# Structural Accessibility and Its Applications to Complex Networks Governed by Nonlinear Balance Equations

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# Structural Accessibility and Its Applications to Complex Networks Governed by Nonlinear Balance Equations

Yu Kawano, *Member*, and Ming Cao, *Senior Member*

***Abstract*—We define and then study the structural controllability and observability for a class of complex networks whose dynamics are governed by the nonlinear balance equations. Although related notions of observability for such complex networks have been studied before and in particular, necessary conditions have been reported to select sensor nodes in order to render such a given network observable, there still remain various challenging open problems, especially from the systems and control point of view. The reason is partly that driver and sensor node selection problems for *nonlinear* complex networks have not been studied systematically, which differs greatly from the relatively comprehensive mathematical development for the linear counterpart. In this paper, based on our refined notions of structural controllability and observability, we construct their necessary conditions for nonlinear complex networks, which are further applied to those networks governed by nonlinear balance equations in order to develop a systematic driver node selection method. Furthermore, we establish a connection between our necessary conditions for structural observability and the conventional sensor node selection method.**

***Index Terms*—Complex networks, nonlinear systems, structural controllability, driver node identification**

## I. Introduction

The challenge of controllability/observability analysis of complex networks is that usually only the existence of couplings between certain pairs of nodes in the networks is known while it is difficult to delineate the dynamical interactions between the related coupled nodes. So the resulting models of complex networks are usually in the form of simplified dynamical processes evolving on graphs. Therefore, one needs to analyze controllability and observability of complex networks from a graphical point of view, which differs significantly from standard analysis for classical dynamical system models. In the linear case, such controllability has been referred to as structural controllability and fully characterized by the concept of the "cactus" of the graph of the complex network [1], [2]. Then, by using the cactus, the paper [3] demonstrated the applicability of structural controllability for understanding the influential nodes of real-life networks, e.g. regulatory and social communication networks.

Motivated by this pioneering work, structural controllability analysis and driver node identification for complex networks have attracted significant attentions from researchers in different fields in the past few years. For example, in biology and chemistry, the concept of structural controllability is employed as a key tool for understanding critical underlying mechanisms or relations of real-life biological and chemical networks [4]–[8]. It is mentioned in [5] that inorganic-organic hybrid materials are more structurally controllable than purely inorganic compounds owing to organic components, and have potential for the construction of functionalized crystalline materials such as molecular conductors. These papers deal with exclusively linear complex networks, and only a few papers [9]–[11] have tried to study nonlinear complex networks. Obviously, nonlinearity is an intrinsic feature of various dynamical processes evolving on real complex networks, such as power flow in energy grids, epidemic processes in human groups [12], gene regulation in multicellular organisms [13], birth-death processes in large populations [14], and oscillations in coupled nonlinear oscillators [15].

For nonlinear complex networks, in particular those governed by balance equations, a sensor node identification method has been presented by [9] and improved in [10]. The method is based on the concept of strongly connected components (SCCs) [9], [16] of the inference diagram of a network. According to [9], if one chooses at least one node from every leaf SCC as a sensor node, then almost all of the balance equations become structurally observable, i.e., a sufficient sensor set is obtained. To further characterize a sufficient sensor set, the analysis in [10] utilizes in addition the property of network symmetry. However, a complete characterization of a sufficient sensor set has not been obtained.

One of the difficulties comes from the fact that there is no systematic analysis method for structurally observability for nonlinear complex networks. It is worth mentioning that in contrast to traditional nonlinear properties such as local accessibility, observability [17], and stability [18], linearization techniques at an equilibrium point cannot be utilized here because the method in [9] fully depends on nonlinearity of balance equations. Moreover, the computation of an equilibrium point itself can be challenging since it can be a function of the unknown parameters representing the couplings between the nodes. In contrast to sensor node identification, driver node identification of the nonlinear complex network has not been studied, and duality between controllability and observability is not very clear yet.

In this paper, we establish systematic controllability and observability analysis methods for nonlinear complex net-

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works. First, we formulate a class of nonlinear systems, which contains the complex networks governed by balance equations studied in [9]. Then, motivated by a concept of structural controllability for linear systems [19] that is different in nature compared to those in [1], [2], we define the concept of structural accessibility and observability for nonlinear complex networks as nonlinear systems having unknown parameters. Next, we present two corresponding necessary conditions: for structural observability, because of specific structures of nonlinear balance equations, one of the necessary conditions is linked to the conventional sensor node selection method in [9]; the other condition can help to analyze the case when the conventional method does not render a sufficient sensor set.

Moreover, we establish a drive node selection method based on a structural accessibility condition. Our results on structural accessibility and observability can be viewed as the dual of each other. However, there are differences in contrast to the linear case. For instance, we can compute lower bounds on driver and sensor nodes based on our necessary conditions. For sensor node selection, the lower bound can be less than the one obtained by subtracting the number of parameters from the number of nodes of the complex network; in comparison, for driver node selection the lower bound can be more than that. This implies that a simple modification of the existing sensor node selection method to select driver nodes cannot give a sufficient driver set if there is a huge gap between the numbers of parameters and nodes in contrast to sensor node selection. This insight can only be gained after our development of systematic analysis.

As mentioned in [9], necessary conditions can help to narrow the candidates of variables that are essential for inhibiting/monitoring some chemical/biological reactions. At least, our structural accessibility analysis can be used to a priori check whether an experimental setups satisfy our constructed necessary conditions. As another possible situation, in chemical reaction networks, directly controlling some specific chemical species may be too difficult or expensive due to technical reasons, but to fully control such networks, these species may be needed to be driver nodes. By identifying *necessary* sets of driver nodes, one may identify such challenging scenario well ahead of real experiments.

Preliminary results for structural observability analysis and sensor node identification have been presented in [20], where however, structural accessibility analysis or driver node identification has not been studied. Some of the results on structural controllability analysis including Theorem 3.2 of this paper do not directly follow from the results on structural observability analysis. Furthermore, as explained, driver and sensor node identifications have different features. Also, in contrast to the preliminary conference version, we provide the complete proofs in this paper.

The remainder of this paper is organized as follows. Section II formulates problems addressed in this paper. Section III gives two necessary conditions for structural accessibility and establishes a driver node selection method based on one of the necessary conditions. Section IV studies the conventional graphical sensor node selection method systematically. Section V demonstrates our methods by a chemical reaction network. Finally, Section VI summarizes this paper.

## II. Problem Formulation

### A. System Description

Let $r_\alpha$, $r_\beta$, and $r_\gamma$ be given positive integers, and define the sets $\sigma_\alpha := \{1, 2, \dots, r_\alpha\}$, $\sigma_\beta := \{1, \dots, r_\beta\}$, and $\sigma_\gamma := \{1, \dots, r_\gamma\}$. In this paper, we consider nonlinear complex networks including those associated with balance equations [9], whose dynamics are governed by the following nonlinear system with unknown parameters $\alpha_i$, $i \in \sigma_\alpha$,

$$\Sigma : \dot{x} = f(x) + \sum_{i \in \sigma_\alpha} \alpha_i h_i(x) p_i,$$

where $x \in \mathbb{R}^n$ denotes the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth vector field, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \sigma_\alpha$, are given smooth functions, and $p_i \in \mathbb{R}^n$, $i \in \sigma_\alpha$, are given vectors.

