Higher derivative gravity and holography

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Introduction

What does the sentence *understanding the universe* mean? It is a very broad concept and the answer to this question can vary a lot, depending on who is answering. For a physicist, understanding the universe means observing the reality that surrounds us, expressing this observation in a mathematical language (a theory), and then testing this interpretation by confronting it with the reality again (performing an experiment). What makes the process fascinating is that the purpose is never to find some ultimate answer but, rather, to grasp the reasons behind physical effects through hypothesis, approximations, and mistakes. In this sense, we can never be sure of the correctness of a theory but, rather, can only be confident that it faithfully describes reality under a given set of assumptions. The only certainty occurs when a theory is recognized as wrong.

The past century was characterized by two major periods: the first half, when revolutionary theories changed our understanding of Nature completely, and the second half, when such theories were used to show an underlying elegance in the fundamental interactions. This is, naturally, an over-simplification of one of the most prolific periods (if not the most prolific period) in the history of physics. However, we can rephrase it as: exceptional minds developed a series of concepts and theories that eventually formed a cornerstone of modern physics, that is quantum field theory, and, later, equally exceptional minds used such cornerstone to explain a wide range of physical effects with a unified description. This process made it possible to describe three fundamental interactions (electromagnetic, weak, and strong) by using the same language or, if you will, by encoding them in the same picture.

Motivated by the enthusiasm derived from observing an increasingly simpler description for a wider and wider class of physical effects, we have been led to attempt to unify the only fundamental interaction left outside of our picture: gravity.

One may wonder if trying to describe the gravitational interaction with the same language we used for the other three is the right thing to do, or even if it makes any sense at all. The honest answer is that we don't know; that's why we try, because, again, we can only be sure of being wrong and, to achieve that, we need to explore every possibility. The hope is to find in the process, regardless of its success, a deeper explanation regarding the nature of this mysterious interaction.

In the search for a unifying theory, a central role has been played by the study of *black holes*: regions of the spacetime where the gravitational field is so strong that not even light can escape. In fact, thanks to the work of Stephen Hawking [1,2] it is believed that, in those extreme regimes, we have to consider both gravitational and quantum effects, making the construction of some sort of unifying theory necessary.

Notwithstanding that every attempt of constructing such a theory still faces major theoretical and technical problems, focusing our attention on black holes has paved the way towards one of the biggest paradigm changes of recent years: the formulation of the holographic principle.

This principle was first proposed by Gerard 't Hooft [3] and interpreted as a property of string theory by Leonard Susskind [4]. It states that we can describe a volume of space by using the information encoded in its boundary. Very much like the holograms, for example, on our bills, which are two-dimensional pictures that look three-dimensional, we can describe a volume of space by observing a geometrical object living in one dimension less.

The inspiration for this came precisely from studies of black hole thermodynamics. Here, it was found that the entropy of a black hole is not proportional to its volume but rather to the area of the surface enclosing the black hole (the event horizon). It is then clear that, if we want to understand the nature of gravity, we have to analyze the holographic principle. In other words, the answer we are seeking may have to pass through the understanding of how these two puzzles, the nature of gravity and the holographic principle, are connected.

The most remarkable realization of the holographic principle is a conjecture called the AdS/CFT correspondence [5]. Also called gauge/gravity duality, it conjectures a relationship between two very different theories. On one side of the correspondence, we have a gravity theory living in N dimensions, while, on the other side, we find a conformal field theory in $N-1$ dimensions. The perspective of connecting a theory of gravity with the much better-understood quantum field theory is a very fascinating one: this would allow us to describe one in terms of the other and vice versa, thus making full use of the holography mentioned above.

Regardless of the fact that a formal proof is still lacking, this correspondence has been extensively studied over the past 20 years, making [5] the second most cited paper in the history of high energy physics. The main reason is that it is a great tool for studying strongly coupled systems. The fact of the matter is that those regimes always represented a challenge due to the difficulties in applying perturbation theory, and it's here that the magic happens: the corresponding theory turns out to be weakly coupled, thus mathematically more treatable.

We can thus find many applications of the gauge/gravity duality across all the fields of physics, such as quantum chromodynamics or condensed matter physics. What is highly interesting is that studying objects such as black holes then became necessary not only for those trying to understand the nature of gravity, but also to those working with particle accelerators or superconductors.

One of the most stimulating questions for a scientist confronting a new fascinating theory is not only *why?* or *how?* but also *can I break it?*. In other words, the correspondence has been applied extensively to a wide range of systems, but what would happen if we step slightly outside of the comfort zone of the original formulation, what limitations would we encounter, and what message would be contained in these limitations? This thesis participating in this quest because we will make use of holography in the context of a modified theory of gravity (in particular, a theory of gravity with higher derivatives), thus stepping outside of the AdS/CFT comfort zone.

Higher derivative gravity theories initially attracted attention since they offer the possibility of extending Einstein's theory by providing a description of more general physical effects such as the propagation of massive gravitons. Moreover, the quest for a renormalizable theory of gravity led to the idea that Einstein's theory as an effective theory was going to receive higher-order corrections. Such corrections would have more and more importance as the energy scales increases [6]. The enthusiasm around these theories grew when it was shown that it was possible to obtain a renormalizable theory if we extend Einstein gravity with all possible curvature-squared terms [7]. The price to be paid was the introduction of ghost-like modes. With the advent of string theory as a potential candidate for a consistent theory of quantum gravity, they even found a more specific use in theoretical physics. Indeed, one interesting result is that perturbative string theory not only contains the pure Einstein-Hilbert action but also terms of a higher order. For example, it was shown that such terms provide a mechanism for supersymmetry breaking [8].

Supersymmetry is one of the pillars of string theory, establishing a symmetry connecting bosons and fermions that, as we will see in Chapter 1, makes it possible to combine spacetime and internal symmetries. It also represents an interesting theoretical (and experimental) challenge because, since it is not observed by the experiments, it should appear as a broken symmetry. Thus the study of a mechanism of supersymmetry breaking by making use of higher derivative terms sheds light on the importance of such theories.

In the context of holography, higher derivative gravities can be a useful instrument for exploring the limitations of the AdS/CFT correspondence. By providing different dynamics and different properties on one side of the duality, one may hope to observe how this is reflected on the other side. Since the long-term goal is to understand the nature of a connection between theories, a broader, more general view may pave the way for a deeper understanding.

Another instrument that we will use in this thesis to reveal potential new features hidden in the duality is Entanglement Entropy (EE). It is well known that, if a group of particles interacts, we cannot describe the quantum states separately but rather the system as a whole, no matter how much the particles get separated. This phenomenon is called *quantum entanglement* and EE is a measurement of it since it expresses the amount of information we lose if part of the quantum system becomes inaccessible.

EE is an interesting quantity that appears in many different fields. For example, this measure plays a crucial role as an order parameter for probing the quantum phase transition in many physical contexts [9]. From a more practical point of view, understanding the amount of correlations has proved useful for efficiently representing a quantum ground state, thus helping our classical computers in the description of quantum systems. Moreover, one can even reverse the logic and thus be able to identify a quantum phase by looking at how computationally complex it is to describe it.

As we will see in Chapter 1, there are a lot of challenges in the practical computation of EE. It is here that holography comes to help by relating this quantum property of matter to a geometric object in a gravity theory, significantly simplifying the computation. To be fair, we can say that EE and holography help each other since EE is a universal quantity that can be defined in any quantum system and thus may be an interesting object for studying holography in different contexts.

The possibility of connecting a geometrical quantity to the quantum properties of matter is then only one of the fascinating aspects of studying the EE of a system. We can also try to better define the edges of this possibility by applying the prescription to a more general context, such as the one provided by higher derivative gravities. We will see that the presence of higher derivatives will not only affect the numerical outcome (as is expected when you modify the conditions of your study) but will also determine a change in the very nature of the geometrical object holographically associated with the EE.

Outline of the thesis

During this introduction, we have already used several terms that may not be in the everyday vocabulary of every reader. For this reason, before jumping into a more technical discussion, we want to dedicate the first two chapters to explaining the context of this thesis in a non-technical matter. Moreover, the beginning of every chapter, including the more technical ones, will be dedicated to providing a description of the strategies and the methods thereby contained with the intention of being as accessible as possible.

Chapter 1 will be dedicated to introducing two recurrent protagonists of this thesis: supersymmetry and entanglement entropy. These will be used as tools to probe our models, to explore what new possibilities we might have, and to get some idea of the limitations we might encounter. As mentioned, we will not go into much detail but rather provide the basic ideas behind these concepts.

Chapter 2 will conclude the introduction to this thesis by presenting the higher derivative theory of gravity we will use as the playground for our exploration, namely New Massive Gravity (NMG). This will be done by following a path that goes from the principles of General Relativity, thus the modern understanding of the gravitational interaction, to the possibilities of respecting such principles in a simpler model. Our path will lead us to simplify the problem by reducing the number of dimensions and then adding higher derivative terms to describe, despite these simplifications, a broader range of physical effects. Only after that we will explore the feature of NMG and be ready to work in this context for the rest of the thesis.

The first original material can be found in Chapter 3. Here, we will be able to construct two supersymmetric extensions of New Massive Gravity. This chapter will provide an overview of the method and extensive details through to the final result. Reading this will be made easier by the fact that we will essentially be following the blueprint of our method. Thus, the reader will be able to see how every intermediate result actually contributes to the construction of the theory.

Once that a new theory has been constructed, it is natural to want to explore its solutions. This will be done in Chapter 4, where we will reveal one technical advantage of working with supersymmetric theories. Instead of directly solving the equations of motion, which can be challenging, one can study other objects, called Killing spinors, that facilitate the classification of the solutions. Therefore, we will first see how we can find such objects and thus find the supersymmetric solutions of the theory constructed in Chapter 3. It is here that we can start to appreciate how the presence of higher derivatives introduces a certain richness into the solutions.

