**Regularization of the Roy Equations with a Smooth Cut-Off**

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The Roy equations for $\pi\pi$ scattering are combined with unitarity to give a nonlinear system of equations
for the determination of the low-energy amplitudes. A Hölder continuous interpolation between the input
high-energy absorptive parts and the output low-energy absorptive parts is implemented; and the resultant
singular equations are regularized by means of an effective inelastic $N/D$ method. If the scattering
lengths, the CDD parameters, and the high-energy absorptive parts satisfy certain constraints, then there
exists a locally unique solution of the system.

## 1. INTRODUCTION

In this paper we continue the study of the system of
nonlinear, singular integral equations that results from
a combination of the Roy equations[1] with elastic uni-
tarity. It is assumed that the partial wave absorptive
parts above a certain point, say $s = s_0$, are given, and
that the $S$-wave scattering lengths are held fixed. The
problem is to prove the existence of solutions for the
partial waves in the domain $4 \le s \le s_0$ and to investigate
the question of the nonuniqueness of such solutions.

It has already been shown[2] that if the input quantities
are small enough, there exists a locally unique solution
in a suitable space of Hölder-continuous functions. How-
ever, it is known that the physically interesting solution
lies outside the scope of the above proof unless one
chooses $s_0$ to be such that all the phase shifts remain
small in $[4, s_0]$. A first step has been made[3] towards
the removal of this limitation, in which a finite-interval
version of the $N/D$ method was used to regularize the
singular equations. The new equations contained the
customary CDD poles, and a further free parameter
entered the solution in some cases, due to the margi-
nally singular nature of the $N$ integral equation.[4] The
fact that this equation is not subject to Fredholm theory
is an embarrassment for numerical work. Although an
explicit integral representation for a resolvent kernel
of the dominant part of the singular equation has been
constructed and the homogeneous equation has been ex-
haustively studied, nevertheless, it is rather awkward
to have to program this resolvent and to use it every
time that the $N$ equation is solved in the course of
iterating the nonlinear system.

These problems are sidestepped in the present paper
by the expedient of introducing a smooth instead of a
sharp cutoff. By means of a Hölder-continuous cutoff
function $h(s)$ we effect a homotopy from the elastically
unitary "output" absorptive part below $s_0$ to the pre-
scribed "input" absorptive part above $s_1$, where $s_1$ is
greater than $s_0$, but still within the domain of validity of
the Roy equations. The equations are again regularized
by means of the $N/D$ method; but the fact that the ampli-
tude is not strictly unitary in $[s_0, s_1]$ leads to a Frye–
Warnock[5] system with an effective elasticity that is a
function of the input absorptive part and of $h(s)$. The

new $N$ integral equation is Fredholm and is eminently
suited to numerical treatment.

In this paper we shall work with the reduced partial-
wave amplitude,

$$f_l^I(s) = \left(\frac{s^{1/2}+2}{s^{1/2}-2}\right)^l F_l^I(s), \tag{1.1}$$

where $F_l^I$ is the projection of the usual invariant scat-
tering amplitude, for isospin $I$, onto the Legendre poly-
nomial. This partial-wave amplitude satisfies elastic
unitarity for $s \in [4, 16]$:

$$\mathrm{Im} f_l^I(s) = q_l(s) |f_l^I(s)|^2, \tag{1.2}$$

where

$$q_l(s) = \left(\frac{s-4}{s}\right)^{1/2} \left[\frac{s^{1/2}-2}{s^{1/2}+2}\right]^l. \tag{1.3}$$

In practice there is little inelasticity below the $K\overline{K}$
threshold, and we shall assume (1.2) to be correct in
the domain of validity of the Roy equations, which in-
cludes the interval $[4, 32]$. The advantage of using the
reduction factor of (1.1) is that this ensures that $F_l^I(s)$
has the correct behavior as $s \to \infty$, if $f_l(s)$ is bounded
uniformly with respect to $s$ and $l$.

The Roy equations for the partial waves can be
written

$$f_l(s) = \frac{s^2}{\pi} \int_4^\infty \frac{ds'}{s'^2(s'-s)} \mathrm{Im} f_l(s') + b_l(s), \tag{1.4}$$

where isospin has been made an implicit variable and
where

$$b_l(s) = C_l(s) + R_l(s) + S_l(s) + U_l(s) + T_l(s), \tag{1.5}$$

where

$$C_l(s) = \frac{1}{4} \left[\frac{s^{1/2}+2}{s^{1/2}-2}\right]^l \left\{\delta_{l0}\left[s - \frac{s-4}{2}(C_{st}+C_{su})\right]\right.$$
$$\left. + \delta_{l1}\frac{s-4}{6}(C_{st}-C_{su})\right\}\alpha, \tag{1.6}$$

$$R_l(s) = \frac{s^2}{\pi} \int_4^\infty \frac{ds'}{s'^2(s'-s)}\left\{\left[\frac{s^{1/2}+2}{s'^{1/2}+2}\right]^{2l} - 1\right\}\mathrm{Im} f_l(s'), \tag{1.7}$$

$$S_l(s) = \left[\frac{s^{1/2}+2}{s^{1/2}-2}\right]^l \frac{s^2}{\pi} \int_4^\infty \frac{ds'}{s'^2(s'-s)} \sum_{l'=l+2}^\infty (2l'+1)$$
$$\times V_{ll'}(s, s')\left[\frac{s'^{1/2}-2}{s'^{1/2}+2}\right]^{l'}\mathrm{Im} f_{l'}(s'), \tag{1.8}$$