It is emphasized that the class of system $\Sigma$ is not restrictive. Since $h_i(x)$ is an arbitrary smooth function, various nonlinear functions can be represented by using combinations of $h_i(x)$. Indeed, not only balance equations, but also plenty of nonlinear systems and networks can be represented in the form $\Sigma$, such as nonlinear mechanical systems, electrical circuits, coupled oscillators [15], Lorenz equations [21], epidemic processes [12], gene regulation [13], and birth-death processes [14]. For these networks, usually only the existence of couplings between certain pairs of nodes is known while it is difficult to delineate exactly the dynamical interactions between the related coupled nodes. These interactions are represented by the unknown parameters $\alpha_i$.

*Remark 2.1:* For these nonlinear systems and networks, the number of unknown parameters is determined by the model. However, there might be several choices of $f(x)$, $h_i(x)$, and $p_i$ in the form $\Sigma$. This choice depends on the choices of parameters $\alpha_i$. Instead of $\alpha_i$, $\bar{\alpha}_i := \alpha_i + c_i$ can be chosen as a parameter for any constant $c \in \mathbb{R}$. More generally, denote $\alpha := [\alpha_1 \ \dots \ \alpha_{r_\alpha}]^{\mathrm{T}}$. Then, for any non-singular constant matrix $G \in \mathbb{R}^{r_\alpha \times r_\alpha}$ and constant vector $c \in \mathbb{R}^{r_\alpha}$, the set of the elements of $\bar{\alpha} = G\alpha + c$ can be chosen as a set of parameters. However, all results obtained in this paper do not depend on the choice of parameters $\alpha_i$.

Nonlinear balance equations [9] can be represented in the form $\Sigma$ with $f = 0$ and has the following specific relationship between $h_i(x)$ and $p_i$.

*Assumption 2.2:* Let $f = 0$. If $\partial h_i(x)/\partial x_j \neq 0$, $i \in \sigma_\alpha$ for some $j$, then the $j$th element of $p_i$ is not zero.

This structure is utilized when we establish the connection between one of our necessary conditions and the inference diagram. However, for system $\Sigma$ itself, we do not assume this relationship.

In this paper, our objective is studying driver/sensor node selection through the controllability/observability analysis. For controllability analysis, we use the system $\Sigma$ with constant input vector fields including unknown parameters $\beta_j$, $j \in \sigma_\beta$,

$$\Sigma_A : \dot{x} = f(x) + \sum_{i \in \sigma_\alpha} \alpha_i h_i(x) p_i + \left( B + \sum_{j \in \sigma_\beta} \beta_j p_{b_j} q_{b_j}^{\mathrm{T}} \right) u,$$

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where $u \in \mathbb{R}^m$ denotes the input, $B \in \mathbb{R}^{n\times m}$ is a given matrix, $p_{b_j} \in \mathbb{R}^n$ and $q_{b_j} \in \mathbb{R}^m$, $j \in \sigma_\beta$, are given vectors.

For observability analysis, we use the system $\Sigma$ with linear output functions including unknown parameters $\gamma_k$, $k \in \sigma_\gamma$,

$$y = \left(C + \sum_{k\in\sigma_\gamma} \gamma_k p_{c_k} q_{c_k}^{\mathrm{T}}\right)x,$$

where $y \in \mathbb{R}^p$ denotes the output, $C \in \mathbb{R}^{p\times n}$ is a given matrix, $p_{c_k} \in \mathbb{R}^p$ and $q_{c_k} \in \mathbb{R}^n$, $k \in \sigma_\gamma$, are given vectors. The system $\Sigma$ with this output is simply denoted by $\Sigma_O$. Note that we have slightly abused the notation, since subscripts $b_j$ and $c_k$ of $\Sigma_A$ and $\Sigma_O$ represent indexes.

In systems $\Sigma_A$ and $\Sigma_O$, there are parameters $\beta_j$ and $\gamma_k$. For driver and sensor node selections, these parameters correspond to the sets of driver and sensor nodes; for more details see Sections III-B and IV below. As mentioned in Remark 2.1, our results do not depend on the choice of parameters $\beta_j$ and $\gamma_k$. When studying these selection problems, the input or output node is directly controlled or measured. Therefore, linear input vector fields and linear outputs are enough, and the following assumptions are satisfied.

*Assumption 2.3:* Let $B = 0$. If $x_i$ is the driver node corresponding to $u_j$, then the $i$th element of $p_{b_j} \in \mathbb{R}^n$ and the $j$th element of $q_{b_j} \in \mathbb{R}^m$ are chosen as 1, and the others are chosen as 0, i.e., the $(i, j)$ element of $p_{b_j} q_{b_j}^{\mathrm{T}}$ is 1; 0 otherwise.

*Assumption 2.4:* Let $C = 0$. If $x_i$ is the sensor node corresponding to $y_k$, then the $k$th element of $p_{c_k} \in \mathbb{R}^p$ and the $i$th element of $q_{c_k} \in \mathbb{R}^n$ are 1, and the others are 0, i.e., the $(k, i)$ element of $p_{c_k} q_{c_k}^{\mathrm{T}}$ is 1; 0 otherwise.

## B. *Structural Accessibility and Observability*

Next, we give definitions of accessibility and observability studied in this paper. For nonlinear systems, controllability and observability are extended as local strong accessibility and local observability, respectively [17]. First we recall these definitions to be self-contained.

*Definition 2.5:* [17] Consider a nonlinear system $\dot{x} = f(x)+\sum_{i=1}^m g_i(x)u_i$. Let $R^{\mathcal{V}}(x_0, T)$ be the reachable set from $x_0$ at time $T > 0$ in a neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of $x_0$, i.e.,

$$\begin{aligned} R^{\mathcal{V}}(x_0, T) = \{x \in \mathbb{R}^n : \exists u : [0, T] \to \mathbb{R}^m \text{ s.t. for } x(0) = x_0, \\ x(t) \in \mathcal{V},\ 0 \le t \le T, \text{ and } x(T) = x\}. \end{aligned}$$

The system is said to be *locally strongly accessible* from $x_0$ if for any neighborhood $\mathcal{V}$ of $x_0$ the set $R^{\mathcal{V}}(x_0, T)$ contains a non-empty open set for any sufficiently small $T > 0$.

*Definition 2.6:* [17] Consider a nonlinear system $\dot{x} = f(x)$, $y = h(x)$. Let $\mathcal{V} \subset \mathbb{R}^n$ be an open set containing $x_0, x_1$. Two states $x_0, x_1 \in \mathbb{R}^n$ are said to be *indistinguishable* (denoted by $x_0 I^{\mathcal{V}} x_1$ ) for the system if the output functions $t \to y(t, x_0)$ and $t \to y(t, x_1)$ of the system for initial states $x(0) = x_0$ and $x(0) = x_1$ are identical on their common domain of definition. The system is said to be *locally observable* at $x_0$ if there exists a neighborhood $\mathcal{W}$ of $x_0$ such that for every neighborhood $\mathcal{V} \subset \mathcal{W}$ of $x_0$ the relation $x_0 I^{\mathcal{V}} x_1$ implies $x_0 = x_1$.

