Partial meet pseudo-contractions

David Santos, Yuri; Bitencourt Matos, Vinícius; Moretto Ribeiro, Márcio; Wassermann, Renata

Published in:
International Journal of Approximate Reasoning

DOI:
10.1016/j.ijar.2018.08.006

International Journal of Approximate Reasoning 103 (2018) 11–27

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Partial meet pseudo-contractions ☆

Yuri David Santos [a,*], Vinícius Bitencourt Matos [b], Márcio Moretto Ribeiro [c], Renata Wassermann [b]

[a] Faculty of Philosophy & Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen, The Netherlands
[b] Department of Computer Science, University of São Paulo (USP), São Paulo, Brazil
[c] School of Arts, Sciences and Humanities (EACH), University of São Paulo (USP), São Paulo, Brazil

ARTICLE INFO

Article history:
Received 14 February 2018
Received in revised form 21 August 2018
Accepted 24 August 2018
Available online 29 August 2018

Keywords:
Belief revision
Limited reasoning
Ontologies
Pseudo-contractions

ABSTRACT

In the AGM paradigm for belief revision, epistemic states are represented by logically closed sets of sentences, the so-called belief sets. An alternative approach uses belief bases, arbitrary sets of sentences. Both approaches have their problems when it comes to contraction operations. Belief bases are more expressive, but, at the same time, they present a serious syntax dependence.
Between those two extremes lie a whole gamut of operations called pseudo-contractions, some of which may be interesting alternatives to the classical ones, providing a good balance between syntax dependence and expressivity.
In this paper we explore some very natural and general constructions for pseudo-contractions, showing some of their properties and giving their axiomatic characterizations. We also illustrate possible practical scenarios where they can be employed.

1. Introduction

Rational agents, be they humans or machines, must have some representation of their knowledge or belief system. *Belief revision* wants to understand how these agents should change these representations when they are faced with new information. As Gärdenfors has claimed [11, preface], there is little use in knowing how to represent knowledge if we do not know how to change these representations. The problem is not trivial, as new information may contradict what the agent previously believed, and it is not always clear how it can be accommodated.

In the most widely accepted theory of belief revision, known as AGM due to the initials of the authors of [1], the belief state of an agent is represented by a set of formulas closed under logical consequence. This representation has allowed for very elegant results, linking mathematical constructions to properties defining rational outcomes. These properties, the *AGM rationality postulates*, describe very intuitive desiderata, such as the *inclusion* postulate for contraction, which states that when the agent is giving up a belief, no new information should be added.

Nevertheless, the logical elegance of the AGM theory brings with it some undesirable effects when applied in practice: logical closure is computationally complex for any reasonably expressive language; in most logics, inconsistency (even if

☆ This paper is part of the Virtual special issue on Defeasible and Ampliative Reasoning, Edited by Ivan Varzinczak, Richard Booth, Giovanni Casini, Szymon Klarman and Gilles Richard.

* Corresponding author.
E-mail addresses: y.david.santos@rug.nl (Y.D. Santos), vinicius.matos@usp.br (V.B. Matos), marciomr@usp.br (M.M. Ribeiro), renata@ime.usp.br (R. Wassermann).

https://doi.org/10.1016/j.ijar.2018.08.006
0888-613X/

momentary) will lead to trivialization; there is no distinction between beliefs that are explicit, or explicitly acquired, and beliefs that were implicitly derived.

In further studies (such as [15]), a generalization of the AGM theory was proposed wherein belief states are represented by arbitrary sets of logical sentences, not necessarily closed, called *belief bases*. Belief bases have a major advantage over closed belief sets: they are more expressive, as many different bases can be representations for one and the same belief state. This is particularly useful in the case of an inconsistent belief state. With logically closed sets, typically an inconsistency will clutter the set with all sentences of the language, making any two inconsistent belief states indistinguishable.

The same mathematical constructions defined for the AGM theory can be applied for belief bases. But here, the inclusion postulate, so intuitive for closed sets, turns out to be too restrictive. Without adding new elements it is not possible to weaken any formula, which could be enough to perform the contraction in consonance with the *success* postulate (i.e., effectively removing the input sentence). This means that more than what is necessary will be removed, considering that we want a *minimal change* (regarded as one of the base principles of belief revision). This problem can also be viewed as the problem of *syntactic dependence*, i.e., for two bases representing the same belief state, the codification of the base (the individual formulas it contains) can lead to different contractions. This phenomenon does not exist in the context of belief sets, since they represent a belief state in the *knowledge level* [27], ignoring syntactic variations.

Thus, on the one hand, we have operations on belief states represented by logically closed sets, which are completely independent of the syntactical form of formulas and, on the other hand, we have base operations where the outcome is completely dependent on the syntax. What we are looking for in this work is a middle ground, where some syntactical distinctions may be blurred. Throughout the paper, we present different examples of what this means. Typically, we are interested in weakening formulas, giving up parts of beliefs. As a motivation, consider the following example:

Example 1. [17] Suppose I believe, for good and independent reasons, that *Andy is son of the mayor* (a) and *Bob is son of the mayor* (b). Then I hear the mayor say: "I certainly have nothing against our youth studying abroad. My only son did it for three years". I then have to retract $a \wedge b$ from my base $B = \{a, b\}$. But it is reasonable to retain a belief that either Andy or Bob is the son of the mayor, i.e., the result of the contraction should be $\{a \vee b\}$.

Hansson called these operations *pseudo-contractions* [17]. Pseudo-contractions are base change operations that violate the inclusion postulate. The goal of this paper is to investigate a particular construction for pseudo-contractions. We first argue for the benefits of this investigation, then we demonstrate some of the properties and relationships of this operation with other pseudo-contractions in the literature, such as the ones proposed by Nebel [26] and by Ribeiro and Wassermann [29]. It is important to note that in departing from the inclusion postulate for bases, we are not advocating for the arbitrary inclusion of formulas, but only for those that would be allowed in the corresponding closed set.

The original motivation for this research program came from databases and ontologies, where the knowledge base is a simplified representation for a more complex set of beliefs.

Example 2. Imagine that we have the knowledge base below:

$$KB = \{\text{likesCold} \wedge \neg\text{flies}, \text{isBird}, \text{isBird} \rightarrow \text{flies}\}.$$

Here, $\{\text{flies}, \neg\text{flies}\} \subseteq Cn(KB)$, that is, KB is inconsistent. It is easy to spot the inconsistency in this case, but one can easily come up with other examples where the chain of reasoning leading to flies is much more complicated, which would be more realistic. It is not uncommon for real life ontologies (KBs) to contain some inconsistencies. This, of course, does not mean that the users of that KB want to treat it as any other inconsistent KB, such as $\{\text{flies}, \neg\text{flies}\}$ or $\mathfrak{L}$ (the whole language) itself: they still find the information in KB meaningful, although strictly speaking $\varphi \in Cn(KB)$ for any φ. So, on the one hand, we want the syntactic form of KB to be taken into account, not only its logical closure ($Cn(KB)$). The bits of information likesCold and isBird, for example, seem to be part of the KB, whereas ¬likesCold and ¬isBird do not (although from a purely logical point of view, which is probably not the user's point of view, ¬likesCold and ¬isBird do follow from the explosion caused by the inconsistency). On the other hand, if we consider KB to be completely in the syntactical realm, a contraction of ¬flies in this case could be accomplished by removing $\text{likesCold} \wedge \neg\text{flies}$, but then we also lose likesCold ($\text{likesCold} \notin Cn(KB - \neg\text{flies})$). Sure, there might be cases where we want likesCold to go with ¬flies: maybe they were entered together and therefore should be deleted together. It is plausible, however, that in some cases the users of this KB (and this is application/domain dependent) would expect likesCold to survive the removal of ¬flies, making $\{\text{likesCold} \wedge \neg\text{flies}, \text{isBird}, \text{isBird} \rightarrow \text{flies}\} - \neg\text{flies} = \{\text{likesCold}, \neg\text{flies}, \text{isBird}, \text{isBird} \rightarrow \text{flies}\} - \neg\text{flies}$. This example shows that an intermediate degree of syntax dependence (not zero but not complete) might be desired.

The above example supports the idea that KBs must neither be treated as belief sets (deductively closed) nor simply as mere sets of formulas, where only the information explicitly represented matters. We come back to ontology engineering as an example of application in Section 5.

The rest of this paper is organized as follows. The subjects of belief revision and pseudo-contractions are approached in Sections 2 and 3, respectively. In Section 4, we introduce a very general pseudo-contraction, show some of its properties and in Section 5, we show examples of where it is used in practice. In Section 6, we present a second construction for pseudo-contraction and show its relation to local change. We conclude and present possibilities for future work in Section 7.

A preliminary version of this paper appeared in [31].

2. Belief change

In this section, we present the necessary background in the theory of Belief Change. We start by introducing the AGM paradigm and proceed to the theory of belief base change.

Before continuing, we will settle some conventions. We denote a language by $\mathfrak{L}$. A consequence relation is a function $C : 2^{\mathfrak{L}} \to 2^{\mathfrak{L}}$. We denote logical sentences by Greek letters ($\alpha, \beta, \dots$) and sets of logical sentences by capital Latin letters ($A, B, \dots$). Propositional atoms are denoted by lowercase Latin letters ($p, q, \dots$). We use Cn for the consequence relation of classical propositional logic and Cn_{FOL} for the consequence relation of first-order logic. Below we list some properties a consequence relation may have.

monotonicity If $A \subseteq B$ then $C(A) \subseteq C(B)$
inclusion $A \subseteq C(A)$
idempotence $C(A) = C(C(A))$
supraclassicality $Cn(A) \subseteq C(A)$
compactness If $\alpha \in C(A)$, then $\alpha \in C(A')$ for some finite $A' \subseteq A$.
deduction $\beta \in C(A \cup \{\alpha\})$ if and only if $\alpha \to \beta \in C(A)$.
subclassicality $C(A) \subseteq Cn_{FOL}(A)$

If a consequence relation satisfies monotonicity, inclusion and idempotence, we say it is *Tarskian*.