where

$$V_{ll'}(s, s') = \tfrac{1}{2} \int_{-1}^{1} dz\, P_l(z)\, P_{l'}(z'),$$ (1.9)

with

$$z = 1 + 2t/(s - 4),$$ (1.10)

$$z' = 1 + 2t/(s' - 4),$$ (1.11)

$$U_l(s) = \left[\frac{s^{1/2} + 2}{s^{1/2} - 2}\right]^l \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'^2} \sum_{l'=0}^\infty (2l' + 1)$$

$$\times W_{ll'}(s, s') \left[\frac{s'^{1/2} - 2}{s'^{1/2} + 2}\right]^{l'} \mathrm{Im} f_{l'}(s'),$$ (1.12)

where

$$W_{ll'}(s, s') = \tfrac{1}{2} C_{su} \int_{-1}^{1} dz\, P_l(z)\, \frac{u^2}{s' - u}\, P_{l'}(z'),$$ (1.13)

$$T_l(s) = \left[\frac{s^{1/2} + 2}{s^{1/2} - 2}\right]^l \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'} \sum_{l'=0}^\infty (2l' + 1)$$

$$\times \left[X_l(s, s') + Y_{ll'}(s, s')\right] \left[\frac{s'^{1/2} - 2}{s'^{1/2} + 2}\right]^{l'}$$

$$\times \mathrm{Im} f_{l'}(s'),$$ (1.14)

where

$$X_l(s, s') = \tfrac{1}{2} C_{st} \int_{-1}^{1} dz\, P_l(z)\, \left[1 + \frac{s}{t - 4}(1 - C_{tu})\right]$$

$$\times \left[\frac{t^2}{s' - t} - \frac{4(t - 4)}{s' - 4}(1 - C_{su}) - \frac{16}{s' - 4}\right],$$ (1.15)

$$Y_{ll'}(s, s') = \tfrac{1}{2} \int_{-1}^{1} dz\, P_l(z)\, \left\{C_{st}\left[1 + \frac{s}{t - 4}(1 - C_{tu})\right]C_{su}\right.$$

$$\left. - \left[C_{su} + \frac{s}{t - 4}(C_{su} - 1)\right] P_{l'}(z')\right\} \frac{(4 - t)^2}{s' - 4 + t},$$ (1.16)

In the above, $C_{st}$, $C_{tu}$, and $C_{su}$ are the usual isospin crossing matrices and $\alpha$ is a constant vector, in which the $I = 0$ and $I = 2$ components are the corresponding $S$-wave scattering lengths, while the $I = 1$ component is zero.

## 2. N/D EQUATIONS WITH A SMOOTH CUTOFF

In this section, we shall replace the abrupt cutoff of Ref. 3 by a gradual one that begins at $s = s_0$ and ends at $s = s_1 > s_0$. To be precise, we write

$$f(s) = \frac{s^2}{\pi} \int_4^\infty \frac{ds'}{s'^2(s' - s)} \mathrm{Im} f(s') + b(s),$$ (2.1)

where we have suppressed the suffix $l$ and where

$$\mathrm{Im} f(s) = h(s) q(s) |f(s)|^2 + [1 - h(s)] a(s).$$ (2.2)

Here $a(s)$ is a function that is specified for $s \geqslant s_0$, which is fixed in advance, and $h(s)$ is a monotonic cutoff function with the properties

$$h(s) = 1, \quad 4 \leqslant s \leqslant s_0$$ (2.3a)

$$h(s) = 0, \quad s \geqslant s_1$$ (2.3b)

$$|h(s_a) - h(s_b)| \leqslant |s_a - a_b|^\mu, \quad s_0 \leqslant s_a < s_b \leqslant s_1, \quad 0 < \mu < 1.$$ (2.3c)

Thus (2.2) effects a Hölder continuous interpolation from the elastic unitarity output expression for $s \leqslant s_0$, to the high-energy input model for $s \geqslant s_1$. For the purposes of the proof we need only the properties (2.3); but numerically it is convenient to make $h(s)$ a thrice-differentiable function for which, in addition to (2.3a) and (2.3b),

$$h'(s_0) = h'(s_1) = h''(s_0) = h''(s_1) = 0.$$ (2.3d)

An example of such a function is

$$h(s) = x^3(4 - 3x), \quad x = \cos\left(\frac{\pi}{2} \frac{s - s_0}{s_1 - s_0}\right),$$ (2.4)

for $s_0 \leqslant s \leqslant s_1$, and this is eminently suited to numerical computations in which cubic splines are employed.

The expression (2.2) has a formal resemblance to the inelastic condition, in which the elasticity function is given. In fact we may rewrite (2.1) in the form

$$g(s) = \frac{1}{\pi} \int_4^\infty \frac{ds'}{s' - s}\left[\rho(s')|g(s')|^2 + \frac{1 - \eta^2(s')}{4\rho(s')}\right] + c(s),$$ (2.5)

where we have written

$$g(s) = s^{-2} f(s)$$ (2.6)

and

$$c(s) = s^{-2} b(s),$$ (2.7)

in order to absorb the subtraction factor $s^2$, and where

$$\rho(s) = s^2 h(s) q(s)$$ (2.8)

and

$$\eta(s) = \{1 - 4h(s)[1 - h(s)]q(s)a(s)\}^{1/2}.$$ (2.9)

Equation (2.5) mimics exactly the standard form of a dispersion relation for a partial wave amplitude, $g(s)$, with born term $c(s)$, for which inelastic unitarity holds, with phase space $\rho(s)$ and elasticity $\eta(s)$.