For linear systems, the concepts of structural controllability and observability [1], [2], [19] are introduced to study controllability and observability of linear complex networks. By combining local strong accessibility/local observability and structural controllability/observability concepts, we introduce the concepts of structural accessibility and observability.

*Definition 2.7:* The system $\Sigma_A$ is said to be *structurally accessible* if there exist $\alpha_i, \beta_j \in \mathbb{R}$, $i \in \sigma_\alpha, j \in \sigma_\beta$ such that the system is locally strongly accessible from almost all $x_0 \in \mathbb{R}^n$.

*Definition 2.8:* The system $\Sigma_O$ is said to be *structurally observable* if there exist $\alpha_i, \gamma_k \in \mathbb{R}$, $i \in \sigma_\alpha, k \in \sigma_\gamma$ such that the system is locally observable at almost all $x_0 \in \mathbb{R}^n$.

The paper [9] studied structural observability in the sense of the above definition, while it has not formally described a class of nonlinear systems and defined its observability. In the linear case, the papers [1], [2] have looked into similar properties for linear systems, where the nonzero elements of matrices $(A, B, C)$ are considered to be independent parameters. The paper [19] has studied the corresponding properties to ours and has provided necessary and sufficient conditions, but these conditions are derived based on the PBH *rank* tests. For nonlinear systems, the PBH tests have not been well studied, while some recent work [22] has tried to provide the PBH *eigenvector* tests. There is still no PBH *rank* test for nonlinear systems, and therefore extending the results in [19] to nonlinear systems is not straightforward at all.

# III. Structural Accessibility

## A. *Necessary Conditions*

In this subsection, we present two necessary conditions for structural accessibility. The first condition is given for system $\Sigma_A$, and the second condition is given for system $\Sigma_A$ with $f = 0$. Note that $\Sigma_A$ with $f = 0$ still contains the class of balance equations studied in [9].

In the standard accessibility analysis, it is known that a system is locally strongly accessible if and only if it satisfies the strong accessibility rank condition, or equivalently, if and only if it does not admit the local accessibility decomposition [17]. One can readily extend the rank condition to structural accessibility and consequently the above necessary and sufficient relations. Therefore, one can conclude that a system $\Sigma_A$ is structurally accessible if and only if it does not admit a structural accessibility decomposition. Different from the non-structural case, there are two possible decompositions $x = \varphi(z)$ and $x = \varphi(\alpha, \beta, z)$. Our first necessarily condition is based on the parameter independent decomposition $x = \varphi(z)$.

*Theorem 3.1:* Let $\sigma_{\alpha_1} := \{i_1, \dots, i_{s_\alpha}\}$ and $\sigma_{\alpha_2} := \{i_{s_\alpha+1}, \dots, i_{r_\alpha}\}$ be disjoint subsets of $\sigma_\alpha$. A system $\Sigma_A$ does not admit a *parameter independent* structural accessibility decomposition $x = \varphi(z)$ if and only if one cannot find $\sigma_{\alpha_i}$, $i = 1, 2$, satisfying all of the following three conditions.

1) the following system is not locally strongly accessible (from any $x_0 \in \mathbb{R}^n$) with respect to inputs $u \in \mathbb{R}^m$, $u_{a_i}, u_{b_j} \in \mathbb{R}$, $i \in \sigma_{\alpha_1}, j \in \sigma_\beta$.

$$\dot{x} = f(x) + \sum_{i\in\sigma_{\alpha_1}} p_i u_{a_i} + Bu + \sum_{j\in\sigma_\beta} p_{b_j} u_{b_j}; \qquad (1)$$

2) the strong accessibility distribution of system (1) is invariant [17] with respect to $p_k$, $k \in \sigma_{\alpha_2}$;
3) the relative degree of (1) with the following output

$$y_k = h_k(x), \ k \in \sigma_{\alpha_2} \tag{2}$$

is infinity.

*Proof:* (Necessity) We prove by contraposition. Suppose that there exist $\sigma_{\alpha_i}$, $i = 1, 2$ such that all of the three conditions hold. If system (1) is not locally strongly accessible, there exists a coordinate transformation $x = \varphi(z)$ for the local strong accessibility decomposition [17]. In the $z$-coordinates, system (1) becomes

$$\begin{aligned}\dot{z}_1 =& \bar{f}_1(z_1, z_2) + \sum_{i \in \sigma_{\alpha_1}} \bar{p}_{1,i}(z_1, z_2) u_{a_i} + \sum_{j \in \sigma_\beta} \bar{p}_{1,b_j}(z_1, z_2) u_{b_j} \\ &+ \bar{B}_1(z_1, z_2) u, \\ \dot{z}_2 =& \bar{f}(z_2)\end{aligned} \tag{3}$$

with suitable functions, where $z_1$ and $z_2$ are respectively the states of locally strongly accessible and non-accessible subsystems. Although the original $B$, $p_i$, and $p_{b_j}$ are constants, new $\bar{B}_1$, $\bar{p}_{1,i}$, $\bar{p}_{1,b_j}$ can become functions of $z_1$ and $z_2$.

Next, we apply the same coordinate transformation $x = \varphi(z_1, z_2)$ to system $\Sigma_A$. Then, we have

$$\begin{aligned}\dot{z}_1 =& \bar{f}_1(z_1, z_2) + \sum_{i \in \sigma_{\alpha_1}} \alpha_i \bar{p}_{1,i}(z_1, z_2) h_i(\varphi(z_1, z_2)) \\ &+ \sum_{k \in \sigma_{\alpha_2}} \alpha_k \bar{p}_{1,k}(z_1, z_2) h_k(\varphi(z_1, z_2)) \\ &+ \left( \bar{B}_1(z_1, z_2) + \sum_{j \in \sigma_\beta} \beta_j \bar{p}_{1,b_j}(z_1, z_2) q_{b_j}^{\mathrm{T}} \right) u, \\ \dot{z}_2 =& \bar{f}(z_2) + \sum_{k \in \sigma_{\alpha_2}} \alpha_k \bar{p}_{2,k}(z_1, z_2) h_k(\varphi(z_1, z_2))\end{aligned} \tag{4}$$

with suitable functions. Note that Conditions 2) and 3) respectively imply that $\bar{p}_{2,k}(z_1, z_2)$ and $h_k(\varphi(z_1, z_2))$, $k \in \sigma_{\alpha_2}$ do not depend on $z_1$. Therefore, the second subsystem does not depend on $z_1$, i.e. it is not structurally accessible.