2.1. The AGM paradigm

Alchourrón, Gärdenfors and Makinson (henceforth AGM) published the paper that is considered the initial hallmark of the area [1], where they proposed to represent epistemic states by logically closed sets of propositions, called *belief sets*, i.e., a set K such that $K = C(K)$ for some consequence relation C (usually Tarskian). Three basic operations were considered in the paper: expansion, contraction and revision. Expansion (denoted by $+$) is the simple addition of a new logical sentence to the belief set, followed by the logical closure of the set, i.e., $K + \alpha = Cn(K \cup \{\alpha\})$. The other two operations were not uniquely defined, but just constrained by a set of *rationality postulates*. Contraction (denoted by $-$) is the removal of a sentence in a way that the resulting set does not imply it (otherwise, since the belief set is logically closed, the sentence would readily reappear). Revision (denoted by $*$) is the addition of a new sentence, like expansion, but with the additional aim of preserving consistency whenever possible.

The following are the basic AGM rationality postulates for contraction.

(closure) $K - \alpha = Cn(K - \alpha)$
(success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin K - \alpha$
(inclusion) $K - \alpha \subseteq K$
(vacuity) If $\alpha \notin K$, then $K - \alpha = K$
(recovery) $K \subseteq (K - \alpha) + \alpha$
(extensionality) If $Cn(\alpha) = Cn(\beta)$, then $K - \alpha = K - \beta$

Two other supplementary postulates were also proposed, but we will not address them here, as they are not as intuitive as the basic ones and are language dependent. Since revisions can be defined from contractions and expansions using the *Levi Identity*, given by $K * \alpha = (K - \neg\alpha) + \alpha$ [1], we will focus on contractions in this paper.

In addition to the postulates of rationality that constrain the space of possible contractions, AGM have proposed a construction for a contraction operation, known as *partial meet contraction*.

Definition 3. [3] Let $B \subseteq \mathfrak{L}$ and $\alpha \in \mathfrak{L}$. The remainder set $B \perp \alpha$ is such that $X \in B \perp \alpha$ iff: $X \subseteq B$; $\alpha \notin Cn(X)$; and for all sets Y, if $X \subset Y \subseteq B$, then $\alpha \in Cn(Y)$.

Definition 4. [1] A function γ is a selection function for the set B if and only if, for all $\alpha \in \mathfrak{L}$:

- If $B \perp \alpha \neq \emptyset$ then $\emptyset \neq \gamma(B \perp \alpha) \subseteq B \perp \alpha$;
- Otherwise, $\gamma(B \perp \alpha) = \{B\}$.

Definition 5. [1] Let γ be a selection function for a set of sentences B. The partial meet contraction of B by a sentence α is given by $B - \alpha = \bigcap \gamma(B \perp \alpha)$.

It turns out that partial meet contraction is exactly equivalent to the aforementioned postulates for contraction, giving us one of the most important results in belief revision:

Theorem 6. *[1] An operator $-$ is a partial meet contraction for a belief set K if and only if for all sentences α the operation $K - \alpha$ satisfies closure, success, inclusion, vacuity, recovery and extensionality.*

2.2. Belief bases

The AGM theory was generalized in different ways, for example for other logics [30] (see Section 2.3). Another generalization was the use of belief bases instead of belief sets to represent epistemic states [16]. Belief bases are arbitrary, not necessarily closed, sets of sentences, which grants them greater expressivity when compared to belief sets.

The same construction of Definition 5 can be applied to belief bases, however not all the AGM postulates are satisfied. In particular, the *recovery* postulate is not satisfied by partial meet base contraction.

Hansson has characterized the operation for bases by means of the following postulates:

(success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B - \alpha)$
(inclusion) $B - \alpha \subseteq B$
(relevance) If $\beta \in B \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq B$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$
(uniformity) If for all $B' \subseteq B$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$

Theorem 7. *[16] An operator $-$ is a partial meet contraction for a belief base B if and only if for all sentences α the operation $B - \alpha$ satisfies success, inclusion, relevance and uniformity.*

Another construction for bases found in the literature is *kernel contraction* (a generalization of *safe contraction* [4]), proposed by Hansson in [18]. Kernel contraction is defined by removing at least one element of each *kernel*, a minimal subset of the base implying the sentence to be contracted. Formally:

Definition 8. [18] Let $B \subseteq \mathfrak{L}$ and $\alpha \in \mathfrak{L}$. The kernel $B \perp\!\!\!\perp \alpha$ is such that $X \in B \perp\!\!\!\perp \alpha$ iff: $X \subseteq B$; $\alpha \in Cn(X)$; and for all sets Y, if $Y \subset X$, then $\alpha \notin Cn(Y)$.

Definition 9. [18] A function σ is an incision function for the set B if and only if:

- $\sigma(B \perp\!\!\!\perp \alpha) \subseteq \bigcup B \perp\!\!\!\perp \alpha$.
- if $\emptyset \neq X \in B \perp\!\!\!\perp \alpha$ then $X \cap \sigma(B \perp\!\!\!\perp \alpha) \neq \emptyset$.

Definition 10. [18] Let σ be a selection function for a set of sentences B. The kernel contraction of B by a sentence α is given by $B -_\sigma \alpha = B \setminus \sigma(B \perp\!\!\!\perp \alpha)$.

The postulates that characterize this contraction function are the same as the ones for partial meet for bases, but replacing relevance by the slightly weaker *core-retainment* postulate [18]:

(core-retainment) If $\beta \in B \setminus (B - \alpha)$, then there is a $B' \subseteq B$ such that $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

A desirable property that partial meet contraction satisfies, but kernel contraction does not, is *relative closure*:

(relative closure) $B \cap Cn(B - \alpha) \subseteq B - \alpha$

This property is a consequence of the postulate of relevance [19, p. 71], which kernel contraction does not satisfy. Due to the lack of this property, kernel contraction violates the principle of minimality of belief change [19, p. 90], as shown in the next example.

Example 11. Consider the logically independent sentences p and q, and let $A = \{p, p \vee q, p \leftrightarrow q\}$. The kernel contraction $A - (p \wedge q) = \{p\}$ is possible, whereas partial meet contraction cannot have this outcome. As observed by Hansson, it is not sensible to give up $p \vee q$, since p was kept.

Nevertheless, a special type of kernel contraction called *smooth kernel contraction* [18] satisfies relative closure.

When studying operations on belief bases, it is interesting to know what happens in the closure of the resulting set (as though we were just using bases to represent the belief sets). Next we define a base-generated operation:

Definition 12. Let $B \subseteq \mathfrak{L}$, $K = Cn(B)$ and $-$ be a contraction operator for B. We say that $\div$ is a contraction operator for K generated from $-$ if for all $\alpha \in \mathfrak{L}$ we have that $K \div \alpha = Cn(B - \alpha)$.

It may be desirable that base-generated operations satisfy the AGM postulates (producing a rational operation on the knowledge level). It turns out that no base-generated contraction operation will satisfy recovery [19, p. 307] (assuming that base contraction is an operation satisfying success and inclusion, the language has infinitely many atoms and is closed under boolean connectives, and the base is finite). In any case, recovery is the most controversial of all the six basic postulates.

2.3. Other generalizations

There have been several generalizations of the AGM theory in order to apply it to different logics. We may notice that, in fact, the AGM paradigm assumes an underlying logic $\langle \mathfrak{L}, Cn\rangle$ where $\mathfrak{L}$ is closed under boolean connectives and Cn is Tarskian and satisfies supraclassicality, compactness and deduction. Both propositional and first-order classical logics satisfy these AGM assumptions, but several interesting fragments do not, such as description logics and Horn logics.

The AGM operation of contraction has been adapted to description logics by Flouris [10] and then refined by several authors. In a subsequent paper, the theory was extended by Ribeiro et al. and the precise assumptions needed were specified [30]. Delgrande and others studied in a series of papers how AGM contraction can be defined for Horn Logics [8,6,9].

The theory of belief base change has been generalized to logics that only need to be compact and monotonic by Hansson and Wassermann [20].

What was missing up to now was a study for scenarios where we need some of the power of belief sets, but not at the cost of completely erasing all syntactical differences. This is what we provide in this paper, after introducing the idea of *pseudo-contractions*.

3. Pseudo-contractions

Partial meet and kernel contractions over belief bases can lead to unnecessary waste of information, largely due to the inclusion postulate. This postulate prevents the weakening of formulas, which can be seen as an argument against its use for belief bases. Moreover, base-generated contractions never satisfy recovery precisely because of inclusion (and success, of course). The next example depicts the effects of inclusion over base contractions.

Example 13. [29] Suppose we have a belief base $\{p \wedge q\}$, which stands for the fact that Cleopatra had a son (p) and that she had a daughter (q). If we want to contract the base by p, i.e., we want to remove the belief that she had a son, we have to give up the whole conjunction, and since the formula q is not included in the base, we give up the belief that Cleopatra had a daughter too.

One can justify that since p and q were together in the base, they should be removed together too. Nonetheless, from a purely logical point of view, there was loss of information, and the set $\{q\}$ should at least be allowed as a possible result for the contraction.

Hansson has proposed a weakening of inclusion, *logical inclusion* [14]. He has called operations satisfying success and logical inclusion *pseudo-contractions*[1] [17].

