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Conditionally complete sponges: new results on generalized lattices

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Abstract

Sponges were recently proposed as a generalization of lattices, focussing on joins/meets of sets, while letting go of associativity/transitivity. In this work we provide tools for characterizing and constructing sponges on metric spaces and groups. These are then used in a characterization of epigraph sponges: a new class of sponges on Hilbert spaces whose sets of left/right bounds are formed by the epigraph of a rotationally symmetric function. Finally, the so-called hyperbolic sponge is generalized to more than two dimensions.

Keywords: lattice, hyperbolic space, Hilbert space, mathematical morphology, orientation, sponge

1. Introduction

Sponges are generalizations of lattices that were recently introduced by van de Gronde [1, 2, 3], as a possible solution to the long-standing problem of applying mathematical morphology to nonscalar data. Morphological theory is based on lattices, but these are fairly restrictive when it comes to defining group-invariant instances on vector spaces or manifolds [4, 5], and are utterly incompatible with periodic spaces (assuming we wish to somehow preserve the periodicity in the lattice structure). As a result, over the years several schemes have been suggested for morphological purposes that let go of lattices, or that try to work around the issue, while retaining something resembling a lattice's join and meet [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Unfortunately most of these schemes lack(ed) a supporting body of theory, making it hard to say much about the behaviour of the resulting filters. Sponges are meant to provide exactly such a framework, and have already been shown to encompass two schemes for vector spaces and hyperbolic spaces [7, 14], to allow the processing of angles in a natural way (without breaking the periodic nature of angles), and to support joins/meets on (hemi)spheres with a designated "lowest" point. While we will give various new examples of sponges, the main goal of this work is to provide additional tools to analyse and identify sponges.

Roughly speaking, an *orientation* is a partial order without transitivity and a *sponge* is a set with an orientation that has meets and joins for all subsets

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satisfying certain conditions. The relevance of a meet or join of a set in the absence of transitivity is due to preservation under isometries (or other kinds of automorphisms) that permute the elements of the set. Indeed, long before the introduction of sponges, the meet of the inner-product sponge was already used for this purpose [17, 18].

As examples of sponges, van de Gronde and Roerdink [3] present the inner-product sponge [7], the one-dimensional angle sponge, and a two-dimensional hyperbolic sponge [14]. These examples are generalized and treated in Sections 3, 5 and 7, respectively. Note that sponges on spheres and hemispheres were also given in [3], but these are (essentially) isomorphic to the inner-product sponge, so we do not treat these here.

In this work we first give a short overview of the main definitions concerning orientations and sponges in Section 2. In Section 3 we briefly revisit the inner-product sponge [7, 3]. Next, in Section 4 we derive a new result that makes it easier to identify sponges in metric spaces. In Section 5 we discuss sponge groups, and in Section 6 we introduce and characterize a new class of sponge (groups) called *epigraph sponges*. In Section 7 we generalize the hyperbolic sponge to the higher dimensional case (previously, only the 2D case was treated [14, 3]). Finally, Section 8 contains a comparison of the various sponges in terms of the geometry of their sets of left and right bounds, as well as the existence of extreme points.

2. Definitions

Let S be a set. An *orientation* of S is a binary relation $\preceq$ on S that satisfies:

reflexivity: $x \preceq x$ for all $x \in S$, and
antisymmetry: $x \preceq y \ \wedge\ y \preceq x \implies x = y$ for all $x, y \in S$.

The pair $(S, \preceq)$ of a set S with an orientation $\preceq$ is called an *oriented set*. A transitive orientation is a partial order. If the orientation $\preceq$ is not transitive, we may consider its reflexive-transitive closure $\preceq^*$. The orientation $\preceq$ is called *acyclic* iff it contains no cycles, which is equivalent to $\preceq^*$ being a partial order.

If P and Q are subsets of an oriented set $(S, \preceq)$, we write $P \preceq Q$ to denote that $p \preceq q$ holds for all $p \in P$ and $q \in Q$. A subset P of S is called *right bounded* iff $P \preceq \{s\}$ for some $s \in S$; it is called *left bounded* iff $\{s\} \preceq P$ for some $s \in S$.[1] Let the set of all right bounds of P be denoted by $R(P)$, and the set of all left bounds by $L(P)$. We abbreviate $R(\{x\})$ by $R(x)$.

Let $J(P)$ and $M(P)$ be subsets of S defined by

$$\begin{aligned} x \in J(P) \quad &\equiv \quad P \preceq \{x\} \ \wedge\ (\forall y \in S : P \preceq \{y\} \implies x \preceq y)\,, \\ x \in M(P) \quad &\equiv \quad \{x\} \preceq P \ \wedge\ (\forall y \in S : \{y\} \preceq P \implies y \preceq x)\,. \end{aligned}$$

[1] Left and right, rather than lower and upper, are used to warn the reader about the lack of transitivity.

If $x, y \in J(P)$, then $P \preceq \{x\}$ and $P \preceq \{y\}$, and hence $x \preceq y$ and $y \preceq x$, and therefore $x = y$ by antisymmetry. This proves that $J(P)$ is always empty or a singleton set [19]. A similar argument proves that $M(P)$ is always empty or a singleton set. If $J(P)$ or $M(P)$ has an element, its unique element is called the *join* or *meet* of P, respectively.

A *sponge* is defined to be an oriented set $(S, \preceq)$ in which every finite, nonempty, right-bounded subset has a join, and every finite, nonempty, left-bounded subset has a meet. If the property holds for joins but not necessarily for meets, we have a join-semisponge (we can define a meet-semisponge analogously). Note that in the original introduction of sponges, J and M were considered partial functions returning a particular element rather than a set of elements. Given that in an orientation J and M always return either the empty set or a singleton set, these views are equivalent.

Alternatively, a sponge can be defined algebraically as a set S with functions J and M, with a domain that includes (at least) all finite, nonempty subsets of S and a range that includes *no more* than all singleton subsets of S, as well as the empty set. To be a sponge, J and M should satisfy (with $y \in S$ and P a finite, nonempty subset of S):

absorption: $\forall x \in P : M(\{x\} \cup J(P)) = \{x\}$,
part preservation: $[\forall x \in P : M(\{x, y\}) = \{y\}]$
$\implies M(P) \neq \emptyset \wedge M(M(P) \cup \{y\}) = \{y\}$,
and the same properties with the roles of J and M reversed.

Note that compared to the original algebraic definition [2, §4.2], absorption is now defined slightly more elegantly, and idempotence now follows from the two absorption laws:

$$M(\{x\}) = M(\{x\} \cup \{x\}) = M(\{x\} \cup J(\{x\} \cup M(P))) = \{x\}$$

for any (finite) $P \supseteq \{x\}$ (the analogous statement $J(\{x\}) = \{x\}$ also holds). In contrast, part preservation needs to explicitly claim that $M(P)$ is nonempty. These changes occur because the empty set behaves differently from the "undefined" value used in the original definition. It has been shown [2, §4.3] that the orientation-based and algebraic definitions are equivalent.

Compared to the algebraic definition of a lattice, the main difference is that we have the somewhat weaker property of part preservation rather than associativity. On the other hand, given that J and M operate on sets rather than being binary operators, commutativity is implied. It should be noted that sponges are closely related to the concept of a weakly associative lattice (WAL) or trellis [20, 21, 22]. However, a WAL requires the join and meet to be defined for all pairs rather than all (finite and nonempty) *bounded* sets. It is known that the former by no means implies the latter [19], and this makes WALs less suited for use in mathematical morphology, as this field relies heavily on the existence of joins and meets of sets. Conversely, in mathematical morphology it often suffices to guarantee the existence of joins and meets of bounded sets, again making sponges a better fit than WALs. Often, it is convenient to be able to consider joins/meets not just over finite sets, but also infinite sets. This is part of our motivation to focus on conditionally complete sponges in the current work (the other part being that all practical examples examined so far belong to this category).

An oriented set $(S, \preceq)$ is called a *conditionally complete sponge* (or *cc sponge* for short) iff, for every nonempty right-bounded subset P of S, the set $J(P)$ is nonempty and, for every nonempty left-bounded subset P of S, the set $M(P)$ is nonempty. It is clear that a cc sponge is a sponge. As the next lemma shows, the requirement on $M(P)$ can be omitted (alternatively, by symmetry, the requirement on $J(P)$ can be omitted).

Lemma 1. *Let $(S, \preceq)$ be an oriented set such that, for every nonempty right-bounded subset P of S, the set $J(P)$ is nonempty. Then $(S, \preceq)$ is a conditionally complete sponge.*

Proof. It suffices to prove that, for every nonempty left-bounded subset P of S, the set $M(P)$ is nonempty. Let P be a nonempty left-bounded subset of S. Define $Q = \{x \mid \{x\} \preceq P\}$. As P is left bounded, Q is nonempty. As P is nonempty, Q is right bounded. Therefore $J(Q)$ is nonempty. Choose $x \in J(Q)$. This means that $Q \preceq \{x\}$ and that

$$\forall y \in S : Q \preceq \{y\} \implies x \preceq y.$$

We claim $x \in M(P)$. For every $y \in P$, we have $Q \preceq \{y\}$ and hence $x \preceq y$; this proves $\{x\} \preceq P$. Now let $y \in S$ have $\{y\} \preceq P$. Then $y \in Q$ and hence $y \preceq x$. This proves $x \in M(P)$. □

Example 1. The set $\mathbb{R}$ of the real numbers with $\leq$ as orientation is a cc sponge (a cc lattice in fact), because every nonempty bounded subset of $\mathbb{R}$ has a supremum.

3. The inner-product sponge

Let E be a real Hilbert space with the inner product denoted by $(\_, \_)$. Let relation $\preceq$ on E be defined by [3, §5.1]

$$x \preceq y \quad \equiv \quad (x, x) \leq (x, y).$$

It is clear that $x \preceq x$ always holds. See Fig. 1 for an illustration of the orientation.

Example 2. Assume $E = \mathbb{R}^2$ with the standard inner product. Consider the four vectors $w = (1, 0)$, $x = (2, 0)$, $y = (2, 1)$, and $z = (1, 3)$. Then we have $w \preceq \{x, y, z\}$, and $x \preceq y$, and $y \preceq z$, but $x \not\preceq z$. It follows that, $x \in M(\{x, y\})$ and $y \in M(\{y, z\})$, but $x \notin M(\{x, y, z\})$.