(Sufficiency) We prove by contraposition. Suppose that one obtains a structurally non-accessible subsystem by $x = \varphi(z)$. The system in the $z$-coordinates can always be described as

$$\begin{aligned}\dot{z}_1 =& \bar{f}_1(z_1, z_2) \\ &+ \sum_{i \in \sigma_{\alpha_1}} \alpha_i \bar{p}_{1,i}(z_1, z_2) \bar{h}_i(z_1, z_2) + \sum_{k \in \sigma_{\alpha_2}} \alpha_k \bar{p}_{1,k}(z_2) \bar{h}_k(z_2) \\ &+ \left( \bar{B}_1(z_1, z_2) + \sum_{j \in \sigma_\beta} \beta_j \bar{p}_{1,b_j}(z_1, z_2) q_{b_j}^{\mathrm{T}} \right) u, \\ \dot{z}_2 =& \bar{f}(z_2) + \sum_{k \in \sigma_{\alpha_2}} \alpha_k \bar{p}_{2,k}(z_2) \bar{h}_k(z_2)\end{aligned} \tag{5}$$

for some $\sigma_i$ $(i = 1, 2)$, where $z_2$ is the state of structurally non-accessible subsystem. By applying the same coordinate transformation to system (1), we have system (3), which is not locally strongly accessible. Moreover, in (5), $\bar{p}_{2,k}$ and $\bar{h}_{2,k}$, $k \in \sigma_{\alpha_2}$ do not depend on $z_1$, which respectively imply that the accessibility distribution of the system (3) is invariant with respect to $\bar{p}_k$, $k \in \sigma_{\alpha_2}$, and the relative degree of the system (3) with any output $y = \bar{h}_{2,k}(z_2)$, $k \in \sigma_{\alpha_2}$ is infinity. Therefore, Conditions 1) – 3) hold in the $z$-coordinates, and these conditions do not depend on the coordinates. This completes the proof. ■

From the proof of Theorem 3.1, if there exists $x = \varphi(z)$ for structurally non-accessible decomposition, then this is nothing but a coordinate transformation for the local strong accessibility decomposition of the system in (1). In other words, if one computes for the local strong accessibility decomposition of (1), then $x = \varphi(z)$ can readily be obtained.

For system (4), we prove that the second subsystem with the states $z_2$ is structurally non-accessible. This does not imply that the first subsystem is structurally accessible because this subsystem can admit structural accessibility decomposition by a parameter dependent coordinate transformation $x = \varphi(z, \alpha_i, \beta_j)$. When $f = 0$, we have a necessary condition for the non-existence of $x = \varphi(z, \alpha_i, \beta_j)$.

*Theorem 3.2:* Let $\sigma_{\alpha_1} := \{i_1, \dots, i_{s_\alpha}\}$ and $\sigma_{\alpha_2} := \{i_{s_\alpha+1}, \dots, i_{r_\alpha}\}$ be disjoint subsets of $\sigma_\alpha$, and let $\sigma_{\beta_1} := \{j_1, \dots, j_{s_\beta}\}$ and $\sigma_{\beta_2} := \{j_{s_\beta+1}, \dots, j_{r_\beta}\}$ be disjoint subsets of $\sigma_\beta$. If system $\Sigma_A$ with $f = 0$ is structurally accessible, then one cannot find $\sigma_{\alpha_i}$, $i = 1, 2$ and $\sigma_{\beta_j}$, $j = 1, 2$ such that

$$n > \mathrm{rank}\bar{B} + \mathrm{dim}H + \mathrm{dim}Q, \tag{6}$$

where

$$\begin{aligned}&\bar{B} := [\ B \quad p_{i_{s_\alpha+1}} \quad \cdots \quad p_{i_{r_\alpha}} \quad p_{b_{j_{s_\beta+1}}} \quad \cdots \quad p_{b_{j_{r_\beta}}}\ ], \\ &H := \mathrm{span}_{\mathbb{R}}\{h_{i_1}(x), \dots, h_{i_{s_\alpha}}(x)\},\ Q := \mathrm{span}_{\mathbb{R}}\{q_{b_{j_1}}, \dots, q_{b_{j_{s_\beta}}}\}.\end{aligned}$$

*Proof:* We prove by contraposition. Suppose that there exist $\sigma_{\alpha_i}$, $i = 1, 2$ and $\sigma_{\beta_j}$, $j = 1, 2$ such that (6) holds. First, define $t_\alpha := \mathrm{dim}H$, and $t_\beta := \mathrm{dim}Q$. Then, there exist the orderings of $h_i(x)$, $i \in \sigma_{\alpha_1}$ and $q_{b_j}$, $j \in \sigma_{\beta_1}$ such that $h_{i_1}(x), \dots, h_{i_{t_\alpha}}(x)$ and $q_{b_{j_1}}, \dots, q_{b_{j_{t_\beta}}}$ are the basis of $H$ and $Q$, respectively. In these orderings, there exist some constants $\delta_{k_\alpha, \ell_\alpha}$ and $\delta_{k_\beta, \ell_\beta}$ such that

$$\begin{aligned}h_{k_\alpha}(x) &= \textstyle\sum_{\ell_\alpha = i_1}^{i_{t_\alpha}} \delta_{k_\alpha, \ell_\alpha} h_{\ell_\alpha}(x),\ k_\alpha = i_{t_\alpha+1}, \dots, i_{s_\alpha}, \\ q_{b_{k_\beta}} &= \textstyle\sum_{\ell_\beta = i_1}^{i_{t_\beta}} \delta_{k_\beta, \ell_\beta} q_{b_{\ell_\beta}},\ k_\beta = j_{t_\beta+1}, \dots, j_{s_\beta}.\end{aligned}$$

Using them, the system $\Sigma_A$ with $f = 0$ can be rewritten as

$$\begin{aligned}\Sigma_A : \dot{x} =& \sum_{\ell_\alpha = i_1}^{i_{t_\alpha}} h_{\ell_\alpha}(x) \xi_{\ell_\alpha} + \sum_{i \in \sigma_{\alpha_2}} \alpha_i h_i(x) p_i \\ &+ Bu + \sum_{\ell_\beta = i_1}^{i_{t_\beta}} \eta_{\ell_\beta} q_{b_{\ell_\beta}}^{\mathrm{T}} u + \sum_{j \in \sigma_{\beta_2}} \beta_j p_{b_j} q_{b_j}^{\mathrm{T}} u,\end{aligned}$$

where

$$\begin{aligned}\xi_{\ell_\alpha} &:= \alpha_{\ell_\alpha} p_{\ell_\alpha} + \textstyle\sum_{k_\alpha = i_{t_\alpha+1}}^{i_{s_\alpha}} \alpha_{k_\alpha} \delta_{k_\alpha, \ell_\alpha} p_{k_\alpha}, \\ \eta_{\ell_\beta} &:= \beta_{\ell_\beta} p_{b_{\ell_\beta}} + \textstyle\sum_{k_\beta = i_{t_\beta+1}}^{i_{s_\beta}} \beta_{k_\beta} \delta_{k_\beta, \ell_\beta} p_{b_{k_\beta}}.\end{aligned}$$

To show that the above system $\Sigma_A$ is not structurally accessible, let us consider the following linear system

$$\dot{x} = \sum_{\ell_\alpha = i_1}^{i_{t_\alpha}} \xi_{\ell_\alpha} u_{a_{\ell_\alpha}} + \sum_{i \in \sigma_{\alpha_2}} p_i u_{a_i} + \sum_{\ell_\beta = i_1}^{i_{t_\beta}} \eta_{\ell_\beta} u_{b_{\ell_\beta}} + \sum_{j \in \sigma_{\beta_2}} p_{b_j} u_{b_j}$$

$+ Bu.$

From (6), the number of inputs $u$, $u_{a_j}$, $j = i_1, \dots, i_{t_\alpha}$, $j \in \sigma_{\alpha_2}$ and $u_{b_k}$, $k = i_1, \dots, i_{t_\beta}$, $k \in \sigma_{\beta_2}$ is less than $n$, which implies that this linear system with $A = 0$ is not controllable. Therefore, one can conclude that the system $\Sigma_A$ with $f = 0$ is not structurally accessible. ■

In the linear case when $f = 0$, it is possible to show that the first and second necessary conditions reduce to Conditions (iiia) and (iiib) of Criterion 1 in [19], respectively, which implies that if these two conditions hold, the linear system is structurally controllable. In the nonlinear case, the sufficiency is not clear yet. Another difficulty in the nonlinear case is extending Theorems 3.2 to the nonzero $f$. These two problems are considered for future work. From the viewpoint of application to driver node identification of the balance equation, a necessary condition when $f = 0$ is useful in itself.