As anticipated, another interesting result of the presence of higher derivative terms can be found by studying the entanglement entropy with a holographic approach. Chapter 5 is dedicated to directly applying the concepts introduced in Chapter 1 to the more complex context of NMG. Here, we will see holography applied to a higher derivative theory, observe which new difficulties emerge from this choice, and see what interesting results are brought about by being able to work with the richer geometries that NMG allows as solutions.

Finally, we will summarize the conclusions of this thesis, keeping the possible developments in our sights. Further details of the calculations presented in the thesis will be provided in four appendices: two dedicated to the formalism necessary for the supersymmetric construction of a theory, and two dedicated to the explicit calculation of EE in two different geometries.

1 The Toolbox

Science is founded on theories and experiments to test those theories. Therefore, every scientist needs several tools to perform the appropriate experiment. Such tools can be a simple screwdriver or an elaborated precise machinery, a pencil or a sophisticated simulation program.

For theoretical physicists, the tools can be physical quantities or even entire theories. You test the model under consideration by inserting different parameters to your equations. The goal is to observe new behaviors and, ultimately, find the edges of the potential knowledge that your study can provide.

In this sense, a theoretical physicist is particularly lucky: the tools themselves are interesting and fascinating. In this thesis, we will use as tools concepts and physical quantities that attracted, and well-deserved, a great volume of studies.

In this chapter, however, we will discuss our primary tools, supersymmetry and entanglement entropy, by providing only the necessary information we will need to perform the investigation here presented.

1.1 Supersymmetry and Supergravity

When, in the early 1960's, scientists were constructing the theory to describe the fundamental particles, the main question was: what are the possible symmetries in particle physics?

Since we are in a regime where special relativity has to be taken into account, our theory has to respect the Poincaré symmetry in the four-dimensional Minkowski spacetime, described by a group called $\mathrm{ISO}(1,3)$. Here, we find:

- the space-time translation symmetries, forming an Abelian Lie group whose generators are called P_a;
- the Lorentz symmetry, describing the invariance under (four-dimensional) rotations and stating the equivalence of inertial reference frames. The resulting non-Abelian group is generated by M_{ab}.

These two groups are combined together to form the so-called Poincaré group, governing the structure of a relativistic system.

In order to describe interactions between particles, we also need a certain set of internal symmetries. Examples are the local U(1) of electromagnetism, or the local SU(3) of quantum chromodynamics. Such symmetries are generated by T_i and form a Lie algebra

$$[T_i\,, T_j] = f_{ij}{}^k\, T_k\,. \tag{1.1}$$

Therefore, a question arises: can these two sets of symmetries be combined in a non-trivial way? The answer comes in the form of a no-go theorem named after Sydney Coleman and Jeffrey Mandula [11]. It states that if we combine the Poincaré and internal symmetries in a non-trivial way (i.e. $[T_i\,, P_a] \neq 0$, $[T_i\,, M_{ab}] \neq 0$), the S matrices for all processes will be zero.

In the search for exceptions to such a restrictive theorem, people came to the conclusion that, in order to evade the theorem, they needed to generalize the notion of Lie algebra to a *graded* Lie algebra. In this generalization, some of the generators are allowed to not obey a commuting law, but rather an anticommuting one.

We call the generators P_a, M_{ab}, and T_i *even generators*, and the new set Q^i_α *odd generators*. Thus, the graded Lie algebra is of the type

$$[\text{even}, \text{even}] = \text{even}, \qquad [\text{even}, \text{odd}] = \text{odd}, \qquad \{\text{odd}, \text{odd}\} = \text{even}.$$

With this structure, the Coleman-Mandula theorem is evaded and we can combine Poincaré and internal symmetries.

To observe what this algebra implies, let's have a look to the following commutator

$$[Q\,,M] \sim Q\,. \tag{1.2}$$

Therefore, Q is in a representation of the Lorentz group and, since we are requiring it to be anticommuting, we have to choose the spinor representation. It is well-known that a spinor field times a boson field gives a spinor field, thus we observe that Q gives a symmetry between bosons and fermions

$$\delta\,\text{boson} = \text{fermion}, \qquad\qquad \delta\,\text{fermion} = \text{boson},$$

and we call it *supersymmetry* [12].

It was shown by Haag, Lopuszanski and Sohnius that supersymmetry is the only candidate to combine Poincaré and internal symmetries non-trivially [13]. Moreover, a supersymmetric model brings several advantages:

- Milder UV divergences due to the cancellations between the contributions from bosonic and fermionic loops.
- Unification of couplings. Indeed, when extrapolated using the renormalization group, the gauge coupling constants of the standard model approach the same value at high energies.
- It provides natural candidates for the particles constituting the cosmological dark matter.

However, supersymmetry requires a spectrum of particles where fermion-boson pairs appear with the same mass. Since it is not observed by the experiments we can conclude that, if such symmetry is realized in Nature, it should appear as a broken symmetry. As of today, the detection of any hints on the validity of such models escaped the experimental proof.

In this section, we will analyze the basic structures that emerge in a supersymmetric theory and see the implications of making such symmetry local. These will be the instruments we will need in Chapter 3.

1.1.1 Algebra and Supermultiplets

Supersymmetry is a spacetime symmetry connecting particles with different spins, i.e. particles that behave differently under rotations.

If we were considering the Poincaré group alone, its irreducible representation would be a particle with a given mass and a given spin. Since a supersymmetric transformation is changing the spin of the particle, we can conclude that the irreducible representation of such group will be a set of particles, called a multiplet (or supermultiplet).

To better understand this concept, we must have a look to the algebra governing supersymmetry: an extension of the Poincaré algebra that includes anticommuting elements.

The first step is to define what a spinor is. The spinor representation of the Lorentz group $\mathrm{SO}(1, d-1)$ is defined by the existence of gamma matrices $(\gamma_\mu)_\alpha{}^\beta$ satisfying the Clifford algebra

$$\{\gamma_\mu\,, \gamma_\nu\} = 2g_{\mu\nu}\,, \tag{1.3}$$

where $g_{\mu\nu}$ is the metric of the $d-$dimensional Minkowski spacetime. Thanks to these matrices we can map spinors into spinors via $(\gamma_\mu)_\alpha{}^\beta \chi_\beta = \tilde{\chi}_\alpha$. These spinors have $2^{[d/2]}$ complex components, but the representation is not irreducible.

The most common representation used in supersymmetry makes use of the so-called Majorana spinors. Such spinors satisfy the reality condition

$$\chi^C \equiv \chi^T C = \bar{\chi} \equiv \mathrm{i}\,\chi^\dagger\,\gamma^0\,, \tag{1.4}$$

where C is the charge conjugation matrix. This matrix can be used to raise and lower indices but, since it is antisymmetric, we must define a convention for the contraction of the indices to take into account that the order matters (i.e. $\chi_\alpha\,\psi^\alpha = -\chi^\alpha\,\psi_\alpha$).

In order to not have particles with spin greater than 2, the maximum number of supersymmetry generators $\mathcal{N}$ is constrained. This influences the dimension of the spinor representation and depends on the dimension of the spacetime we are using for out theory. Therefore, a general description of the algebra would be rather heavy. In this section, we are going to explore the simplest example with the promise of giving any necessary detail for the more complex structure in Chapter 3. The more curious reader will find a comprehensive exposition in [12].

Let's consider a superalgebra containing $\mathcal{N}=1$ spinor charge Q_α in three dimensions. The subalgebra of bosonic charges P_μ and $M_{\mu\nu}=-M_{\nu\mu}$ is the Lie algebra of the Poincaré group, while the new relations are

$$\begin{aligned}
\left[P_\mu\,, Q_\alpha\right] &= 0\,,\\
\left[M_{\mu\nu}\,, Q_\alpha\right] &= -\frac{1}{2}\,(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta\,,\\
\{Q_\alpha\,, \bar{Q}^\beta\} &= -\frac{1}{2}\,(\gamma_\mu)_\alpha{}^\beta P^\mu\,.
\end{aligned} \tag{1.5}$$

We notice that, due to the first commutator, particles in the same multiplet will have the same mass.

To construct a supermultiplet, the guiding principle is to match the number of fermionic and bosonic degrees of freedom. The matching must happen both on-shell and off-shell. To see that, let's consider the off-shell counting for one of the simplest supermultiplet: the scalar multiplet (in three dimensions). Here, we find

- One scalar A, with 1 bosonic degree of freedom;
- One spinor χ, with spin 1/2 and 2 fermionic degrees of freedom;
- An auxiliary scalar F, providing the necessary extra bosonic degree of freedom.

The auxiliary field has an algebraic equation of motion and will not contribute to the counting in the on-shell matching. In this case, indeed, we find that the spinor will lose half of his degrees of freedom due to the equations of motion and it will thus match the number of bosonic degrees of freedom of the scalar A.

The presence of auxiliary fields is necessary to match the number of degrees of freedom (and thus closing the algebra). In more complex cases, such as the one presented in Chapter 3, we will have the possibility of choosing different sets of auxiliary fields: the only requirement is to respect the algebra both off-shell and on-shell.

From a more practical point of view, the multiplets are often presented by giving the transformation rules of their components. The advantage is that it is then possible to construct a supersymmetric invariant action. For example, in the case of the scalar multiplet we have

$$\begin{aligned}
\delta A &= \frac{1}{\sqrt{2}}\,\bar{\epsilon}\chi\,, \\
\delta\chi &= \frac{1}{\sqrt{2}}\,(\partial\!\!\!/ A + F)\,\epsilon\,, \\
\delta F &= \frac{1}{\sqrt{2}}\,\bar{\epsilon}\,\partial\!\!\!/\,\chi\,, \qquad\qquad \partial\!\!\!/ = \gamma^\mu\partial_\mu\,.
\end{aligned} \tag{1.6}$$

Presenting the multiplet with its transformation rules gives the opportunity to see how supersymmetry maps bosons to fermions and vice versa. The spinor parameter ϵ is a constant Majorana spinor. To have a full understanding of what we are going to see in Chapter 3, we need to take another step further and gauge this symmetry, i.e. sending $\epsilon \to \epsilon(x)$.