Lemma 2. *Let $x \preceq y$ and $x \neq y$. Then $\|x\| < \|y\|$.*

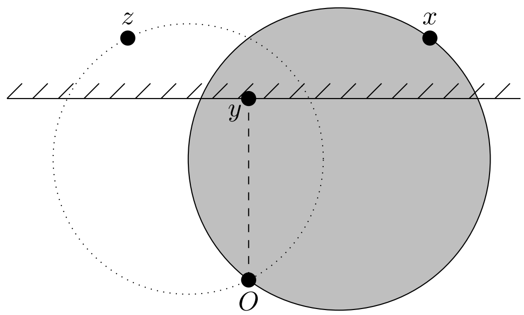

Figure 1: Illustration of the inner-product orientation. We have $x \preceq y$, as well as $y \preceq z$, but x and z are incomparable. The origin is marked as O. Sets of lower bounds are disks (illustrated for x and z). Sets of upper bounds are closed half-spaces (illustrated for y).

Proof. If $x = 0$, the assertion holds trivially. We may therefore assume that $x \neq 0$. Therefore $\|x\| > 0$. By Cauchy-Schwarz, $|(x, y)| \leq \|x\| \cdot \|y\|$ with equality if and only if y is a multiple of x. On the other hand, $\|x\|^2 = (x, x) \leq |(x, y)|$ because $x \preceq y$. It remains to consider the case that y is a multiple of x, say $y = \lambda x$. As $\|x\| > 0$ and $(x, x) \leq (x, y)$, this implies $\lambda \geq 1$, and hence $y = x$ or $\|x\| < \|y\|$. $\square$

Corollary 1. *Relation $\preceq$ is an acyclic orientation on E.*

Theorem 1. *The pair $(E, \preceq)$ is a cc sponge with a least element.*

Proof. It can be verified that 0 is less than (or equal to) every element in E. By Lemma 1, It therefore suffices to show that every nonempty subset of E has a meet. Let P be a nonempty subset of E. It suffices to show that P has a meet. We have

$$\{x\} \preceq P \quad \equiv \quad P \subseteq R(x), .$$

If $x = 0$ then $R(x) = E$. In all other cases, $R(x)$ is the closed halfspace $\{y \mid (x, x) \leq (x, y)\}$. The intersection of *all* closed halfspaces that contain P is the closed convex hull $\mathrm{cv}(P)$ of P, i.e., the topological closure of the convex hull of P.

We distinguish two cases. First, assume there is no $x \neq 0$ with $\{x\} \preceq P$. Then it is easily seen that 0 is the meet of P. Otherwise, there exists $x \neq 0$ with $\{x\} \preceq P$. Then all elements of $\mathrm{cv}(P)$ are farther from the origin then x. As $\mathrm{cv}(P)$ is closed, convex, and nonempty in the Hilbert space E, there is a unique point $z \in \mathrm{cv}(P)$ with smallest distance $\|z\|$ to the origin. We claim that z is the meet of P.

We first prove $\{z\} \preceq P$. Indeed, for any $p \in P$, the line segment between z and p is contained in $\mathrm{cv}(P)$; therefore all its points have a distance to the origin $\geq \|z\|$; therefore the angle between the vectors $p - z$ and $0 - z$ is not sharp, i.e. $(p - z, 0 - z) \leq 0$, and hence $(z, z) \leq (z, p)$, i.e. $z \preceq p$.

It remains to observe that, for any vector y with $\{y\} \preceq P$, we have $z \in \mathrm{cv}(P) \subseteq R(y)$, so that $z \preceq y$. This proves that z is the meet of P. $\square$

We remark that although every nonempty subset of E has a meet, not every nonempty subset also has a join. In particular, the subset needs to be contained in a ball with the origin on its boundary to even be right bounded. It is, however, possible to extend E with an extra element so that every nonempty set is right bounded [3, §5.3].

4. A sponge in a complete metric space

We consider a *topological orientation* to be a topological space S with an orientation, such that the orientation relation is a closed subset of the product space $S \times S$. In other words, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, and $x_n \preceq y_n$ for all $n \in \mathbb{N}$, then $x \preceq y$. This is in line with the concept of a topological lattice used by Birkhoff [23, §X.11], but slightly stricter than the analogous concept of a partially ordered topological space considered by Ward [24] (who only requires sets of left and right bounds to be closed). Note that Birkhoff shows that the weaker concept is equivalent to the stronger concept in complete lattices; Lemma 3 below shows that at least in some cases something similar holds for orientations as well.

Let S be a complete metric space with distance function d. Let $\preceq$ be a topological orientation on S. A function $h : S \to \mathbb{R}$ is called a *discriminator* iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in S : x \preceq y\ \wedge\ h(y) < h(x) + \delta \implies d(x, y) < \varepsilon.$$

This condition implies that h is strictly monotonic, in the sense that $x \preceq y\ \wedge\ x \neq y$ implies $h(x) < h(y)$. The following theorem shows how in a complete metric space cc sponges can be characterized by the existence of meets of all left-bounded *pairs* rather than all left-bounded nonempty sets. This is similar in spirit to what Birkhoff has shown for lattices [23, §X.10 Thm. 16].

Theorem 2. *Assume that S is a complete metric space, that $(S, \preceq)$ is a topological orientation, that every left-bounded pair in S has a meet, and that $h : S \to \mathbb{R}$ is continuous and a discriminator. Then $(S, \preceq)$ is a cc sponge.*

Proof. Let P be a nonempty right-bounded subset of S. It suffices to prove that P has a join. Write $Q = \{x \mid P \preceq \{x\}\}$. As P is right bounded, Q is nonempty. For every $p \in P$, $q \in Q$, we have $h(p) \le h(q)$. Every pair of elements of Q is left bounded (by an element of P), and therefore has a meet, which is easily seen to be in Q.

Let H be the infimum of $h(q)$ over all $q \in Q$. It holds that $-\infty < H < \infty$ because Q is nonempty and left bounded. This implies that there is an infinite sequence $(q_n)_{n\in\mathbb{N}}$ in Q with $\lim_{n\to\infty} h(q_n) = H$. We first prove that this sequence is a Cauchy sequence. Let $\varepsilon > 0$ be given. As h is a discriminator, there is a number $\delta > 0$ such that, for all $x, y \in Q$ with $x \preceq y$ and $h(y) < h(x) + \delta$, $d(x, y) < \frac{1}{2}\varepsilon$. As $\lim_{n\to\infty} h(q_n) = H$, there is a number m such that $h(q_n) < H + \delta$ for all $n \ge m$. For indices $i, j \ge m$, the pair $\{q_i, q_j\}$ in Q has a meet $z_{ij} \in Q$. Therefore, $H \le h(z_{ij})$. It follows that both $h(q_i)$ and $h(q_j)$ are less than $h(z_{ij}) + \delta$.

As $\{z_{ij}\} \preceq \{q_i, q_j\}$, this implies that $d(q_i, q_j) \leq d(q_i, z_{ij}) + d(z_{ij}, q_j) < \varepsilon$. This proves that $(q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Because S is a complete metric space, the Cauchy sequence has a limit, say r. As function h is continuous, $h(r) = H$. On the other hand, $r \in Q$ holds because relation $\preceq$ is topologically closed. For every $q \in Q$, the pair q, r has a meet $z \in Q$, with $h(r) = H \leq h(z)$. As h is strictly monotonic and $z \preceq r$, it follows that $r = z \preceq q$. This proves $\{r\} \preceq Q$, and hence that r is the join of P. □

Note that the above proof implies that r in no way depends on the precise choice of the sequence $(q_n)_{n\in\mathbb{N}}$. Also, it should be clear that we could just as easily have shown the dual statement, so for completeness:

Corollary 2. *Assume that S is a complete metric space, that $(S, \preceq)$ is a topological orientation, that every right-bounded pair in S has a join, and that $h : S \to \mathbb{R}$ is continuous and a discriminator. Then $(S, \preceq)$ is a cc sponge.*

Corollary 3. *Let $\preceq$ be a topological orientation on $\mathbb{R}$, which implies $\leq$. Assume every left-bounded pair for $\preceq$ has a meet in $\mathbb{R}$, and that the distance function is given by $d(x, y) = |x - y|$. Then $(\mathbb{R}, \preceq)$ is a cc sponge.*

Proof. Theorem 2 is applied to $S := \mathbb{R}$ with for h the identity function. It is clear that h is continuous. It is a discriminator because $x \leq y < x + \varepsilon$ implies $d(x, y) < \varepsilon$. □

5. Sponge groups

In this section, we investigate the possibility to combine the structures of sponges and groups in a useful manner. For the sake of generality, the theory is developed for not necessarily commutative groups. In such a group, the group operation is denoted by $\cdot$, and the neutral element by $\mathbf{1}$. Note that some of the results shown here have been shown earlier for weakly associative lattices by Rachůnek [25].[2]

5.1. Oriented groups and sponge groups

An *oriented group* G is defined to be a group with an orientation $\preceq$, such that

$$\forall x, y, z \in G : \quad x \preceq y \implies x \cdot z \preceq y \cdot z \ \wedge \ z \cdot x \preceq z \cdot y. \tag{1}$$

It is called a *cc sponge group* iff moreover $(G, \preceq)$ is a cc sponge. Note that inversion reverses the orientation: $x \preceq y \quad \equiv \quad y^{-1} \cdot x \preceq \mathbf{1} \quad \equiv \quad y^{-1} \preceq x^{-1}$.

In an oriented group $(G, \preceq)$ with unit element $\mathbf{1}$, the *positive cone* is the subset C of G of the elements $x \in G$ with $\mathbf{1} \preceq x$. It is easy to see that this set satisfies

$$\begin{aligned} &C \cap C^{-1} = \{\mathbf{1}\}, \\ &\forall x \in G, y \in C : x \cdot y \cdot x^{-1} \in C. \end{aligned} \tag{2}$$

[2] Rachůnek referred to orientations as semi-orders; unfortunately, this term is also used for other concepts.

The second condition says that C is invariant under conjugation.

Conversely, if G is a group with a subset C that satisfies the properties in Eq. (2), then one can define the orientation $\preceq$ on G by

$$x \preceq y \quad \equiv \quad x^{-1} \cdot y \in C.$$

This makes $(G, \preceq)$ an oriented group. Indeed, relation $\preceq$ is reflexive because $\mathbf{1} \in C$. It is antisymmetric because, if $x \preceq y$ and $y \preceq x$, then $x^{-1} \cdot y \in C \cap C^{-1} = \{\mathbf{1}\}$, so that $x = y$. Equation (1) is easily seen to hold. This proves that $(G, \preceq)$ is an oriented group. The orientation is a partial order if and only if $C \cdot C \subseteq C$.