In the next subsection, we study graphical driver node identification based on Theorem 3.1. The conditions constructed in [19] for linear systems have not been linked to the graphical structures of networks. As we show below, nonlinearity of the complex network is a key factor to connect structural accessibility (controllability) conditions and driver node identification.

### B. Driver Node Selection for Balance Equations

We develop a driver node selection method of the balance equation based on the existing sensor node selection method for it [9] and Theorem 3.1. Then, we discuss Theorem 3.2 from the viewpoint of driver node selection.

Our driver node selection method is based on the strongly connected component (SCC) [16] of the inference diagram of the system $\Sigma_A$, which is obtained as follows:

1) Draw its inference diagram, which contains a directed edge $x_j \to x_i$ if $x_j$ appears in $x_i$'s differential equation.
2) Decompose the inference diagram into SCCs, where an SCC is a maximal strongly connected subgraph.

To establish the connection between driver node selection and Theorem 3.1, we introduce two concepts for SCCs. First, as mentioned in Section II-A, the balance equation satisfies Assumption 2.2. This structure implies that the set of nodes $x_j$ satisfying $\partial h_i(x)/\partial x_j \neq 0$ is an SCC. Moreover, every node of this SCC has a self loop. We call this type of SCC the *strictly SCC* corresponding to $h_i(x)$ or simply *strictly* $\mathrm{SCC}_{h_i}$.

Second, an SCC is said to be a *root SCC* if the SCC has no incoming path from the other SCCs. It is possible to show that every inference diagram has at least one root SCC.

*Remark 3.3:* In [9], the directions of edges in the inference diagram are the opposite of what has been defined here. However, our representation follows the flow of information more naturally because $x_j \to x_i$ implies that $x_j$ affects the dynamics of $x_i$. Since the directions of edges are opposite, a root SCC used for sensor node selection in [9] is a different concept from our root SCC, and thus we call a root SCC in [9] a *leaf* SCC. That is, an SCC is said to be a leaf SCC if the SCC has no outgoing path to the other SCCs. A root and leaf SCCs are the dual concepts of each other.

The sensor node selection method proposed in [9] is to choose at least one node of every leaf SCC as a sensor node. This method can simply be extended to driver node selection, but this extension is not enough for driver node identification as shown in the following main result of this paper.

*Assumption 3.4:* If $\partial h_k(x)/\partial x_\ell \neq 0$, $\ell = j_1, \dots, j_i$, then $(\partial h_k/\partial x_\ell)v \neq 0$ for any non-zero $v := [v_{j_1} \ \cdots \ v_{j_i}] \in \mathbb{R}^\ell$.

*Theorem 3.5:* Consider the inference diagram of system $\Sigma_A$. Under Assumptions 2.2, 2.3 and 3.4, a system $\Sigma_A$ does not admit a *parameter independent* structural accessibility decomposition $x = \varphi(z)$ if and only if a set of driver nodes is chosen such that both of the following conditions hold.

i) at least one node from every root SCC is a driver node;
ii) the following $P$ has the full rank.

$$P := \begin{bmatrix} P_\alpha & P_\beta \end{bmatrix}, \tag{7}$$
$$P_\alpha := \begin{bmatrix} p_1 & \cdots & p_{r_\alpha} \end{bmatrix},\ P_\beta := \begin{bmatrix} p_{b_1} & \cdots & p_{b_{r_\beta}} \end{bmatrix}.$$

*Proof:* The proof is based on Theorem 3.1. In particular, we show that one cannot find $\sigma_{\alpha_i}$, $i = 1, 2$ such that all Conditions 1) – 3) in Theorem 3.1 hold if and only if Conditions i) and ii) of this theorem hold.

First, we prove the only if part by contraposition. If Condition i) does not hold, i.e., if there is a root SCC not having a driver node, then every node in this root SCC is not structurally accessible according to the definition of root SCC. Furthermore, in the $x$-coordinate, one can find $\sigma_{\alpha_i}$, $i = 1, 2$ such that all of Conditions 1) – 3) hold. Next, if Condition ii) does not hold, then for $\sigma_{\alpha_1} = \sigma_\alpha$ Condition 1) holds, and in this case Conditions 2) and 3) are automatically satisfied.

Then, we consider the if part. If Condition ii) holds, Condition 1) does not hold for $\sigma_{\alpha_1} = \sigma_\alpha$. It remains to consider the case $\sigma_{\alpha_1} \neq \sigma_\alpha$. We show that if Condition i) holds, Condition 3) does not hold for $\sigma_{\alpha_1} \neq \sigma_\alpha$ by contraposition. Suppose that there exist $\sigma_{\alpha_i}$, $i = 1, 2$, $\sigma_{\alpha_1} \neq \sigma_\alpha$ such that the relative degree of system (1) with output (2) is infinity. Then, for any $k \in \sigma_{\alpha 2}$, we have

$$\frac{\partial h_k(x)}{\partial x} p_i = 0,\ \forall i \in \sigma_{\alpha_1},\ \frac{\partial h_k(x)}{\partial x} p_{b_j} = 0,\ \forall j \in \sigma_\beta. \tag{8}$$

From Assumption 2.2 and structure of the system $\Sigma$, if there is a path from a strictly $\mathrm{SCC}_{h_i}$ to node $x_\ell$, then the $\ell$th element of $p_i$ is non-zero. The first equality in (8) and Assumption 3.4 imply that if the $\ell$th element of $p_i$ is non-zero, then $\partial h_k(x)/\partial x_\ell = 0$. That is, any strictly $\mathrm{SCC}_{h_k}$, $k \in \sigma_{\alpha 2}$ does not contain node $x_\ell$. Therefore, there is no path from a strictly $\mathrm{SCC}_{h_i}$, $i \in \sigma_{\alpha_1}$ to a strictly $\mathrm{SCC}_{h_k}$, $k \in \sigma_{\alpha 2}$.

