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ABSTRACT

Motivated by recent work in this area we expand on a generalization of port-Hamiltonian systems that is obtained by replacing the Hamiltonian function representing energy storage by a *Lagrangian subspace*. This leads to a new class of algebraic constraints and DAE systems in physical systems modeling. It is shown how Dirac structures and Lagrangian subspaces allow for similar representations, and how this can be exploited to convert algebraic constraints originating from Dirac structures into algebraic constraints corresponding to Lagrangian subspaces, and conversely.

1. Introduction

It is well-known [1–3] that port-Hamiltonian system dynamics may exhibit *algebraic constraints* in the state variables, leading to mixtures of *differential and algebraic equations* (DAEs). From a network modeling perspective these algebraic constraints arise from interconnection of the subsystems composing the overall system. The presence of such algebraic constraints is reflected in the properties of the underlying *Dirac structure* of the system. In fact, the Dirac structure is determined by the composition of the Dirac structures of the subsystems, and need not be a mapping from the co-energy variables to the flow variables but instead a *relation* between them; see e.g. [4,3,5] for more details. In this latter case, there are constraints between the co-energy variables of the system, which, via the Hamiltonian function, translate into algebraic constraints in the state variables. Examples include kinematic constraints in mechanical systems, and voltage or current constraints in electrical circuits.

On the other hand it was recently observed in [6], see also the subsequent work [7–10], that by *generalizing* the definition of linear port-Hamiltonian systems algebraic constraints may arise in *different* ways as well. At the same time in [11], motivated primarily by considerations in the geometric formulation of Lagrangian systems, systems with kinematic constraints, as well as optimal control, the definition of port-Hamiltonian systems was generalized by replacing the gradient of the Hamiltonian function in the port-Hamiltonian dynamics by a Lagrangian submanifold which is *not* necessarily the graph of the gradient of a Hamiltonian. This leads to algebraic constraints in the state variables which are of a different nature than those originating from Dirac structures.

In the present paper we will elaborate on the algebraic constraints of generalized port-Hamiltonian systems defined by Dirac structures *as well as* by Lagrangian subspaces; thus elucidating and complementing earlier contributions. For simplicity of exposition we will concentrate on *linear time-invariant finite-dimensional systems*, and moreover on the *lossless* case (no energy-dissipation) without external variables (inputs/outputs). For developments concerning *time-varying* or *infinite-dimensional* linear port-Hamiltonian DAE systems we refer to [7,6], and for the nonlinear case to [11].

Conceptually, the current paper is closest to [11] by emphasizing the geometric definition of a generalized port-Hamiltonian system as a pair of a Dirac structure and a Lagrangian subspace, while some constructions (as well as the emphasis on the linear case) are inspired by [6,12]. The paper is structured as follows. In Section 2 we give the geometric definition of linear generalized port-Hamiltonian DAE systems (without energy-dissipation and external variables), entailing algebraic constraints due to the Dirac structure as well as to the Lagrangian subspace. Inspired by [6,12] we give an explicit coordinate representation in terms of a parametrizing state vector. The end of Section 2 provides a number of simple, but illustrative, examples of algebraic constraints corresponding to either the Dirac structure or to the Lagrangian subspace. In Section 3 we zoom in on algebraic constraints and the underlying geometry of Dirac structures and Lagrangian subspaces. We show, and illustrate by examples, how algebraic constraints corresponding to Dirac structures may be converted into algebraic constraints corresponding to Lagrangian subspaces on an extended state space, and conversely. Section 4 contains the conclusions.

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2. Definition of generalized port-Hamiltonian DAE systems

An *unconstrained* linear lossless port-Hamiltonian system without external variables on an n-dimensional linear state space $\mathcal{X}$ is described by a system of ordinary differential equations (ODEs)

$$\dot{x} = JQx, \tag{1}$$

where $J : \mathcal{X}^* \to \mathcal{X}, J = -J^T$, is a skew-symmetric mapping (also called Poisson structure map), and the symmetric matrix $Q = Q^T$ defines a *Hamiltonian* function $H(x) = \frac{1}{2}x^T Qx$. Obviously by skew-symmetry of J

$$\frac{d}{dt}H(x) = x^T QJQx = 0, \tag{2}$$

expressing energy-conservation. On the other hand, in network modeling of physical systems, the dynamics is *not* always in ODE form (1), but instead involves *algebraic equations* in the state vector x. This was formalized in the standard definition of a port-Hamiltonian system by generalizing the skew-symmetric map J to a general (constant) *Dirac structure*, defined as follows [13,4,3]. Consider the product $\mathcal{X} \times \mathcal{X}^*$, with projections $\pi : \mathcal{X} \times \mathcal{X}^* \to \mathcal{X}$ and $\pi^* : \mathcal{X} \times \mathcal{X}^* \to \mathcal{X}^*$. Define on $\mathcal{X} \times \mathcal{X}^*$ the bilinear form

$$\langle\langle (f_1, e_1), (f_2, e_2)\rangle\rangle_+ := \langle e_1 \mid f_2\rangle + \langle e_2 \mid f_1\rangle, \tag{3}$$

with $(f_i, e_i) \in \mathcal{X} \times \mathcal{X}^*, i = 1, 2$, and $\langle e \mid f\rangle$ denoting the duality product between $f \in \mathcal{X}$ and $e \in \mathcal{X}^*$.