In a commutative group G, the operation is usually written as addition $+$, with neutral element $\mathbf{0}$. By the commutativity of the group operation $(+)$, Eq. (1) reduces to

$$x \preceq y \implies x + z \preceq y + z. \tag{1'}$$

The positive cone C is characterized by

$$C \cap -C = \{\mathbf{0}\}. \tag{2'}$$

So $(G, \preceq)$ is an oriented additive group if and only if there is a set C that satisfies Eq. (2').

Example 3. The easiest example is the additive group $(\mathbb{R}, +)$. The order $\leq$ is an orientation that satisfies condition (1) because $x \leq y$ implies $x + z \leq y + z$. The triple $(\mathbb{R}, +, \leq)$ is a cc sponge, because every nonempty left-bounded subset of $\mathbb{R}$ has an infimum and every nonempty right-bounded subset has a supremum.

Similarly, the vector space $\mathbb{R}^n$ becomes a cc sponge when $x \leq y$ is defined to mean $\forall i : x_i \leq y_i$.

Example 4. Most interesting sponge groups are commutative. A noncommutative one is the group G of the real matrices

$$g(s, t, u) = \begin{pmatrix} s & u \\ 0 & t \end{pmatrix}$$

with s, $t > 0$. Let C be the subset containing the matrices $g(1, 1, u)$ with $u \geq 0$. Then $(G, \preceq)$ is a sponge: a nonempty subset V of G is left bounded iff there are $s, t > 0$ and $a \in \mathbb{R}$ with $V \subseteq \{g(s, t, u) \mid a \leq u\}$. Its meet is $g(s, t, b)$ for $b = \inf\{u \mid g(s, t, u) \in V\}$.

Example 5. Let $G = \mathrm{GL}_n(\mathbb{R})$, the group of the invertible real $n \times n$ matrices. Let C be the set of diagonalizable matrices with all eigenvalues real and ≥ 1. The set C satisfies the properties in Eq. (2). It therefore induces an orientation $\preceq$ that makes $(G, \preceq)$ an oriented group.

If $n > 1$, then $(G, \preceq)$ is not a sponge. For $n = 2$, this is shown as follows. Assume that it is a sponge, and consider the elements

$$h(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \qquad g(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

For $1 < t$, we have $h(u) \preceq g(t)$ because $h(u)^{-1} \cdot g(t) \in C$. As $(G, \preceq)$ is a sponge, the pair $h(0)$ and $h(1)$ has a join, say k. For all $t > 1$, we have $\mathbf{1} = h(0) \preceq k \preceq g(t)$. This implies that $k \in C$ and $\det(k) = 1$. The identity is the only element of C with determinant 1. This proves that $k = \mathbf{1}$. This implies $h(1) \preceq \mathbf{1}$, a contradiction. □

The following result is used in the next section.

Lemma 3. *Assume that G is a topological group and that $(G, \preceq)$ is an oriented group (but not necessarily a topological orientation). Then, the positive cone C is closed if and only if the relation $\preceq$ is closed.*

Proof. If a function g is continuous, the preimage of a closed set under g is closed as well. Now, note that $\preceq$ can be identified with the set $\{(x, y) \in G^2 \mid x^{-1}{\cdot}y \in C\}$, the preimage of C under the function $g_1(x, y) = x^{-1} \cdot y$. Owing to G being a topological group, g_1 is continuous, and $\preceq$ is closed if C is closed. Next, note that $C = \{x \in G \mid \mathbf{1} \preceq x\} = \{x \in G \mid (\mathbf{1}, x) \in \preceq\}$, the preimage of $\preceq$ under the continuous function $g_2(x) = (\mathbf{1}, x)$. We thus also have that C is closed if $\preceq$ is closed. This concludes the proof. □

5.2. Refining the orientation

Let $(G, \preceq)$ be an oriented group. Refining the orientation $\preceq$ means replacing its positive cone by a subset of it. The following lemma gives a sufficient (but not necessary) condition that refining preserves cc sponge groups. For a subset C of G, consider the condition

$$y \in C \ \wedge\ \mathbf{1} \preceq x \preceq y \implies x \in C \quad \text{for all } x, y \in G \tag{3}$$

One might rephrase this condition as "C has no gaps".

Lemma 4. *Let $(G, \preceq)$ be a cc sponge group. Let C be a subset of G, invariant under conjugation, with $\mathbf{1} \in C$ and $\{\mathbf{1}\} \preceq C$. Assume that Eq. (3) holds. Let $\sqsubseteq$ be the relation on G defined by $x \sqsubseteq y \equiv x^{-1} \cdot y \in C$. Then $(G, \sqsubseteq)$ is a cc sponge group.*

Proof. The first formula of Eq. (2) holds because of $\mathbf{1} \in C$ and $\{\mathbf{1}\} \preceq C$, and antisymmetry of $\preceq$. The second one holds by assumption. Therefore, $(G, \sqsubseteq)$ is an oriented group.

By Lemma 1, it remains to consider a nonempty subset P of G with $\{x\} \sqsubseteq P$ for some x, and to prove that P has a meet with respect to $\sqsubseteq$. By Eq. (1), we may assume that $x = \mathbf{1}$.

Assume P is nonempty and satisfies $\{\mathbf{1}\} \sqsubseteq P$. It suffices to prove that P has a meet for $\sqsubseteq$. By the definition of $\sqsubseteq$, we have $P \subseteq C$. As $(G, \preceq)$ is a cc sponge and $\mathbf{1} \preceq C$, the set P has a meet for $\preceq$, say y. We claim that y is the meet of P for $\sqsubseteq$.

To prove $\{y\} \sqsubseteq P$, let p be an arbitrary element of $P \subseteq C$. Then $\mathbf{1} \preceq y \preceq p$ and $p \in C$. From $\mathbf{1} \preceq y$ it follows that $y^{-1} \preceq \mathbf{1}$, and thus $y^{-1}{\cdot}p \preceq p$. From $y \preceq p$

it follows that $\mathbf{1} \preceq y^{-1} \cdot p$. Combining, we have $\mathbf{1} \preceq y^{-1} \cdot p \preceq p$. Equation (3) therefore implies that $y^{-1} \cdot p \in C$, so that $y \sqsubseteq p$. This proves $\{y\} \sqsubseteq P$.

For any z with $\{z\} \sqsubseteq P$, we need to prove $z \sqsubseteq y$. The assumption $\{z\} \sqsubseteq P$ means that $z^{-1} \cdot P \subseteq C$. For every $p \in P$, we therefore have $z^{-1} \cdot p \in C$, and hence $\mathbf{1} \preceq z^{-1} \cdot p$, and hence $z \preceq p$. As y is the meet of P for $\preceq$, this implies $z \preceq y$. It follows that $\mathbf{1} \preceq z^{-1} \cdot y \preceq z^{-1} \cdot p \in C$ for all $p \in P$. Equation (3) implies that $z^{-1} \cdot y \in C$ and hence $z \sqsubseteq y$. This concludes the proof that $(G, \sqsubseteq)$ is a cc sponge. □

Condition (3) is sufficient but not necessary. For instance, consider the additive group $\mathbb{R}$ with operation $+$ and neutral element 0 (see Example 3). Let C be the subset

$$C = \{n + t \mid n \in \mathbb{N} \wedge 0 \leq t \leq f(n)\}$$

for some descending function $f : \mathbb{N} \to \mathbb{R}$ with $0 \leq f(0)$. Let $\preceq$ be the associated orientation of $\mathbb{R}$. Then every right-bounded pair has a join because

$$C \cap (y + C) \neq \emptyset \implies \exists z : C \cap (y + C) \subseteq z + C.$$

Using the automorphism $x \mapsto -x$, it follows that every left-bounded pair has a meet. Therefore, Corollary 3 implies that $(\mathbb{R}, \preceq)$ is a sponge. In fact, it is a sponge group.

5.3. Orienting quotient sets and factor groups

Let $(G, \preceq)$ be an oriented group and let H be a subgroup of G. Recall that the (right) quotient set G/H consists of the residue classes $\overline{x} = x \cdot H$ for all $x \in G$. The group G has a left action on the quotient G/H defined by $g \cdot \overline{x} = \overline{g \cdot x}$ for all $g \in G$.

Let C be the positive cone of $(G, \preceq)$, and consider the relation $\sqsubseteq$ on G/H defined by

$$\overline{x} \sqsubseteq \overline{y} \quad \equiv \quad x^{-1} \cdot y \in C \cdot H,$$

and the property

$$q \in C \ \wedge \ r \in C \ \wedge \ q \cdot r \in H \implies q \in H \ \wedge \ r \in H \ \text{ for all } q, r \in G \tag{4}$$

In words, if H contains the product of two positive elements, it contains the elements as well.

Note that $x \preceq y \implies \overline{x} \sqsubseteq \overline{y}$, because $\mathbf{1} \in H$.

Lemma 5. *Assume $(G, \preceq)$ is an oriented group, and H a subgroup of G. Then $(G/H, \sqsubseteq)$ is an oriented set if and only if Eq. (4) is satisfied. If H is a normal subgroup of G, it is an oriented group.*

Proof. For $(G/H, \sqsubseteq)$ to be an oriented set, $\sqsubseteq$ must be reflexive and antisymmetric. Now, since $\preceq$ is reflexive and $\mathbf{1} \in H$, $\sqsubseteq$ is reflexive as well. It remains to show that $\sqsubseteq$ is antisymmetric. Assume $\overline{x} \sqsubseteq \overline{y}$ and $\overline{y} \sqsubseteq \overline{x}$. Then $x^{-1} \cdot y \in C \cdot H$ and $y^{-1} \cdot x \in C \cdot H$. Put $z = x^{-1} \cdot y$. Then $z \in C \cdot H$ and $z^{-1} \in C \cdot H$. This implies that H has elements h, k with $h \preceq z$ and $k \preceq z^{-1}$. It follows that $1 \preceq h^{-1}z$ and

$1 \preceq z^{-1} \cdot k^{-1}$. As $h^{-1} \cdot z \cdot z^{-1} \cdot k^{-1} \in H$, Eq. (4) implies that $h^{-1} \cdot z \in H$. It follows that $z \in H$ and hence $\overline{x} = \overline{y}$. As a result, $(G/H, \sqsubseteq)$ is an oriented set if Eq. (4) holds.