The second equality of (8), Assumption 3.4 and the definition of $p_{b_j}$ imply $\partial h_k(x)/\partial x_i = 0$, i.e., a strictly $\mathrm{SCC}_{h_k}$, $k \in \sigma_{\alpha 2}$ does not contain a driver node $x_i$. Therefore, any SCC containing a strictly $\mathrm{SCC}_{h_k}$, $k \in \sigma_{\alpha 2}$ does not have a driver node. ■

Assumption 3.4 implies that if there is a path from a strictly $\mathrm{SCC}_{h_k}$ to a strictly $\mathrm{SCC}_{h_j}$, then the information is conveyed without cancellation. This assumption holds if $h_i(x)$ depends on only one node $x_j$ or $h_i(x)$ is not a linear function. This is a mathematical explanation, which has not been explained by [9], for the importance of nonlinearity when one studies driver and sensor node selection of the balance equation based on its inference graphical.

According to Theorem 3.5, for driver node selection, we need to choose at least one node from a root SCC as a driver node such that $P$ in (7) has the full rank. This rank condition gives a lower bound on the number of driver nodes $r_\beta \geq n - \mathrm{rank} P_\alpha (\geq n - r_\alpha)$. Since $r_\alpha$ is the number of strictly SCCs, the lower bound on the number of driver nodes $n - \mathrm{rank} P_\alpha$ becomes large if a strictly SCC consists of many nodes. In such a case, the concept of a root SCC is not enough to determine a sufficient driver set. In fact, as explained in Section IV below, this observation is specific for driver node selection.

As mentioned in [9], an (educated) brute-force search may be used to inspect a minimum set of driver nodes but is a computationally prohibitive task for large complex networks. For $2^n$ driver node combinations, and a randomly chosen set of parameters $\alpha_i$ and $\beta_j$, we need to verify local strong accessibility. As a necessary and sufficient condition, the local strong accessibility rank condition [17] is known. To check this condition, we need to compute Lie brackets of the system, that is to compute partial derivatives of the nonlinear functions. This may be doable, but is not always easy for large scale systems. In contrast, by using our Theorem 3.5, we obtain at least a necessary driver set that seems to be sufficient in many cases because of the reasons explained in the next paragraph. Our method only requires the computations of the rank of a constant matrix $P$ and the SCC decomposition, and for the latter there is a linear time algorithm [16]. To verify whether our method gives a sufficient sensor set, a brute-force search may be used. If we combine our method and a brute-force search, the considered driver node combinations is reduced from $2^n$ to $2^{n-r_\beta}$, where $r_\beta$ is the number of driver nodes determined by our method. Even though our method may not give a sufficient driver set, the method is useful to reduce computational complexity of a brute-force search.

In the previous subsection, we obtained two necessary conditions, and the first condition is connected with driver node selection. Let us consider the second necessary condition in Theorem 3.2. When $h_i(x)$, $i \in \sigma_\alpha$ and $q_{b_j}$, $j \in \sigma_\beta$ are linearly independent, condition (6) does not hold if the matrix $P$ in (7) has the full rank. For driver node selection, $q_{b_j}$ are chosen to be linearly independent. Thus, we need to take care of the linear dependence of nonlinear functions $h_i(x)$, $i \in \sigma_\alpha$. However, nonlinear functions are not linearly dependent in general unless $h_i(x) = h_j(x)$, $i \neq j$. Therefore, the first condition is more relevant than the second when we study driver node selection of nonlinear complex networks.

## IV. Structural Observability

In this subsection, we analyze the existing graphical approach for sensor node identification [9]. First, we connect the graphical approach with structural observability decomposition by a parameter independent coordinate transformation as done for structural accessibility. Then, we show that a parameter dependent transformation characterizes the cases when the existing method does not give a sufficient sensor set in contrast to driver node selection.

We have the following graphical necessary condition for structural observability. We omit the proof because it is similar to that of Theorem 3.5 for structural accessibility.

*Theorem 4.1:* Consider the inference diagram of system $\Sigma_O$. Under Assumptions 2.2, 2.4 and 3.4, a system $\Sigma_O$ does not admit a *parameter independent* structural observability decomposition $x = \varphi(z)$ if and only if a set of sensor nodes is chosen such that both of the following conditions hold.

i) at least one node from every leaf SCC is a sensor node;
ii) the following system is locally observable:

$$\begin{cases} \dot{x} = \sum_{i \in \sigma_\alpha} p_i u_{a_i}, \\ y_j = h_j(x), \ j \in \sigma_\alpha, \\ y_{c_k} = q_{c_k}^{\mathrm{T}} x, \ k \in \sigma_\gamma. \end{cases} \tag{9}$$

Condition 2) gives a lower bound on the number of sensor nodes $r_\gamma$. Let $r_O$ be the maximum dimension of the observability codistribution [17] of system (9) with outputs $y_j$, $j \in \sigma_\alpha$. Then, we need to choose $y_{c_k}$, $k \in \sigma_\gamma = \{1, \dots, r_\gamma\}$ such that system (9) becomes locally observable. Therefore, the number of sensor nodes $r_\gamma$ is lower bounded on $n - r_O$, where $r_O$ can be greater than $r_\alpha$, i.e., the lower bound can be less than $n - r_\alpha$ in contrast to driver node selection.

Next, we provide a necessary condition for the non-existence of parameter dependent structural observability decomposition. This condition is more complicated than the accessibility condition in Theorem 3.2 because in contrast to accessibility, the number of outputs does not directly relate to structural observability even if $f = 0$.

*Theorem 4.2:* Let $\sigma_{\alpha_1} := \{i_1, \dots, i_{s_\alpha}\}$ and $\sigma_{\alpha_2} := \{i_{s_\alpha+1}, \dots, i_{r_\alpha}\}$ be disjoint subsets of $\sigma_\alpha$, and let $\sigma_{\gamma_1} := \{j_1, \dots, j_{s_\gamma}\}$ and $\sigma_{\gamma_2} := \{j_{s_\gamma+1}, \dots, j_{r_\gamma}\}$ be disjoint subsets of $\sigma_\gamma$. Next, define $t_\alpha$ and $t_\gamma$ as

$$t_\alpha := \dim \mathrm{span}_{\mathbb{R}} \{p_{i_1}, \dots, p_{i_{s_\alpha}}\},$$
$$t_\gamma := \dim \mathrm{span}_{\mathbb{R}} \{q_{c_{j_1}}, \dots, q_{c_{j_{s_\gamma}}}\}.$$

Suppose that $p_k$, $k \in \sigma_1$ and $q_{c_k}$, $k \in \gamma_1$ are ordered such that $p_{i_1}, \dots, p_{i_{t_\alpha}}$ and $q_{c_{j_1}}, \dots, q_{c_{j_{t_\gamma}}}$ are the basis of the above subspaces respectively. Then, if system $\Sigma_O$ with $f = 0$ is structurally observable, one cannot find $\sigma_{\alpha_i}$ and $\sigma_{\gamma_j}$, $i, j = 1, 2$ such that

$$\frac{\partial h_i(x)}{\partial x} p_j = 0, \ i \in \sigma_{\alpha_k}, j \in \sigma_{\alpha_\ell}, k \neq \ell, \tag{10}$$
$$n > v_1 + v_2 + v_3 \tag{11}$$

for the following $v_k$ $(k = 1, 2, 3)$:

