To show the reduction of communication, we counted the numbers of open-loop phases m_j^* among the 100 simulations. There are 342, 105 and 462 times for $m_j^* = 1$, $m_j^* = 2$ and $m_j^* = 3$, respectively, which implies that around 45% of communication is saved.

V. Conclusion

In this letter, a self-triggered stochastic MPC scheme for stochastic linear systems subject to probabilistic constraints has been presented. The proposed scheme has achieved the reduction of the communication load and controller update rate. By taking into consideration the disturbance during the open-loop phase, a set of tightening constraints are enforced on the nominal states enabling to have a new computationally tractable problem formulation. Recursive feasibility and mean-square stability analysis have been provided together with the simulation results to show the efficacy of the proposed self-triggered stochastic MPC scheme.

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