Conversely, assume $(G/H, \sqsubseteq)$ is an oriented set. Assume there exist a q and r in G such that $\mathbf{1} \preceq q$, $\mathbf{1} \preceq r$, and $q \cdot r \in H$. Then $\overline{\mathbf{1}} \sqsubseteq \overline{q}$ because $\mathbf{1} \in H$. On the other hand, $q \cdot r \in H$ and $\mathbf{1} \preceq r$ together imply that $q \preceq (q \cdot r)$, so that $\overline{q} \sqsubseteq \overline{\mathbf{1}}$. Since $\sqsubseteq$ is an orientation, we have $\overline{\mathbf{1}} = \overline{q}$, as well as $q \in H$ and $r \in H$. So, $(G/H, \sqsubseteq)$ is an oriented set only if Eq. (4) holds.

Finally, if H is a normal subgroup of G, G/H is a group with $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$, so if Eq. (1) holds, $(G/H, \sqsubseteq)$ is an oriented group. Now, assume $\overline{x} \sqsubseteq \overline{y}$. Then $x^{-1} \cdot y \in C \cdot H$, so $x^{-1} \cdot z^{-1} \cdot z \cdot y \in C \cdot H$ as well: $\overline{z} \cdot \overline{x} \sqsubseteq \overline{z} \cdot \overline{y}$. Similarly, since $(G, \preceq)$ is an oriented group, we can use the second property of Eq. (2) together with the normality of H to see that $z^{-1} \cdot x^{-1} \cdot y \cdot z \in C \cdot H$, and thus $\overline{x} \cdot \overline{z} \sqsubseteq \overline{y} \cdot \overline{z}$. □

Example 6. Consider the additive cc sponge group $(\mathbb{R}, +, \leq)$ and the subset $C = \{x \in \mathbb{R} \mid 0 \leq x < c\}$ for some constant $c > 0$. Lemma 4 is applicable because $0 \in C$ and $\{0\} \leq C$, and the additive version of condition (3) holds: if $y \in C$ and $0 \leq x \leq y$ then $x \in C$. Lemma 4 therefore implies that $(\mathbb{R}, +, \preceq)$ is a cc sponge, where $x \preceq y \equiv y - x \in C$.

Now consider the (normal) subgroup $\mathbb{Z}$ of $\mathbb{R}$. Let $\sqsubseteq$ be the induced relation on the factor group $\mathbb{R}/\mathbb{Z}$ as defined above. By Lemma 5, relation $\sqsubseteq$ is an orientation if and only if the additive version of condition (4) holds: if $x \in C$ and $y \in C$ and $x + y \in \mathbb{Z}$, then $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. This is true if and only if $c \leq \frac{1}{2}$.

5.4. Quotient sponges

We now give a sufficient (but not necessary) condition for the orientation on G/H to also be a cc sponge:

$$\forall z \in G : \exists h \in H : R(z) \cap C \cdot H \subseteq R(h) \tag{5}$$

Here, recall that $R(z) = \{y \mid z \preceq y\}$, and note that $R(z) = C \cdot \{z\}$. Intuitively, this condition captures the notion

Lemma 6. *Let $(G, \preceq)$ be a cc sponge group. Let H be a subgroup of G that satisfies Eqs. (4) and (5). Then $(G/H, \sqsubseteq)$ is a cc sponge. If H is a normal subgroup of G, it is a cc sponge group.*

Proof. By Lemma 1, it suffices to prove that every nonempty left-bounded subset P of G/H has a meet. By translation invariance, we may assume that the left bound of P is $\overline{\mathbf{1}}$. So, we have $\{\overline{\mathbf{1}}\} \sqsubseteq P$. This implies that G has a nonempty subset Q with $\{\mathbf{1}\} \preceq Q$ and $P = \{\overline{q} \mid q \in Q\}$. Now Q has a meet, say $m \in G$. It satisfies $m \preceq q$ for all $q \in Q$. Therefore $\overline{m} \sqsubseteq p$ for all $p \in P$.

Moreover, let $y \in G$ be such that $\{\overline{y}\} \sqsubseteq P$. Then $\overline{y} \sqsubseteq \overline{q}$ for all $q \in Q$. This implies that $y^{-1} \cdot Q \subseteq C \cdot H$. On the other hand, by translation invariance, $y^{-1} \cdot Q \subseteq R(y^{-1})$. Equation (5) with $z := y^{-1}$ now implies that H has an element h with $y^{-1} \cdot Q \subseteq R(h)$. It follows that $\{y \cdot h\} \preceq Q$. As m is the meet of

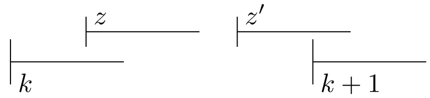

Figure 2: Two possible configurations for z in Example 7. Note that the case that $C + \{z\}$ overlaps with neither $C + \{k\}$ nor $C + \{k + 1\}$ is taken care of implicitly in the proof, as it merely relies on *at most* one of A and B being nonempty.

Q, this implies $y \cdot h \preceq m$ and hence $\overline{y} \sqsubseteq \overline{m}$. This proves that $\overline{m}$ is the meet of P in G/H.

Finally, if H is a normal subgroup of G, $(G/H, \preceq)$ is an oriented group by Lemma 5. Since we have just shown it is also a cc sponge, it is a cc sponge group. □

Example 7. Continuing Example 6: for $(\mathbb{R}/\mathbb{Z}, \sqsubseteq)$ with $c \leq \frac{1}{2}$, condition (5) holds. This is shown as follows. Let $z \in \mathbb{R}$ be given. We have to find $h \in \mathbb{Z}$ with $(C + \{z\}) \cap (C + \mathbb{Z}) \subseteq (C + \{h\})$. Put $k = \lfloor z \rfloor \in \mathbb{Z}$. Then, given that $c \leq \frac{1}{2}$, $(C + \{z\}) \cap (C + \mathbb{Z}) = A \cup B$, where $A = (C + \{z\}) \cap (C + \{k\})$ and $B = (C + \{z\}) \cap (C + \{k + 1\})$. Since by construction $k \leq z < k + 1$, we have $A = \{y \mid y \in \mathbb{R} \wedge z \leq y < k + c\}$ and $B = \{y \mid y \in \mathbb{R} \wedge k + 1 \leq y < z + c\}$. Now, if $z + c < k + 1$, then $B = \emptyset$ and $(C + \{z\}) \cap (C + \mathbb{Z}) = A \subseteq (C + \{k\})$. Otherwise, $A = \emptyset$ since $0 < c \leq \frac{1}{2}$ and $z \geq k + 1 - c \geq k + c$, so that $(C + \{z\}) \cap (C + \mathbb{Z}) = B \subseteq (C + \{k + 1\})$. See Fig. 2 for possible configurations. We can therefore choose $h := k$ if $z < k + c$, and $h := k + 1$ otherwise. For $c = \frac{1}{2}$, the resulting cc sponge group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the angle sponge introduced in [2, 3].

Example 8. Consider the additive group $\mathbb{R}^2$ with the orientation given by

$$(x_1, x_2) \preceq (y_1, y_2) \ \equiv\ x_1 \leq y_1 < x_1 + \tfrac{1}{2} \ \wedge\ x_2 \leq y_2 < x_2 + \tfrac{1}{2}\ .$$

Use Lemma 4 to prove that this is a cc sponge group. Equations (4) and (5) hold for the grid $H = \mathbb{Z}^2$. They also hold if H is one of the two coordinate axes. Equation (4) fails if H is the line given by $x_1 = x_2$. If H is the line $x_1 + x_2 = 0$, Eq. (4) holds and Eq. (5) fails. In this case G/H is a sponge, but the projection $G \to G/H$ does not preserve meets.

6. Epigraph sponges

Let E be a real Hilbert space with inner product $(\_, \_)$. We assume that $\dim(E) \geq 2$. Let h be a unit vector in E and let $H = h^\perp$ be the hyperplane orthogonal to h. For any $x \in E$, we can write $x = x_h h + x_\perp$, where $x_\perp$ is orthogonal to h and $x_h \in \mathbb{R}$. Note that $x_h = (x, h)$, because $x_\perp$ is orthogonal to h and because h is a unit vector.

A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is used to define a positive cone by comparing the values of x_h and $||x_\perp||$. Let C_f be the set of points on or above the graph (the *epigraph*) of $f \circ \|\cdot\|$ evaluated on H. So, it is given by

$$x \in C_f \iff f(\|x_\perp\|) \leq x_h .$$

Proposition 1. *$(E, \preceq_f)$, with $x \preceq_f y \quad \equiv \quad y - x \in C_f$, is an oriented group if and only if $f(0) = 0$ and $f(d) > 0$ for all $d > 0$.*

Proof. It is not too difficult to see that C_f satisfies Eq. (2') if and only if $f(0) = 0$ and $f(d) > 0$. This concludes the proof. $\square$

From now on, by convention, we assume that $f(d) > 0$ iff $d > 0$. By Proposition 1, we thus have an oriented group $(E, \preceq_f)$ with

$$x \preceq_f y \quad \equiv \quad y - x \in C_f \quad \equiv \quad f(\|y_\perp - x_\perp\|) \leq y_h - x_h.$$

In the remainder of this section, two questions are treated. 1. Is the orientation $\preceq_f$ a partial order? 2. Is $(E, \preceq_f)$ a sponge?

6.1. Is the orientation $\preceq_f$ a partial order?

Recall from Section 5.1 that the orientation $\preceq$ is a partial order (i.e. is transitive) if and only if the positive cone is closed under the group operation $+$.

Recall that a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *subadditive* iff $f(x + y) \leq f(x) + f(y)$ for all nonnegative real x, y. Let us call f *interval subadditive* iff $f(z) \leq f(x) + f(y)$ for all nonnegative real x, y, z with $0 \leq x - y \leq z \leq x + y$. Note that, if f is interval subadditive, then it is subadditive, because one can take $z = x + y$ and interchange x and y if $x < y$. On the other hand, if f is subadditive and ascending, then f is interval subadditive.

Example 9. The function f given by $f(0) = 0$, $f(u) = 2$ if $0 < u \leq 1$, and $f(u) = 1$ if $1 < u$, is interval subadditive, but not ascending.

Theorem 3. *(a) Assume the function f is interval subadditive. Then the orientation $\preceq_f$ is a partial order.*
(b) Assume that $\preceq_f$ is a partial order and that $\dim(E) \geq 3$. Then function f is interval subadditive.