Definition 2.1 (*Constant Dirac Structure [14]*)**.** A Dirac structure is a subspace $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ such that $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp +}$, where $^{\perp\!\!\!\perp +}$ denotes the orthogonal companion with respect to the bilinear form $\langle\cdot, \cdot\rangle_+$.

Remark 2.2 (*[5,3]*)**.** An equivalent definition of a Dirac structure can be stated as follows. A subspace $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ is a Dirac structure iff $\langle\cdot, \cdot\rangle_+$ restricted to $\mathcal{D}$ is *zero*, and $\mathcal{D}$ is *maximal* with respect to this property. The dimension of any Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ is equal to $\dim \mathcal{X}$. Furthermore, by taking $f_1 = f_2 = f, e_1 = e_2 = e$ in (3) it follows that $\langle e \mid f\rangle = 0$ for any $(f, e) \in \mathcal{D}$, expressing power conservation, and generalizing skew-symmetry.

A linear *port-Hamiltonian DAE system* with Hamiltonian $H(x) = \frac{1}{2}x^T Qx$, briefly pH DAE system, is now geometrically given as[1]

$$(-\dot{x}, Qx) \in \mathcal{D} \tag{4}$$

Note that the graph of a skew-symmetric map $-J$

$$\mathcal{D}_J := \{(f = -Je, e) \in \mathcal{X} \times \mathcal{X}^* \mid e \in \mathcal{X}^*\} \tag{5}$$

is a special type of Dirac structure. In fact, a Dirac structure $\mathcal{D}$ can be represented into the form (5) for some skew-symmetric J if and only if $\pi^*(\mathcal{D}) = \mathcal{X}^*$. On the other hand, if $\pi^*(\mathcal{D}) \neq \mathcal{X}^*$ then the dynamics (4) gives rise to the algebraic constraints

$$e = Qx \in \pi^*(\mathcal{D}) \tag{6}$$

This type of algebraic constraints will be referred to as *Dirac algebraic constraints*. They arise as constraints on the variables e, called in port-based modeling terminology the *co-energy* (or *effort*) variables. Through the specification of the Hamiltonian $H(x) = \frac{1}{2}x^T Qx$ they translate into the algebraic constraints $Qx \in \pi^*(\mathcal{D})$ on the state variables x.

Recently, and from different points of view [6,11], it was noted that a *second type* of algebraic constraints can be formulated by generalizing the gradient Qx of the Hamiltonian $H(x) = \frac{1}{2}x^T Qx$ to a *Lagrangian subspace* of $\mathcal{X} \times \mathcal{X}^*$. This latter notion is defined as follows, resembling[2] the previous definition of a Dirac structure. Consider on $\mathcal{X} \times \mathcal{X}^*$ the alternate bilinear form

$$\langle\langle (x_1, e_1), (x_2, e_2)\rangle\rangle_- := \langle e_1 \mid x_2\rangle - \langle e_2 \mid x_1\rangle, \tag{7}$$

with $(x_i, e_i) \in \mathcal{X} \times \mathcal{X}^*, i = 1, 2$.

Definition 2.3 (*Lagrangian Subspace*)**.** A Lagrangian subspace is a subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ such that $\mathcal{L} = \mathcal{L}^{\perp\!\!\!\perp -}$, where $^{\perp\!\!\!\perp -}$ denotes the orthogonal companion with respect to the bilinear form $\langle\cdot, \cdot\rangle_-$.

Remark 2.4. Alternatively, a Lagrangian subspace is defined as a *maximal* subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ on which $\langle\cdot, \cdot\rangle_-$ is zero. Similarly to Dirac structures, the dimension of any Lagrangian subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ is equal to $n = \dim \mathcal{X}$.

Note that the gradient of the Hamiltonian $H(x) = \frac{1}{2}x^T Qx$ defines the special type of Lagrangian subspace

$$\mathcal{L}_Q := \{(x, Qx) \in \mathcal{X} \times \mathcal{X}^* \mid x \in \mathcal{X}\}, \tag{8}$$

i.e., the graph of the symmetric mapping Q. Furthermore, a Lagrangian subspace $\mathcal{L}$ can be put into the form (8) for a certain symmetric Q if and only if $\pi(\mathcal{L}) = \mathcal{X}$, while if $\pi(\mathcal{L}) \neq \mathcal{X}$ then the following algebraic constraints in the state x arise

$$x \in \pi(\mathcal{L}) \tag{9}$$

This type of algebraic constraints will be referred to as *Lagrange algebraic constraints*, since they are determined by the Lagrangian subspace $\mathcal{L}$.

A generalized port-Hamiltonian DAE system is now defined by a *pair* $(\mathcal{D}, \mathcal{L})$ as follows.

Definition 2.5 (*Generalized pH DAE System*)**.** Consider a Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ and a Lagrangian subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$. This defines the *generalized port-Hamiltonian DAE system* (briefly, gpH DAE system) $(\mathcal{D}, \mathcal{L})$, with dynamics given by

$$(-\dot{x}, e) \in \mathcal{D}, \; (x, e) \in \mathcal{L} \tag{10}$$

Here (10) should be read as follows. Consider any (feasible) $x \in \mathcal{X}$ for which there exist $e \in \mathcal{X}^*$ and $f \in \mathcal{X}$ such that $(x, e) \in \mathcal{L}$ and $(f, e) \in \mathcal{D}$. Then[3] minus the velocity $-\dot{x}$ is given as any such f.