(logical inclusion) $Cn(B - \alpha) \subseteq Cn(B)$

In fact, any operation on bases satisfying logical inclusion will give rise to a base-generated operation that satisfies inclusion. Since the idea of the inclusion postulate is to prevent addition of new information (not the addition of new formulas), logical inclusion seems to be far more adequate than its stronger variant for bases. This is a strong argument in favor of pseudo-contractions, constructions which can keep most of the advantages of base contraction and at the same time minimize informational loss.

One of the first such operations was Nebel's pseudo-contraction [26], whose corresponding generated contraction on belief sets satisfies all the six basic AGM postulates.

Definition 14. Let $\bigwedge B$ be the conjunction of all elements of B, and γ a selection function for B. Nebel's pseudo-contraction for the set B is the operator $-$ such that for all sentences α:

$$B - \alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset); \\ \bigcap \gamma(B \perp \alpha) \cup \{\alpha \to \bigwedge B\} & \text{otherwise.} \end{cases}$$

As already remarked by Ribeiro and Wassermann [29], this construction seems to have been conceived with the sole purpose of generating an operation on belief sets that satisfies recovery, fixing a problem of base contractions. Besides the restoration of recovery, it is difficult to find another intuition behind Nebel's operation. In the same paper, they also note that it adds to the belief base more information than necessary to attain recovery, and show that $\{\alpha \to \beta \mid \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\}$ is

[1] Sometimes contractions are regarded as operations satisfying success and inclusion. Naturally, all contractions defined in this way are also pseudo-contractions. In general, when we use the term pseudo-contraction we are referring specially to those operations that do not satisfy inclusion.

enough for that purpose (let us call this variant *restricted Nebel's pseudo-contraction*). The next example shows that sometimes Nebel's operator can have an odd behaviour.

Example 15. [29] Returning to the context of the previous Cleopatra example, Nebel's pseudo-contraction (or its restricted version) would make $\{p \wedge q\} - p = \{p \to p \wedge q\}$, which means that if we do not believe anymore that Cleopatra had a son, we do not know anything about her having a daughter.

Following the criticism of Nebel's operator, Ribeiro and Wassermann define a new operator, called *general partial meet pseudo-contraction* (from now on abbreviated to GPMPC) based on the notion of extension of a selection function.

Definition 16. [29] Let γ be a selection function for a set B and let B^* contain B. An extension of γ to B^* is a selection function γ^* such that for every $Y \in \gamma^*(B^* \perp \alpha)$ there is an $X \in \gamma(B \perp \alpha)$ such that $X \subseteq Y$.

Definition 17. [29] Let B be a finite belief base, $\alpha \in \mathfrak{L}$ and γ a selection function for B. The general partial meet pseudo-contraction $B - \alpha$ is given by[2]:

$$B - \alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma^*(Cn^*(B) \perp \alpha) & \text{otherwise} \end{cases}$$

where $Cn^*(B) = B \cup f_\alpha(B \setminus \bigcap \gamma(B \perp \alpha))$, γ^* is an extension of γ to $Cn^*(B)$ and f_α is a consequence relation.

Despite having a quite complicated definition, GPMPC has a simple intuition behind it: the weakening of removed formulas. The function f_α expands the original set of beliefs with some of the classical consequences of the elements that would have been removed by a partial meet contraction (which has its selection function γ, as usual). Then, with an extension of γ, we make a partial meet contraction over this expanded base. If instead of an extension of γ we had used an arbitrary selection function, the whole construction would make no sense, since the purpose of the construction is to weaken the formulas that *would* have been removed (therefore the extension should remove them in any case, and it does exactly that).

Ribeiro and Wassermann showed that this operator satisfies success, logical inclusion, logical vacuity (if $\alpha \notin Cn(B)$, then $B - \alpha = B$), extensionality and core-retainment. The corresponding base-generated operation satisfies all basic AGM postulates except recovery (this type of operation has been called *withdrawal* [22]). If Cn^* is subclassical, the operator also satisfies logical relevance[3]:

(logical relevance) If $\beta \in B \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn(B)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

Pseudo-contractions in general are quite flexible. Ribeiro and Wassermann discussed the balance between "degrees" of inclusion and recovery. If the base contraction respects inclusion, the base-generated operation will not satisfy recovery. If, on the other extreme, a pseudo-contraction adds all that is allowed by logical inclusion, we have a belief set, and thus the contraction will satisfy recovery. Notice that this is not possible with GPMPC, since only the set of removed elements is expanded. The in-between cases present only "degrees" of inclusion and recovery. In order to limit the additions to the base, Ribeiro and Wassermann defined a postulate called *core-addition*:

(core-addition) If $\beta \in (B - \alpha) \setminus B$, then there is a $\beta' \in B \setminus (B - \alpha)$ and a $B' \subseteq B - \alpha$ such that $\alpha \to \beta' \notin Cn(B')$ but $\alpha \to \beta' \in Cn(B' \cup \{\beta\})$

This postulate is not satisfied by GPMPC in general, but it is not difficult to find cases of the operator satisfying it, depending on the Cn^* used (for example, if $f_\alpha(B \setminus \bigcap \gamma(B \perp \alpha)) = \{\alpha \to \beta \mid \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\}$, rendering GPMPC as restricted Nebel's operation).

We want to further our understanding of this complicated operation. First, we have found out whether it satisfies some important additional properties.

Proposition 18. *If a GPMPC operation satisfies inclusion, then it satisfies relevance.*

For the proof of Proposition 18, we first need the following lemma.

Lemma 19. *Let $A^* = A \cup X$, γ be a selection function for A and γ^* be an extension of γ to A^*. If $\bigcap \gamma^*(A^* \perp \alpha) \subseteq A$, then $\bigcap \gamma^*(A^* \perp \alpha) = \bigcap \gamma(A \perp \alpha)$.*

[2] We have made slight modifications in the definition to avoid confusion. Where we have written $Cn^*(B)$ was originally B^*, and f_α was Cn^*.

[3] Different from what would be expected, this postulate for a base operation is not equivalent to relevance for the base-generated operation (as is the case for inclusion and logical inclusion).

Proof. If $\varepsilon \in \bigcap \gamma(A \perp \alpha)$, then $\varepsilon \in X$ for all $X \in \gamma(A \perp \alpha)$. Since, by definition of γ^*, for all $Y \in \gamma^*(A^* \perp \alpha)$, there is an $X \in \gamma(A \perp \alpha)$ such that $X \subseteq Y$, we conclude that $\varepsilon \in Y$ for all $Y \in \gamma^*(A^* \perp \alpha)$, therefore $\bigcap \gamma(A \perp \alpha) \subseteq \bigcap \gamma^*(A^* \perp \alpha)$.

For the other direction of the proof, let us assume $\bigcap \gamma^*(A^* \perp \alpha) \nsubseteq \bigcap \gamma(A \perp \alpha)$. So, there must be an $\varepsilon \in \bigcap \gamma^*(A^* \perp \alpha)$ such that $\varepsilon \notin \bigcap \gamma(A \perp \alpha)$. Thus, $\varepsilon \in Y$ for all $Y \in \gamma^*(A^* \perp \alpha)$, but $\varepsilon \notin X$ for some $X \in \gamma(A \perp \alpha)$, accordingly, for all those Y's and some of those X's, $\varepsilon \in Y \setminus X$. Since $X \in A \perp \alpha$, for any $\varepsilon' \in A \setminus X$ we have $\alpha \in Cn(X \cup \{\varepsilon'\})$, and since $\alpha \notin Cn(Y)$, it must be the case that $\varepsilon \notin A$. But we know that $\bigcap \gamma^*(A^* \perp \alpha) \subseteq A$, therefore this ε cannot exist and so we come to a contradiction. □

Proof of Proposition 18. Let $-$ be a GPMPC and $-_\gamma$ be a partial meet contraction. If α is a tautology, $B - \alpha = B = B -_\gamma \alpha$, so let us consider the main case when $\alpha \notin Cn(\emptyset)$.

Since inclusion holds, $\bigcap \gamma^*(Cn^*(B) \perp \alpha) \subseteq B$, and thus, by Lemma 19, $\bigcap \gamma^*(Cn^*(B) \perp \alpha) = \bigcap \gamma(B \perp \alpha)$. We know relevance holds for partial meet contraction, so it must hold for GPMPC in this case as well. □

Proposition 20. *GPMPC does not satisfy uniformity (in general).*

Proof. We give a counterexample:

$$
\begin{aligned}
&A = \{p, q, r\} && Cn^*(A) = A \cup \{p \vee x\} \\
&A \perp p = \{\{q, r\}\} && Cn^*(A) \perp p = \{\{q, r, p \vee x\}\} \\
&A \perp (p \vee x) = \{\{q, r\}\} && Cn^*(A) \perp (p \vee x) = \{\{q, r\}\}. \quad \square
\end{aligned}
$$

Proposition 21. *GPMPC does not satisfy core-addition (in general).*

Proof. Again, we can look at a counterexample:

Let $A = \{p \wedge q\}$, $A \perp p = \emptyset$. Let $A^* = \{p \wedge q, q \vee x\}$, so $A^* \perp p = \{q \vee x\}$. The sentence $q \vee x$ was legally added to A^* but $p \wedge q \notin Cn(\{q \vee x\} \cup \{p\})$. □

The next natural step is to first analyse what happens if we just take the partial meet contraction of the base expanded by some weak consequence relation, i.e., $Cn^*(B)$.