Proof. (a) Let p, $q \in C_f$ and $r = p + q$. It suffices to prove that $r \in C_f$. By symmetry, we may assume that $||q_\perp|| \leq ||p_\perp||$. As $r_\perp = p_\perp + q_\perp$, we have $0 \leq ||p_\perp|| - ||q_\perp|| \leq ||r_\perp|| \leq ||p_\perp|| + ||q_\perp||$. interval subadditivity now implies that $f(||r_\perp||) \leq f(||p_\perp||) + f(||q_\perp||) \leq p_h + q_h = r_h$. This proves that $r \in C_f$.

(b) As $\dim(E) \geq 3$, one can choose two orthogonal unit vectors u, v both orthogonal to h. Let nonnegative real x, y, z with $0 \leq x - y \leq z \leq x + y$ be given. Consider the unit vectors $v(\varphi) = (\cos\varphi)u + (\sin\varphi)v$ for real φ. Put $p = xu + f(x)h$ and $q(\varphi) = yv(\varphi) + f(y)h$. Then $p \in C_f$ and $q(\varphi) \in C_f$. As $\preceq_f$ is transitive, it follows that $r(\varphi) = p + q(\varphi) \in C_f$. The expression $||r(\varphi)_\perp|| = ||xu + yv(\varphi)||$ can take all values in the range from $x - y$ to $x + y$. In particular, one can choose φ such that $||r(\varphi)_\perp|| = z$. Then it holds that $f(z) = f(||r(\varphi)_\perp||) \leq r_h = p_h + q_h = f(x) + f(y)$. $\square$

Remark 1. In the case of $\dim(E) = 2$, it can be proved in the same way that the orientation $\preceq_f$ is a partial order if and only if $f(z) \leq f(x) + f(y)$ for all nonnegative real x, y, z with $z = x \pm y$.

6.2. The main result on epigraph sponges

Before introducing the main theorem of this section, recall that a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *superadditive* iff $f(x+y) \geq f(x) + f(y)$ for all x, y. We define f to be *square-superadditive* iff $f(\sqrt{x^2+y^2}) \geq f(x) + f(y)$ for all x, y. Note that f is square-superadditive iff the function $\varphi(x) = f(\sqrt{x})$ is superadditive. Furthermore, if f is superadditive or square-superadditive, it is also ascending (due to the nonnegativity of f), or increasing if f is positive for all nonzero arguments. Finally, if f is square-superadditive, then f is superadditive, due to $\sqrt{x^2+y^2} \leq x+y$ for all nonnegative x and y, and the ascendingness of square-superadditive functions. Also, recall that f is lower semicontinuous iff, for every d, $a \in \mathbb{R}_{\geq 0}$ with $f(d) > a$, there exists $\varepsilon > 0$ such that for all $e \in \mathbb{R}_{\geq 0}$ with $|e-d| < \varepsilon$ it holds that $f(e) > a$. If f is continuous it is lower semicontinuous.

We are now ready to state the main theorem of this section.

Theorem 4. *Assume that* $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *satisfies* $f(d) > 0$ *iff* $d > 0$.
(a) Let $\dim(E) \geq 3$. *Then* $(E, \preceq_f)$ *is a cc sponge if and only if function* f *is square-superadditive and lower semicontinuous.*
(b) Let $\dim(E) = 2$ *and* f *is lower semicontinuous. Then* $(E, \preceq_f)$ *is a cc sponge if and only if* f *is superadditive.*

In dimension 2 the condition for sponges is weaker than in higher dimensions. This seems to be because the hyperplane H is only one dimensional and therefore has less space for sets without meet or join.

We conjecture that part (b) of the theorem can be strengthened in the sense that, if $(E, \preceq_f)$ is a sponge on a two-dimensional space, then f is lower semicontinuous.

Example 10. Let f be the identity function $f(u) = u$. The positive cone of the oriented group $(E, \preceq_f)$ consists of the vectors x in E with $||x_\perp|| \leq x_h$. If $\dim(E) = 3$, this is a classical solid cone with the top in the origin. Function f is additive, ascending, and continuous. Therefore, it is subadditive, superadditive, and lower semicontinuous. Theorem 3 therefore implies that $\preceq_f$ is a partial order. Theorem 4 implies that $(E, \preceq_f)$ is a cc sponge if $\dim(E) = 2$. As f is not square-superadditive, the theorem also implies that $(E, \preceq_f)$ is not a cc sponge if $\dim(E) \geq 3$. The positive cone is sketched in Fig. 3.

Example 11. Let f be given by $f(u) = u^2$. The positive cone consists of the vectors x with $||x_\perp||^2 \leq x_h$. This is a solid paraboloid. Function f is square-superadditive and continuous. Therefore it is also superadditive and lower semicontinuous. Theorem 4 therefore implies that $(E, \preceq_f)$ is a cc sponge (as $\dim(E) \geq 2$ by convention). The positive cone is sketched in Fig. 3. As f is not subadditive, Theorem 3 implies that $\preceq_f$ is not a partial order.

Example 12. Let f be given by $f(0) = 0$ and $f(u) = u$ for $0 \leq u \leq 1$, and $f(u) = 2u$ for $u > 1$. One can verify that f is superadditive and lower semicontinuous. Assume $\dim(E) = 2$. Theorem 4 implies that $(E, \preceq_f)$ is a cc sponge. The positive cone is sketched in Fig. 3.

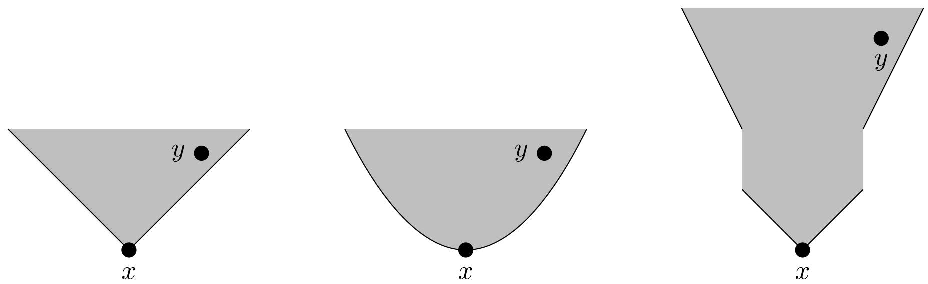

Figure 3: Positive cones of Examples 10 to 12: $x \preceq y$.

The rest of this section is devoted to the proof of this theorem. Section 6.3 prepares the ground by investigating the oriented group $(E, \preceq_f)$. The only-if parts of the theorem are proved in Section 6.4. The if parts are proved with Corollary 2 in Section 6.5.

6.3. Properties of the oriented group

As a preparation of the proof of Theorem 4, we investigate the oriented group $(E, \preceq_f)$ introduced in Proposition 1.

Lemma 7. *Relation $\preceq_f$ is topologically closed in E^2 if and only if the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is lower semicontinuous.*

Proof. As $(E, \preceq_f)$ is an oriented group, $\preceq_f$ is closed if and only if the epigraph $C_f = \{w \mid f(\|w_\perp\|) \leq (w, h)\}$ is closed (Lemma 3). By convention, $\dim(E) \geq 2$. We can therefore choose a unit vector u orthogonal to h.

Assume that C_f is closed, and that $f(d) > a$. Then the vector $w = d\,u + a\,h$, is not in C_f. As C_f is closed, there exists $\varepsilon > 0$ such that the ball around w with radius ε does not meet C_f. It follows that $f(e) > a$ for all real numbers e with $|e - d| < \varepsilon$. This proves that f is lower semicontinuous.

Conversely, assume that f is lower semicontinuous. To show that C_f is closed, consider $w \notin C_f$. This means that $(w, h) < f(\|w_\perp\|)$. Choose a number a with $(w, h) < a < f(\|w_\perp\|)$. As f is lower semicontinuous, there is a number $\varepsilon > 0$ such that $f(x) > a$ for all numbers x with $|x - \|w_\perp\|| < \varepsilon$. Furthermore, the point w has an open neighborhood N in E such that all points $w' \in N$ satisfy $(w', h) < a$ and $|\|w'_\perp\| - \|w_\perp\|| < \varepsilon$. As a result, $(w', h) < f(\|w'_\perp\|)$, and hence $w' \notin C_f$. This proves that C_f is topologically closed. $\square$

Proposition 2. *Every finite subset P of E is left and right bounded in $(E, \preceq_f)$.*

Proof. For real t, the vector $t\,h$ is a right bound of $p \in P$ if and only if $f(\|p_\perp\|) + p_h \leq t$. Therefore, $t\,h$ is a right bound of P when t is larger than the maximum of the numbers $f(\|p_\perp\|) + p_h$ with p ranging over P. A left bound can be constructed analogously. $\square$

We next observe that, owing to $f(d) > 0$ for $d \neq 0$, relation $\preceq_f$ satisfies

$$x \preceq_f y \ \wedge \ x \neq y \implies (x, h) < (y, h).$$

Recalling that $f(0) = 0$, the above directly implies:

Proposition 3. *Let P be a subset of E that has a join x. Then x is the unique lowest point with respect to h of the set of right bounds of P, i.e. $(x, h) < (y, h)$ for all right bounds $y \neq x$ of P.*

Proposition 4. *Let V be a finite-dimensional linear subspace of E that contains h. Let P be a subset of V that has a join (or meet) x. Then $x \in V$.*

Proof. As V is finite-dimensional, the space E is the direct sum $V \oplus V^{\perp}$. Let $\zeta : E \to E$ be the linear mapping given by $\zeta(v + w) = v - w$ for all $v \in V$, $w \in V^{\perp}$. Then ζ is an isometry of E which preserves h. It therefore preserves $\preceq_f$, and joins and meets for $\preceq_f$. Hence, it keeps x invariant because it keeps P invariant. Finally, $\zeta(x) = x$ implies $x \in V$. $\square$

Proposition 5. *Let x and y be two points in E, satisfying $x_h = y_h$ and $x_\perp = -y_\perp$. Then, if x and y have a join, it is the point $(x_h + f(\|x_\perp\|))h$.*

Proof. By Proposition 4, if x and y have a join, it is an element of the subspace spanned by x, y and h. We can also see that the join of x and y has to be a multiple of h. This is trivially true if $\dim(E) = 1$. For $\dim(E) \geq 2$ it is also true, as otherwise the symmetry of the problem would imply the existence of two equally valid candidates, contradicting Proposition 3. For real t, the multiple $t\,h$ is a right bound of $\{x, y\}$ if and only if $t \geq x_h + f(\|x_\perp\|) = y_h + f(\|y_\perp\|)$. Therefore, by Proposition 3, the join, if it exists, is indeed equal to $(x_h + f(x))h$. $\square$

6.4. Properties of epigraph sponges

Having looked at some of the properties of the oriented group $(E, \preceq_f)$, we now consider what happens if it is in fact a sponge.