Remark 2.6. *Uniqueness* of $-\dot{x} = f$ given a feasible x can be guaranteed only under extra assumptions. E.g., if the Lagrangian subspace is the graph of a symmetric map Q which is *positive definite*, then the corresponding pH DAE system has index 1 with unique $-\dot{x} = f$ and corresponding solution [2]. On the other hand, Lagrange algebraic constraints typically have index 2 or higher.

A *coordinate representation* of the dynamics (10) of the gpH DAE system $(\mathcal{D}, \mathcal{L})$ can be obtained as follows. As shown in [4,14], any Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ for an n-dimensional linear space $\mathcal{X}$ can be represented in kernel representation as

$$\mathcal{D} = \{(f, e) \in \mathcal{X} \times \mathcal{X}^* \mid Kf + Le = 0\} \tag{11}$$

for $n \times n$ matrices K, L satisfying

$$KL^T + LK^T = 0, \; \operatorname{rank}\begin{bmatrix} K & L \end{bmatrix} = n \tag{12}$$

[1] Substitute $f = -\dot{x}, e = Qx$. The minus sign in $f = -\dot{x}$ ensures consistent power flow sign convention.

[2] It should be noted that the definitions of Lagrangian subspaces and Dirac structures *diverge* in the nonlinear case, with Dirac structures on a manifold $\mathcal{X}$ still defining pointwise a linear subspace of the product $T_x\mathcal{X} \times T_x^*\mathcal{X}, x \in \mathcal{X}$, while Lagrangian subspaces generalize to Lagrangian *submanifolds* of the cotangent bundle $T^*\mathcal{X}$.

[3] Note that strictly speaking f is in the *tangent space* to $\mathcal{X}$ at x, which however by linearity of $\mathcal{X}$ can be identified with $\mathcal{X}$.

Analogously, see Proposition A.1 for a proof, any Lagrangian subspace admits a kernel representation

$$\mathcal{L} = \{(x, e) \in \mathcal{X} \times \mathcal{X}^* \mid S^T x - P^T e = 0\} \tag{13}$$

for $n \times n$ matrices P, S satisfying

$$S^T P = P^T S, \ \operatorname{rank} \begin{bmatrix} S^T & P^T \end{bmatrix} = n \tag{14}$$

Equivalently, the Lagrangian subspace $\mathcal{L}$ can be represented in image representation as

$$\mathcal{L} = \{(x, e) \in \mathcal{X} \times \mathcal{X}^* \mid \exists z \in \mathcal{Z} = \mathbb{R}^n \text{ s.t. } \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} P \\ S \end{bmatrix} z\} \tag{15}$$

It follows that the dynamics of the gpH DAE system defined by the pair $(\mathcal{D}, \mathcal{L})$ is obtained by setting $f = -\dot{x}$ in (11), yielding $K\dot{x} = Le$ with $(x, e) \in \mathcal{L}$. Using the image representation (15) of $\mathcal{L}$ this implies the following DAE system in the parametrizing state vector $z \in \mathcal{Z}$

$$KP\dot{z} = LSz \tag{16}$$

In case of Lagrange algebraic constraints the matrix P is not of full rank, inducing algebraic constraints in z, while in case of Dirac algebraic constraints the matrix K is not of full rank; also inducing algebraic constraints. A Hamiltonian function for the coordinate representation (16), in terms of the parametrizing state vector z, is defined by (note that $S^T P = P^T S$ by (14))

$$H(z) := \frac{1}{2} z^T S^T P z \tag{17}$$

In fact, along solutions of (16)

$$\frac{d}{dt} H(z) = z^T S^T P \dot{z} = e^T \dot{x} = 0, \tag{18}$$

since $e^T f = 0$ for all $(f, e) \in \mathcal{D}$. The model (16) together with the expression (17) was already postulated in [12].

(15) shows that the parametrizing state vector z can be always taken to be a mixture of the x and e variables; i.e., a mixture of *energy* and *co-energy* variables. This can be formalized as follows. Consider any Lagrangian subspace $\mathcal{L} \subset \mathcal{X} \times X^*$. Then, see Proposition A.2 for a proof, there always exists a sub-vector x_1 of $x \in \mathcal{X}$, and a complementary sub-vector e_2 of $e \in \mathcal{X}^*$, such that $\mathcal{L}$ is represented as

$$\mathcal{L} = \{(x, e) \in \mathcal{X} \times \mathcal{X}^* \mid \begin{bmatrix} e_1 \\ x_2 \end{bmatrix} = \widehat{Q} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix}\} \tag{19}$$

Particular cases are $x_1 = x$ and e_2 void, in which case $\widehat{Q} = Q$, or $e_2 = e$ and x_1 void, in which case $\widehat{Q} = Q^{-1}$ if Q is invertible, and the *co-energy function* $\frac{1}{2} e^T Q^{-1} e$ is the Legendre transform of $H(x) = \frac{1}{2} x^T Q x$.

An alternative, and in some sense *dual*, coordinate representation of a generalized port-Hamiltonian DAE system $(\mathcal{D}, \mathcal{L})$ can be obtained as follows. Consider based on (11) and (12) the *image* representation of $\mathcal{D}$ given as

$$\mathcal{D} = \operatorname{im} \begin{bmatrix} L^T \\ K^T \end{bmatrix}, \tag{20}$$

and the *kernel* representation (13) of $\mathcal{L}$. Substitution of $-\dot{x} = f = L^T v, e = K^T v$, with v an alternative parametrizing state vector, then leads to the DAEs

$$S^T L^T v + P^T K^T \dot{v} = 0 \tag{21}$$

By pre-multiplying (21) by z^T, and performing integration by parts on the second term $z^T P^T K^T \dot{v}$, this results in the previously obtained coordinate expression (16) in the parametrizing state vector z. Thus (21) can be considered as *a dual* (or *adjoint*) representation to (16).