In order to find all the solutions of (2.5), for a given $c(s)$, we apply the standard Frye–Warnock method,[5] in which one writes

$$g(s) = N(s)/D(s).$$ (2.10)

Here

$$D(s) = 1 + \sum_n \frac{r_n}{s - t_n} - \frac{1}{\pi} \int_4^\infty \frac{ds'}{s' - s} \rho(s')n(s')$$ (2.11)

and

$$N(s) = \frac{1 + \eta(s)}{2} n(s) + i \frac{1 - \eta(s)}{2\rho(s)} \mathrm{Re} D(s),$$ (2.12)

for $s \geqslant 4$, in which the real function $n(s)$ is the solution of the nonsingular integral equation

$$\eta(s)n(s) = \bar{c}(s) + \sum_n r_n \frac{\bar{c}(s) - \bar{c}(t_n)}{s - t_n}$$

$$+ \frac{1}{\pi} \int_4^\infty ds' \frac{\bar{c}(s') - \bar{c}(s)}{s' - s} \rho(s')n(s')$$ (2.13)

and

$$\bar{c}(s) = c(s) + \frac{P}{\pi} \int_4^\infty \frac{ds'}{s'-s} \frac{1-\eta(s')}{2\rho(s')} . \qquad (2.14)$$

The $t_n$ are the positions of the CDD poles, $r_n$ being the residues. For the sake of formal elegance, and to allow us to take over the standard formulas without change, we have written all integrals from $s=4$ to $s=\infty$, but because of the support properties of $h(s)$, actually $\rho(s)$ vanishes for $s \geq s_1$, so the integration domain in (2.11) and (2.13) is $4 \leq s \leq s_1$. According to (2.9), $1-\eta(s)$ vanishes both for $s \leq s_0$ and for $s \geq s_1$, and we have to define the integrand in (2.14) by continuity at the point $s=s_1$. In fact

$$\frac{1-\eta(s)}{2\rho(s)} = \frac{2[1-h(s)]a(s)s^{-2}}{1+\{1-4h(s)[1-h(s)]q(s)a(s)\}^{1/2}}, \quad (2.15)$$

and one can see that this expression changes continuously from 0 for $s \leq s_0$ to $s^{-2}a(s)$ for $s \geq s_1$. Thus $\bar{c}(s)$ is also continuous at $s=s_1$, and so by means of the smooth cutoff we have removed the logarithmic singularity that complicated the earlier method.

Since $c(s)$ is assumed to be known for $4 \leq s \leq s_1$ and $a(s)$ is known for $s \geq s_0$, we know $\eta(s)$ and hence $\bar{c}(s)$ for $4 \leq s \leq s_1$, and $\bar{c}(s)$ is in fact Hölder continuous on this interval. It may be shown[6] that any Hölder continuous solution of the nonlinear equation (2.5) has a representation of the form (2.10)–(2.14), on condition that the phase shift of the solution tends to a limit as $s \to \infty$, and on condition that $\eta(s)$ has no zeros in $[4, \infty)$, since such zeros would introduce singularities of the third kind[7] into the integral equation (2.13). In our case, $\eta(s)=1$ for $s \leq s_0$ and for $s \geq s_1$, so any possible zero can only lie in the interval $(s_0, s_1)$. However, since $h$ falls monotonically from 1 to 0, it is easy to see that $4h(1-h) \leq 1$, the equality being reached only once, at the point at which $h = \frac{1}{2}$. On the other hand, $qa \leq 1$, the equality being reached at the position of an elastic resonance. Hence we need only choose $s_0$ and $s_1$ in such a way that the given function, $a(s)$, is not equal to $1/q(s)$ at precisely the point for which $h(s)=\frac{1}{2}$, in order to ensure that

$$\eta(s) > 0 \qquad (2.16)$$

for $s_0 \leq s \leq s_1$. The simplest way to do this is to choose the interval $[s_0, s_1]$ in such a way that $a(s)$ does not have a resonance in it. Thus we can avoid third kind singularities and make (2.13) Fredholm. Conversely, one can show that any solution of (2.10)–(2.14) satisfies the original nonlinear equation (2.5), on condition that $D(s)$ has no zeros in the cut plane.

To conclude this section, let us study more closely the connection between the method of this paper and that of Ref. 3. Suppose that an amplitude $f(s)$ is given that satisfies (2.1), and which is elastically unitary up to $s_1$. Such an amplitude is a solution of the sharp cutoff equations of Ref. 3, if the cutoff is placed at $s_1$ and if the CDD parameters are chosen appropriately. It has been shown that the amplitude, considered as a solution of the sharp cutoff equations, is embedded in a continuum of solutions of dimension

$$d = [2\delta(s_1)/\pi]. \qquad (2.17)$$

The reason for this is that the sharp cutoff equations contain $n$ CDD poles, where

$$n = [\delta(s_1)/\pi], \qquad (2.18)$$

and each CDD pole carries two real parameters, the position and residue. Further, if $\alpha \geq \frac{1}{2}$, where

$$\alpha = \delta(s_1)/\pi - n, \qquad (2.19)$$

then an additional degree of freedom arises, because an arbitrary multiple of the homogeneous solution of the marginally singular $N$ integral equation is allowed.