Lemma 8. *Assume that $(E, \preceq_f)$ is a sponge. Then $f(d) + f(e) \leq \max(f(d + e), f(|d - e|))$ for all $d, e \in \mathbb{R}_{\geq 0}$. If $\dim(E) \geq 3$, then f is square-superadditive.*

Proof. As $\dim(E) \geq 2$, we can choose a unit vector $u \in E$, orthogonal to h. Let d and e be given. Consider the doubleton set $P = \{d\,u, -d\,u\}$ in E. By Proposition 2, the set P has a right bound. As $(E, \preceq_f)$ is a sponge, P has a join. By Proposition 5, the join is $f(d)\,h$. This implies that

$$\forall w \in E: \quad d\,u \preceq_f w \ \wedge \ -d\,u \preceq_f w \implies f(d)\,h \preceq_f w. \tag{6}$$

Applying Eq. (6) to $w = e\,u + a\,h$ (for arbitrary $a \in \mathbb{R}$), we observe:

$$\begin{aligned}
& \forall a: \quad d\,u \preceq_f e\,u + a\,h \ \wedge \ -d\,u \preceq_f e\,u + a\,h \implies f(d)\,h \preceq_f e\,u + a\,h \\
\equiv \quad & \forall a: \quad f(|d - e|) \leq a \ \wedge \ f(d + e) \leq a \implies f(e) \leq a - f(d) \\
\equiv \quad & \forall a: \quad \max(f(d + e), f(|d - e|)) \leq a \implies f(d) + f(e) \leq a \\
\equiv \quad & f(d) + f(e) \leq \max(f(d + e), f(|d - e|)).
\end{aligned}$$

This concludes the first part of the proof.

Now, if $\dim(E) \geq 3$, we can choose a unit vector v orthogonal to both h and u. Equation (6) is now applied to $w = e\,v + a\,h$ for arbitrary $a \in \mathbb{R}$, and we have

$$\begin{aligned} & \forall a: \quad d\,u \preceq_f e\,v + a\,h \ \wedge \ -d\,u \preceq_f e\,v + a\,h \implies f(d)\,h \preceq_f e\,v + a\,h \\ \equiv\ & \forall a: \quad f(\sqrt{d^2+e^2}) \leq a \implies f(e) \leq a - f(d) \\ \equiv\ & f(d) + f(e) \leq f(\sqrt{d^2+e^2}). \end{aligned}$$

This proves that f is square-superadditive. This concludes the proof. □

Proposition 6. *Assume that $(E, \preceq_f)$ is a sponge. Then f is ascending if and only if it is increasing.*

Proof. If f is ascending and not increasing, then there is some interval $[d_1, d_2]$ over which f has a constant value, say w. Choose e with $0 < e < \frac{1}{2}(d_2 - d_1)$. Then $f(e) > 0$ and $f(d_2) = f(d_2 - e) = f(d_2 - 2e) = w$, so that Lemma 8 applied to $d_2 - e$ and e gives

$$w < f(d_2 - e) + f(e) \leq \max(f(d_2), f(d_2 - 2e)) = w.$$

This is clearly a contradiction, so we conclude that if f is ascending, it is also increasing. The converse implication is trivial. □

Assume that f is ascending. Then all discontinuities of f are “of the first kind” (jump discontinuities) [26, Corollary to Thm. 4.29]. That is, even if f is discontinuous in d, the limits $f^-(d) = \lim_{e \uparrow d} f(e)$ and $f^+(d) = \lim_{e \downarrow d} f(e)$ exist, and $f^-(d) \leq f(d) \leq f^+(d)$. Function f is lower semicontinuous if and only if $f^-(d) = f(d)$ for all d.

Lemma 9. *Assume that $(E, \preceq_f)$ is a sponge. Then f is ascending if and only if it is lower semicontinuous.*

Proof. First assume that f is ascending. Let d be an argument where f is not continuous. As before, choose a unit vector u orthogonal to h. Let the vectors x and y be given by $x = d\,u + f^+(d)\,h$ and $y = -d\,u + f^-(d)\,h$. For real numbers z_h and e, the vector $z = e\,u + z_h\,h$ is a right bound of $\{x, y\}$ if and only if

$$z_h \geq r_e = \max(f(|d - e|) + f^+(d), f(|d + e|) + f^-(d)).$$

By Propositions 3 and 4, the join of $\{x, y\}$ is the lowest such right bound, so it is $z^* = e\,u + r_e\,h$ with $e = \arg\min_e r_e$. Now note that for $e < 0$, $f^+(d) < f(|d - e|)$, so that

$$f^+(d) + f^-(d) < 2\,f^+(d) < r_e \quad \text{for all } e < 0.$$

For $e > 0$, $f^+(d) < f(|d + e|)$, so that

$$f^+(d) + f^-(d) < r_e \quad \text{for all } e > 0.$$

Furthermore, for $0 < e < 2d$, $f(|d-e|) < f^-(d)$, so that $r_e = f(d+e) + f^-(d)$. It follows that $\lim_{e \downarrow 0} = f^+(d) + f^-(d)$. Clearly, this limit must be the height of the lowest right bound z^* of $\{x, y\}$. Now, since $r_e = f(d) + f^+(d)$ for $e = 0$, the existence of the join implies that $f(d) = f^-(d)$. This proves that f is lower semicontinuous.

Now, assume that f is lower semicontinuous, and that f is *not ascending*. So, there should be real numbers u and v such that $0 \leq u < v$ and $f(v) < f(u)$. As f is lower semicontinuous, the set $G = \{d \in \mathbb{R}_{\geq 0} \mid f(d) \leq f(v)\}$ is topologically closed. It follows that its subsets $G_0 = \{d \in G \mid d \leq u\}$ and $G_1 = \{d \in G \mid u \leq d\}$ are also closed. G_0 is bounded from above by u and G_1 is bounded from below by u. Therefore, G_0 has a greatest element g_0, and G_1 has a smallest element g_1. It is clear that $g_0 < u < g_1$, and that $d = \frac{1}{2}(g_0 + g_1) \notin G$, so that $f(d) > f(v)$. Putting $e = \frac{1}{2}(g_1 - g_0)$, we have $0 < e \leq d$ and $\max(f(d+e), f(d-e)) = \max(f(g_1), f(g_0)) \leq f(v) < f(d)$. This contradicts Lemma 8, so if f is lower semicontinuous it is also ascending, completing the proof. $\square$

Combining the above lemmas, we find the following:

Corollary 4. *Assume that $(E, \preceq_f)$ is a sponge. Then the following are equivalent:*

1. *relation $\preceq_f$ is topologically closed,*
2. *f is lower semicontinuous,*
3. *f is ascending,*
4. *f is increasing,*
5. *f is superadditive.*

If $\dim(E) \geq 3$*, all of the aforementioned properties hold.*

Proof. Lemma 7 shows that the first two properties are equivalent (even if $(E, \preceq_f)$ is just an orientation). Proposition 6 shows that in the current context the third and fourth property are equivalent. Lemma 9 shows that the second and third property are equivalent. We also noted already that if f is superadditive, it is also ascending. This leaves only one implication to prove: that if f is ascending, it is also superadditive. If f is ascending, Lemma 8 now implies that $f(d) + f(e) \leq f(d+e)$ for all nonnegative reals d and e, since those satisfy $|d-e| \leq d+e$. This implies that f is superadditive. Finally, when $\dim(E) \geq 3$, Lemma 8 tells us that f is square-superadditive, and thus superadditive. $\square$

Necessity. If $(E, \preceq_f)$ is a sponge and $\dim(E) \geq 3$, then f is square-superadditive and lower semicontinuous by Lemma 8 and Corollary 4. This implies the only-if part of Theorem 4(a). If $(E, \preceq_f)$ is a sponge, $\dim(E) = 2$, and f is lower semicontinuous, then f is superadditive by Corollary 4. This implies the only-if part of Theorem 4(b). Note that conditional completeness is not needed for these implications.

6.5. Sufficiency

In this section the if parts of Theorem 4 is proved by means of Corollary 2. We therefore assume that f is lower semicontinuous and square-superadditive (or just superadditive if $\dim(E) = 2$). It follows that f is increasing, because it is superadditive, and $f(d) > 0$ for all $d > 0$.

Lemma 10. *The covector $h^* : E \to \mathbb{R}$, defined by $h^*(x) = (h, x)$, is continuous and a discriminator.*

Proof. Being a linear bounded functional, h^* is continuous [27, Thm. 1.18]. To see that it is also a discriminator, consider $\varepsilon > 0$ to be given. We can now pick a $\delta > 0$ that is both less than $\frac{1}{2}\varepsilon$ and less than $f(\frac{1}{2}\varepsilon)$. Clearly, recalling that f is increasing, any element $x \in C_f$ for which $h^*(x) = x_h < \delta$ satisfies $\|x\| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. Since $y_h - x_h < \delta \equiv h^*(y) < h^*(x) + \delta$, it follows that for any two elements $x, y \in E$, $x \preceq y$ and $h^*(y) < h^*(x) + \delta$ imply that $d(x, y) = \|y - x\| < \varepsilon$. □

Recall that function f is lower semicontinuous and increasing. It is easy to see that $f(x) \leq f^+(x) < f(y)$ whenever $0 \leq x < y$. We also have

$$f^+(x) + f(y) \leq f(x + y) \quad \text{whenever } x \geq 0 \text{ and } y > 0. \tag{7}$$

This follows from the fact that f is superadditive, increasing, and lower semicontinuous, as well as the fact that $f(x+\varepsilon) + f(y-\varepsilon) \leq f(x+y)$ holds for all ε with $0 < \varepsilon < y$.