Other related work Before continuing with our study of pseudo-contractions, we will briefly mention some related approaches. Very close to Nebel's approach, both in goals (saving recovery) and construction, is one of Nayak's contractions found in [25]. In the attempt to regain recovery, the author altered one of the contractions proposed in the paper to expand the base with beliefs of the form $\alpha \to \beta$, where α is the contracted element and β is a formula that was removed.

One of the most important studies concerning a pseudo-contraction (although the authors did not use this term) was the one started in [24] by Meyer et al. and extended in [23]. The idea was to use the additional structure provided by belief bases to improve the results of theory contraction. The operation they define weakens some formulas, instead of only removing, and is uniquely defined for each input formula and "infobase" (as they call their belief bases).

Despite some affinity, our work greatly differs from theirs in terms of general outlook. Their approach is more focused on semantics; infobase contraction is not concerned with the syntactic form of the formulas in the base, as they mention. Clearly our approach has a syntactic aspect. The drawback of focusing in the semantics is that, for example, the result of the infobase contraction (in Example 2 of [24]) $\{p, q\} - (p \wedge q)$ is the unwieldy set $\{p \vee ([p \wedge \neg q \wedge \neg(p \wedge q)] \vee [q \wedge \neg p \wedge \neg(p \wedge q)]), q \vee ([p \wedge \neg q \wedge \neg(p \wedge q)] \vee [q \wedge \neg p \wedge \neg(p \wedge q)])\}$. Each of these formulas are logically equivalent to $p \vee q$, which is the result to be intuitively expected, but one can argue that the syntactical form of the base matters (specially for practical purposes).

Our approach is more general and abstract, focusing on consequence relations, and our intention is to give a better understanding of pseudo-contractions in general, not of one construction in particular. Another difference in their paper is the language, which was restricted to a two-valued language containing $\perp$, $\top$ and closed under the standard connectives.

4. Cn*-pseudo-contraction

In this section, we present our proposal for a pseudo-contraction operation that depends on the kind of formulas that we are allowed to add when removing a formula. We first introduce the operation and give some examples of use, and then show its formal properties. This section and Section 6 contain the main results of the paper.

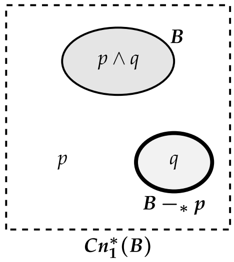

Fig. 1. Cn_1^* in Example 23.

4.1. Definition and examples

We want to study the effects of partial meet contraction over weakly closed sets,[4] i.e., sets closed by a consequence relation Cn^* which, in general,[5] "produces" fewer consequences than the classical Cn. We can formally define this operator as follows:

Definition 22. Let B be a set of sentences, Cn^* a consequence relation and γ a selection function for $Cn^*(B)$. The operator $-_*$ is such that, for all sentences α:

$$B -_* \alpha = \bigcap \gamma(Cn^*(B) \perp \alpha).$$

We will call this $-_*$ operator Cn*-pseudo-contraction. If we take $Cn^*(A) = A$, Cn*-pseudo-contraction becomes partial meet for belief bases, and if $Cn^*(A) = Cn(A)$, it becomes partial meet for belief sets. So, this operator is in fact a generalization of those approaches, allowing us to get results between them if we make Cn^* vary in the range between A and $Cn(A)$, which correspond to the thresholds for satisfaction of inclusion and logical inclusion, respectively.

Before continuing with formal definitions of Cn*-pseudo-contractions, we should compare it with the most known and accepted approaches. We are going to compare it with three of them: the traditional AGM paradigm, base change and AGM with a weaker underlying logic (with the same Cn^* subclassical consequence relation), hereafter referred to as *restricted-AGM*. This last approach is the one used, for example, in [8,9] for Horn logics and in [30] for more general logics.

First, addressing traditional AGM contractions, the same arguments in favor of belief base contractions hold for our approach. Belief bases are more expressive. In the AGM paradigm one cannot distinguish two inconsistent states. Despite Cn*-pseudo-contraction expanding the base before contraction, Cn^* is in general subclassical and may not imply everything from the contradiction.

Now, one might ask what is the difference between our approach and restricted-AGM. In fact, the epistemic states would be equally closed under the same subclassical consequence relation. However, in the restricted-AGM approach the remainder sets would also be computed under the same restricted logic, which means that the success postulate would be checked against this weaker logic, and thus the contracted element would not be entailed by the resulting set with respect to this logic, but it could be classically entailed. In our pseudo-contraction, success is still classical, which means that in this case the contracted sentence would not be classically entailed by the resulting set. Moreover, if the logic used by the restricted-AGM approach is less expressive, the belief base will be encoded in this poorer language, whereas in our approach this is not necessarily the case.

We give some examples of the behaviour of this pseudo-contraction.

Example 23. Consider again Example 13 and the belief base $B = \{p \wedge q\}$, where p stands for *Cleopatra had a son* and q, *Cleopatra had a daughter*. If we want to contract by p, applying a base partial meet contraction produces $B - p = \emptyset$. This is not always the expected result, because the loss of faith in the belief that Cleopatra had a son also made us lose faith in the belief that she had a daughter.

In this first example, we can define Cn_1^* as:

$$Cn_1^*(X) = X \cup \{\alpha_i \mid \text{there is an } I \text{ s.t. } i \in I \text{ and } \bigwedge_{j \in I} \alpha_j \in X\}.$$

This consequence relation "breaks the conjunctions into conjuncts" producing $Cn_1^*(B) = \{p \wedge q, p, q\}$, and then the result of the Cn*-pseudo-contraction is $\{p \wedge q\} -_* p = \{q\}$ as illustrated in Fig. 1. Here, we gain a small degree of syntax independence comparing to base contraction, for the presence of the separate formulas p and q is not necessary anymore if

[4] The term "closure" is somewhat misused in this paper, as Cn^* is just a function, not necessarily one that closes the set.
[5] Not all results presented here depend on Cn^* being subclassical.

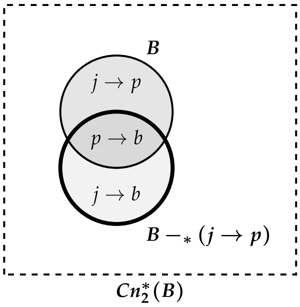

Fig. 2. Cn_2^* in Example 24.

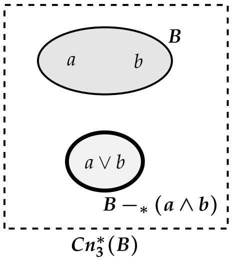

Fig. 3. Cn_3^* in Example 1.

we have their conjunction. This means that several pseudo-contraction operations based on this Cn^* will not change their results if we exchange some formulas for their conjunction in the base. Thus, these contractions ignore *some* syntax details, giving them *some* syntax independence.

Example 24. Suppose I believe that *the town of Juazeiro do Norte is located in the state of Pernambuco* $(j \rightarrow p)$ and that *the state of Pernambuco is located in Brazil* $(p \rightarrow b)$. Talking to a colleague, I find out that this town is not located in his state (Pernambuco), that is, I contract $j \rightarrow p$ from my base. The outcome is $B - (j \rightarrow p) = \{p \rightarrow b\}$. So, I no longer know whether Juazeiro do Norte is located in Brazil.

In this second example, we can use the following Cn_2^*:

$$Cn_2^*(X) = X \cup \{\alpha \rightarrow \beta \mid \alpha \rightarrow \delta \in X \text{ and } \delta \rightarrow \beta \in X, \text{ for some } \delta\}.$$

This function gives us one step of the transitivity of the implication, and yields $Cn_2^*(B) = \{j \rightarrow p, p \rightarrow b, j \rightarrow b\}$ with the outcome $B -_* (j \rightarrow p) = \{p \rightarrow b, j \rightarrow b\}$ as shown in Fig. 2.

In [17], Hansson discussed contraction on disjunctively closed belief bases, motivated by Example 1 (Section 1). The operation he has studied is a special case of the Cn*-pseudo-contraction. A closure similar to the following Cn_3^* is used:

$$Cn_3^*(X) = X \cup \{\bigvee_{i \in I} \alpha_i \mid \text{ for all } j \in I, \alpha_j \in X\}.$$

This consequence operation introduces disjunctions of formulas in the base, producing $Cn_3^*(B) = \{a, b, a \vee b\}$. As depicted in Fig. 3, one of the possible results of the pseudo-contraction is then $B -_* (a \wedge b) = \{a \vee b\}$, as Hansson wished.

4.2. Formal properties

As usual, we give a characterization for this operator, but before that we need to introduce the following "starred" version of some postulates.

(inclusion*) $B - \alpha \subseteq Cn^*(B)$
(relevance*) If $\beta \in Cn^*(B) \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn^*(B)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$
(uniformity*) If for all $B' \subseteq Cn^*(B)$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$

It is not surprising that, in order to characterize this generic operator, which embeds two different logics, we would need to mention this second logic Cn^* in the postulates. The representation theorem is the following:

Theorem 25. *Provided that Cn^* satisfies inclusion, idempotence and subclassicality, an operator is a $-_*$ operator if and only if it satisfies success, inclusion*, relevance* and uniformity*.*

Proof. *Construction-to-postulates:* We know that $A -_* \alpha = \bigcap \gamma(Cn^*(A) \perp \alpha) = Cn^*(A) -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction. We also know that $-_\gamma$ satisfies success, inclusion, relevance and uniformity. So, we have:

- If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(Cn^*(A) -_\gamma \alpha)$;
- $Cn^*(A) -_\gamma \alpha \subseteq Cn^*(A)$;
- If $\beta \in Cn^*(A) \setminus (Cn^*(A) -_\gamma \alpha)$, then there is a B' such that $Cn^*(A) -_\gamma \alpha \subseteq B' \subseteq Cn^*(A)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$;
- If for all $B' \subseteq Cn^*(A)$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $Cn^*(A) -_\gamma \alpha = Cn^*(A) -_\gamma \beta$.