2.1. Examples

Dirac algebraic constraints arise from the interconnection of subsystems. On the other hand, Lagrange algebraic constraints reflect degeneracies in the definition of energy-storage. This is illustrated by the following examples. The first two are standard examples of Dirac algebraic constraints, while the last three show how Lagrange algebraic constraints arise in physical systems modeling.

Example 2.7 (*Mechanical Systems with Kinematic Constraints*)**.** Consider a mechanical system with position coordinates $q \in \mathbb{R}^n$, momenta $p = M\dot{q} \in \mathbb{R}^n$, and mass matrix $M = M^T > 0$, subject to constant kinematic constraints $A^T \dot{q} = 0$, where A is an $n \times k$ matrix. Consider a Hamiltonian function $H(q, p) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T K q$ with K a matrix defining the elastic energy. The Dirac structure $\mathcal{D}$ is given as (see [4,3,2,5])

$$\begin{aligned} \mathcal{D} \;=\; & \{(f_q, f_p, e_q, e_p) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \mid \exists \lambda \in \mathbb{R}^k \text{ s.t.} \\ & \begin{bmatrix} f_q \\ f_p \end{bmatrix} = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} - \begin{bmatrix} 0 \\ A \end{bmatrix} \lambda, \ A^T e_p = 0\} \end{aligned}$$

Substitution of $e_p = M^{-1} p$ leads to the Dirac algebraic constraints $A^T M^{-1} p = 0$. Note that $A\lambda$ is the vector of constraint forces.

Example 2.8 (*LC-circuits*)**.** Dirac algebraic constraints are ubiquitous in electrical circuits; cf. [15] for the port-Hamiltonian modeling. Concentrating on LC-circuits, such constraints arise in two ways. The first case corresponds to the occurrence of a cycle in the circuit graph whose edges only contain *capacitors*. By Kirchhoff's voltage law the sum of the voltages across these capacitors is identically zero, leading to an algebraic constraint between the charges of those capacitors. The second case corresponds to the existence of a node in the circuit graph whose adjacent edges only contain *inductors*. By Kirchhoff's current law the sum of the currents entering this node is equal to zero, thus leading to an algebraic constraint between the flux linkages of those inductors.

Example 2.9 (*Mass–spring System with Zero Mass*)**.** Consider a mass–spring system with Hamiltonian $\widehat{H}(q, p) = \frac{1}{2} kq^2 + \frac{p^2}{2m}$, with m mass and k spring constant. For $m \neq 0$ the graph of the gradient of $\widehat{H}$ is given as the Lagrangian subspace

$$\begin{bmatrix} q \\ p \\ e_q \\ e_p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & m \\ k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

with $z_1 = q$ the *position* of the mass (an energy variable), and $z_2 = \frac{p}{m}$ its *velocity* (a co-energy variable). Now let m converge to zero. Then the Lagrangian subspace converges to the Lagrangian subspace

$$\begin{bmatrix} q \\ p \\ e_q \\ e_p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

which is not the graph of a symmetric map $Q : \mathcal{X} \to \mathcal{X}^*$ anymore, with resulting Lagrange algebraic constraint $p = 0$. We obtain the gpH DAE system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

with Hamiltonian $H(z_1, z_2) = \frac{1}{2} k z_1^2$. (Note that in this simple example the resulting DAE system is trivial, since necessarily $z_1 = q = 0$ whenever $k \neq 0$.)

Example 2.10 (*Mechanical Systems with Strong Constraining Force*). Consider a two-dimensional mass–spring system with Hamiltonian $\widehat{H}(q_1, q_2, p_1, p_2)$

$$\widehat{H} = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_{12}(q_2 - q_1)^2 + \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2$$

being the series interconnection of two masses m_1, m_2 and two springs with spring constants k_1, k_{12}. This defines the Lagrangian subspace given in image representation as

$$\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ e_{q_1} \\ e_{q_2} \\ e_{p_1} \\ e_{p_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{k_{12}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},$$

where we have chosen the parametrizing state vector z as the following mixture of energy and co-energy variables:

$$z_1 = q_1, \; z_2 = k_{12}(q_2 - q_1), \; z_3 = p_1, \; z_4 = p_2$$

(thus z_2 equals the elastic force of the second spring). Letting $k_{12} \to \infty$ (corresponding to the replacement of the second spring by a *rigid rod*) yields the Lagrangian subspace

$$\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ e_{q_1} \\ e_{q_2} \\ e_{p_1} \\ e_{p_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} =: \begin{bmatrix} P \\ S \end{bmatrix} z, \tag{22}$$

with P singular, entailing the algebraic constraint[4] $q_1 = q_2$. This leads to the gpH DAE system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \begin{bmatrix} k_1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

where $\frac{z_3}{m_1} = \frac{z_4}{m_2}$ (equality of velocity of the first and the second mass, linked by a rigid rod). Note that z_2 (whose derivative does not appear in the DAE system) represents the constraint force exerted by the rigid rod on the masses m_1 and m_2 (with opposite sign).