1.1.2 Supergravity

The discoveries in the field of particle physics were mainly due to the use of *gauge symmetries*: a symmetry that acts differently at any given point of the spacetime. It was thus natural to extend this concept to supersymmetry. Since, as we shall see in a moment, such gauging implies the inclusion of gravity, we call *supergravity* the theory of local supersymmetry.

In the 80's, the important results in the field of superstring theory led to the idea that a consistent theory of quantum gravity should be a superstring theory. In such a theory, particles are described as excitations of an extended object called string. It is very interesting to notice that, in a low energy regime, the theory reduces to a supergravity field theory.

There are many equivalent ways to look at supergravity. Most commonly it is referred as the theory that combines general relativity and supersymmetry (or a supersymmetric theory of gravity). Since we are talking about an algebraic structure, nothing could express this concept better than a commutator. Looking at the anticommutator in (1.5), we notice that

$$\{Q, Q\} \sim P\,. \tag{1.7}$$

This indicates that having a general coordinate transformation is equivalent to having a local supersymmetry. It is also true the converse: any supersymmetric theory which includes gravity shall require a local realization of supersymmetry.

More technically, in order to have transformation rules compatible with a theory of gravity with fermions, we must require the spinor parameter to be local, $\epsilon(x)$.

When one constructs a gauge theory, we know that we must introduce gauge fields in order to compensate the contributions coming from the derivatives of the local parameter. In the case of supergravity, such gauge field is a spin 3/2 vector-spinor called *gravitino* and it is the superpartner of the graviton (spin 2).

In the context of supergravity, due to the presence of spinors, it is more convenient to use the vierbein formulation of Einstein gravity. Since it is a gauge theory, we also need the gauge field corresponding to the local rotations M_{ab} and the local translations P_a. These are, respectively, the spin connection ω_μ^{ab} and the vierbein e_μ^a, where a, b are the Lorentz gauge group indices, and μ, ν are the spacetime indices. The corresponding field strengths are the torsion $C_{\mu\nu}^a$ and the Riemann curvature tensor $R_{\mu\nu}^{ab}$. In Chapter 3, we will use this kind of curvature invariants to construct the desired supersymmetric theory.

Conformal Symmetry and a Strategy

The construction of a supergravity theory can be, however, rather challenging, especially if one wants to include matter multiplets. The systematic approach we will use in this thesis is called *superconformal method*. Here, we will briefly see its underlying strategy and then see an explicit example in Chapter 3.

The idea is to build a theory with a larger symmetry, which gives us more control on the construction, and then eliminate the extra symmetries by imposing the appropriate constraints.

In particular, we first formulate a theory governed by the superconformal algebra, this is done with the help of so-called compensating matter fields. Then, we eliminate the compensating fields to obtain the desired theory respecting the Poincaré supersymmetric subalgebra. The cancellation of such fields will happen by gauge-fixing the extra symmetries.

Thanks to this method, it is easier to construct a matter-coupled supergravity theory because we are supported by the structure coming from the conformal symmetry. Although this symmetry will be a simple tool for the construction of the theory, it deserves a small description.

The conformal symmetry is an extension of the Poincaré group including dilatations and the so-called special conformal transformations. The transformations are respectively defined by

$$x^\mu \to \lambda x^\mu, \qquad\qquad x^\mu \to \frac{x^\mu - a^\mu x^2}{1 - 2a \cdot x + a^2 x^2}\,, \tag{1.8}$$

where λ and a^μ are the parameters describing the transformations.

We call D the generator of the dilatations and K_μ the one of the special conformal transformations. Then, the commutation relations involving this extension read

$$\begin{aligned}
\big[D\,, P_\mu\big] &= \mathrm{i}\,P_\mu\,, \\
\big[D\,, K_\mu\big] &= -\mathrm{i}\,K_\mu\,, \\
\big[P_\mu\,, K_\nu\big] &= 2\mathrm{i}\,(M_{\mu\nu} - \eta_{\mu\nu} D), \\
\big[K_\mu\,, M_{\nu\rho}\big] &= \mathrm{i}\,(\eta_{\mu\nu} K_\rho - \eta_{\mu\rho} K_\nu),
\end{aligned} \tag{1.9}$$

and the other commutators vanish. We thus conclude that D is a scalar and K_μ is a vector under Lorentz transformations.

The superconformal algebra [14] combines this symmetry with the algebra given in (1.5). In this extension, a new fermionic generator S^α appears: the special supersymmetry charge. The important anticommutators involving this new generator are

$$\{Q, S\} \sim 0\,, \qquad\qquad \{S, S\} \sim K\,, \tag{1.10}$$

thus, in the same way as two supersymmetry transformations produce a translation, two special supersymmetry transformations produce a special conformal transformation.

In general, the superconformal transformation will be a matrix of the form [12]

$$\begin{bmatrix} \text{conformal algebra} & Q, S \\ Q, S & \text{R-symmetry} \end{bmatrix}$$

where the R-symmetry rotates the supercharges and contributes to the closure of the algebra. We will first construct a theory invariant under this kind of transformations and then remove the conformal symmetry by gauge-fixing it.

We now have all the concepts to understand the procedure exposed in Chapter 3, where this method will be explicitly applied to a matter-coupled theory with extended supersymmetry, i.e. a theory with more than one supersymmetry.

1.2 Entanglement Entropy

Entanglement Entropy (EE) is the second tool we will need in this thesis. It measures the quantum correlation between two systems since it encodes the amount of information loss when one of the two systems becomes inaccessible.

This quantity fascinates a broad range of physicists since a good part of a decade. It is used in condensed matter to guess the ground state wave functions, it can be used as an order parameter to describe phase transitions and thermalization. Moreover, and this is the main interest of this thesis, it is related to an area law and thus to the geometry of a specific surface, as we will see later in this section and more extensively in Chapter 5.

In this section, we will take every necessary step to define EE, observe the difficulties in its computation, and finally explore the holographic method introduced to overcome such difficulties. A comprehensive review of the topic can be found in [17].

In order to define entanglement entropy, let us consider a bipartite system described by a well-defined Hilbert space $\mathcal{H}_{tot}$ such that it can be factorized into two disjoint Hilbert spaces of the subsystems A and B as,

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{1.11}$$

Let us characterize the full system with the density matrix ρ_{tot}. For the observer who has access only to the region A, the system is effectively represented by the reduced density matrix ρ_A,

$$\rho_A = \mathrm{Tr}_B\ \rho_{tot}, \tag{1.12}$$

where the partial trace is performed over $\mathcal{H}_B$. Analogously, you can define ρ_B. Since the reduced density matrix doesn't keep track of the correlations between A and B, some information is lost and, in general, we will have $\rho_{tot} \neq \rho_A \otimes \rho_B$.

The entanglement entropy of the subsystem A is defined after the von Neumann entropy of the reduced density matrix ρ_A,

$$S_A = -\mathrm{Tr}_A\ \rho_A \log \rho_A. \tag{1.13}$$

It is called *entropy* because if you take ρ to be a thermal state (i.e. $\rho \sim \exp(-\beta H)$, where H is the Hamiltonian and $\beta = T^{-1}$) you have the usual formula for the thermal entropy.

This quantity has several properties:

- if ρ is a pure state on the total system AB, then $S_A = S_B$. This shows that EE is not an extensive quantity[1]. The relation doesn't hold at finite temperature.
- Subadditivity, i.e. $S_{AB} \leqslant S_A + S_B$, where S_{AB} is the EE of the system given by unifying A and B. Moreover, we can define another interesting quantities: the *mutual information*

$$I(A:B) \equiv S_A + S_B - S_{AB} \geqslant 0\,, \tag{1.14}$$

  which focuses more on the correlations between the two systems.
- Strong subadditivity, i.e. $S_A + S_C \leqslant S_{AB} + S_{BC}$, where the systems A, B and C do not intersect each other.

However, as we can see in eq. (1.13), we notice that the actual computation of entanglement entropy involves a logarithm of a matrix that can be, in principle, arbitrarily complicated. For this reason, in order to make the calculation easier (or even doable at all), we can define a new object called *Rényi entropy*

$$S_A^n = \frac{1}{n-1} \log \operatorname{Tr} \rho_A^n\,, \qquad \lim_{n\to 1} S_A^n = S_A\,, \tag{1.15}$$

where the limit makes use of the fact that $\operatorname{Tr} \rho_A = 1$. This quantity plays a crucial role in the calculation of EE in the context of quantum field theories.

[1] An extensive quantity is additive. For example, the volume, the energy, the mass, and the thermal entropy are extensive quantities.

1.2.1 EE in Quantum Field Theories

Despite the simplicity of the formula and its successful application to simple quantum mechanical systems, it is extremely difficult to generalize the prescription (1.13) to perturbative quantum field theories in arbitrary dimensions. In this case, we have an infinite amount of degrees of freedom and hence we expect the EE to present UV divergences. Therefore, we regularize our result by introducing a UV cut-off [9, 10].

Interestingly, since the leading term of the divergence will come from the short-distance correlations, we notice that the degrees of freedom contributing the most to the EE are going to be those near the boundary of the entangling regions. Thus, we can expect the leading term to be proportional to the area of this boundary, paving the way for the area law we will see at the end of this section.

However, this area law does not describe the scaling of EE in general: as we will see in this section and, more precisely, in Chapter 5, for two-dimensional CFT the scaling happens to be logarithmic.

For two-dimensional conformal field theories, the symmetry structure of the theory encourages us to apply the *replica trick* [9]. Here, instead of focusing only on the system, we can compute the Rényi entropy (1.15) of n copies of the system and then take the limit $n \to 1$ to obtain the EE.

Although for higher dimensional conformal field theories the replica method can be applicable only for certain topologies of the entangling region, it is instructive to see how the method is implemented. Our main obstacle is to evaluate the term $\mathrm{Tr}_A\, \rho_A^n$, appearing in the equation (1.15), in the context of quantum field theory. To do so, we rely on the path-integral formalism.