Since $Cn^*(A) -_\gamma \alpha = A -_* \alpha$, we are done.

Postulates-to-construction: This part is almost trivially obtained from the proof of the representation theorem for partial meet contraction for bases, which can be found in [19].

Let $-_*$ be an operation for A that satisfies success, inclusion*, relevance* and uniformity*. From the last proof and Observation 28 (ahead) we conclude that $-_*$ also satisfies logical relevance and uniformity. Let γ be a function such that:

- If $Cn^*(A) \perp \alpha = \emptyset$, then $\gamma(Cn^*(A) \perp \alpha) = \{Cn^*(A)\}$;
- Otherwise $\gamma(Cn^*(A) \perp \alpha) = \{X \in Cn^*(A) \perp \alpha \mid A -_* \alpha \subseteq X\}$.

We need to show that (1) γ is a well-defined function, (2) γ is a selection function and (3) $\bigcap \gamma(Cn^*(A) \perp \alpha) = A -_* \alpha$ for all α.

Part 1: For γ to be a well-defined function, for all α and β, if $Cn^*(A) \perp \alpha = Cn^*(A) \perp \beta$, we must have $\bigcap \gamma(Cn^*(A) \perp \alpha) = \bigcap \gamma(Cn^*(A) \perp \beta)$. Suppose that $Cn^*(A) \perp \alpha = Cn^*(A) \perp \beta$. It follows from observation 1.39 in [19] that any subset of $Cn^*(A)$ implies α if and only if it implies β. By uniformity, $Cn^*(A) -_* \alpha = Cn^*(A) -_* \beta$. By the definition of γ we have $\gamma(Cn^*(Cn^*(A)) \perp \alpha) = \gamma(Cn^*(Cn^*(A)) \perp \beta)$. Since Cn^* is Tarskian, by idempotence, the result follows.

Part 2: For γ to be a selection function it remains to be proven that if $Cn^*(A) \perp \alpha$ is not empty, then neither is $\gamma(Cn^*(A) \perp \alpha)$. Then, assuming $Cn^*(A) \perp \alpha \neq \emptyset$, we know that there is at least one $X \in Cn^*(A) \perp \alpha$, and we must show that at least one of these X contains $A -_* \alpha$. Since $Cn^*(A) \perp \alpha$ is not empty, $\alpha \notin Cn(\emptyset)$, and by success, $\alpha \notin Cn(A -_* \alpha)$. By inclusion*, $A -_* \alpha \subseteq Cn^*(A)$, then, by the upper bound property [2], there is an A' such that $A -_* \alpha \subseteq A'$ and $A' \in Cn^*(A) \perp \alpha$. By the construction of γ, $\gamma(Cn^*(A) \perp \alpha)$ is non-empty.

Part 3: Case 1, $\alpha \in Cn(\emptyset)$. Then, by logical relevance, since there is no A' such that $\alpha \notin Cn(A')$, no element is in $A \setminus (A -_* \alpha)$, then, using inclusion*, $A \subseteq A -_* \alpha \subseteq Cn^*(A)$. We know that $Cn^*(A) \perp \alpha = \emptyset$, then $\bigcap \gamma(Cn^*(A) \perp \alpha) = Cn^*(A)$. We need to show that $Cn^*(A) \subseteq A -_* \alpha$. By relevance*, we know that $Cn^*(A) \setminus (A -_* \alpha) = \emptyset$, then $Cn^*(A) \subseteq A -_* \alpha$.

Case 2, $\alpha \notin Cn(\emptyset)$. We have that $Cn^*(A) \perp \alpha$ is non-empty and, by part 2, $\gamma(Cn^*(A) \perp \alpha)$ is non-empty as well. Since $A -_* \alpha$ is a subset of all elements of $\gamma(Cn^*(A) \perp \alpha)$, $A -_* \alpha \subseteq \bigcap \gamma(Cn^*(A) \perp \alpha)$. We need to show that $\bigcap \gamma(Cn^*(A) \perp \alpha) \subseteq A -_* \alpha$.

Take $\varepsilon \notin A -_* \alpha$. If $\varepsilon \notin Cn^*(A)$, obviously $\varepsilon \notin \bigcap \gamma(Cn^*(A) \perp \alpha)$. If $\varepsilon \in Cn^*(A) \setminus (A -_* \alpha)$, then by relevance* there is an A' such that $A -_* \alpha \subseteq A' \subseteq Cn^*(A)$, $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\varepsilon\})$. It follows from the upper bound property that there is an A'' such that $A \subseteq A''$ and $A'' \in Cn^*(A) \perp \alpha$. From $A \subseteq A''$, $\alpha \in Cn(A' \cup \varepsilon)$ and $\varepsilon \in A''$ we conclude that $\alpha \in Cn(A'')$, so we must have $\varepsilon \notin A''$. By our definition of γ, $A'' \in \gamma(Cn^*(A) \perp \alpha)$, and since $\varepsilon \notin A''$, we conclude that $\varepsilon \notin \bigcap \gamma(Cn^*(A) \perp \alpha)$. □

Here we prove that this operation satisfies success indeed, and not just a starred version of it (as would be the case for restricted-AGM).

(success*) If $\alpha \notin Cn^*(\emptyset)$, then $\alpha \notin Cn^*(B - \alpha)$

Despite the fact that the non-starred uniformity is not always satisfied, we have the weaker extensionality. Relevance is not satisfied, nor its weakening, core-retainment.

Proposition 26. *The $-_*$ operator satisfies extensionality.*

Proof. Follows directly from the fact that partial meet contraction for bases satisfies extensionality. □

Proposition 27. *The $-_*$ operator does not satisfy core-retainment.*

Proof. We give a counter-example:

$$B = \{\beta, \beta \wedge \alpha\}, Cn^*(B) = \{\beta, \beta \wedge \alpha, \alpha, \beta \to \alpha\}.$$

We can have a contraction $B -_* \alpha = \{\beta \to \alpha\}$. □

Depending on some properties of Cn^*, we can assign additional properties to the operation. If Cn^* satisfies inclusion and subclassicality, particularly, then logical inclusion and uniformity will hold.

Observation 28. *If Cn^* satisfies*

- *subclassicality, then an operation satisfying inclusion* also satisfies logical inclusion.*
- *inclusion, then an operation satisfying uniformity* also satisfies uniformity.*
- *subclassicality and inclusion, then an operation satisfying relevance* also satisfies logical relevance.*

From this observation it follows that Cn^* being subclassical is a sufficient condition for $-_*$ to be properly called a pseudo-contraction (success is granted by the partial meet).

As already mentioned, the fact that kernel contraction does not satisfy relevance causes it to violate relative closure, an important property when it comes to rational contractions. Since the Cn*-pseudo-contraction also violates relevance, we have to check whether it still preserves relative closure.

Proposition 29. *If Cn^* satisfies inclusion, the $-_*$ operator satisfies relative closure.*

Proof. We know that $A -_* \alpha = Cn^*(A) -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction. Since partial meet satisfies relative closure [19], $Cn^*(A) \cap Cn(Cn^*(A) -_\gamma \alpha) \subseteq Cn^*(A) -_\gamma \alpha$ is valid. From this we have $Cn^*(A) \cap Cn(A -_* \alpha) \subseteq A -_* \alpha$. By the inclusion property of Cn^* (which is Tarskian) and set theory we get $A \cap Cn(A -_* \alpha) \subseteq Cn^*(A) \cap Cn(A -_* \alpha)$. □

Cn*-pseudo-contraction operates in an intermediate level between the complete syntactic independence of classical belief revision and complete syntactic dependence of belief base operations. Uniformity* describes the kind of syntactic independence that can be expected for the proposed operation. This characteristic has important consequences. For example, vacuity (a principle of minimal change which states that, if you do not believe in the input sentence, its contraction will not change your beliefs) is not satisfied. This violation is related to the satisfaction of another property, which we call enforced closure*, that illustrates this balance on syntactic independence.

(enforced closure*) $B - \alpha = Cn^*(B - \alpha)$

Proposition 30. *If Cn^* is Tarskian and satisfies subclassicality, an operator that satisfies inclusion* and relevance* also satisfies enforced closure*.*

Proof. Since Cn^* is Tarskian, by inclusion, $A - \alpha \subseteq Cn^*(A - \alpha)$. We want to show that $Cn^*(A - \alpha) \subseteq A - \alpha$. Suppose by contradiction that $\beta \in Cn^*(A - \alpha) \setminus (A - \alpha)$. From inclusion*, monotonicity and idempotence of Cn^* we obtain $\beta \in Cn^*(A) \setminus (A - \alpha)$. Relevance* guarantees that there is an A' such that $A - \alpha \subseteq A' \subseteq Cn^*(A)$, $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$. By subclassicality of Cn^* and $\beta \in Cn^*(A - \alpha)$ we have $\beta \in Cn(A - \alpha)$. By $A - \alpha \subseteq A'$ and by the inclusion property of Cn, $\beta \in Cn(A')$. So, we have $Cn(A') = Cn(A' \cup \{\beta\})$, which is a contradiction. □

Enforced closure* forces the base to be closed, even when the input is not in the original set of beliefs. Nevertheless, the $-_*$ operator at least satisfies a weaker variant of vacuity, *vacuity**.