Example 2.11 (*Ideal Transformer*). An electrical transformer is a magnetic energy storage element consisting of two coils coupled by a magnetic core. Its constitutive relations define the Lagrangian subspace given by

$$S^T \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = P^T \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

in the magnetic fluxes φ_1, φ_2 and currents i_1, i_2 corresponding to the two coils. Here $S = \left(\frac{\mathcal{R}_m}{N_1 N_2}\right) I_2$, with I_2 the 2×2 identity matrix, and

$$P = \begin{bmatrix} \frac{N_1}{N_2}\left(1 + \frac{\mathcal{R}_m}{\mathcal{R}_{l1}}\right) & 1 \\ 1 & \frac{N_2}{N_1}\left(1 + \frac{\mathcal{R}_m}{\mathcal{R}_{l2}}\right) \end{bmatrix}$$

with reluctances $\mathcal{R}_{l1}, \mathcal{R}_{l2}$ and $\mathcal{R}_m$, and N_1, N_2 the number of turns of the two coils. In case of an *ideal* transformer, $\frac{\mathcal{R}_m}{\mathcal{R}_{li}} \to 0$ for $i = 1, 2$, and the rank of the matrix P drops from 2 to 1, leading to Lagrange algebraic constraints and the well-known transformer ratio.

Now connect the transformer at port 1 to a capacitor with electrical charge q, voltage v_C and capacitance C, and at port 2 to an inductor with flux Φ, current i_L and inductance L. Adding the constitutive relations of the capacitor $q = C\, v_C$ and of the inductor $\Phi = L\, i_L$, one obtains the extended Lagrangian subspace $\mathcal{L}_{tot}$ represented by

$$P_{tot} = \operatorname{diag}\left(\begin{bmatrix} \frac{N_1}{N_2} & 1 \\ 1 & \frac{N_2}{N_1} \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \right), \; S_{tot} = \operatorname{diag}(S, \; I_2)$$

in the energy variables $\varphi_1, \varphi_2, q, \Phi$, and co-energy variables v_1, v_2, i_C, v_L. The total Dirac structure $\mathcal{D}_{tot}$ of the system is given by the matrices $K_{tot} = I_4$ and

$$L_{tot} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

This defines a gpH DAE system given as in (16) or (21).

3. Algebraic constraint representations

In this section we further analyze Dirac and Lagrange algebraic constraints. First we elaborate on different representations of them.

Consider a Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$. Denote as before by $\pi : \mathcal{X} \times \mathcal{X}^* \to \mathcal{X}$ the projection on $\mathcal{X}$, and by $\pi^* : \mathcal{X} \times \mathcal{X}^* \to \mathcal{X}^*$ the projection on $\mathcal{X}^*$. The subspace $\mathcal{D} \cap (0 \times \mathcal{X}^*)$ defines the *conserved quantities* of any corresponding pH DAE system [5,3], while $\pi^*(\mathcal{D})$ defines the Dirac algebraic constraints.

Following [14] the bilinear form Δ on the subspace $\pi^*(\mathcal{D}) \subset \mathcal{X}^*$ given as

$$\Delta(\pi^*(v), \pi^*(w)) := \langle \pi^*(v) \mid \pi(w) \rangle, \quad v, w \in \mathcal{D} \tag{23}$$

is well-defined and skew-symmetric. Conversely [14], any skew-symmetric form on a subspace of $\mathcal{X}^*$ defines a Dirac structure $\mathcal{D}$. Thus Dirac structures are in one-to-one correspondence with skew-symmetric forms defined on subspaces of $\mathcal{X}^*$. Furthermore, it follows [4] that any Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$ can be *embedded* into the graph of a skew-symmetric map on an *extended* space.

Proposition 3.1. *Consider any Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$, and suppose $\pi^*(\mathcal{D}) \subset \mathcal{X}^*$ is $(n-k)$-dimensional. Define $\Lambda^* := \mathbb{R}^k$. Then there exists a full-rank $n \times k$ matrix G and a skew-symmetric $n \times n$ matrix J extending the skew-symmetric form Δ in (23), such that $\mathcal{D}$*

[4] This can be called a *geometric constraint*, although the set-up is *different* from the standard approach to geometric constraints following from the *integration* of kinematic constraints $A^T \dot{q} = 0$ as in Example 2.7 to $A^T q = c$, with the vector c determined by the initial condition of the system.

is given as the set of all points $(f, e) \in \mathcal{X} \times \mathcal{X}^*$ *satisfying for some* $\lambda \in \Lambda^*$

$$-f = Je + G\lambda, \quad 0 = G^T e \tag{24}$$

In fact, G is such that $\ker G^T = \pi^*(\mathcal{D})$*. Conversely, any such equations for a skew-symmetric map* $J : \mathcal{X}^* \to \mathcal{X}$ *define a Dirac structure.*

Hence any Dirac structure $\mathcal{D}$ extends to a Dirac structure $\widetilde{\mathcal{D}} \subset \mathcal{X} \times \Lambda \times \mathcal{X}^* \times \Lambda^*$ given by $(\widetilde{K}, \widetilde{L})$ defined as

$$\widetilde{K} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \; \widetilde{L} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} : \mathcal{X}^* \times \Lambda^* \to \mathcal{X} \times \Lambda \tag{25}$$

Analogously, cf. Proposition A.3, any Lagrangian subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ gives rise to the well-defined *symmetric* bilinear form on $\pi(\mathcal{L})$

$$\Sigma(\pi(v), \pi(w)) := \langle \pi^*(v) \mid \pi(w) \rangle, \quad v, w \in \mathcal{L} \tag{26}$$

Conversely any symmetric bilinear form on a subspace of $\mathcal{X}$ defines a Lagrangian subspace $\mathcal{L}$. Thus Lagrangian subspaces are in one-to-one correspondence with symmetric forms defined on subspaces of $\mathcal{X}$.