Proposition 7. *Let V be a real Hilbert space, and let p, $q \in V$ be such that $(p, q) \geq 0$ and $q \neq 0$. Assume that f is square-superadditive, or that f is super-additive and $\dim(V) = 1$. Then $f^+(\|p\|) + f(\|q\|) \leq f(\|p + q\|)$.*

Proof. First assume that f is square-superadditive. Let φ be the superadditive function given by $\varphi(x) = f(\sqrt{x})$. Equation (7) implies that

$$f^+(\|p\|) + f(\|q\|) = \varphi^+(\|p\|^2) + \varphi(\|q\|^2) \leq \varphi(\|p\|^2 + \|q\|^2).$$

On the other hand, we have $\varphi(\|p\|^2 + \|q\|^2) \leq \varphi(\|p + q\|^2) = f(\|p + q\|)$, as $(p, q) \geq 0$, and φ is increasing.

If $\dim(V) = 1$, we have $\|p + q\| = \|p\| + \|q\|$ because $(p, q) \geq 0$. If, moreover, f is superadditive, then Eq. (7) gives $f^+(\|p\|) + f(\|q\|) \leq f(\|p\| + \|q\|) = f(\|p + q\|)$. □

Lemma 11. *Every pair of elements of E has a join in E.*

Proof. If x and y are comparable by $\preceq_f$, one of them is their join. We may therefore assume that they are not comparable. Therefore, the difference vector $x - y$ is not a multiple of h. We may translate the origin in the hyperplane $h^\perp$ of vectors orthogonal to h to the point $\frac{1}{2}(x_\perp + y_\perp)$, and thus assume that $x_\perp = -y_\perp \neq 0$.

Let e be the unit vector $\|y_\perp\|^{-1}\,y_\perp$. Let S be the linear subspace spanned by h and e. This subspace contains x and y. The vectors h and e form an orthonormal basis of it. We abbreviate the inner products with h and e by $u_h = (u, h)$ and $u_e = (u, e)$. Note that $x_e < 0 < y_e$.

Let U be the set of right-bounds of $\{x, y\}$, so that $U = (x+C_f)\cap(y+C_f)$. As x and y are not comparable, we have $x \notin U$ and $y \notin U$. Given the assumptions made at the start of this section, the set U is topologically closed. We observe that

$$\begin{array}{cl} & u \in U \\ \equiv & x \preceq_f u \ \wedge \ y \preceq_f u \\ \equiv & f(\|u_\perp - x_\perp\|) \le u_h - x_h \ \wedge \ f(\|u_\perp - y_\perp\|) \le u_h - y_h \\ \equiv & \max(x_h + f(\|u_\perp - x_\perp\|), y_h + f(\|u_\perp - y_\perp\|)) \le u_h. \end{array} \tag{8}$$

In particular, for $u \in S$, we have

$$u \in U \quad \equiv \quad \max(x_h + f(|u_e - x_e|), y_h + f(|u_e - y_e|)) \le u_h.$$

Because $y \notin U$, the lowest point of $U \cap S$ above y is

$$y' = y_e\, e + y'_h\, h, \text{ where } y'_h = x_h + f(y_e - x_e) > y_h.$$

Let S' be the rectangle of the points $z \in S$ with $x_e \le z_e \le y_e$ and $y_h \le z_h \le y'_h$. As S' is compact and U is closed, the intersection $U \cap S'$ is compact. It is nonempty because $y' \in U \cap S'$. Therefore, there is $z \in U \cap S'$ with $z_h \le u_h$ for all $u \in U \cap S'$. We claim that $z_h \le u_h$ for all $u \in U \cap S$; it suffices to consider $u \in U \cap S \setminus S'$. In that case, if $x_e \le u_e \le y_e$, then $z_h \le y'_h < u_h$. If $y_e < u_e$, then $z_h \le y'_h < u_h$ because f is increasing. The case $u_e < x_e$ is treated in the same way. This proves that $z_h \le u_h$ for all $u \in U \cap S$.

As z is a lowest point of $U \cap S$ and $x_e \le z_e \le y_e$, we have

$$z_h = \max(x_h + f(z_e - x_e), y_h + f(y_e - z_e)).$$

At first sight, one might expect the two terms of the maximum to be equal, but this need not be the case because of the semicontinuity of f. Instead, we claim that

$$z_h \le x_h + f^+(z_e - x_e) \ \wedge \ z_h \le y_h + f^+(y_e - z_e). \tag{9}$$

The lefthand inequality is treated first. If $z_e = y_e$, then $z = y'$ and $z_h = x_h + f(y_e - x_e)$, which is less than $x_h + f^+(y_e - x_e)$. Otherwise, it holds that $x_e \le z_e < y_e$. Assume that $x_h + f^+(z_e - x_e) < z_h$. Then there is a real number t with $z_e < t < y_e$ and $s = x_h + f(t - x_e) < z_h$. As f is increasing, we also have $s' = y_h + f(y_e - t) < z_h$. If we put $s'' = \max(s, s')$, the vector $u = t\,e + s''\,h$ satisfies $u \in U$ and $u_h = s'' < z_h$, contradicting the minimality of z_h. This proves the lefthand inequality of (9). The other one follows by symmetry.

It remains to prove that every element $u \in U$ is a right-bound of z. Let $u \in U$ be given. As we need to compare the vectors $u_\perp$ and $z_\perp$, we define $q = u_\perp - z_\perp$.

First assume that $q = 0$. This implies that $u_\perp = z_\perp = z_e\, e$. It follows that $u \in U \cap S$, and hence $z_h \le u_h$, and hence $z \preceq_f u$.

It remains to assume that $q \ne 0$. Two cases are distinguished: $q_e \ge 0$ or $q_e \le 0$. Assume $q_e \ge 0$. Put $p = z_\perp - x_\perp$. Then $(p, q) \ge 0$. We use Proposition 7 with $E := h^\perp$, and p and q as chosen just now. The relation $z \preceq_f u$ is proved in

$$
\begin{aligned}
& z \preceq_f u \\
\equiv\quad & \{ \text{ definition } \preceq_f \} \\
& f(\|u_\perp - z_\perp\|) \le u_h - z_h \\
\Leftarrow\quad & \{ \text{ (8) gives } x_h + f(\|u_\perp - x_\perp\|) \le u_h \} \\
& z_h + f(\|u_\perp - z_\perp\|) \le x_h + f(\|u_\perp - x_\perp\|) \\
\equiv\quad & \{ \text{ choices of } p \text{ and } q \} \\
& z_h + f(\|q\|) \le x_h + f(\|q + p\|) \\
\Leftarrow\quad & \{ q \ne 0, \text{ Proposition 7 with } E := h^\perp, \text{ and choice of } p \} \\
& z_h \le x_h + f^+(\|z_\perp - x_\perp\|) \\
\equiv\quad & \{ x_\perp = x_e\, e,\ z_\perp = z_e\, e, \text{ and Eq. (9)} \} \\
& \text{true} .
\end{aligned}
$$

The case $q_e \le 0$ is treated in the same way with $p = z_\perp - y_\perp$. □

The if parts of Theorem 4 are now obtained by collecting the results. Assume that f is lower semicontinuous and square-superadditive (superadditive if $\dim(E) = 2$). Then Lemma 11 implies that every pair has a join in $(E, \preceq_f)$. As f is superadditive and satisfies $f(d) > 0$ for all $d > 0$, it is increasing. The relation $\preceq_f$ is closed because of Lemma 7. Therefore, Lemma 10 implies that $(E, \preceq_f)$ has a discriminator. Therefore, Corollary 2 implies that $(E, \preceq)$ is a cc sponge. This concludes the proof of Theorem 4.

7. The hyperbolic sponge

Let h be a unit vector in a real Hilbert space E with $\dim(E) \ge 2$. Let $H = h^\perp$ be the hyperplane orthogonal to h and let $H^+ = \{x \in E \mid 0 < (h, x)\}$ be the (open) half space in direction h. We again have $x_h = (h, x)$ and $x_\perp = x - (h, x)h$.

It is known that H^+ can be considered a model for hyperbolic space [28, §7]: the Poincaré half-space model. In this model, the distance between two points in H^+ is

$$d_{\mathcal{H}}(x, y) = \operatorname{arcosh}\left(1 + \frac{\|x - y\|^2}{2\, x_h\, y_h}\right).$$

Where $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$. It can be checked [14, §12.2.1] that for two points x, y such that $x_\perp = y_\perp$, $d_{\mathcal{H}}(x, y) = |\ln(x_h) - \ln(y_h)|$. Because of this, we put

$$h_{\mathcal{H}}(x) = \ln(x_h).$$

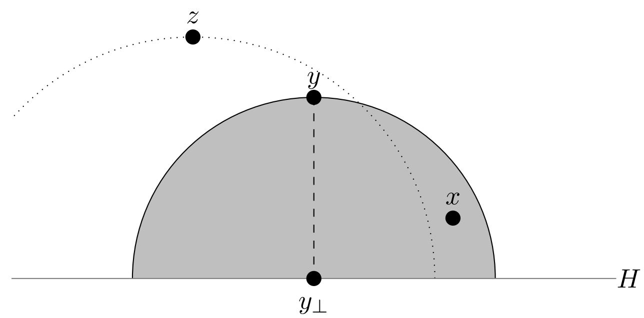

Figure 4: Illustration of the hyperbolic orientation in the Poincaré half-plane model. We have $x \preceq y$, as well as $y \preceq z$, but x and z are incomparable. Note how the set of left bounds of y is half of a closed disk in this model, and that only points strictly above H correspond to points in hyperbolic space.

Since $x \in H^+$, $x_h > 0$, and the above is well-defined. Note that while H^+ is an open subset of E, $(H^+, d_{\mathcal{H}})$ is, in fact, a complete metric space. Furthermore, the metric spaces $(H^+, d_{\mathcal{H}})$ and (H^+, d_E) with $d_E(x, y) = \|x-y\|$ are homeomorphic, as the identity function is a homeomorphism between the two.

Relation $\preceq$ on H^+ is defined by

$$x \preceq y \quad \equiv \quad \|x - y_\perp\| \leq y_h.$$

Note that $x \preceq x$ holds because $\|x - x_\perp\| = x_h$. We observe that the above definition corresponds to saying that $x \preceq y$ if and only if y lies between x and the highest point of the geodesic through x and y in the half-space model of hyperbolic space [28, Thm. 9.3]; this is the converse of the original formulation [14, §12.4.4], but for the current exposition it was much more convenient to use the convention that "higher" values are larger.