(vacuity*) If $\alpha \notin Cn(B)$, then $B - \alpha = Cn^*(B)$

Proposition 31. *If Cn^* satisfies subclassicality, an operator that satisfies inclusion* and relevance* also satisfies vacuity*.*

Proof. Assume $\alpha \notin Cn(B)$.

By inclusion* we already have $B - \alpha \subseteq Cn^*(B)$. To finish the proof it is sufficient to prove that $Cn^*(B) \setminus (B - \alpha) = \emptyset$.

By relevance*, if $\beta \in Cn^*(B) \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn^*(B)$ and $\alpha \in Cn(B' \cup \{\beta\})$. From this and subclassicality of Cn^* we get $B' \cup \{\beta\} \subseteq Cn^*(B) \subseteq Cn(B)$ and then by monotony and idempotence of Cn we have $Cn(B' \cup \{\beta\}) \subseteq Cn(B)$. Since $\alpha \notin Cn(B)$ by assumption, we cannot have such α. □

Notice that the satisfaction of vacuity by GPMPC comes at a cost: it has a less "uniform" behaviour as compared to the Cn*-pseudo-contraction, as it does not satisfy in general any version of uniformity. Cn*-pseudo-contraction, on the other hand, satisfies uniformity* under fairly weak assumptions on the properties of Cn^*.

We can circumvent Cn*-pseudo-contraction's lack of vacuity by defining the operation to do nothing in the case where the input sentence is not entailed by the base. However, with this we lose relevance* and only "partially" get rid of enforced closure*. These effects show the need for a way to limit the additions to the base, and one way to codify this limit is via the postulate of core-addition. In Section 6 we will introduce a variant of Cn*-pseudo-contraction which can easily satisfy the core-addition postulate (if the right consequence relation is used), but first we will show two practical examples of Cn^* which occur in the literature.

5. Applications

In this section, we present two examples of applications with different notions of Cn^*. The first one was the original motivation for this work: revising and maintaining ontologies described in description logics. Ontology engineers typically represent part of the relations between classes and take a form of closure for granted. The second application is a form of approximate reasoning where only part of the language is considered for inferences and which has already been used for approximating belief revision.

5.1. Ontology engineering

Ontologies[6] are explicit specifications of some body of knowledge, designating a set of concepts and their properties, as well as relationships between these concepts. Usually these representations are implemented through a knowledge base consisting of a set of sentences of some *description logic*, like $\mathcal{ALC}$.[7] Ontologies have a wide variety of applications today.

Ontology engineering, the process of creating and maintaining ontologies, is often done with the aid of ontology editing software, such as *Protégé*.[8] With this kind of software, the users can specify an ontology and query it.

The knowledge bases used in ontology engineering are represented as belief bases: they are not closed by deduction. Due to that, the response power of queries is reduced. As a way to bypass this limitation, weak closures like Cn^* are commonplace in this field. Inference engines in ontology engineering software (for example, *HermiT*[9]) are used to expand the set of beliefs in the knowledge base, in order to support user queries.

It is not uncommon that these ontologies present inconsistencies, or become inconsistent with the addition of new information. If, with the goal of avoiding these inconsistencies, we apply a contraction operation on the expanded belief base, the operation is exactly as the Cn*-pseudo-contraction depicted before. Here we face the question whether we should do the contraction over the original base or over the expanded one (which brings us back to the same base contraction versus pseudo-contraction dilemma). These operations have been carried out in practice without much theoretical background. We now hope to have shed some light in what happens behind the curtains in these operations.

In the following, based on the observed behaviour of HermiT inside the Protégé environment, we show that Cn^*_H, the consequence function used by the HermiT reasoner, satisfies all properties necessary to label it as an example of the Cn^* studied here. Given an ontology (or knowledge base) KB, HermiT is expected to generate an inferred ontology $Cn^*_H(KB)$ that is the smallest set for which the following properties hold[10]:

- If an axiom is in KB, it is in $Cn^*_H(KB)$.
- For all pairs of user defined concepts C, D, if $KB \models C \sqsubseteq D$, then $C \sqsubseteq D \in Cn^*_H(KB)$.
- For all user defined individuals a and concepts C, if $KB \models a : C$, then $a : C \in Cn^*_H(KB)$.
- For all user defined individuals x, y and roles r, if $KB \models (x, y) : r$, then $(x, y) : r \in Cn^*_H(KB)$.

Proposition 32. *The Cn^*_H defined above satisfies inclusion, idempotence, monotonicity and subclassicality.*

Proof. The first observation warrants that Cn^*_H satisfies inclusion.

It is easy to see that if one of the three last rules is used to generate a consequence over KB, this same consequence would be generated for any KB' such that $KB \subseteq KB'$, since the usual underlying consequence relation $\models$ for description logics is monotonic. Thus, Cn^*_H is also monotonic.

Since all the consequences generated by the four rules above are also consequences of Cn_{FOL} (considering the standard translation of description logics to first order logic), Cn^*_H is subclassical.

Applying Cn^*_H twice is the same as applying it once, that is, $Cn^*_H(KB) = Cn^*_H(Cn^*_H(KB))$. Since $Cn^*_H(KB) \subseteq Cn^*_H(Cn^*_H(KB))$ (by the first rule), we must only prove that $Cn^*_H(KB) \supseteq Cn^*_H(Cn^*_H(KB))$. We prove by contradiction. Suppose there exists some $\varphi \in Cn^*_H(Cn^*_H(KB)) \setminus Cn^*_H(KB)$. As φ is in $Cn^*_H(Cn^*_H(KB))$ but not in $Cn^*_H(KB)$, it follows from the minimality of $Cn^*_H(Cn^*_H(KB))$ that φ must be generated by one of the three last rules, all of which imply that $Cn^*_H(KB) \models \varphi$. Since Cn^*_H is subclassical, $KB \models Cn^*_H(KB)$. Hence, as $\models$ is transitive, $KB \models \varphi$, and because $Cn^*_H(Cn^*_H(KB))$ was generated by one of the three last rules, it has one of the corresponding formats. Thus, $\varphi \in Cn^*_H(KB)$, which is a contradiction. Therefore, Cn^*_H is idempotent. □

Thus, we have a Cn^*_H that satisfies inclusion, idempotence, monotonicity and subclassicality, which makes it a perfect example of our weak closure Cn^* described in the previous sections. Therefore, all those properties hold for Cn^*_H as well, and a pseudo-contraction based on it would have the aforementioned properties.

[6] For an introduction on the subject see [12], [13].

[7] An introduction to description logics can be found in [5].

[8] http://protege.stanford.edu/.

[9] http://www.hermit-reasoner.com/.

[10] Unfortunately, we can neither prove nor find any documentation about these properties. Either way, this should be the standard behaviour.

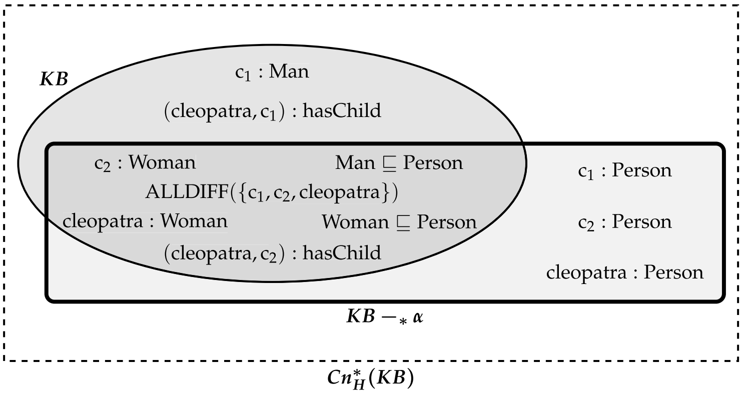

Fig. 4. Cn^*_H in Example 33.

Example 33. Consider Example 13 in a Description Logics context, modelled as follows:

- Concepts: Person, Man and Woman.
- Role: hasChild (domain: Person; range: Person).
- Individuals (assumed different): cleopatra, c_1 and c_2.
- TBox axioms: $\mathrm{Man} \sqsubseteq \mathrm{Person}$, $\mathrm{Woman} \sqsubseteq \mathrm{Person}$.
- ABox axioms: $\mathrm{c}_1 : \mathrm{Man}$, $\mathrm{c}_2 : \mathrm{Woman}$, $\mathrm{cleopatra} : \mathrm{Woman}$, $(\mathrm{cleopatra}, \mathrm{c}_1) : \mathrm{hasChild}$, $(\mathrm{cleopatra}, \mathrm{c}_2) : \mathrm{hasChild}$.

Let KB be the ontology above, and let α be the sentence $\mathrm{cleopatra} : \exists\mathrm{hasChild}.\mathrm{Man}$. We have[11]

$$Cn^*_H(KB) = KB \cup \{\mathrm{c}_1 : \mathrm{Person},\ \mathrm{c}_2 : \mathrm{Person},\ \mathrm{cleopatra} : \mathrm{Person}\}$$

and

$$Cn^*_H(KB) \perp \alpha = \{Cn^*_H(KB) \setminus \{\mathrm{c}_1 : \mathrm{Man}\},\ Cn^*_H(KB) \setminus \{(\mathrm{cleopatra}, \mathrm{c}_1) : \mathrm{hasChild}\}\}.$$

Thus, depending on the selection function γ, the operation $-_*$ may remove our belief that c_1 is a man, our belief that c_1 is Cleopatra's child, or both.