Furthermore, analogously to the Dirac structure case, cf. Proposition A.3, any Lagrangian subspace can be embedded into the graph of a symmetric mapping on an extended space.

Proposition 3.2. *For any Lagrangian subspace* $\mathcal{L}$ *there exists full-rank* $n \times k$ *matrix M and a symmetric* $n \times n$ *matrix Q extending the symmetric form* Σ *in (26) such that* $\mathcal{L}$ *is given as the set of all points* $(x, e) \in \mathcal{X} \times \mathcal{X}^*$ *satisfying for some* $\mu \in \mathcal{M} := \mathbb{R}^k$

$$e = Qx + M\mu, \quad 0 = M^T x \tag{27}$$

In fact, M is such that $\ker M^T = \pi(\mathcal{L})$*. Conversely, any such equations for a symmetric map* $Q : \mathcal{X} \to \mathcal{X}^*$ *define a Lagrangian subspace.*

Hence any Lagrangian subspace extends to a Lagrangian subspace $\widetilde{\mathcal{L}} \subset \mathcal{X} \times \mathcal{M} \times \mathcal{X}^* \times \mathcal{M}^*$ given by a pair $(\widetilde{P}, \widetilde{S})$ defined as

$$\widetilde{P} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \; \widetilde{S} = \begin{bmatrix} Q & M \\ M^T & 0 \end{bmatrix} : \mathcal{X} \times \mathcal{M} \to \mathcal{X}^* \times \mathcal{M}^* \tag{28}$$

Note that one can associate with (27) the constrained optimization problem of extremizing $\frac{1}{2}x^T Qx$ under the constraint $M^T x = 0$, or, using Lagrange multipliers, the unconstrained optimization (over x, μ) of $\frac{1}{2}x^T Qx + \mu^T M^T x$, with $\pi(\mathcal{L})$ the constrained state space and $\mathcal{L} \cap (\mathcal{X} \times 0)$ the set of constrained extrema.

3.1. From Dirac to Lagrange constraints, and back

These results can be employed as follows. Consider any gpH DAE system $(\mathcal{D}, \mathcal{L})$, with $\mathcal{D}$ given by (K, L) and $\mathcal{L}$ given by (P, S). Then we can convert its Dirac algebraic constraints into Lagrange algebraic constraints as follows. Define the extended Dirac structure $\widetilde{\mathcal{D}}$ given by $(\widetilde{K}, \widetilde{L})$ as in (25). Furthermore, define the extended Lagrangian subspace $\widetilde{\mathcal{L}} \subset \mathcal{X} \times \Lambda \times \mathcal{X}^* \times \Lambda^*$ by specifying

$$\widetilde{P} := \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \; \widetilde{S} := \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}, \tag{29}$$

i.e., $\widetilde{\mathcal{L}} = L \times 0 \times \Lambda^*$. This corresponds to the parametrizing extended state vector $\tilde{z} = \begin{bmatrix} x \\ \lambda \end{bmatrix}$, and a Hamiltonian $\widetilde{H}(\tilde{z})$ given as

$$\widetilde{H}(\tilde{z}) = \frac{1}{2}\tilde{z}^T \widetilde{S}^T \widetilde{P} \tilde{z} = \frac{1}{2} x^T S^T P x \tag{30}$$

(thus reducing in value to the original Hamiltonian function). The resulting gpH DAE system on the extended space is given as

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{31}$$

It is directly checked that any solution of (31) projects to a solution of the original gpH DAE system, and conversely any solution of the original gph DAE system is the projection of a solution of (31). Thus the gpH DAE system with Dirac and Lagrange algebraic constraints has been converted into a gpH DAE system in the extended state vector $\tilde{z}$ with *only Lagrange algebraic constraints*. This underlies some of the examples in [6], and shows that the framework adopted in [6] (which is not employing Dirac algebraic constraints) is in this sense general enough for the analysis of gpH DAE systems.

Analogously, we may as well convert the *Lagrange* algebraic constraints of the gpH DAE system $(\mathcal{D}, \mathcal{L})$ into additional *Dirac* algebraic constraints on an extended space. Consider for this purpose the extended Lagrangian subspace $\widetilde{L}$ given by $(\widetilde{P}, \widetilde{S})$ as in (28), corresponding to the Hamiltonian $\widetilde{H}(\tilde{x}) = \frac{1}{2}x^T Qx + x^T M\mu$. Then define the extended Dirac structure $\widetilde{\mathcal{D}} \subset \mathcal{X} \times \mathcal{M} \times \mathcal{X}^* \times \mathcal{M}^*$ by specifying

$$\widetilde{K} := \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}, \; \widetilde{L} := \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}, \tag{32}$$

i.e., $\widetilde{\mathcal{D}} = \mathcal{D} \times \mathcal{M} \times 0$. The resulting gpH DAE system is given as

$$\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q & M \\ M^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} \tag{33}$$

Thus, dually, we have converted the gpH DAE system with Dirac and Lagrange algebraic constraints into a pH DAE system on an extended space with *only Dirac algebraic constraints*. These two conversions, using 'generalized Lagrange multipliers' λ or μ, are especially useful for *simulation* of gpH DAE systems; see also [6].