1) $v_1 = \mathrm{rank}[C^{\mathrm{T}} \ q_{c_{i_1}} \ \cdots \ q_{c_{i_{t_\alpha}}} \ q_{c_{i_{s_\gamma+1}}} \ \cdots \ q_{c_{i_{r_\gamma}}}]$;
2) $v_2$ is the maximum dimension of the observability codistribution of the following system,
$$\begin{cases} \dot{x} = \sum_{i \in \sigma_{\alpha_2}} p_i u_{a_i}, \\ y_{c_k} = h_k(x), \ k \in \sigma_{\alpha_2}; \end{cases} \tag{12}$$
3) If for any $h_\ell(x)$, $\ell \in \sigma_{\alpha_1}$, there exists constant $\mu_{h_\ell}$ and uniformly exist constants $v$ and $\mu_0, \mu_1, \dots, \mu_v$ such that the Lie derivatives satisfy

$$L_{p_{j_1}, \dots, p_{j_{v+1}}} h_\ell(x)$$

$$= \mu_{h_\ell} + \mu_0 h_\ell(x) + \sum_{i=1}^{v} \mu_i L_{p_{j_1},\dots,p_{j_i}} h_\ell(x),$$

for any $j_1, \dots, j_{v+1} \in \{i_1, \dots, i_{t_\alpha}\}$, then denote $\hat{v}$ by the minimum number of $v$. Define $v_3 = t_\alpha(\hat{v}+1)$. If there does not exist $v$, then $v_3 := \infty$.

*Proof:* We prove by contraposition. Suppose that there exist $\sigma_{\alpha_i}$ and $\sigma_{\gamma_j}$, $i, j = 1, 2$ such that all conditions holds. First, there exist some constants $\delta_{k_\alpha,\ell_\alpha}$ and $\delta_{k_\gamma,\ell_\gamma}$ such that

$$p_{k_\alpha} = \textstyle\sum_{\ell_\alpha=i_1}^{i_{t_\alpha}} \delta_{k_\alpha,\ell_\alpha} p_{\ell_\alpha},\ q_{c_{k_\gamma}} = \sum_{\ell_\gamma=j_1}^{j_{t_\gamma}} \delta_{k_\gamma,\ell_\gamma} q_{c_{\ell_\gamma}},$$
$$k_\alpha = i_{t_\alpha+1}, \dots, i_{s_\alpha}, \quad k_\gamma = i_{t_\gamma+1}, \dots, i_{s_\gamma}.$$

By using them, the system $\Sigma_O$ can be rewritten as

$$\Sigma_O : \begin{cases} \dot{x} = \displaystyle\sum_{\ell_\alpha=i_1}^{i_{t_\alpha}} \xi_{\ell_\alpha}(x) p_{\ell_\alpha} + \sum_{i\in\sigma_2} \alpha_i h_i(x) p_i, \\ y = \left( C + \displaystyle\sum_{\ell_\gamma=j_1}^{j_{t_\gamma}} \eta_{c_{\ell_\gamma}} q_{c_{\ell_\gamma}}^{\mathrm{T}} + \sum_{j\in\sigma_{\gamma_2}} \gamma_j p_{c_j} q_{c_j}^{\mathrm{T}} \right) x, \end{cases}$$

where

$$\xi_{\ell_\alpha}(x) := \alpha_{\ell_\alpha} h_{\ell_\alpha}(x) + \textstyle\sum_{k_\alpha=i_{t_\alpha+1}}^{i_{s_\alpha}} \alpha_{k_\alpha} \delta_{k_\alpha,\ell_\alpha} h_{k_\alpha}(x),$$
$$\eta_{c_{\ell_\gamma}} := \gamma_{\ell_\gamma} p_{c_{\ell_\gamma}} + \textstyle\sum_{k_\gamma=i_{t_\gamma+1}}^{i_{s_\gamma}} \gamma_{k_\gamma} \delta_{k_\gamma,\ell_\gamma} p_{c_{k_\gamma}}.$$

For the above system $\Sigma_O$, we show that the maximum dimension (with respect to $x$ and parameters) of its observability codistribution is not more than $v_1 + v_2 + v_3$. Then, condition (11) implies that the system $\Sigma_O$ is not structurally observable.

Since the output function of the system $\Sigma_O$ is linear, its time derivative $\dot{y}$ is a linear combination of $h_i(x)$, $i \in \sigma_\alpha$. For any parameters, the observability codistribution of $\Sigma_O$ is contained in the sum (in the sense of the linear subspace) of

$$\begin{aligned} \mathrm{span}_{\mathbb{R}}\{ & d(Cx), d(q_{c_{i_1}}^{\mathrm{T}} x), \cdots, d(q_{c_{i_{t_\alpha}}}^{\mathrm{T}} x), \\ & d(q_{c_{i_{s_\gamma+1}}}^{\mathrm{T}} x), \cdots, d(q_{c_{i_{r_\gamma}}}^{\mathrm{T}} x)\} \end{aligned} \quad (13)$$

and the observability codistribution of the following system.

$$\begin{cases} \dot{x} = \displaystyle\sum_{\ell_\alpha=i_1}^{i_{t_\alpha}} \xi_{\ell_\alpha}(x) p_{\ell_\alpha} + \sum_{i\in\sigma_2} \alpha_i h_i(x) p_i, \\ y_{c_j} = h_j(x),\ j \in \sigma_{\alpha_2}, \\ y_{c_{k_\alpha}} = \xi_{k_\alpha}(x),\ k_\alpha = i_1, \dots, i_{t_\alpha}. \end{cases} \quad (14)$$

Moreover, from (10), the observability codistribution of the system (14) is the sum of the observability codistributions of the following two systems

$$\begin{cases} \dot{x} = \displaystyle\sum_{i\in\sigma_{\alpha_2}} \alpha_i h_i(x) p_i, \\ y_{c_k} = h_k(x),\ k \in \sigma_{\alpha_2}; \end{cases} \quad (15)$$

$$\begin{cases} \dot{x} = \displaystyle\sum_{\ell_\alpha=i_1}^{i_{t_\alpha}} \xi_{\ell_\alpha}(x) p_{\ell_\alpha}, \\ y_{c_{k_\alpha}} = \xi_{k_\alpha}(x),\ k_\alpha = i_1, \dots, i_{t_\alpha}. \end{cases} \quad (16)$$

From 1), the dimension of (13) is $v_1$. Next, from the structure of (15), the observability codistributions of the systems (12) and (15) are the same. Then, from 2), the maximum dimension of the observability codistributions of the system (15) is $v_2$.

It suffices to consider system (16). From its structure, the observability codistribution of system (16) is spanned by the differential one-forms of $\xi_{\ell_\alpha}(x)$, $\ell_\alpha = i_1, \dots, i_{t_\alpha}$ and its Lie derivatives $L_{p_{j_1},\dots,p_{j_k}} \xi_{\ell_\alpha}(x)$, where $j_1, \dots, j_k \in \{i_1, \dots, i_{t_\alpha}\}$ and $k = 1, 2, \dots$. Since $\xi_{\ell_\alpha}(x)$ is a linear combination of $h_i(x)$, $i \in \sigma_{\alpha_1}$, from 3) there exists constant $\mu_{\xi_{\ell_\alpha}}$ such that

$$\begin{aligned} & L_{p_{j_1},\dots,p_{j_{\hat{v}+1}}} \xi_{\ell_\alpha}(x) \\ & \quad = \mu_{\xi_{\ell_\alpha}} + \mu_0 \xi_{\ell_\alpha}(x) + \textstyle\sum_{i=1}^{\hat{v}} \mu_i L_{p_{j_1},\dots,p_{j_i}} \xi_{\ell_\alpha}(x) \end{aligned}$$

for any $j_1, \dots, j_{\hat{v}+1} \in \{i_1, \dots, i_{t_\alpha}\}$. Note that the number of outputs of the system (16) is $t_\alpha$. Then, it is possible to show that the maximum dimension of its observability codistribution is not more than $v_3 = t_\alpha(\hat{v}+1)$. ■

For driver node selection, we mentioned that Theorem 3.2 may not be relevant. In contrast, Theorem 4.2 characterizes the cases when the existing sensor node selection method does not give a sufficient sensor set.