We have implemented[12] the operation $-_*$ using HermiT's Cn^*_H operator. In the example above, considering γ defined as

$$\gamma\left(Cn^*_H(KB) \perp \alpha\right) = \begin{cases} Cn^*_H(KB) \perp \alpha & \text{if this set is not empty (which is the case)} \\ \{Cn^*_H(KB)\} & \text{otherwise,} \end{cases}$$

our program outputs the following ontology, as expected:

$$\begin{aligned}&\{\mathrm{Man} \sqsubseteq \mathrm{Person},\ \mathrm{Woman} \sqsubseteq \mathrm{Person},\\ &\ \mathrm{c}_2 : \mathrm{Woman},\ \mathrm{cleopatra} : \mathrm{Woman},\\ &\ (\mathrm{cleopatra}, \mathrm{c}_2) : \mathrm{hasChild},\\ &\ \mathrm{c}_1 : \mathrm{Person},\ \mathrm{c}_2 : \mathrm{Person},\ \mathrm{cleopatra} : \mathrm{Person},\\ &\ \mathrm{ALLDIFF}(\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{cleopatra}\})\}.\end{aligned}$$

Fig. 4 illustrates this example.

5.2. Limited inference

Another application of our framework is the limited inference that Levesque introduced to deal with explicit beliefs [21] (but we will follow the presentation of this logic given by [32]). He distinguished between an agent's explicit and implicit beliefs and argued that the set of explicit beliefs should actually be closed under a very weak logic, avoiding, for example, belief in $p \wedge q$ without belief in $q \wedge p$.

11 Aside from tautologies containing top and bottom concepts.

12 The source code is available at https://github.com/viniciusbm/srwpseudocontraction.

Levesque proposed a formal definition for a multivalued logic convenient to model limited reasoning. The syntax of the logic follows that of propositional logic. The difference lies in the interpretations: valuations are not defined over the atoms of the language, but over the set of literals. The restriction is that an atom and its negation should not be both mapped into 0. Hence, differently from classical valuation, for each atom p, instead of two different options it is possible to have three possibilities: $v(p) = 1$ and $v(\neg p) = 0$; $v(p) = 0$ and $v(\neg p) = 1$; and $v(p) = 1$ and $v(\neg p) = 1$ (in the same vein as Priest's *Logic of Paradox* [28]).

A set of formulas in conjunctive normal form (CNF) Γ *3-entails* a formula in CNF α ($\Gamma \vDash_3 \alpha$) if every valuation that maps Γ into 1 also maps α into 1. The consequence operator $C_3(\Gamma) = \{\alpha \mid \Gamma \vDash_3 \alpha\}$ is Tarskian and subclassical which makes it a possible candidate for Cn^*. Furthermore, checking if $\alpha \in C_3(\Gamma)$ can be done in polynomial time over the size of Γ and α.

Finally, $\vDash_3$ has an intuitive axiomatization given by classical proof theory without modus ponens. Hence, for example, $p \wedge q \vDash_3 q \wedge p$ and $p \vee q \vDash_3 q \vee p$, but $p, \neg p \vee q \nvDash_3 q$.

For the above mentioned reasons, C_3 is an interesting candidate for Cn^* in our framework. Consider the following example: $B = \{\neg p \vee q, p\}$, $B' = \{q \vee \neg p, p\}$. Both these bases are classically equivalent and, although $B \nvDash_3 q$, notice that q classically follows from both, i.e. $B \vDash q$ and $B' \vDash q$. Hence, the contraction $B - q$ is not trivial. However, since B and B' are not syntactically identical, there is no guarantee that $B - q = B' - q$. For example, we could have $B - q = \{\neg p \vee q\}$ and $B' - q = \{q\}$. On the other hand, using our framework with $Cn^* = C_3$, since $C_3(B) = C_3(B')$, we would have that $B -_* q = B' -_* q = \bigcap \gamma(C_3(B) \perp q)$.

This example gives an interesting balance between syntactic independence and computational complexity. The closure over C_3 is easy to compute and it keeps finite bases finite. Furthermore, it guarantees some sort of syntactic independence which prevents some of the drawbacks of the belief base approach.

Levesque's logic has been extended by Schaerf and Cadoli [32] to deal with approximations of classical logic. The idea is to add as a parameter to the $\vDash_3$ relation a set S of atoms. This is the set of atoms for which the valuation should behave classically, i.e., for which the atom and its negation should be assigned opposite values. Hence, for these atoms, the third possibility is excluded. If S is empty, we have the original logic suggested by Levesque and if S is the full language, we are back to classical logic. This logic has already been used to model approximate belief revision in [7]. It is not hard to see that for any fixed S, $\vDash_3^S$ is also Tarskian and subclassical.

6. Two-place Cn*-pseudo-contraction

In this section, we introduce another kind of Cn*-pseudo-contraction, which also takes into account the input formula. We then show that the operation of local contraction, which focuses in the part relevant to the input, can be seen as a special case of the two-place Cn*-pseudo-contraction defined.

6.1. A more general operation

The Cn*-pseudo-contraction operator introduced in Definition 22 makes use of a consequence relation Cn^* which depends only on the belief base. As a result of that, the base is expanded with formulas that are not necessarily useful as weakenings of the elements that will be given up. We can look at the more general case where the Cn^* relation used depends also on the input sentence. We use pseudo-contractions as a way to weaken formulas in base contraction. This two-place generalization allows us to weaken only the formulas related to the contracted element, which brings us closer to a core-addition compliant operation.

Definition 34. Let B be a set of sentences, $Cn^* : \mathfrak{L} \times 2^{\mathfrak{L}} \to 2^{\mathfrak{L}}$ and γ a selection function for $Cn^*(\alpha, B)$. The operator $\dot{-}$ is such that, for all sentences α:

$$B \dot{-} \alpha = \bigcap \gamma(Cn^*(\alpha, B) \perp \alpha).$$

We will call this operator *two-place Cn*-pseudo-contraction*. We can see these two-place Cn^* as functions generating one consequence relation Cn^*_α for each sentence α. If all Cn^*_α satisfy a given property of consequence relations of Section 2 (e.g., monotonicity), then we say that Cn^* satisfies said property.

This operator is a generalization of Nebel's pseudo-contraction and of GPMPC. These two operators cannot be seen as specialized versions of (one-place) Cn*-pseudo-contraction since for $B -_* \alpha$, $Cn^*(B)$ is the same for all α, whereas those operators add different things to the base depending on the formula to be removed. Note that some properties which hold for Nebel's pseudo-contraction and GPMPC may not hold in general, but can be easily obtained by adding restrictions to the definition of the particular $Cn^*(\alpha, B)$. Vacuity, for example, can be obtained by adding the restriction $Cn^*(\alpha, B) = B$ if $\alpha \notin Cn(B)$.

Since two-place Cn*-pseudo-contractions are generalizations of the Cn*-pseudo-contractions, it is natural that some of the properties showed in the previous section do not hold for the former. Particularly, they satisfy success, inclusion* and relevance*, but uniformity* is lost. This happens because this postulate relates two operations, each of which using a particular Cn^*_α. Uniformity* is then replaced by the odd condition:

If for all $B' \subseteq Cn^*(\alpha, B)$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $\bigcap \gamma(Cn^*(\alpha, B) \perp \alpha) = \bigcap \gamma(Cn^*(\alpha, B) \perp \beta)$.

The expression in the left-hand side of the equals sign is equivalent to $B \dot{-} \alpha$, but the expression in the right-hand side is not the same as $B \dot{-} \beta$, since α was used in the parameter of Cn^* instead of β. For the same reason this pseudo-contraction also violates the postulate of extensionality. Nevertheless, we can recover this postulate if we restrict the two-place Cn^* making it always use a canonical form of the formula (which could be obtained for formulas of classical propositional logic, for instance, by using CNF and a lexicographic ordering on atoms):

Proposition 35. *If for all α, β, B it is the case that $Cn^*(\alpha, B) = Cn^*(f(\alpha), B)$, where $f(\alpha) = f(\beta)$ if $Cn(\alpha) = Cn(\beta)$, then the $\dot{-}$ operator satisfies extensionality.*

Proof. If $Cn(\alpha) = Cn(\beta)$, then by extensionality of partial meet contraction, $\bigcap \gamma(Cn^*(f(\alpha), B) \perp \alpha) = \bigcap \gamma(Cn^*(f(\alpha), B) \perp \beta)$, but since $Cn(\alpha) = Cn(\beta)$, we have that $f(\alpha) = f(\beta)$ and thus $B \dot{-} \alpha = B \dot{-} \beta$. □

This proposition explains why GPMPC satisfies extensionality despite being a two-place Cn*-pseudo-contraction: the only place in which its weak closure takes the input α into account is inside a partial meet contraction, and if two formulas are logically equivalent, by extensionality of partial meet contraction, the result of the contraction is the same. So, $Cn^*(\alpha, B)$ in this case is the same for all equivalent α.

The first and third items of Observation 28 and Propositions 29, 30 and 31 are also valid and applicable to two-place Cn*-pseudo-contractions.

This time we can restore vacuity without losing relevance* (different from the situation with the one-place Cn^*). We can preclude Cn^* from adding anything if $\alpha \notin Cn(B)$, since α and B are now parameters of the consequence function, forcing $Cn^*(\alpha, B) = B$ when $\alpha \notin Cn(B)$, so that $B - \alpha = Cn^*(\alpha, B) = B$, and thus keeping relevance*. Also, despite enforced closure* being satisfied, its effect can be nullified for we can now generate only consequences related to the removed element, which could have been lost in the contraction and therefore are not just useless derivative formulas.

As an example of the use of a two-place Cn*-pseudo-contraction, we now consider the idea of local change, proposed in [20].