First, we need to define the density matrix in the Euclidean plane (t_E, x) where our $2D$ QFT is defined; the region A of our interest is going to be an interval $x \in [x_1, x_2]$ at $t_E = 0$. The ground state wave functional can be found by performing a path integral from $t_E = -\infty$ to $t_E = 0$ in the Euclidean formalism

$$\Psi(\varphi_0(x)) = \int_{\varphi(t_E=-\infty,x)}^{\varphi_0(x)} D\varphi\, e^{-S(\varphi)}, \tag{1.16}$$

where $\varphi(t_E, x)$ is the field which defines the QFT. Moreover, we indicate with $\varphi_0(x)$ the values of the field at the time $t_E = 0$ where we the region A is defined. Clearly, those values depend only on the coordinate x. The total density matrix is given by two copies of the wave functional, with fields φ_0 and φ_0' respectively,

$$\rho|_{\varphi_0\,\varphi_0'} = \Psi(\varphi_0(x))\bar{\Psi}(\varphi_0'(x)) \tag{1.17}$$

where the complex conjugate $\bar{\Psi}(\varphi_0'(x))$ is obtained by path-integrating from $t_E = \infty$ to $t_E = 0$.

To obtain the reduced density matrix ρ_A, we have to integrate φ_0 on the complement of A with the condition $\varphi_0(x) = \varphi_0'(x)$ if $x \notin A$.

$$\rho_A|_{\varphi_0\,\varphi_0'} = Z^{-1}\int_{t_E=-\infty}^{t_E=\infty} D\varphi\, e^{-S(\varphi)} \prod_{x\in A} \delta(\varphi(0_+,x) - \varphi_+(x))\,\delta(\varphi(0_-,x) - \varphi_-(x))\,, \tag{1.18}$$

where Z is the vacuum partition function introduced to normalize ρ_A, and $\varphi_\pm$ are boundary conditions.

Finally, to compute $\mathrm{Tr}_A\,\rho_A^n$, we prepare n copies of the system described by (1.18) as shown in Fig. 1

$$\rho_A|_{\varphi_{1+}\,\varphi_{1-}}\ \rho_A|_{\varphi_{2+}\,\varphi_{2-}}\ \ldots\rho_A|_{\varphi_{n+}\,\varphi_{n-}}\ , \tag{1.19}$$

and then take the trace. As shown in Figure 1, this is realized by properly

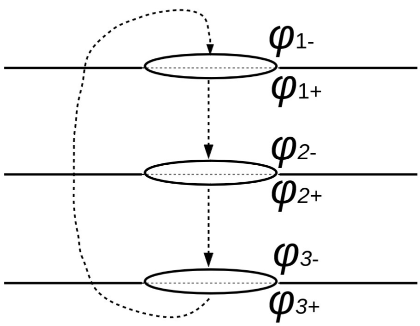


FIGURE 1
An example of replica trick with $n = 3$. The different replicas are glued together by matching the boundary conditions $\varphi_{i\pm}$, the desired quantity is then obtained by integrating over them.

gluing together the multiple copies (i.e. imposing $\varphi_{i-} = \varphi_{(i+1)+}$ as boundary conditions) and then integrating $\varphi_{i+}(x)$. This means that the path integral is now performed on an n-sheeted Riemann surface commonly referred to as $\mathcal{R}_n$

$$\mathrm{Tr}_A\,\rho_A^n = Z^{-n}\int_{(t_E,x)\in\mathcal{R}_n} D\varphi\, e^{-S(\varphi)} \equiv \frac{Z_n}{Z^n}\,, \tag{1.20}$$

where Z_n is the partition function computed on $\mathcal{R}_n$. If we generalize this procedure to higher-dimensional spaces, $\mathcal{R}^n$ is going to be a singular space obtained by gluing n copies of the original space along ∂A, the boundary of the region under investigation.

The singularity of such a space can be seen already in the two-dimensional case. Let's imagine we wish to walk around x_1, one of the extremes of the interval A. Initially, when we rotate around x_1 by an angle of 2π we come back to the starting point. However, since we are taking multiple copies of the system and gluing them together as described, after 2π we simply end on the next copy. Therefore, we now need to rotate by an angle of $2\pi n$ to come back to our original position. We call this singularity a *conical singularity*, located along the surface ∂A with a deficit angle of $2\pi(1-n)$.

Let's have a look at the curvature singularity of this spacetime. We define r as the coordinate expressing the distance from the boundary ∂A (in our case, it is the distance from x_1), and $0 \leqslant \phi \leqslant 2\pi n$ as the angle. To simplify the calculation, we define $\phi = n\psi$ so that the spacetime is described by

$$ds^2 = dr^2 + r^2 n^2 d\psi^2. \tag{1.21}$$

We can now regularize the geometry by smoothing out the tip of the cone, such that

$$ds^2 = dr^2 + f^2(r) d\psi^2\,, \tag{1.22}$$

with $f(r) \sim r$ for $r \to 0$, and $f(r) \sim nr$ for $r \to \infty$. It is easy to see that

$$R = -\frac{2f''}{f}\,, \qquad \text{and} \qquad \int dr\, d\psi\, \sqrt{g}\, R = -4\pi(n-1). \tag{1.23}$$

We thus observe that, regardless of the choice of $f(r)$ (although we require it to be linear away from the tip), the curvature singularity is completely located at the tip of the cone, namely

$$R \sim -4\pi\,\delta^2(x_1)\,(n-1)\,. \tag{1.24}$$

1.2.2 A Holographic Proposal

The analysis of entanglement entropy in strongly coupled quantum systems requires techniques beyond the perturbative regime. However, these obstacles can be overcome by considering a holographic realization of Entanglement Entropy (HEE) originally proposed by Ryu and Takayanagi (RT) in their seminal work [15, 16]. According to their proposal, the entanglement entropy S_A of a region A in a d dimensional boundary theory corresponds holographically to a geometrical quantity, i.e. the area of a co-dimension-2 spacelike minimal surface γ_A in the $(d+1)$-dimensional dual gravity theory. The minimal surface is anchored to the boundary in such a way that it satisfies the homology constraint $\partial\gamma_A = \partial A$, as shown in Fig. 2. The exact statement of their proposal is astonishingly simple and reads as

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}}\,, \tag{1.25}$$

where $G_N^{(d+1)}$ is the $(d+1)$-dimensional Newton constant. The generalization of this formula for asymptotically AdS static spacetimes has been achieved in [17]. Furthermore, the covariant version of the RT proposal for time-dependent background has been formulated in [18].

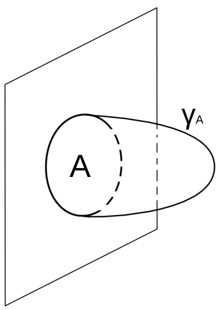


FIGURE 2
The Ryu-Takayanagi proposal: if we want to compute the EE of the region A living on a time slice of the d-dimensional boundary, we need to compute the area of the surface γ_A extending deeper in the bulk where a $d+1$-dimensional gravity theory is defined. The surface is anchored to the boundary satisfying the constraint $\partial\gamma_A = \partial A$.

We saw in (1.24) that, due to the replica trick, we have a geometry with a singularity at the boundary. Following [19], we assume that the codimension-2 surface γ_A defined in the bulk is singular too, presenting the same deficit angle of $2\pi(1-n)$. Therefore, the Ricci scalar will be of the form

$$R = -4\pi\,\delta^2(\gamma_A)\,(n-1) + R^{(0)}, \tag{1.26}$$

where $R^{(0)}$ is the one of the pure AdS_{d+1} spacetime. Looking now at the bulk Einstein-Hilbert action, we observe that

$$\begin{aligned} S_{AdS} &= -\frac{1}{16\pi G_N^{(d+1)}} \int dx^{d+1}\,\sqrt{g}\,(R+\Lambda) \\ &= -n\,S^{vac}_{Ads} - \frac{1}{16\pi G_N^{(d+1)}}\,\big(4\pi(1-n)\big)\int_\gamma \sqrt{g}, \end{aligned} \tag{1.27}$$

where S^{vac}_{Ads} is the vacuum contribution and it is simplified once that we normalize in order to obtain $\mathrm{Tr}\,\rho = 1$. Therefore, we easily compute that

$$\mathrm{Tr}\,\rho_A^n = \exp\left(\frac{1}{4G}\int_\gamma \sqrt{g}\,(1-n)\right), \tag{1.28}$$

which yields

$$\lim_{n\to 1}\frac{1}{1-n}\log\,\mathrm{Tr}\,\rho_A^n = \frac{1}{4G}\int_\gamma \sqrt{g}\,. \tag{1.29}$$

The action principle in the gravity theory requires γ_A to be the surface that minimizes the area, thus giving an intuitive proof of the validity of the formula (1.25).

The assumptions here taken are valid in three-dimensional pure gravity since a solution of the Einstein equation should be locally equivalent to AdS_3 [20]. However, the generalization to higher dimensions or to more complicated theories is not obvious.

Recently, a holographic proof of the RT proposal has been expounded by Lewkowycz and Maldacena (LM) in [21]. The essence of the proof is established by implementing the n-copy replica trick in the dual bulk geometry. The metric of this replicated bulk geometry acquires a Z_n singularity on the hypersurface. The powerfulness of this method reveals that, by imposing the limit $n \to 1$, the hypersurface converges to the usual minimal surface in the RT proposal.

To sum up, we overcome the technical difficulties of computing the logarithm of a matrix in equation (1.13) by implementing the replica trick (1.15). Finding ourselves in trouble to analyze more complicated systems, we recur to an holographic technique that simplifies the problem by reducing to the computation of the area of a specific surface (1.25).

Simplicity is not the only wonderful feature of this proposal and not even the most interesting one. Indeed, one can interpret the equation (1.25) as an indication that the quantum properties of matter are deeply related to a geometrical object on the other side of the correspondence. Such indication yields a fascinating perspective in the context of emergent spacetimes [22, 23].