*Example 4.3:* Consider the nonlinear balance equation [9] that can be represented as the system $\Sigma$ with

$$h_1 = x_1 x_2,\ p_1 = \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}^{\mathrm{T}},$$
$$h_2 = x_3 x_4,\ p_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^{\mathrm{T}}.$$

If we apply the method in [9], at least one node is chosen as a set of sensor nodes. For instance, we choose $x_1$ as the sensor node, i.e., $y_{\gamma_1} = \gamma_1 [1\ 0\ 0\ 0]^{\mathrm{T}} x$. First, one can check that Conditions i) and ii) in Theorem 4.1 hold. Next, we show that the necessary condition in Theorem 4.2 does not hold. Let $\sigma_{\alpha_1} := \sigma_\alpha$ and $\sigma_{\gamma_1} := \sigma_\gamma = \{1\}$. Then, we obtain $v_1 = 1$, and $v_2 = 0$. It suffices to compute $v_3$ in 3). Since $p_1 = p_2$, we have $t_\alpha = 1$. To find $\hat{v}$, we compute the Lie derivatives of $h_1(x)$ and $h_2(x)$. They satisfy

$$L_{p_1} L_{p_1} h_i(x) = 2,\ i = 1, 2.$$

Then, $\hat{v} = 1$, and consequently $v_3 = t_\alpha(\hat{v}+1) = 2$. In summary, $v_1 + v_2 + v_3 = 1 + 0 + 2 < 4 = n$. From Theorem 4.2, if we choose $x_1$ as the sensor node, the balance equation is not structurally observable. Actually, we have a structural observability decomposition as follows:

$$\dot{z} = \begin{bmatrix} z_2 \\ -z_2 z_3 \\ 2(\alpha_1 - \alpha_2) z_2 \\ z_2 \end{bmatrix},\ y = z_1,$$

where

$$z = \begin{bmatrix} x_1 \\ -\alpha_1 x_2 x_2 + \alpha_2 x_3 x_4 \\ \alpha_1(x_1 + x_2) + \alpha_2(x_3 + x_4) \\ x_2 \end{bmatrix}.$$

Therefore, $z_1, z_2, z_3$ and $z_4$ are the states of the structurally observable and unobservable subsystems, respectively. Note that if we choose arbitrary two distinct nodes from $x_i$ ($i = 1, \dots, 4$) as sensor nodes, then $v_1 = 2$, and the necessary condition in Theorem 4.2 holds. Actually, one can check that the balance equation becomes structurally observable. For this balance equation, every necessary and sufficient sensor set is clarified by our methods in contrast to the existing method [9] and its improvement [10].

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## V. Example

Consider the chemical reaction system with 11 species involved in four reactions [9].

$$\begin{aligned} x_1 + x_2 + x_3 &\rightarrow x_4 + x_6 + x_{10} \\ x_4 &\leftrightarrow x_5 \\ x_8 + x_9 &\leftrightarrow x_7 \\ x_{10} + x_{11} &\rightarrow x_7 + x_8 \end{aligned}$$

Its state space equation is

$$\begin{aligned} \dot{x}_i &= -\dot{x}_6 = -\alpha_1 x_1 x_2 x_3, \ i = 1,2,3, \\ \dot{x}_4 &= \alpha_1 x_1 x_2 x_3 - \alpha_2 x_4 + \alpha_3 x_5, \\ \dot{x}_5 &= \alpha_2 x_4 - \alpha_3 x_5, \\ \dot{x}_7 &= \alpha_4 x_8 x_9 - \alpha_5 x_7 + \alpha_6 x_{10} x_{11}, \\ \dot{x}_8 &= -\alpha_4 x_8 x_9 + \alpha_5 x_7 + \alpha_6 x_{10} x_{11}, \\ \dot{x}_9 &= -\alpha_4 x_8 x_9 + \alpha_5 x_7, \\ \dot{x}_{10} &= \alpha_1 x_1 x_2 x_3 - \alpha_6 x_{10} x_{11}, \\ \dot{x}_{11} &= -\alpha_6 x_{10} x_{11}, \end{aligned}$$

and Fig. 1 shows its inference diagram.

The inference diagram has three leaf SCCs, and thus at least one node from each leaf SCC is chosen as a sensor node. For instance, in [9], $x_5$, $x_6$, and $x_7$ are chosen. Then, the chemical reaction network becomes structurally observable indeed [9].

We consider driver node selection. First, the inference diagram has only one root SCC. Next, one can check $\mathrm{rank} P_\alpha = 4$. Then, we need at least 7 driver nodes, which is not clear only from the inference diagram, and there is a significant difference between the number of driver nodes and root SCCs. For instance, if we choose nodes $x_i$ $(i = 1,2,3,5,8,10,11)$ as driver nodes, $P$ has the full rank. Indeed, the chemical reaction network becomes structurally accessible. In this case, there is a gap between driver and sensor node identification.

## VI. Conclusion

In this paper, we have defined structural accessibility and observability for nonlinear complex networks governed by balance equations. Then, we have developed a driver node selection method based on a necessary condition for structural accessibility. Our driver node selection method can be viewed as the dual of the existing sensor node selection method, but there are differences. There can be a huge gap between the numbers of driver nodes and root SCCs in contrast to the difference between the numbers of sensor nodes and leaf SCCs as demonstrated by a chemical reaction network. Currently we are interested in developing a graphical approach for more general nonlinear networks based on our necessary conditions.

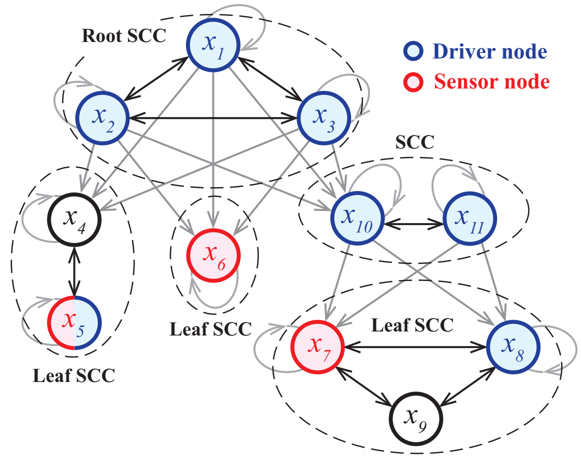


Fig. 1. Inference diagram of chemical reaction network

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