6.2. Application: local change

In [20], Hansson and Wassermann proposed a local consequence operator. This operator looks at the relevant part of the belief base with respect to a certain formula (or set of formulas) and extracts the consequences of only this relevant subset of the base. To find this relevant part of the base, they used the concept of *compartments around a formula*, defined as the elements of the minimal consistent subsets of the base that either imply the formula or its negation:

Definition 36. [20] The compartment of A relevant to α is defined by:

$$c(\alpha, A) = \begin{cases} \emptyset, & \text{if } \alpha \in Cn(\emptyset) \text{ or } \neg\alpha \in Cn(\emptyset) \\ \bigcup((A \perp\!\!\!\perp \alpha) \cup (A \perp\!\!\!\perp \neg\alpha) \setminus (A \perp\!\!\!\perp \bot)) & \text{otherwise.} \end{cases}$$

Hansson and Wassermann define a local consequence operator as $Cn_\alpha(B) = Cn(c(\alpha, B))$. As this is meant to generate only the inferences relevant to α, it is easy to see that this operator is not Tarskian, as it does not satisfy inclusion.

Based on this idea, we propose the following two-place Cn^*:

Definition 37. $Cn^*(\alpha, A) = A \cup Cn(c(\alpha, A))$.

This weak closure operation expands the original belief base with respect only to beliefs that can be considered relevant to the sentence being contracted. Now we want to verify what properties this Cn^* satisfies.

Proposition 38. *Cn^* of Definition 37 satisfies inclusion, monotonicity and subclassicality, but does not satisfy idempotence.*

The only result in this paper that applies to two-place Cn*-pseudo-contractions and depends on idempotence is Proposition 30, which is no surprise, as we have already noted that two-place Cn*-pseudo-contractions in general do not satisfy enforced closure*. Observation 28 applies to Cn^* of Definition 37 except for the second item, because uniformity* is not satisfied by two-place Cn*-pseudo-contractions. Propositions 29 and 31 are also applicable to this construction.

Fig. 5 shows how this study helps to complete the picture of (pseudo-)contraction operations, by illustrating how the two operations introduced here comprise a backbone of their hierarchy. In particular, we can see that most of the operations “in the middle” of the diagram are not unrelated generalizations of AGM contraction, but are actually all siblings via (one or two-place) Cn*-pseudo-contractions. Further investigation is necessary to decide whether all Local Change operations are GPMPC operations (the converse is not the case).

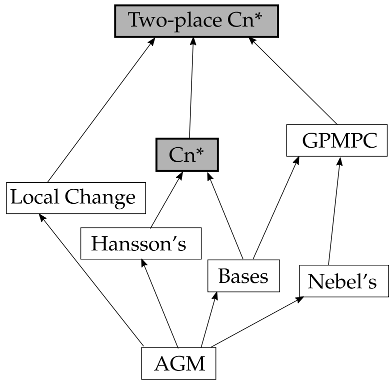

Fig. 5. $A \rightarrow B$ means that all operations of type A are also operations of type B. The operations depicted above are: *Two-place Cn***-pseudo-contraction*; *Cn**-pseudo-contraction; *Local Change* (a two-place Cn*-pseudo-contraction using the Cn^* of Definition 37); *Hansson's* pseudo-contraction (Fig. 3); partial meet contraction for belief *Bases*; partial meet contraction for belief sets (*AGM*); restricted *Nebel's* pseudo-contraction; *GPMPC*.

7. Conclusions and future work

In this paper, we explored the relatively unknown space of operations between AGM and belief base contractions, the so-called pseudo-contractions.

The AGM paradigm assumes that the knowledge of an agent is represented by belief sets. Although the AGM constructions can also be applied to belief bases, the resulting operations can lead to some unwanted waste of formulas, because the inclusion postulate forbids the addition of weakened versions of existing formulas to a belief base. Pseudo-contractions satisfy logical inclusion instead, reducing informational loss when compared to base contractions while maintaining a level of expressivity higher than that of belief sets (at least avoiding trivialization in case of contradiction, i.e. distinct inconsistent bases may be treated differently).

We have proposed two different forms of pseudo-contraction. The first one, Cn*-pseudo-contraction, relies on a transformation function Cn^* that enriches a belief base with some of its consequences. We provided a representation theorem that relates our pseudo-contraction construction to AGM partial-meet contraction and postulates and we proved some properties of the construction based on the properties of the particular Cn^* chosen. We also presented two examples of applications, illustrating specific useful Cn^*s.

The second operation, two-place Cn*-pseudo-contraction, depends not only on the belief base, but also on the input formula. This is necessary if we want to capture some notion of relevance in the operation. Together with the (one-place) Cn*-pseudo-contraction, this operation helps to complete the backbone of the hierarchy of pseudo-contraction operations, showing how all of them are related (Fig. 5).

There is still a lot to be done. Revision operations and kernel constructions were only touched upon. In future work, we would like to extend the results to kernel operations and investigate under which conditions revisions can be defined using the Levi identity and when it is necessary to define them with specific constructions.

A complete study and characterization of two-place Cn*-pseudo-contractions is another topic left for future work.

References

[1] C.E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change, J. Symb. Log. 50 (2) (1985) 510–530.
[2] C.E. Alchourrón, D. Makinson, New studies in Deontic Logic, in: Hierarquies of Regulation and Their Logic, Reidel Publishing Company, 1981, pp. 125–148.
[3] C.E. Alchourrón, D. Makinson, On the logic of theory change: contraction functions and their associated revision functions, Theoria 48 (1982) 14–37.
[4] C.E. Alchourrón, D. Makinson, On the logic of theory change: safe contraction, Stud. Log. 44 (4) (1985) 405–422.
[5] F. Baader, I. Horrocks, U. Sattler, Description logics, Handb. Knowl. Represent. 3 (2008) 135–179.
[6] R. Booth, T.A. Meyer, I.J. Varzinczak, R. Wassermann, On the link between partial meet, kernel, and infra contraction and its application to horn logic, J. Artif. Intell. Res. (JAIR) 42 (2011) 31–53.
[7] S. Chopra, R. Parikh, R. Wassermann, Approximate belief revision, Log. J. IGPL 9 (6) (2001) 755–768.
[8] J.P. Delgrande, Horn clause belief change: contraction functions, in: G. Brewka, J. Lang (Eds.), Proceedings of the 11th International Conference of Principles of Knowledge Representation and Reasoning, AAAI Press, 2008, pp. 156–165.
[9] J.P. Delgrande, R. Wassermann, Horn clause contraction functions, J. Artif. Intell. Res. (JAIR) 48 (2013) 475–511.
[10] G. Flouris, On Belief Change and Ontology Evolution, PhD thesis, University of Crete, 2006.
[11] P. Gärdenfors, Knowledge in Flux – Modeling the Dynamics of Epistemic States, MIT Press, 1988.
[12] T.R. Gruber, A translation approach to portable ontology specifications, Knowl. Acquis. 5 (2) (1993) 199–220.
[13] T.R. Gruber, Toward principles for the design of ontologies used for knowledge sharing, Int. J. Hum.-Comput. Stud. 43 (5) (1995) 907–928.
[14] S.O. Hansson, New operators for theory change, Theoria 55 (2) (1989) 114–132.
[15] S.O. Hansson, A dyadic representation of belief, in: P. Gärdenfors (Ed.), Belief Revision, in: Cambridge Tracts in Theoretical Computer Science, vol. 29, Cambridge University Press, 1992, pp. 89–121.
[16] S.O. Hansson, Reversing the Levi identity, J. Philos. Log. 22 (1992) 637–639.
[17] S.O. Hansson, Changes of disjunctively closed bases, J. Log. Lang. Inf. 2 (4) (1993) 255–284.

[18] S.O. Hansson, Kernel contraction, J. Symb. Log. 59 (3) (1994) 845–859.
[19] S.O. Hansson, A Textbook of Belief Dynamics, Kluwer Academic, 1999.
[20] S.O. Hansson, R. Wassermann, Local change, Stud. Log. 70 (1) (2002) 49–76.
[21] H.J. Levesque, A logic of implicit and explicit belief, in: Proceedings of the National Conference on Artificial Intelligence, AAAI'84, 1984.
[22] D. Makinson, On the status of the postulate of recovery in the logic of theory change, J. Philos. Log. 16 (1987) 383–394.
[23] T.A. Meyer, Basic infobase change, Stud. Log. 67 (2) (2001) 215–242.
[24] T.A. Meyer, W.A. Labuschagne, J. Heidema, Infobase change: a first approximation, J. Log. Lang. Inf. 9 (2000) 353–377.
[25] A.C. Nayak, Foundational belief change, J. Philos. Log. 23 (5) (1994) 495–533.
[26] B. Nebel, A knowledge level analysis of belief revision, in: Proceedings of the 1st International Conference of Principles of Knowledge Representation and Reasoning, 1989.
[27] A. Newell, The knowledge level, Artif. Intell. 18 (1) (1982) 87–127.
[28] G. Priest, The logic of paradox, J. Philos. Log. 8 (1) (1979) 219–241.
[29] M.M. Ribeiro, R. Wassermann, Degrees of recovery and inclusion in belief base dynamics, in: 12th International Workshop on Non-Monotonic Reasoning, NMR 2008, 2008.
[30] M.M. Ribeiro, R. Wassermann, G. Flouris, G. Antoniou, Minimal change: relevance and recovery revisited, Artif. Intell. 201 (2013) 59–80.
[31] Y.D. Santos, M.M. Ribeiro, R. Wassermann, Between belief bases and belief sets: partial meet contraction, in: Proceedings of the International Workshop on Defeasible and Ampliative Reasoning, DARe-15, 2015.
[32] M. Schaerf, M. Cadoli, Tractable reasoning via approximation, Artif. Intell. 74 (2) (1995) 249–310.