In this thesis we will focus on theories of gravity in three dimensions. Here, the procedure is facilitated even more by the fact that the quantity to be computed will be just the length of a line. The peculiarity of this case is observed also on the other side of the correspondence where a two-dimensional CFT lives. As anticipated, the EE in such theories do not respect an area law and takes the form

$$S_A = \frac{c}{3} \log\left(\frac{\ell}{\epsilon}\right) , \tag{1.30}$$

where c is the central charge, ℓ the size of the entangling region, and ϵ the UV cut-off. Further details and an interesting extension of this result will be given in chapter 5.

One more consideration before concluding this introductory chapter. It is evident from eq. (1.25) that it is structurally very similar to the Bekenstein-Hawking (BH) formula for the black hole entropy. Interestingly, this striking similarity was one of the primary inspirations to the authors of [15]. However, the fact that the EE is proportional to the number of matter fields and is ultraviolet divergent substantially differentiates the nature of the RT proposal from the BH formalism. Furthermore in [24], Jacobson gives a more comprehensive connection between the two formalisms by stating that the entanglement entropy describes the quantum correction to the black hole entropy in the presence of matter fields.

2 The Laboratory: New Massive Gravity

In Chapter 1, we have introduced the tools that we will use in this thesis. The only thing left to be done before using them is to set up a place where we can expect to observe something interesting: a laboratory.

In our case, such place is not a room full of machinery, nor a telescope in some exotic place. The laboratory we will need is a model describing a theory of massive gravity called *New Massive Gravity*.

In this chapter, we will walk through the conceptual steps that led to the formulation of New Massive Gravity (NMG). Then, we shall present the theory and its properties, exploring the implications that such theory brings with it, and thus completing the set up of our laboratory.

2.1 General Relativity

Gravity fascinates philosophers and scientists since the ancient times. Humankind is naturally inclined to wonder and question the reality that surrounds us, in particular why some of the stars in the sky appear to be moving. The search for the answer went through a series of twists and turns in our understanding of Nature, crushing, at times violently, with the existent knowledge and the religious beliefs.

The first concrete attempt of grasping the nature of gravity can be found in the revolutionary work of Sir Isaac Newton of 1687, *Philosophiæ Naturalis Principia Mathematica*. His law of universal gravitation succeeded in describing the motion of the planets observed with the technology available at that time. Moreover, observing some changes in the orbit of Uranus, the predictions of this law were crucial for the discovery of Neptune. However, two centuries later, the technology allowed the astronomers to notice irregularities in the orbit of Mercury, paving the way to questioning the range of validity of Newton's law predictions. Another issue that was not addressed by Newton and puzzled the scientists for centuries (and it still does) is questioning the very nature of an interaction affecting everything in the universe.

The history of our understanding of the gravitational interaction is the perfect example of the natural conflict between the scientific method and the principle of authority. Notwithstanding, the final steps that led to the modern description of gravitation, namely *General Relativity* (GR), were not originated by the necessity of challenging an existent theory to explain a new observation.

In 1905, Albert Einstein presented the theory of special relativity, providing a theoretical framework where Maxwell's equations of electromagnetism were describing the same physics in every inertial frame. The theory was a radical change of paradigm that revolutionized, among other things, our concepts of time and space. Indeed, we can't treat time and space separately anymore, being them the interconnected parts of a *spacetime* with 3 spatial dimensions and 1 time dimension.

Two years later, he began his attempt of extending the principle of relativity to physical systems where you can't define an inertial frame as everything is accelerated by the gravitational force. The theory was finally presented in 1915 and shaped our modern understanding of gravitation: not an instantaneous interaction violating the principles of special relativity anymore, but rather an interplay between matter and the structure of spacetime. The presence and the motion of matter bend the spacetime, curving it and changing its geometry, and the curvature of the spacetime determines the motion of matter.

The driving force of this theory is the *equivalence principle* that states that effects of gravity and acceleration are indistinguishable. The mathematical structure of GR is rather challenging and the most appropriate reference to have a comprehensive understanding of the subject is [25]. Nevertheless, we will try to have a glimpse of its elegance.

In Newtonian gravity, the dynamical variable is the gravitational potential. On the other side, in GR the dynamical variable is the object encoding the geometrical properties of the spacetime: the metric $g_{\mu\nu}$. This object obeys a set of second order differential equations called *Einstein's equations* that, in natural units, read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = T_{\mu\nu}\,. \tag{2.1}$$

These equations represent the interplay between matter and geometry. On the left-hand side, we find the Einstein tensor $G_{\mu\nu}$, expressing the geometry of the spacetime and its dynamics. This is done through the particular, divergence-free, combination of the metric and the Ricci tensor, which is composed of derivatives of the metric. On the other side, we have the stress-energy tensor $T_{\mu\nu}$ that describes the distribution and the motion of matter. We thus observe how geometry and matter are connected and influence each other.

Being constructed with tensors, GR is characterized by a manifest general covariance: the physics described and its predictions do not depend on the coordinate system. Furthermore, there is no reference to a preferred invariant background structure. Therefore, the theory expresses the deep concept that the laws of physics are the same for every observer. A local realization of this concept is the equivalence principle mentioned before.

It is interesting to notice that in the small-curvature regime, thus for weak gravitational fields and for speeds significantly smaller than the speed of light, the predictions of the theory coincide with the ones given by Newton's law. Moreover, the predictions of the theory clarified the observations on Mercury's orbit, thus far unexplained by Newton's law.

Another consequence of this theory is that, if the spacetime is in fact curved by the presence of matter, we should observe that even a light ray has to be deflected by a gravitational field. This is because light should cover the shortest distance between two points and, in a curved spacetime, this path is described by a geodesic that is not necessarily a straight line. Sir Arthur Eddington verified this prediction during a solar eclipse in 1919 by observing how the light of stars was bent by the gravitational field of the Sun.

Once we admit that light is affected by the presence of matter due to the curvature of spacetime, we can imagine regions of spacetime where the curvature is so high that not even light can escape. In fact, this is arguably the most fascinating prediction of GR that allow the existence of such regions, called *black holes*. The name derives precisely from the impossibility of light to escape from them. The surface confining this region is called *event horizon* and, according to GR, everything crossing this surface is doomed to remain in the black hole. Despite this unhappy fate, an infalling observer will never notice, from local observations, if he or she is crossing the event horizon.

Black holes are also characterized by the presence of a gravitational singularity, a region of the spacetime with infinite curvature. Although the appearance of a singularity indicates a breakdown of the theory, this is actually the beginning of the most exciting challenge for theoretical physics. At those distances and energies, one cannot ignore the quantum effects anymore and we thus need a theory that combines quantum and gravitational effects. This question is still unanswered and encodes all the interest that these mysterious objects attracted in the last sixty years.

The first example of a black hole[1] came from Karl Schwarzschild a few months after Einstein's publication, while he was serving at the front in Russia during WWI. It is a spherically symmetric solution of the form (in natural units)

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{2.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the angular part of the metric, and $r_s = 2M$ is the Schwarzschild radius of an object of mass M. It might appear that the metric is singular at $r = r_s$ but, as Eddington showed in 1924 [26], the singularity disappears by changing coordinates. The only physical singularity is located at $r = 0$, where the curvature goes to infinity.

Another astonishing prediction of this elegant theory is the existence of *gravitational waves*. Indeed, by studying how perturbations of the geometry propagate, one observes that they obey a wave equation. If in Newtonian gravity the interaction was transmitted instantaneously, in GR it propagates at the speed of light as a wave in the very fabricate of the spacetime. Due to

[1] Actually, Schwarzschild's solution is the first exact solution to Einstein's equations.

the fact that this physical effect is very small and difficult to be detected, we had to wait till 2015 to have the first experimental evidence of the existence of such waves [27].

Around the same time that Einstein developed and published his field equations for the metric, the mathematician David Hilbert got interested in Einstein's work and published almost simultaneously an action from which one can derive those field equations. The action reads

$$S = \int d^4x \sqrt{g} \left(\alpha R + \beta + \mathcal{L}_{\text{matter}} \right) , \tag{2.3}$$

where $g = \det(g_{\mu\nu})$, the first term (now called Einstein-Hilbert term) gives rise to Einstein's equation, and $\mathcal{L}_{\text{matter}}$ is the matter contribution. Hilbert, respecting the postulates of GR, included also another term. The effect of this term is to modify Einstein's equations by introducing what nowadays we call the *cosmological constant* Λ,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu} \, . \tag{2.4}$$

This constant is the value of the density of energy of vacuum. Therefore, its presence curves the spacetime even in the absence of matter.

The history of this constant is linked with our knowledge of the structure of the universe. Einstein initially gave it a physical interpretation and introduced it because his equations did not allow a static universe as a solution since gravity would have caused the universe to contract. He then rejected the possibility and removed the constant when Hubble observed in 1929 that the universe was expanding. However, with the observation of a distant supernova in 1998, developments in the field of observational cosmology allowed (among other great discoveries) to detect an acceleration in the expansion of the universe, a result compatible with the presence of a positive cosmological constant in the field equations.

Understanding the nature and the value of the cosmological constant is one of the big open problems in cosmology. Observations seem to indicate a small positive value for it, while theoretical predictions based on quantum field theory give a much larger value. The discrepancies go from 40 to 100 orders of magnitude, depending on the assumptions.

One recurrent spacetime in this thesis will be the Anti-de Sitter (AdS) spacetime. It is a maximally symmetric solution of the Einstein's equations with a negative cosmological constant. It can be expressed in the Poincaré coordinate patch as

$$ds^2 = \frac{L^2}{r^2}\big(- dt^2 + dr^2 + dx^2 + dy^2\big)\,, \tag{2.5}$$

where L is the AdS radius. If the de Sitter spacetime, the analogous solution with a positive cosmological constant, has a curvature that induces an expansion, the AdS spacetime describes a situation where two points will tend to attract each other over time.

As we have seen in the introduction, this spacetime is of a great utility and interest in the context of holography and will appear as a solution in Chapter 3 and as a starting point in Chapter 5.