The above amplitude is also a solution of the equations of this section, again if the CDD pole parameters are chosen appropriately (because of the generality of the $N/D$ representation). In this special case, there will be no difference between Im$f(s)$ and $a(s)$ for $s$ in $(s_0, s_1)$, since they are both equal to $q(s)|f(s)|^2$, and one can always find a $\bar{\delta}(s)$ such that

$$q(s)f(s) = \exp(i\delta(s))\sin\delta(s) = \frac{\eta(s)\exp(2i\bar{\delta}(s)) - 1}{2ih(s)}, \quad (2.20)$$

where $\delta(s)$ is the actual phase shift. For $4 \leq s \leq s_0$, $\bar{\delta}(s)$ is the same as $\delta(s)$, but since $h(s_1)=0$ and $\eta(s_1)=1$, it follows that $\bar{\delta}(s_1)=n_c\pi$, where $n_c$ is an integer. In fact (2.20) can be solved for $\eta$ and $\bar{\delta}$, yielding

$$\eta^2(s) = 1 - 4h(s)[1-h(s)]\sin^2\bar{\delta}(s), \qquad (2.21)$$

which is consistent with (2.9), and

$$\tan[2\bar{\delta}(s)] = \frac{h(s)\sin[2\delta(s)]}{1 - h(s)\{1 - \cos[2\delta(s)]\}}. \qquad (2.22)$$

Now we have agreed to choose $s_0$ and $s_1$ such that $\delta(s)$ does not attain a multiple of $\pi/2$ in the interval $[s_0, s_1]$. Then we define the integer

$$n = [\delta(s_1)/\pi] = [\delta(s_0)/\pi], \qquad (2.23)$$

and a fraction

$$\alpha = \delta(s_1)/\pi - n. \qquad (2.24)$$

Then if $\alpha < \frac{1}{2}$, $\bar{\delta}(s_1) = n\pi$, whereas if $\alpha > \frac{1}{2}$, $\bar{\delta}(s_1) = (n+1)\pi$. Now we define in the standard manner

$$D(s) = \exp\left[ -\frac{1}{\pi} \int_4^{s_1} \frac{ds'}{s'-s} \bar{\delta}(s') \right]$$
$$\underset{s \sim s_1}{\sim} (s-s_1)^{-n_c}, \qquad (2.25)$$

where

$$n_c = n + \theta(\alpha - \tfrac{1}{2}). \qquad (2.26)$$

Then the $D$ function that satisfies (2.11) is

$$D(s) = (s-s_1)^{n_c} \left[ \prod_{n=1}^{n_c} (s-t_n) \right]^{-1} D(s) \qquad (2.27)$$

where the $t_n$, the CDD pole positions, are $n_c$ distinct points for which $\sin\bar{\delta}(s)$ vanishes. In the case $\alpha < \frac{1}{2}$, there are at least $n_c$ points below $s_0$ where this happens; however, if $\alpha > \frac{1}{2}$, there may be only $n_c - 1$ such points, but in this case we can always take the $n_c$th point to be $s_1$. For definiteness we stipulate that no CDD poles are to be placed in the interval $[s_0, s_1]$ if $\alpha < \frac{1}{2}$, and that just one is to be placed in this interval, at $s_1$, if $\alpha \geq \frac{1}{2}$. It is interesting to note that, for a given $c(s)$ and a given $a(s)$ for $s \geq s_0$, the dimension of the manifold of solutions of

the $N/D$ system is $2n_c$ if $\alpha < \frac{1}{2}$, since each CDD pole carries two real parameters (the position and residue), but the dimension is $2n_c - 1$ if $\alpha > \frac{1}{2}$, since in this case the position of the last CDD pole is frozen at $s = s_1$. In general the dimension is then

$$2n_c - \theta(\alpha - \tfrac{1}{2}) = 2[\delta(s_1)/\pi] + \theta(\alpha - \tfrac{1}{2}) = [2\delta(s_1)/\pi]. \tag{2.28}$$

This is a satisfying result, since it agrees precisely with the dimension found in the case of the sharp cutoff.[3,8]

It must be stressed that the above discussion is somewhat artificial, in the sense that most solutions of the equations of this section will not be elastically unitary for $s_0 < s < s_1$. Our purpose was simply to make contact with the earlier results. In an autonomous application of the present method, the final amplitude would be in general unitary only below $s_0$. In the interval $(s_0, s_1)$, the amplitude would neither be unitary, nor would its imaginary part be equal to the input function $a(s)$.

## 3. SOLUTION OF THE NONLINEAR SYSTEM

In this section we shall treat the nonlinear system, incorporating the $N/D$ equations, as a nonlinear mapping, on the assumption that the following input quantities have been specified: the scattering lengths, the CDD parameters, and $a_l(s)$ for $s \geq s_0$. We shall show that the mapping is contractive if the inputs satisfy certain conditions.

It is of some importance to choose a well-behaved function as the basic quantity to be determined. Let us write

$$b_l(s) = \tilde{b}_l(s) + \hat{b}_l(s), \tag{3.1}$$

where $\tilde{b}_l(s)$ corresponds to the expressions (1.5)–(1.16), in which however $\mathrm{Im} f_l(s)$ is replaced by $[1 - h(s)] a_l(s)$, the known input quantity, and where $\hat{b}_l(s)$ is the remainder, namely the corresponding formula (1.5) [omitting $C_l(s)$], in which $\mathrm{Im} f_l(s)$ is replaced by $h(s) q_l(s) |f_l(s)|^2$. Now $\tilde{b}_l(s)$ is wholly known, and we shall seek to make $\hat{b}_l(s)$ uniformly small.