Lemma 12. *Let $x \preceq y$ and $x \neq y$. Then $h_{\mathcal{H}}(x) < h_{\mathcal{H}}(y)$.*

Proof. As x_h is the height of x above H, and $y_\perp \in H$, we have $x_h \leq \|x - y_\perp\|$, with equality if and only if $x_\perp = y_\perp$. On the other hand, $x \preceq y$ implies $\|x - y_\perp\| \leq y_h$. Considering that the logarithm is increasing on the positive reals, the inequality follows, unless $x_\perp = y_\perp$.

Therefore, assume that $x_\perp = y_\perp$. We then have $x = x_h h + x_\perp$ and $y = y_h h + x_\perp$. Given that $x \preceq y$, we see that $\|x - y_\perp\| = x_h \leq y_h$. As $x \neq y$, we see that $x_h < y_h$, and thus $h_{\mathcal{H}}(x) < h_{\mathcal{H}}(y)$. □

Corollary 5. *Relation $\preceq$ is an acyclic orientation on H^+.*

Proposition 8. *(a) Every finite subset P of H^+ has a right bound in $(E, \preceq)$.*
(b) A pair x, $y \in H^+$ is left bounded if and only if $\|x_\perp - y_\perp\| < x_h + y_h$.

Proof. (a) Choose $\lambda > 0$ with $\|p\| \leq \lambda$ for all $p \in P$. Then $y = \lambda\, h$ is a right bound of P, because $\|p - y_\perp\| = \|p\| \leq \lambda = y_h$ for every $p \in P$.

(b) Let p be a left bound of x and y. Then $p \in H^+$. Therefore p is not on the line segment between $x_\perp$ and $y_\perp$. This implies $\|x_\perp - y_\perp\| < \|x_\perp - p\| + \|p - y_\perp\| \leq x_h + y_h$. The converse follows from considering when two (hemi)spheres overlap. □

Remark 2. It follows that there are pairs without a left bound. For example, let e be a unit vector orthogonal to h, take $x = h$ and $y = h + 2\,e$. The pair x, y has no left bound because $\|x_\perp - y_\perp\| = 2$ and $x_h = y_h = 1$. This implies that inversion of the orientation gives a completely different oriented set.

Lemma 13. *The relation $\preceq$ on the complete metric space $(H^+, d_{\mathcal{H}})$ is closed.*

Proof. We first show that the relation is closed on the metric space (H^+, d_E) with the Euclidean metric. To this end, consider the function $f : H^+ \times H^+ \to \mathbb{R}$ given by $f(x, y) = y_h - \|x - y_\perp\|$. As f is continuous under the Euclidean metric, and $\preceq$ is the preimage of the closed set $\{t \mid t \geq 0\}$, $\preceq$ is closed. Since $(H^+, d_{\mathcal{H}})$ and (H^+, d_E) are homeomorphic, $\preceq$ is closed in $(H^+, d_{\mathcal{H}})$ as well. □

Lemma 14. *On the complete metric space $(H^+, d_{\mathcal{H}})$, $h_{\mathcal{H}}$ is a discriminator.*

Proof. In order to prove that function $h_{\mathcal{H}}$ is a discriminator, we try, given $y \in H^+$ and $\delta > 0$, to bound the distance $d_{\mathcal{H}}(x, y)$ for all vectors x in the set

$$\begin{aligned} L_\delta(y) &= \{x \in H^+ \mid x \preceq y \ \wedge \ h_{\mathcal{H}}(y) < h_{\mathcal{H}}(x) + \delta\} \\ &= \{x \in H^+ \mid \|x - y_\perp\| \leq y_h \ \wedge \ \ln(y_h) < \ln(x_h) + \delta\}. \end{aligned}$$

In view of the formula for $d_{\mathcal{H}}(x, y)$, the maximal value of this distance is obtained by maximizing $\|x - y\|$ and minimizing x_h. The maximal distance is therefore reached when $\|x - y_\perp\| = y_h$ and $\ln(y_h) = \ln(x_h) + \delta$. This maximal distance is not reached in $L_\delta(y)$, however, but only on its boundary. In any case, such vectors x give the least upper bound of the distance. If we write $x_u = \|x_\perp - y_\perp\|$, these two equations become $x_h^2 + x_u^2 = y_h^2$ and $y_h = x_h e^\delta$. After some calculation, one finds that $d_{\mathcal{H}}(x, y) = \operatorname{arcosh}(e^\delta)$ holds because

$$1 + \frac{\|x - y\|^2}{2\,x_h\,y_h} = \frac{2\,x_h\,y_h + (x_h - y_h)^2 + x_u^2}{2\,x_h\,y_h} = \frac{2\,y_h^2}{2\,x_h\,y_h} = e^\delta.$$

Using continuity of the arcosh function for $\delta \downarrow 0$, one finds that $h_{\mathcal{H}}$ is a discriminator. □

Theorem 5. *The pair $(H^+, \preceq)$ is a cc sponge.*

Proof. We have already shown that $\preceq$ is closed, and that $h_{\mathcal{H}}$ is a discriminator. It is also clear that $h_{\mathcal{H}}$ is continuous. Thus, if we can show that every left-bounded pair in H^+ has a meet, we can apply Theorem 2. Now, let x, y be a left-bounded pair in H^+. If x and y are comparable, one of them is the meet. We may therefore assume that x and y are not comparable. It follows that $x_\perp \neq y_\perp$.

Let e be a unit vector pointing from $x_\perp$ to $y_\perp$. Let $a = \|y_\perp - x_\perp\|$. Then $a > 0$ and $y_\perp = x_\perp + a\,e$. Proposition 8(b) implies that $a < x_h + y_h$.

Let S be the plane that contains the points x, $x_\perp$, y, $y_\perp$. Let $S^+ = S \cap H^+$. In the halfplane S^+, the set of left bounds of x is the half disk with center $x_\perp$ and radius x_h; similarly for y. These half disks intersect because $a < x_h + y_h$. As x and y are not comparable, it is not the case that one of the half disks is contained in the other. Therefore, the corresponding circles meet in two points, one of which is in S^+. Assume that the circles meet in $z \in S^+$. As z is contained in both half disks, it is a left bound of both x and y. We claim that z is the meet of x and y.

The projection $z_\perp$ is on the line through $x_\perp$ and $y_\perp$, and can therefore be written $z_\perp = x_\perp + b\,e$. As x and y are not comparable, we have $0 < b < a$. Let $c = a - b$. Then $y_\perp = z_\perp + c\,e$. It follows that

$$\left.\begin{aligned} b^2 + z_h^2 &= x_h^2 \\ c^2 + z_h^2 &= y_h^2 \\ b + c &= a. \end{aligned}\right\} \tag{10}$$

In order to prove that z is the meet of x and y, it remains to prove that any left bound u of x and y is a left bound of z. Let $z_\perp + t\,e$ be the orthogonal projection of u onto the line through $x_\perp$ and $y_\perp$, and let s be the distance of u to this line. We now have

$$\begin{array}{ll}
 & u \preceq x \ \wedge\ u \preceq y \\
\equiv & \{ \text{ definition } \} \\
 & (t+b)^2 + s^2 \leq x_h^2 \ \wedge\ (t-c)^2 + s^2 \leq y_h^2 \\
\equiv & \{ \text{ Equation (10) } \} \\
 & t^2 + 2bt + s^2 \leq z_h^2 \ \wedge\ t^2 - 2ct + s^2 \leq z_h^2 \\
\Rightarrow & \{\ b > 0 \text{ and } c > 0, \text{ and hence } 2bt \geq 0 \text{ or } 2ct \leq 0\ \} \\
 & t^2 + s^2 \leq z_h^2 \\
\equiv & \{ \text{ definition } \} \\
 & u \preceq z.
\end{array}$$

This proves that z is the meet of x and y. Considering Lemmas 13 and 14, and observing that $h_{\mathcal{H}}$ is continuous, we can now apply Theorem 2 to conclude that $(H^+, \preceq)$ is a cc sponge. □

8. The geometry of the various sponges

In order to compare the various sponges we constructed, it is useful to examine the left cones $L(x)$ and the right cones $R(x)$ of elements in the different sponges.

In a sponge group, all left and right cones are isomorphic because $R(x) = x \cdot R(0)$ and $L(x) = x \cdot L(0)$ and $L(0) = R(0)^{-1}$. Also, there can be no left- or right-extreme points: suppose that x_0 is a left-extreme point, we then have

$x_0 \cdot L(0) = \{x_0\}$, meaning that x_0 cannot have an inverse (which it has to have, since it is a group element).

In the inner-product sponge of Section 3, every right cone $R(x)$ for $x \neq 0$, is a half space, while the left cone $L(x)$ is the ball centered at $\frac{1}{2}x$ with radius $\frac{1}{2}\|x\|$. In the hyperbolic sponge of Section 7, every left cone is a half ball, while the right cone is bounded by a component of a hyperboloid.

The inner-product sponge has precisely one left-extreme point, viz. the origin of the space, and no right-extreme points. The hyperbolic sponge has no extreme points.

In the inner-product sponge, every nonempty subset has a meet, which can be the origin. In the hyperbolic sponge, every finite or bounded subset has a join.

In the inner-product sponge, the right cones $R(x)$ and $R(y)$ are disjoint if and only if $x \neq 0$ and $y = \lambda x$ for some $\lambda < 0$. In the hyperbolic sponge, the left cones $L(x)$ and $L(y)$ are disjoint if and only if $x_h + y_h \leq \|x_\perp - y_\perp\|$.

9. Summary and future work

This work aims to extend the toolbox of those interested in the extra freedom that (conditionally complete) sponges provide compared to lattices. To this end Theorem 2 shows that in a complete metric space, it suffices to show (under mild conditions) that an orientation has joins and meets for pairs to prove that it is a sponge, and even a conditionally complete one. Section 5 shows how for sponge groups, the additional structure helps to identify new sponges derived from a known sponge group. Theorem 4 in its turn gives a fairly satisfying characterization of epigraph sponges, which are applicable to any Hilbert space with a "preferred" direction. Examples of the latter include matrices with the Frobenius inner product and the (scaled) identity matrix playing the role of h, but also colour spaces for instance, where white would play the role of h. Section 7 uses Theorem 2 to establishes that the hyperbolic sponge identified previously [2, 3] generalizes to arbitrary dimension (earlier, only the two dimensional case was studied).

In future, it would be interesting to not just look at sponge groups, but also sponges with groups acting on them: both the inner product sponge and the epigraph sponge deal with some kind of rotation invariance, but we do not yet have an overarching theory to constrain and characterize sponges with given group invariances (and we do expect that at least some interesting things can be said). One could also consider further generalizations, for example by including semigroups.

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