To summarize, there are many open questions when gravity is involved in our studies. There are difficulties related to the observational aspect: since very precise measurements are required, we have to constantly develop our technology accordingly and we saw that, in some cases, the wait can be even a century long. Other issues address a more abstract level. The ability of a theory in making verifiable predictions gives, in a sense, a measure of how good the theory is and any discrepancy reveals how little we know about the history of our universe, regardless the impressive amount and quality of the discoveries.

In particular, all these issues and questions are connected by the necessity of finding an underlying fundamental theory that reveals the true nature of this interaction. Any effort of overcoming the open issues is, directly or indirectly, aimed towards the seeking of the answer to this fundamental question. One possibility is to create theoretical models that simplify the problem and allow us to have some insights and thus face such a big challenge.

2.2 Three-Dimensional Gravity

Thus far we have seen a few aspects of GR and had a glimpse to some famous solutions to the Einstein's equations. It is easy to notice that such solutions describe a four-dimensional world. This is because GR was created to describe the universe as we perceive it: a four-dimensional spacetime, with three spatial dimensions and one time dimension.

However, the principles of General Relativity do not refer to a specific dimensionality. Nothing forbids us to study spacetimes where we have at least a spatial and a time dimension. The solutions will vary dramatically for different dimensions and, in some cases, will describe intriguing physical situations.

To overcome some technical obstacle, deriving from the mathematical structure of differential equations in four dimensions, one can *reduce* the number of dimensions and study a $(2+1)$-dimensional spacetime. The price to pay for such simplification of the equations is that all the cosmological motivations mentioned in the previous section are now removed since any model constructed with a three-dimensional gravity theory will not be realistic. While for many people this could be a reason to not analyze such models, there are several advantages to be taken into consideration.

The first consequence of lowering the number of dimensions is that we now have a nice playground to test ideas, face simpler equations, and thus perform exact computations within the desired gravitational model. Although there is no guarantee that a three-dimensional model will still be valid in the more realistic four-dimensional case, the results can be used to extract insights on what is worth trying, what kind of result one could expect, and potentially reveal some fundamental information. The most direct example that can be found in this thesis is the computation performed in Chapter 5 to find the entanglement entropy that, in some cases, can be even performed by hand.

The second advantage of working with a three-dimensional gravity theory is that, when the manifold admits a boundary, we can study a two-dimensional conformal field theory on this boundary. This can be extremely useful in the context of studying the validity and the limitation of the holographic principle, as we have seen in the introduction.

Gravity in $(2+1)$ dimensions is peculiar and this can be seen from the symmetry properties of the Riemann tensor. This tensor can be decomposed, in D dimension, in the following way

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{D-2}\big(g_{\mu[\rho}\,R_{\sigma]\nu} - g_{\nu[\rho}\,R_{\sigma]\mu}\big) - \frac{2}{(D-1)(D-2)}\,R\,g_{\mu[\rho}\,g_{\sigma]\nu}\,, \tag{2.6}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor that vanishes in three dimensions. Therefore, setting $D=3$, we are left with simply

$$R_{\mu\nu\rho\sigma} = 2\big(g_{\mu[\rho}\,R_{\sigma]\nu} - g_{\nu[\rho}\,R_{\sigma]\mu}\big) - R\,g_{\mu[\rho}\,g_{\sigma]\nu}\,. \tag{2.7}$$

We thus see that the Riemann tensor is completely determined by the Ricci tensor. Being a symmetric two tensor, we can conclude that the Riemann tensor has the same amount of independent components as the Einstein tensor. Those components are completely fixed by the Einstein's equations and thus we can say that *gravity does not propagate any local degrees of freedom* in three dimensions.

This restriction seems to be too heavy to make three-dimensional gravity an interesting theory. However, as soon as we allow the existence of a non-vanishing cosmological constant, interesting geometries appear among the solutions of the theory. In fact, local triviality does not forbid the theory to display surprising properties when we look at the global structure of the spacetime.

An interesting example of that was first discovered by Bañados, Teitelboim, and Zanelli (BTZ) [28]. If we consider GR with a negative cosmological constant, the theory admits a black hole solution. The line element of a rotating BTZ black hole is given by

$$\begin{aligned} ds^2 = & - \left(-M + \frac{r^2}{L^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{L^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 \\ & + r^2 \left(d\phi - \frac{J}{2\,r^2}\,dt\right)^2, \end{aligned} \tag{2.8}$$

where $L^2 = -1/\Lambda$ is the AdS radius, M is the mass of the black hole, and J is its angular momentum. The BTZ black hole is locally isomorphic to the AdS spacetime, but it exhibits the global properties of a black hole. Indeed, we can identify the presence of an event horizon at $r = r_+$ and of an inner horizon at $r = r_-$. The presence of two horizons is typical of a rotating black hole and their location can be found by looking for the singularities in the metric. A simple calculation gives

$$r_{\pm} = L\left[\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{M\,L}\right)^2}\right)\right]^{\frac{1}{2}}, \tag{2.9}$$

or, in other words, the mass and the angular momentum of the black hole determine the size of the horizons as [28]

$$M = \frac{r_+^2 + r_-^2}{L^2}, \qquad\qquad J = \frac{2\,r_+\,r_-}{L}. \tag{2.10}$$

The condition for the existence of the two horizons is that $|J| \leqslant ML$, where the equality means that the two horizons merge and we have a so-called extremal black hole. The constraint is imposed in order to avoid the presence of black holes without any horizon, i.e. naked singularities. These objects represent a conceptual challenge and it is conjectured (cosmic censorship hypothesis [29]) that no realistic process can produce such singularity, i.e. a horizon should form to hide it. If we wish to study a non-rotating black hole, we can set $J = 0$ and the inner horizon disappears.

Extremal black holes will appear as solutions of the theory constructed in Chapter 3. In Chapter 4, we will have the opportunity of studying these objects with a richer geometrical structure. In order to finally focus on our laboratory, New Massive Gravity, we first need to briefly explore the field of Massive Gravities.

2.3 Massive Gravity

We have seen in the previous section that some of the challenges one encounters in the study of the gravitational interaction can be softened by lowering the number of dimensions. The price to be paid in order to work with more manageable equations is the loss of propagating local degrees of freedom.

Since one of the goals of studying such theories is to have insights on the four-dimensional theory, which presents propagating local degrees of freedom, this oversimplification makes the model rather irrelevant for this purpose. Instead of discharging the model, we can modify it by adding degrees of freedom and richness to our theory.

In particular, we want a theory that describes a *massive* graviton because this will introduce extra degrees of freedom (see Table 1). We have already seen in the introduction some physical reasons to consider a theory of massive gravity but, for the purposes of this discussion, the main goal of this deformation of GR is to introduce some richness in the dynamics that the theory describes.

	Off-Shell	On-shell
Massless	$D(D-1)/2$	$D(D-3)/2$
Massive	$D(D+1)/2$	$D(D-1)/2-1$

TABLE 1
The counting of degrees of freedom of a spin-2 particle in D dimensions. Notice how a massless spin-2 particle does not propagate any degree of freedom in three dimensions and how the situation is improved by considering a massive particle.

We talk about a deformation of GR because, in the search of a more general theory of gravity, any model should naturally maintain the good features of GR. As General Relativity reduces to Newtonian gravity in the non-relativistic regime, this kind of deformations should reduce to GR in the regimes where it has been proven (via observations and experiments) to be correct. The parameter controlling the deformation is the mass parameter and we expect to reproduce the results predicted by GR if we look at the regime where it is arbitrarily small. This, as we will see, turns out to be not always the case.

The difficulties in constructing a theory of massive gravity arise as soon as one tries to introduce a mass term in the action. This can be done in two ways: by introducing an explicit mass term with an extra two-tensor, or by introducing terms containing higher derivatives of the metric. Although our interest is the study of New Massive Gravity, a higher-derivative theory, we will briefly explore the first possibility because it can clarify some issues related to massive gravities.

2.3.1 Explicit Mass Term

A common term we are used to seeing in actions describing massive particles is some kind of contraction of the fields describing them. For example, for a vector field A_μ (spin 1) we usually take the contraction $m^2 A_\mu A^\mu$ to obtain an explicit mass term. In the case of a gravitational theory, however, the dynamical field is the metric $g_{\mu\nu}$ and this approach does not work: the contraction $g_{\mu\nu}g^{\mu\nu}$ is just a constant.

In 1939, Fierz and Pauli wrote down a theory for spin-2 particles with a non-vanishing mass [30] in the flat Minkowski spacetime. If we start from the action of GR

$$S = \frac{1}{16\pi\, G} \int d^4x \, \sqrt{g}\, R \,, \tag{2.11}$$

and we consider small fluctuations $h_{\mu\nu}$ of the metric around the Minkowski spacetime, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the same action will read, at second order, as

$$S = \frac{1}{16\pi\, G} \int d^4x \, \mathcal{L}^{(2)}_{EH} \,, \tag{2.12}$$

where

$$\mathcal{L}^{(2)}_{EH} = \frac{1}{4} \left(\partial^2 h_{\mu\nu} + 2\, \partial^\rho \, \partial_{(\mu} h_{\nu)\rho} + 2\, \partial_{(\mu}\partial_{\nu)} h - h\, \partial^2 h \right) \, h^{\mu\nu}, \tag{2.13}$$

and h is the trace of $h_{\mu\nu}$. It is interesting to notice that this action inherits from the full theory a linearized diffeomorphism invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$ that will reduce the number of degrees of freedom.

Taking this linearized version of the Einstein-Hilbert action, we can add an explicit mass term to obtain the Fierz-Pauli (FP) action, namely

$$\mathcal{L}_{FP} = \mathcal{L}^{(2)}_{EH} - \frac{1}{4} m^2 \, (h_{\mu\nu}h^{\mu\nu} - h^2). \tag{2.14}$$

The mass term breaks the gauge symmetry of linearized GR mentioned above, thus altering the counting of degrees of freedom: in the four dimensional case, the degrees of freedom goes from 2 (massless case) to 5 (massive case), as Table 1 shows.