Example 3.3. Consider the system in Example 2.10, where we additionally impose as in Example 2.7 the kinematic constraint $\dot{q}_1 = 0$. The skew-symmetric map $\widetilde{L}$ as in (25) is given as

$$\widetilde{L} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

where the scalar λ corresponds to the constraint force for the kinematic constraint $\dot{q}_1 = 0$. The extended Lagrangian subspace $\widetilde{\mathcal{L}}$ as in (29) is specified by (cf. the expressions of P, S in (22))

$$\widetilde{S} = \begin{bmatrix} k_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; \widetilde{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This yields the following gpH DAE system as in (31)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ \lambda \end{bmatrix}$$

Example 3.4 (*Singular Optimal Control*)**.** This example does not originate from physical system modeling, but instead from *optimal* *control*, and is partly based on [11] (see also [2] and references quoted in there). Consider the totally singular optimal control problem of minimizing a quadratic cost criterion $\frac{1}{2}\int_0^T q^T(t)Gq(t)dt$, with $G = G^T \geq 0$, over the control system $\dot{q} = Aq + Bu$ with $q \in \mathbb{R}^n, u \in \mathbb{R}^m$. Define the optimal control Hamiltonian

$$H(q,p,u) = p^T(Aq+Bu) + \frac{1}{2}q^TGq$$

with $p \in \mathbb{R}^n$ the co-state vector. Application of Pontryagin's Maximum principle leads to the consideration of the pH DAE system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} G & A^T & 0 \\ A & 0 & B \\ 0 & B^T & 0 \end{bmatrix} \begin{bmatrix} q \\ p \\ u \end{bmatrix}$$

involving *Dirac* algebraic constraints. On the other hand, the system can be rewritten as a gpH DAE system in the (q,p) variables, having only *Lagrange* algebraic constraints, with Dirac structure $\mathcal{D}$ given by the graph of $-J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ and Lagrangian subspace

$$\mathcal{L} = \{\left(\begin{bmatrix} q \\ p \end{bmatrix}, \begin{bmatrix} e_q \\ e_p \end{bmatrix}\right) \mid \begin{bmatrix} e_q \\ e_p \end{bmatrix} = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u, B^Tp = 0\}$$

with u ranging over $\mathbb{R}^m$ (playing the role of the generalized Lagrange multiplier vector μ).

4. Conclusions

Following [11], and inspired by [6], we have elaborated on a geometric definition of generalized port-Hamiltonian DAE systems, defined by pairs of Dirac structures and Lagrangian subspaces. For physical system models, the Dirac structure corresponds to the interconnection structure of the system, while the Lagrangian subspace corresponds to the definition of their energy. This generalizes the classical definition of port-Hamiltonian systems by symmetrizing the role of energy and co-energy variables, and allowing for degenerate energy or co-energy functions. In particular we analyzed their algebraic constraints and representations as DAE systems using the kernel or image representations of both the Dirac structure and the Lagrangian subspace. As a result we showed how systems with both Dirac and Lagrange algebraic constraints can be converted, through the use of generalized Lagrange multipliers, to a system involving only Lagrange or only Dirac algebraic constraints. The study of gpH DAE systems, e.g. their regularity and index properties, appears to be of great interest; see already [6,9,7,10,2].

Although for clarity of exposition we restricted attention to systems without energy-dissipation and external variables, the extension is straightforward by replacing the Hamiltonian function in the standard definition of a pH DAE system by a Lagrangian subspace. Important further extensions concern the generalization to *distributed-parameter* systems, and to *nonlinear* systems, replacing Lagrangian subspaces by Lagrangian submanifolds (see [11]).

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Appendix

Proposition A.1. *A subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ with $\dim \mathcal{X} = n$ is a Lagrangian subspace if and only if there exist $n \times n$ matrices P, S satisfying*

$$S^TP = P^TS, \; \operatorname{rank}\begin{bmatrix} S^T & P^T \end{bmatrix} = n \tag{34}$$

such that (see (15))

$$\mathcal{L} = \{(x,e) \in \mathcal{X} \times \mathcal{X}^* \mid \exists z \in \mathcal{Z} = \mathbb{R}^n \text{ s.t. } \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} P \\ S \end{bmatrix} z\} \tag{35}$$

Proof. The 'if' direction follows by checking that $\langle\langle (x_1,e_1),(x_2,e_2)\rangle\rangle_- = 0$ for any two pairs (x_i,e_i) with $x_i = Pz_i, e_i = Sz_i, i = 1,2$, and P, S satisfying (34).

For the 'only if' direction we note that any n-dimensional subspace $\mathcal{L}$ can be written as in (35) for certain $n \times n$ matrices P, S satisfying $\operatorname{rank}\begin{bmatrix} S^T & P^T \end{bmatrix} = n$. Then take any two pairs $(x_i, e_i) \in \mathcal{L}$ with $x_i = Pz_i, e_i = Sz_i, i = 1,2$. Since $\mathcal{L}$ is Lagrangian it follows that

$$\begin{aligned} 0 \;=\; & \langle\langle (x_1,e_1),(x_2,e_2)\rangle\rangle_- = z_2^TS^TPz_1 - z_1^TS^TPz_2 = \\ & -z_1^T(S^TP - P^TS)z_2 \end{aligned} \tag{36}$$

for all z_1, z_2, implying that $S^TP = P^TS$. ■

Proposition A.2. *Consider any Lagrangian subspace $\mathcal{L} \subset \mathcal{X} \times X^*$ with kernel representation (see the previous Proposition A.1)*