Consider the following mapping for $\hat{b}_l(s)$, at fixed $\tilde{b}_l(s)$:

$$\hat{b}_l'(s) = \Phi[\hat{b}; l, s] \equiv \hat{R}_l(s) + \hat{S}_l(s) + \hat{U}_l(s) + \hat{T}_l(s), \tag{3.2}$$

where $\hat{R}_l$, $\hat{S}_l$, $\hat{U}_l$, and $\hat{T}_l$ are written as in (1.7)–(1.16), but with

$$h(s) q_l(s) s^4 |N_l(s)/D_l(s)|^2 \tag{3.3}$$

in place of $\mathrm{Im} f_l(s)$. Here $\rho_l(s)$ and $\eta_l(s)$ are defined as in (2.8)–(2.9), and $n_l(s)$ is the solution of (2.13), where

$$\bar{c}_l(s) = s^{-2}[\tilde{b}_l(s) + \hat{b}_l(s)] + \frac{P}{\pi} \int_4^\infty \frac{ds'}{s' - s} \frac{1 - \eta_l(s')}{2\rho_l(s')} ; \tag{3.4}$$

moreover, $D_l(s)$ is defined by (2.11), and $N_l(s)$ by (2.12). We have reinstated the angular momentum suffix, but isospin remains implicit.

Suppose that $\hat{b}_l(s)$ belongs to the Banach space of sequences of functions that have continuous second derivatives, with norm

$$\|\hat{b}\| = \sup_{I, l, s} |\hat{b}_l^I(s)| + \sup_{I, l, s} \left| \frac{d}{ds} \hat{b}_l^I(s) \right| + \sup_{I, l, s} \left| \frac{d^2}{ds^2} \hat{b}_l^I(s) \right|, \tag{3.5}$$

where the suprema are taken over $I = 0, 1, 2$ and $l = 0, 1, 2, \cdots$, and $s \in [4, s_1]$. We suppose that the input quantities $a_l(s)$, $s \geq s_0$, are such that $\|\tilde{b}\|$ is finite. This is not unreasonable, since $\tilde{b}_l(s)$ is actually analytic in a neighborhood of $4 \leq s \leq s_0 - \epsilon$, $\epsilon > 0$, and although $\tilde{b}_l(s)$ certainly has branch points at $s = s_0$ and $s = s_1$, it will have a bounded second derivative at these points if the given function $a_l(s)$ is sufficiently smooth. We shall show in this section that if $\hat{b}_l(s)$ belongs to our space and is small enough in norm, then $\hat{b}_l'(s)$ also belongs to the space. In view of the quadratic nature of the mapping $\Phi$, one can then show easily that the contraction mapping theorem applies if the inhomogeneities are small enough.

The first step in the proof consists in showing that (2.13) has a unique solution $n_l(s)$, given $\hat{b}_l(s)$ and so $\bar{c}_l(s)$. Let us consider this linear equation as a mapping on the subsidiary Banach space of sequences of continuous functions with norm

$$\|n\|_1 = \sup_{I, l, s} |n_l^I(s)|. \tag{3.6}$$

Since

$$\left| \frac{1}{\pi} \frac{\bar{c}(s') - \bar{c}(s)}{s' - s} \rho(s') \right|$$
$$\leq \frac{1}{\pi} \left| \int_s^{s'} dx \frac{d}{dx} \bar{c}(x) \right| \cdot |s' - s|^{-1} s'^2 h(s') \leq \frac{1}{\pi} s_1^2 \|\bar{c}\|, \tag{3.7}$$

it follows that if

$$\|\bar{c}\| < \pi s_1^{-2} \inf_{s_0 \leq s \leq s_1} \eta(s), \tag{3.8}$$

then (2.13) defines a contraction mapping on the space (3.6), and so the solution $n_l(s)$ is unique in this space. This solution is not merely continuous, but is actually differentiable, as we can see by differentiating both sides of (2.13) and by using the fact that

$$\left| \frac{d}{ds} \frac{\bar{c}(s') - \bar{c}(s)}{s' - s} \right| \leq \frac{1}{2} \|\bar{c}\|. \tag{3.9}$$

Hence the singular integral in (2.11) is well defined, and in fact $D_l(s)$ is not merely bounded on $[4, s_1]$, but is Hölder continuous as well.

It is not sufficient that $D_l(s)$ is bounded; we must show that it has no zeros on the first Riemann sheet. In the case that there are no CDD poles, this is easy enough. When there are CDD poles however, we expect that the real part of $D_l(s)$ will have zeros on the real axis. If the CDD pole residues are small, there will be one zero close to each pole, at the mass squared of a resonance. $D_l(s)$ will itself have a complex zero nearby, and we must ensure that such a zero is on the second Riemann sheet.

If we choose $\hat{b}_l(s)$ and $\tilde{b}_l(s)$ to be small enough in

norm, then the solution $n_l(s)$ of (2.13) will be dominated by the inhomogeneous term,

$$\{\bar{c}_l(s) + \sum_n r_{l,n}[\bar{c}_l(s) - \bar{c}_l(t_{l,n})]/(s - t_{l,n})\}/\eta_l(s). \quad (3.10)$$