The mass term in equation (2.14) is not the only possibility we have, a priori. However, the relative factor between $h_{\mu\nu}h^{\mu\nu}$ and h^2 is the result of a fine tuning aimed to the description of a massive spin-two particle. A different tuning, in fact, will influence the counting of degrees of freedom. To see that, let us consider a generic tuning

$$\mathcal{L}_{FP} = \mathcal{L}^{(2)}_{EH} - \frac{1}{4}m^2\,(h_{\mu\nu}h^{\mu\nu} - \alpha h^2), \tag{2.15}$$

where α is a parameter governing the tuning. The equations of motion are

$$m^2\,(\partial^\nu\, h_{\mu\nu} - \alpha\partial_\nu\, h) = 0\,, \tag{2.16}$$

and, taking the divergence and its trace, we obtain the following two equations

$$m^2\,(\partial^\nu\partial^\mu\, h_{\mu\nu} - \alpha\Box\, h) = 0\,, \qquad \Box h - \partial^\mu\partial^\nu\, h_{\mu\nu} - \frac{4\alpha - 1}{2}m^2 h = 0. \tag{2.17}$$

When $\alpha = 1$, thus in the case of the FP mass term, these three equations constraint $h_{\mu\nu}$ to be traceless and divergence-free. In this way we reduce the number of degrees of freedom to the right amount in order to describe a massive spin-two particle (5 in the four-dimensional case). However, without imposing $\alpha = 1$ there is no way to impose the tracelessness condition, revealing the presence of an extra degree of freedom, which turns out to be a scalar ghost, i.e. a field with negative kinetic energy that leads to instabilities at classical level and to non-unitarity at the quantum level.

As we mentioned before, we would expect the FP theory to reproduce the results of GR in the limit where the mass parameter goes to zero. However, this is not the case if you consider the predictions made by the two theories of the light-bending angle: the disagreement is of about 25%. Moreover, taking the massless limit of FP coupled to a conserved energy-momentum tensor does not lead to GR, but rather to the linearized Einstein gravity with extra degrees of freedom. This is the so-called vDVZ discontinuity, named after its discoverers van Dam, Veltman, and Zakharov [31, 32], which is related to the difference in the gauge symmetries that the two theories respect.

This issue can be solved by considering nonlinear extensions of the theory, which should cure the discontinuity. However, it was shown be Boulware and Deser that any nonlinear completion of the FP theory leads to a ghost [33]. To have a complete review of the models constructed to overcome this problem, see for example [34].

2.3.2 3D Topologically Massive Gravity

The previous discussion on the emergence of ghost fields gives us a further motivation for concentrating on three-dimensional theories. Since a massless graviton does not propagate any degree of freedom (Table 1), any potential ghost mode connected to such field would be harmless[2].

We will focus on theories describing a massive graviton where the necessary extra degrees of freedom are introduced using terms containing higher derivatives of the metric. The main motivation to do so has to be searched in the renormalizability properties of such theories. Since this discussion goes beyond the scope of this thesis, the curious reader can further investigate this motivation in [7].

The simplest example of such theories in three dimensions is the so-called Topologically Massive Gravity (TMG) [35, 36], where the name is coming from the fact that the extra term is constructed entirely out of the connection $\Gamma^{\sigma}_{\mu\nu}$. The theory has a third derivative, parity non-invariant action given as

$$\begin{aligned} I &= \frac{1}{16\pi G}\int d^3x\sqrt{-g}\Big[\sigma\Big(R-2\Lambda\Big) \\ &\qquad +\frac{1}{2\mu}\,\eta^{\mu\nu\alpha}\Gamma^{\beta}{}_{\mu\sigma}\Big(\partial_\nu\Gamma^{\sigma}{}_{\alpha\beta}+\frac{2}{3}\Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\alpha\beta}\Big)\Big], \end{aligned} \tag{2.18}$$

where σ is dimensionless and μ has the dimension of mass.

TMG has a single massive spin-2 excitation (with +2 helicity for $\mu > 0$) in the bulk with a mass-squared given as

$$m_g^2 = \mu^2\sigma^2 + \Lambda. \tag{2.19}$$

In the $\Lambda \to 0$ limit, the single massive degree of freedom remains intact, with a mass $m_g = |\mu\sigma|$ and a positive kinetic energy as long as $\sigma < 0$, which is opposite to the one used in Einstein's theory.

[2] This was, historically, the first reason to look at three-dimensional theories, rather than the motivations here presented.

Since one of the central topics of this thesis is holography, an interesting analysis involves the study of the boundary theory dual to such theory. In particular, we want a theory where unitarity in the bulk is compatible with the unitarity conditions of the dual CFT on the boundary. To achieve that, a necessary condition is the positivity of the central charge.

Imposing the Brown-Henneaux [37] boundary conditions[3] for asymptotically AdS_3 spacetime leads to two copies of the Virasoro algebra with the left and right central charges given as

$$c_{L,R} = \frac{3L}{2G}\left(\sigma \mp \frac{1}{\mu L}\right), \qquad \Lambda \equiv -\frac{1}{L^2}. \tag{2.20}$$

In the $\mu \to \infty$ limit, these reduce to the ones given in [37] for the pure cosmological Einstein's theory with the choice $\sigma = 1$. It was shown in [38] that the positivity of the energy of the bulk excitations with Brown-Henneaux boundary conditions requires the theory to be in the so-called chiral limit, where $\sigma^2\mu^2 = 1/L^2$. However, a closer investigation performed in [39] showed that the theory in the chiral limit allows log-modes as solutions which have finite but negative energy, albeit with weaker boundary conditions [40].

To summarize, TMG suffers from the so-called *bulk-boundary clash*: there is no region of the parameter space where both the bulk energy excitations and the boundary central charges are positive at the same time. The only exception is the chiral point where one of the central charges vanishes. At this point, new problematic modes with negative energy appear in the spectrum. This has led to the conjecture that TMG at its chiral point is dual to a logarithmic conformal field theory (LCFT) [41–43].

[3]The asymptotic symmetries are crucial for the definition of the global charges of a theory. The authors of [37] showed that the global charges of a gauge theory may lead to nontrivial extensions of the asymptotic symmetry algebra.

2.4 New Massive Gravity

We finally have all the elements we need to set up our laboratory and thus dedicate this section to the formulation of New Massive Gravity [44]. Following [45], we begin by taking the Lagrangian of the Fierz-Pauli theory (2.14), which leads to the equations of motion

$$(\Box - m^2) h_{\mu\nu} = 0 \,, \tag{2.21}$$

and two conditions that constraint $h_{\mu\nu}$ to be traceless and divergence-free, i.e.

$$h = 0 \,, \qquad\qquad \partial^\mu h_{\mu\nu} = 0 \,. \tag{2.22}$$

A key component of the FP Lagrangian (2.14) is the linearized Einstein tensor

$$G^{(1)}_{\mu\nu} = \frac{1}{2} \left(\partial^2 h_{\mu\nu} + 2\, \partial^\rho \, \partial_{(\mu} h_{\nu)\rho} + 2\, \partial_{(\mu} \partial_{\nu)} h - h\, \partial^2 h \right) \,, \tag{2.23}$$

and we can use this object to solve for the constraint by increasing the number of derivatives. Namely, we express $h_{\mu\nu}$ in terms of a new spin-two field $h'_{\mu\nu}$ by substituting

$$h_{\mu\nu} = G^{(1)}_{\mu\nu}(h') \,. \tag{2.24}$$

An immediate consequence is that the divergence-free condition becomes redundant, being it automatically implied, and we are left with the equations

$$(\Box - m^2) G^{(1)}_{\mu\nu}(h') = 0 \,, \qquad\qquad G^{(1)}(h') = 0 \,, \tag{2.25}$$

where $G^{(1)}(h')$ is the trace of $G^{(1)}_{\mu\nu}(h')$. We can thus interpret these equations as the equations of motion for a metric perturbation $h'_{\mu\nu}$ in a four-derivative theory.

As we have seen in equation (2.7), the trace of the linearized Einstein tensor is proportional to the linearized Ricci scalar $R^{(1)}$. Therefore, we can rewrite both equations into a unique equation of motion given by

$$G^{(1)}_{\mu\nu}(h') - \frac{1}{m^2} \left[\Box G^{(1)}_{\mu\nu}(h') - \frac{1}{4} \left(\partial_\mu \, \partial_\nu \, - \eta_{\mu\nu} \, \Box \right) R^{(1)}(h') \right] = 0 \,. \tag{2.26}$$

Such theories, as we have seen in the previous section, suffer of the vDVZ discontinuity and we mentioned that the issue can be cured by considering nonlinear extensions of the theory. In other words, we want an action that reproduces these equations of motion at the linear level. The action was found in 2009 by Bergshoeff, Hohm and Townsend and takes the name of New Massive Gravity (NMG)

$$\begin{aligned} S_{NMG} &= \frac{1}{\kappa^2}\int d^3x\sqrt{g}\left[\sigma R - 2\lambda m^2 + \frac{1}{m^2}K\right], \\ & K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2, \end{aligned} \tag{2.27}$$

where $[\kappa] = -1/2$ in fundamental units, and $\sigma = \pm 1$. In order to obtain the equations of motion (2.26) at the linear level, we must set the parameters $\sigma = -1$ and $\lambda = 0$.

In opposition to TMG (section 2.3.2), NMG is a parity-preserving theory propagating two massive modes with helicites ± 2. Being a four-derivative theory, one may expect to see more modes propagating but the massless spin-two mode does not propagate since we are in three dimensions. This is a fortunate circumstance since, by choosing $\sigma = -1$, we have the Einstein-Hilbert term with the 'wrong' sign. Moreover, the tuning of the relative coefficient in K has the purpose of eliminating a massive spin-0 mode. All these features concur to the result that this theory is remarkably ghost-free.