$$\mathcal{L} = \{(x,e) \in \mathcal{X} \times \mathcal{X}^* \mid S^Tx - P^Te = 0\} \tag{37}$$

for $n \times n$ matrices P, S satisfying (34). Suppose $\operatorname{rank} P = m \leq n = \dim \mathcal{X}$. Then there exists an m-dimensional sub-vector x_1 of $x \in \mathcal{X}$, and a complementary $n-m$-dimensional sub-vector $e_2 \in \mathcal{X}^*$ such that $\mathcal{L}$ is represented as

$$\mathcal{L} = \{(x,e) \in \mathcal{X} \times \mathcal{X}^* \mid \begin{bmatrix} e_1 \\ x_2 \end{bmatrix} = \widehat{Q} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix}\} \tag{38}$$

with

$$\widehat{Q}^T \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix} \widehat{Q} \tag{39}$$

Proof. The proof resembles the proof of a similar statement for Dirac structures in [16]. Write, possibly after row permutations of $P, P^T = \begin{bmatrix} P_1^T & P_2^T \end{bmatrix}$ with P_1 having m rows and $\operatorname{rank} P_1 = \operatorname{rank} P$. Then $\operatorname{im} P_2^T \subset \operatorname{im} P_1^T$. Furthermore, $S^TP = P^TS$ yields

$$S_1^TP_1 + S_2^TP_2 = P_1^TS_1 + P_2^TS_2 \tag{40}$$

Combined with surjectivity of P_1 and $\operatorname{im} P_2^T \subset \operatorname{im} P_1^T$ this yields $\operatorname{im} S_1^T \subset \operatorname{im} P_1^T + \operatorname{im} S_2^T$. Hence

$$\operatorname{rank}\begin{bmatrix} S_2^T & P_1^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} S_1^T & S_2^T & P_1^T & P_2^T \end{bmatrix} = n, \tag{41}$$

thus implying that $\begin{bmatrix} S_2^T & P_1^T \end{bmatrix}$ is invertible. In view of (35) we have

$$\begin{bmatrix} e_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} S_1 \\ P_2 \end{bmatrix} z, \; \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ S_2 \end{bmatrix} z \tag{42}$$

implying that

$$\begin{bmatrix} e_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} S_1 \\ P_2 \end{bmatrix} \left(\begin{bmatrix} P_1 \\ S_2 \end{bmatrix}\right)^{-1} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} =: \widehat{Q} \begin{bmatrix} x_1 \\ e_2 \end{bmatrix} \tag{43}$$

Since $\mathcal{L}$ is Lagrangian it follows that for all $(x_j, e_j) \in \mathcal{L}, j = a, b$

$$x_b^Te_a = x_a^Te_b \tag{44}$$

Writing out $x_j = \begin{bmatrix} x_{j1} \\ x_{j2} \end{bmatrix}$ and $e_j = \begin{bmatrix} e_{j1} \\ e_{j2} \end{bmatrix}$ this yields

$$x_{b1}^Te_{a1} - x_{a2}^Te_{b2} = x_{a1}^Te_{b1} - x_{b2}^Te_{a2} \tag{45}$$

implying equality (39). ■

Proposition A.3. *Let $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ be a Lagrangian subspace. Then*

$$\Sigma(\pi(v), \pi(w)) := \langle \pi^*(v) \mid \pi(w) \rangle, \quad v, w \in \mathcal{L} \tag{46}$$

is a well-defined and symmetric bilinear form on $\pi(\mathcal{L})$. Furthermore, the symmetric map induced by Σ can be extended to the symmetric map $\widetilde{S}$ as in (28) with $\ker M^T = \pi(\mathcal{L})$, in such a way that $\mathcal{L}$ is given by (27).

Proof. In order to prove that Σ is *well-defined* let v_1, v_2 be such that $\pi(v_1) = \pi(v_2)$. Then $v := v_1 - v_2 \in \mathcal{L}$ satisfies $\pi(v) = 0$, and thus for any $w \in \mathcal{L}$

$$\langle \pi^*(v) \mid \pi(w)\rangle = \langle \pi^*(w) \mid \pi(v)\rangle = 0 \tag{47}$$

showing that indeed $\langle \pi^*(v_1) \mid \pi(w)\rangle = \langle \pi^*(v_2) \mid \pi(w)\rangle$ for any $w \in \mathcal{L}$. Symmetry of Σ directly follows from $\langle \pi^*(v) \mid \pi(w)\rangle = \langle \pi^*(w) \mid \pi(v)\rangle$ for any two $v, w \in \mathcal{L}$. As done in [4] for the Dirac structure case we may extend the symmetric map induced by Σ to the symmetric map Q as in the left-upper block of (28). Since $\mathcal{L}$ is Lagrangian it easily follows that $\mathcal{L} \cap (0 \times \mathcal{X}^*) = \pi(\mathcal{L})^\perp$ with $^\perp$ denoting the orthogonal complement with respect to the duality pairing between $\mathcal{X}$ and $\mathcal{X}^*$. Define M such that $\ker M^T = \pi(\mathcal{L})$. Now, let $(x, e) \in \mathcal{L}$. Then $x \in \ker M^T = \pi(\mathcal{L})$ and $e = Qx$ modulo $(\ker M^T)^\perp = \operatorname{im} M$, and thus $\mathcal{L}$ is indeed given by (27). ■

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