For $\|\tilde{b}\|$ sufficiently small, we can be sure that the norm of the fixed point, $\hat{b}_l(s)$, which is quadratic in $\|\tilde{b}\|$, is still smaller. It then follows from (3.4) that the dominant part of (3.10) will be obtained by replacing $\bar{c}_l(s)$ by

$$\tilde{c}_l(s) = s^{-2}\tilde{b}_l(s) + \frac{P}{\pi}\int_4^\infty \frac{ds'}{s' - s} \frac{1 - \eta_l(s')}{2\rho_l(s')}, \quad (3.11)$$

and this is a known input function. We are free to choose this known function to be such that the dominant part of (2.11),

$$1 + \sum_n \frac{r_{l,n}}{s - t_{l,n}} - \frac{1}{\pi}\int_4^\infty \frac{ds'}{s' - s} \rho_l(s')\tilde{c}_l(s'), \quad (3.12)$$

has no zeros on the first Riemann sheet. For our purposes it is even necessary to suppose that $\tilde{c}_l(s)$ has been chosen such that the modulus of (3.12) has a positive lower bound which is uniform with respect to $l$. Then it is clear that we can arrange that $|D_l(s)|$ has also such a uniform positive lower bound. Hence it is possible to rule out ghosts by restricting the input suitably.

The fact that $D_l(s)$ is dominated by the known function, (3.12), means that we can exclude first-sheet zeros of $D_l(s)$, but not zeros of $\mathrm{Re}D_l(s)$ on the real axis. Generally, for small values of the $r_{l,n}$, there will be a zero of $\mathrm{Re}D_l(s)$ near each CDD pole. However, since we have agreed that it is possible to choose the function (3.12) in such a way that $|D_l(s)|$ has a uniform lower bound, it follows that, at a zero of $\mathrm{Re}D_l(s)$, $|n_l(s)|$ is uniformly bounded below, since it is simply $|\mathrm{Im}D_l(s)|$. Hence we have no difficulty in obtaining an estimate of the Lipschitz coefficient in our contraction mapping proof. Detailed conditions which ensure that $n_l(s)$ does not have zeros near the zeros of $\mathrm{Re}D_l(s)$ have been given in the literature[9]; but for our purposes such fulsomeness is unnecessary, since we are not trying to calculate the maximal radius of a ball on which the mapping $\Phi$ is contractive. We are content to show that the radius is nonzero if the inhomogeneities are small enough, and if they are chosen such that the modulus of the function (3.12) has a uniform lower bound.

Consider now the expression (3.3), which has to be injected into $\hat{R}_l$, $\hat{S}_l$, $\hat{U}_l$, and $\hat{T}_l$ in order finally to yield $\hat{b}_l'$, the image of $\hat{b}_l$ under the mapping $\Phi$. We have now to show that

$$\|\hat{b}'\| \leq \kappa\|\hat{b} + \tilde{b}\|^2, \quad (3.13)$$

where $\kappa$ is a constant. Since we know that $N_l(s)$ has a uniform upper bound, that is proportional to $\|\hat{b} + \tilde{b}\|$, and we have ensured that $|D_l(s)|$ has a uniform positive lower bound, it will be enough if we can show that $P_l$, $S_l$, $U_l$, and $T_l$ are bounded in norm if we replace $\mathrm{Im}f_{l'}(s')$ by

$$h(s')\left(\frac{s' - 4}{s'}\right)^{1/2}\left[\frac{s'^{1/2} - 2}{s'^{1/2} + 2}\right]^{l'}, \quad (3.14)$$

in Eqs. (1.7), (1.8), (1.12), and (1.14). The rest of this section and the Appendix are devoted to this demonstration.

All the integrals are over the finite domain, $4 \leq s \leq s_1$, and none of them are singular. The only nontrivial point concerns the infinite $l'$ series, and we can prove convergence only if $s_1 < 32$, this being the limit of validity of the Roy equations. The term $R_l(s)$ is trivial in this respect, since it only contains one partial wave. At first sight it looks as though $R_l(s)$ might not be uniformly bounded as $l \to \infty$. However, in view of the bound (3.14), what one has to maximize is

$$[(s^{1/2} + 2)^2(s'^{1/2} - 2)/(s'^{1/2} + 2)^3]^l \quad (3.15)$$

for $s$ and $s'$ in $[4, s_1]$. This quantity remains smaller than unity for $s \lesssim 70$, and any $s' \geq 4$. For $s < 32$, it is less than $(16/27)^l$, which is certainly bounded as $l \to \infty$. In fact we have to estimate

$$\left[\left(\frac{s^{1/2} + 2}{s'^{1/2} + 2}\right)^{2l} - 1\right]\Big/(s' - s), \quad (3.16)$$

which involves differentiating the numerator with respect to $s$. Since we must also consider the second derivative of $R_l(s)$, in order to be able to bound $\|R\|$, we have finally to majorize the first three derivatives of the numerator in (3.16). Aside from trivial factors, these derivatives involve (3.15) again, with, however, factors of $l$, up to the third power. Clearly these powers are tamed by the bound $(16/27)^l$, and we conclude that $|R_l(s)|$, $|R_l'(s)|$, and $|R_l''(s)|$ are uniformly bounded.

Let us consider next $S_l(s)$. For $l' \geq l + 2$, it is clear from the definition (1.9) that $V_{ll'}(s', s') = 0$, and so if we define

$$\bar{V}_{ll'}(s, s') = \frac{1}{s' - s}\left[\frac{s^{1/2} + 2}{s^{1/2} - 2}\right]^l V_{ll'}(s, s'), \quad (3.17)$$

then it is easy to see that

$$\left(\frac{\partial}{\partial s}\right)^n \bar{V}_{ll'}(s, s') = -\int_0^1 y^n dy \left(\frac{\partial}{\partial x}\right)^{n+1}$$
$$\times \left[\left(\frac{x^{1/2} + 2}{x^{1/2} - 2}\right)^l V_{ll'}(x, s')\right]_{x = s' + y(s - s')}, \quad (3.18)$$

for $n = 0, 1, 2$. In the Appendix, we prove that

$$\left|\left(\frac{\partial}{\partial s}\right)^n\left[\left(\frac{s^{1/2} + 2}{s^{1/2} - 2}\right)^l V_{ll'}(s, s')\right]\right|$$
$$\leq \kappa \exp(-\epsilon l')\left[\frac{s'^{1/2} + 2}{s'^{1/2} - 2}\right]^{2l'}, \quad (3.19)$$

for any $s \in [4, s_1]$, with $s_1 < 32$, where $\kappa$ is a constant and $\epsilon$ is a small positive constant, both independent of $s$, $s'$, $l$, and $l'$. Hence

$$\left|\left(\frac{d}{ds}\right)^n S_l(s)\right| \leq \kappa \int_4^{s_1}\left[\frac{s' - 4}{s'}\right]^{1/2} \sum_{l' = l + 2}^\infty (2l' + 1)$$
$$\times \left|\left(\frac{\partial}{\partial s}\right)^n[s^2\bar{V}_{ll'}(s, s')]\right|\left[\frac{s'^{1/2} - 2}{s'^{1/2} + 2}\right]^{2l'}$$
$$\leq \kappa \int_4^{s_1} ds'\left(\frac{s' - 4}{s'}\right)^{1/2} \sum_{l' = 0}^\infty (2l' + 1)e^{-\epsilon l'}, \quad (3.20)$$

for $n = 0, 1, 2$, and this is clearly bounded.

The terms $U_l(s)$ and $T_l(s)$, defined in (1.12) and (1.14), are somewhat easier to treat, since there is no vanishing denominator. It can be shown that $W_{ll'}(s, s')$ and $Y_{ll'}(s, s')$ satisfy inequalities of the type (3.19), while

$$\left| \left(\frac{\partial}{\partial s}\right)^n \left[ \left(\frac{s^{1/2} + 2}{s^{1/2} - 2}\right)^l X_l(s, s') \right] \right| \leqslant \kappa, \qquad (3.21)$$

for $n = 0, 1, 2$. The methods are similar to those given in the appendix, and details may be found in Ref. 10. These inequalities suffice to bound $U_l(s)$ and $T_l(s)$, and their first two derivatives; and this concludes the proof that $\hat{b}_l'(s)$ belongs to the Banach space. Thus $\Phi$ is contractive if the inhomogeneities are small enough.

## ACKNOWLEDGMENT

## APPENDIX

We shall sketch the derivation of certain bounds[10] for the functions $V$, $W$, $X$, and $Y$, defined in (1.9), (1.13), (1.15), and (1.16). From (1.10) and (1.11), we can write

$$z' = 1 + (z - 1)/\alpha, \qquad (A1)$$

where

$$\alpha = (s' - 4)/(s - 4), \qquad (A2)$$

and so $P_{l'}(z')$ is in fact a polynomial in $z$. Let us write a Cauchy integral for this polynomial around the following elliptical contour in the $z$ plane:

$$\partial \epsilon(z_1) = \{z : |z + (z^2 - 1)^{1/2}| = z_1 + (z_1^2 - 1)^{1/2}, z_1 > 1\}. \qquad (A3)$$

The $z$ integration in (1.9) can be performed under the contour integral, the result being

$$V_{ll'}(s, s') = \frac{\alpha}{2\pi i} \oint_{\partial \epsilon(z_1)} d\xi \, P_{l'}(\xi) \, Q_l(\overline{\xi}), \qquad (A4)$$

where $Q_l$ is the Legendre function of the second kind and where

$$\overline{\xi} = 1 + \alpha(\xi - 1). \qquad (A5)$$

If $\xi \in \partial \epsilon(z_1)$, it may be shown from the Laplace representations that $|P_{l'}(\xi)|$ is bounded above by $P_{l'}(z_1)$ and $|Q_l(\xi)|$ by $Q_l(z_1)$. However, we have to majorize $|Q_l(\overline{\xi})|$ and not $|Q_l(\xi)|$, and $\overline{\xi}$ describes an ellipse that lies wholly outside $\partial \epsilon(z_1)$ if $\alpha > 1$, and wholly inside $\partial \epsilon(z_1)$ if $\alpha < 1$. In the former case, for a given $\xi$ on $\partial \epsilon(z_1)$, $|Q_l(\overline{\xi})|$ is bounded by $Q_l(\eta)$, where $\eta$ is the rightmost extremity of the ellipse with foci at $\pm 1$ that passes through $\overline{\xi}$. For any $\xi \in \partial \epsilon(z_1)$, it is easy to check that the corresponding $\eta$ satisfies

$$\alpha z_1 - (\alpha - 1) \leqslant \eta \leqslant \alpha z_1 + (\alpha - 1), \qquad (A6)$$

and since $Q_l(z)$ is a monotonically decreasing function of $z$, for $z > 1$, it follows that $|Q_l(\overline{\xi})|$ is majorized by $Q_l(\alpha z_1 - \alpha + 1)$ for all $\xi \in \partial \epsilon(z_1)$. In the case $\alpha < 1$, $\overline{\xi}$ describes an ellipse that lies wholly within $\partial \epsilon(z_1)$, and we may now conclude that the corresponding $\eta$ satisfies

$$\alpha z_1 - (1 - \alpha) \leqslant \eta \leqslant \alpha z_1 + (1 - \alpha); \qquad (A7)$$

and so $|Q_l(\overline{\xi})|$ is bounded by $Q_l(\alpha z_1 + \alpha - 1)$, on condition that the argument of the latter function is greater than unity. We may combine both results as follows:

$$\sup_{\xi \in \partial \epsilon(z_1)} |Q(\overline{\xi})| = Q_l(\alpha z_1 - |\alpha - 1|), \qquad (A8)$$

for any $\alpha > 2/(1 + z_1)$. Hence we have

$$|V_{ll'}(s, s')| \leqslant \alpha z_1 P_{l'}(z_1) Q_l(\tilde{z}), \qquad (A9)$$

where we have majorized the circumference of the ellipse by that of its circumscribing circle and where

$$\tilde{z} = \alpha z_1 - |\alpha - 1| = [(s' - 4) z_1 - |s' - s|]/(s - 4). \qquad (A10)$$

We wish now to motivate a choice for $z_1$, in order to make (A9) as useful as possible, and to maximize $s_1$, the largest value of $s$ for which the inequalities hold. In the first place we write

$$z_1 = 2 z_0^2 - 1, \qquad (A11)$$

and require

$$z_0 \leqslant \left(\frac{s' + 4}{s' - 4}\right) \exp(-\epsilon/2), \qquad (A12)$$

where $\epsilon$ is a small, positive constant. This ensures that

$$P_{l'}(z_1) \leqslant [z_0 + (z_0^2 - 1)^{1/2}]^{2l'}$$
$$\leqslant \exp(-\epsilon l') \left[\frac{s'^{1/2} + 2}{s'^{1/2} - 2}\right]^{2l'}. \qquad (A13)$$

Suppose further that $z_1$ is such that

$$\tilde{z} \geqslant \frac{s + 4}{s - 4} \exp(\epsilon). \qquad (A14)$$

Then

$$Q_l(\tilde{z}) \leqslant [\tilde{z} + (\tilde{z}^2 - 1)^{1/2}]^{-l} Q_0(\tilde{z})$$
$$\leqslant \exp(-\epsilon l) \left(\frac{s^{1/2} - 2}{s^{1/2} + 2}\right)^l \frac{s - 4}{8}. \qquad (A15)$$

Now in the case $s' \leqslant s$, the inequality (A14) implies the following constraint upon $s$:

$$s \leqslant [1 + \exp(\epsilon)]^{-1}[(s' - 4) z_1 + s' - 4 \exp(\epsilon)]. \qquad (A16)$$

The minimum value of the right-hand side, as a function of $s'$, leads to

$$s \leqslant s_1 \equiv [\exp(\epsilon) + 1]^{-1} \cdot [64 \exp(-\epsilon) - 4(\exp(\epsilon) - 1)], \qquad (A17)$$

in view of (A11) and (A12). The limit of this bound as $\epsilon \to 0$ is 32, and this is the maximum value of $s$ for which the Roy equations are valid. Since we wish to retain the exponential factor in (A15), we need $\epsilon > 0$, and this means that $s_1$ will be less than 32, although we can make it as close to 32 as we like by making $\epsilon$ small enough

The inequality (A14) implies

$$z_1 \geqslant [s(\exp(\epsilon) - 1) + s' + 4 \exp(\epsilon)]/(s' - 4) \qquad (A18)$$

in the case $s' \geqslant s$; and this can be combined with (A11)

and (A12) to yield

$$\frac{(s'+4)^2}{s'-4}\exp(-\epsilon) \geq \tfrac{1}{2}(s'-4)(z_1+1) \geq s' + \frac{s+4}{2}(\exp(\epsilon)-1).$$

$$(A19)$$

The minimum of the left term is $32\exp(-\epsilon)$, so if we choose $z_1$ such that the middle term is equal to this constant, then the first inequality is satisfied also, on condition that we restrict both $s'$ and $s$ to the interval $[4, s_1]$ where $s_1$ was defined in (A17). Thus we choose

$$z_1 = -1 + \frac{64}{s'-4}\exp(-\epsilon),$$

$$(A20)$$

and on combining (A9), (A13), (A15), and (A20), we find

$$\left| \left[\frac{s^{1/2}+2}{s^{1/2}-2}\right]^{\,l} V_{ll'}(s,s') \right|$$

$$\leq 8\exp[-\epsilon(l+l')]\left(\frac{s'^{1/2}+2}{s'^{1/2}-2}\right)^{2l'},$$

$$(A21)$$

for any $s, s' \in [4, s_1]$.

By similar techniques one can show that

$$\left| \left(\frac{\partial}{\partial s}\right)^n \left[\left(\frac{s^{1/2}+2}{s^{1/2}-2}\right)^{\,l} V_{ll'}(s,s')\right] \right|$$

$$\leq \kappa\, l^n \exp[-\epsilon(l+l')]\left(\frac{s'^{1/2}+2}{s'^{1/2}-2}\right)^{2l'},$$

$$(A22)$$

for $n = 1, 2, 3,$ where $\kappa$ is a constant. To prove this, $Q_l(\bar{\xi})$ must be differentiated repeatedly with respect to $s$ under the integral in (A4). The recurrence relations for the Legendre functions are then used to make all cancellations explicit. Details of some of the necessary calculations are to be found in Ref. 10; the remainder are very similar, and we shall not reproduce them here. The factor $l^n$ arises because of the $n$th order derivative with respect to $s$; but it can be removed, thanks to the term $\exp(-\epsilon l)$, at the expense of an adjustment of the constant $\kappa$. The inequality (3.19) then follows immediately.

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