The equations of motion for NMG are

$$2m^2\, G_{\mu\nu} + K_{\mu\nu} = 0\,, \tag{2.28}$$

where $K_{\mu\nu}$ is the contribution coming from the higher derivatives and it is given by

$$\begin{aligned} K_{\mu\nu} &= 2\,\Box R - \frac{1}{2}\left[\partial_\mu\partial_\nu R + g_{\mu\nu}\,\Box R\right] - 8R_\mu{}^\rho R_{\rho\nu} \\ &\quad + \frac{9}{2}R\,R_{\mu\nu} + g_{\mu\nu}\left[R_\mu{}^\rho R_{\rho\nu} - \frac{13}{12}R^2\right]. \end{aligned} \tag{2.29}$$

We can look for maximally symmetric solutions of this theory, i.e. vacua such that

$$G_{\mu\nu} = -\Lambda\, g_{\mu\nu}\,, \tag{2.30}$$

for which we have the de Sitter spacetime if $\Lambda > 0$, and the anti-de Sitter solution for $\Lambda < 0$. As it is shown in [44], such configurations solve the equations of motion if the following equation is satisfied

$$\Lambda^2 + 4m^2\sigma\Lambda - 4\lambda m^4 = 0\,, \tag{2.31}$$

or, in other words, if

$$\Lambda = -2m^2\left[\sigma \pm \sqrt{1+\lambda}\right]\,. \tag{2.32}$$

In this way, we can find a range of parameters to have a Minkowsky, dS, or AdS vacua.

The main reason why this theory is of a great interest for the purposes of this thesis is that, although the theory is three dimensional and thus provides a simplified scenario, it admits a great variety of solutions. In other words, regardless of the fact that it lives in lower dimensions, it can still produce rich geometries [45] that will be very useful for our purposes. We will see a supersymmetric extension of the theory in Chapter 3, a classification of the solutions of our interest in Chapter 4, and use some of those geometries in Chapter 5 in the context of a holographic calculation of the Entanglement Entropy.

2.4.1 Bulk Modes

We have the possibility of reducing the number of derivatives from four to two by introducing an auxiliary symmetric tensor $f_{\mu\nu}$ [44]. The resulting second-order action for NMG is then

$$S[g,f] = \frac{1}{\kappa^2}\int d^3x\sqrt{g}\left[\sigma R - 2\lambda m^2 + f^{\mu\nu}G_{\mu\nu} - \frac{m^2}{4}\left(f^{\mu\nu}f_{\mu\nu} - f^2\right)\right]\,. \tag{2.33}$$

The action (2.27) is then recovered after eliminating f by using its algebraic field equations

$$f_{\mu\nu} = \frac{2}{m^2}S_{\mu\nu} \qquad\qquad S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}R\,g_{\mu\nu}\,, \tag{2.34}$$

where $S_{\mu\nu}$ is called Schouten tensor.

In order to study the holographic properties of this theory, we can observe what happens at the linearized level when we expand about a maximally-symmetric vacuum of the theory. In other words, we write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\,h_{\mu\nu} \tag{2.35}$$

where $h_{\mu\nu}$ is the small perturbation about the background metric $\bar{g}_{\mu\nu}$. The bar quantities will always refer to the background metric and, to expand about a maximally-symmetric background, we set

$$\bar{G}_{\mu\nu} = -\Lambda\, \bar{g}_{\mu\nu}\,. \tag{2.36}$$

We can also expand the auxiliary field by setting

$$f_{\mu\nu} = \frac{1}{m^2}\Big[\Lambda\,(\bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu}) - \epsilon\, k_{\mu\nu}\Big] + \mathcal{O}(\epsilon^2)\,, \tag{2.37}$$

where $k_{\mu\nu}$ is an independent symmetric tensor expressing the fluctuation.

As we have seen in the previous section, it is important to not change the set of gauge symmetries governing the theory. Indeed, the full nonlinear theory is invariant under

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= 2\, D_{\{\mu}\xi_{\nu\}}\,, \\ \delta_\xi f_{\mu\nu} &= \xi^\rho \partial_\rho\, f_{\mu\nu} + 2\,\partial_{\{\mu}\xi^\rho f_{\nu\}\rho}\,, \end{aligned} \tag{2.38}$$

where D_μ is the full covariant derivative. We can then expand the diffeomorphism parameter following $\xi_\mu = \bar{\xi}_\mu + \epsilon\,\zeta_\mu$. Since the background metric is non-dynamical and we want to keep $\bar{g}_{\mu\nu}$ invariant, we constraint the diffeomorphisms such that they are also isometries of $\bar{g}_{\mu\nu}$. The constraint is thus translated in the request that $\bar{\xi}_\mu$ has to be a background Killing vector field. The expansion of the auxiliary field was constructed such that $k_{\mu\nu}$ is gauge invariant. On the other hand, the metric fluctuations are affected by a gauge transformation following

$$\delta_\zeta\, h_{\mu\nu} = 2\,\nabla_{\{\mu}\,\zeta_{\nu\}}\,, \tag{2.39}$$

where ∇_μ is the covariant derivative with respect to the background metric. Under these gauge transformations, the linearized Einstein tensor is not invariant, but rather requires a combination of terms appearing in the linearized field equations, namely

$$\mathcal{G}_{\mu\nu}(h) \equiv G^{(1)}_{\mu\nu}(h) + \Lambda\, h_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)}\,\bar{g}_{\mu\nu} - 2\Lambda\, h_{\mu\nu} + \Lambda\, h\, g_{\mu\nu} \tag{2.40}$$

We can finally write down the linearized action of NMG with this parametrization and obtain

$$\begin{aligned} \mathcal{L}^{(2)}_{NMG} &= \frac{(\Lambda - 2m^2\sigma)}{4m^2} h^{\mu\nu}\mathcal{G}_{\mu\nu}(k) \\ &- \frac{1}{m^2} k^{\mu\nu}\mathcal{G}_{\mu\nu}(h) - \frac{1}{4m^2}(k^{\mu\nu}k_{\mu\nu} - k^2). \end{aligned} \tag{2.41}$$

If $\Lambda \neq 2m^2\sigma$ we can decouple the fields h and k by a simple field redefinition

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{2\, k_{\mu\nu}}{\Lambda - 2m^2\sigma}\,, \tag{2.42}$$

which yields

$$\begin{aligned}\mathcal{L}^{(2)}_{NMG} &= \frac{(\Lambda - 2m^2\sigma)}{4m^2}\bar{h}^{\mu\nu}\mathcal{G}_{\mu\nu}(\bar{h}) \\ &- \frac{k^{\mu\nu}}{m^2(\Lambda - 2m^2\sigma)}\mathcal{G}_{\mu\nu}(k) - \frac{1}{4m^2}(k^{\mu\nu}k_{\mu\nu} - k^2).\end{aligned} \tag{2.43}$$

Here, the first term can be ignored since it is the Einstein-Hilbert action linearized about the vacuum and does not propagate any degree of freedom. The other terms, in the second line of the equation, are more interesting: they constitute the Fierz-Pauli action describing a spin-two field with a mass $M^2 = -\sigma m^2 + \Lambda/2$ in AdS spacetime. In order to have positive energies and avoid ghosts, we require that

$$m^2\,(\Lambda - 2m^2\sigma) > 0\,. \tag{2.44}$$

The other request that we have is the non-existence of tachyons, i.e. particles traveling faster than the speed of light that would compromise the causality properties of the theory. If about the flat space we can just require that $M^2 \geqslant 0$, in the AdS case it was shown that unitarity allows scalar fields to have a negative mass squared, provided that the Breitenlohner-Freedman (BF) bound [46] is satisfied. Namely, we impose a constraint on the mass squared as follows

$$M^2 \geqslant \Lambda\,, \tag{2.45}$$

that is equivalent to asking $\Lambda \leqslant -2m^2\sigma$. It has been argued that the same bound holds for spin-2 fields [45].

If we consider $\Lambda = 2m^2\sigma$ instead, the fields h and k cannot be decoupled anymore and we find ourself in the so-called critical point. The linearized action (2.41) will then read

$$\mathcal{L}^{(2)}_{NMG} = -\frac{1}{m^2}h^{\mu\nu}\mathcal{G}_{\mu\nu}(k) - \frac{1}{4m^2}(k^{\mu\nu}k_{\mu\nu} - k^2). \tag{2.46}$$

It is clear that the metric perturbation h has now the role of Lagrangian multiplier for the constraint $\mathcal{G}_{\mu\nu}(k) = 0$. We can solve for this constraint by setting

$$k_{\mu\nu} = 2\nabla_{(\mu}A_{\nu)}\,, \tag{2.47}$$

that leads to the linearized Lagrangian

$$\mathcal{L}_2 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + 4\sigma m^2 A^\mu A_\mu \,, \qquad F_{\mu\nu} = 2\partial_{(\mu} A_{\nu)} \,. \tag{2.48}$$

Therefore, we see that the Lagrangian describes a massive spin 1 field A with mass squared $-8\sigma m^2$.

2.4.2 Central Charges

We can now have a look at the degrees of freedom at the boundary of the AdS_3 spacetime. As shown in [37], gravity theories with an AdS_3 vacuum admit an asymptotic symmetry group consisting of two copies of the Virasoro algebra. This asymptotic symmetry corresponds to a conformal symmetry of a two-dimensional dual field theory. The relative central charges can be expressed in terms of the parameters characterizing the three-dimensional gravitational theory in the bulk. For example, in the case of pure Einstein gravity we have

$$c_L = c_R = c = \frac{3L}{2\,G_3} \,, \tag{2.49}$$

where L is the AdS radius and G_3 is the Newton constant in three dimensions. These charges are relevant, among other reasons, because they can be used to express the entropy of a black hole. Using the Cardy's formula [47], the entropy of the BTZ black hole is given by

$$S = \frac{A_{BTZ}}{6\,L} c \,, \tag{2.50}$$

where A_{BTZ} is the area of the black hole.

We can apply the same method to determine the central charges for NMG with an AdS_3 background. Indeed, it was shown that for a parity-preserving theory with an AdS vacuum, such as NMG, the value of the central charges can be derived by the general formula [48–50]

$$c = \frac{L}{2\,G_3}\, g_{\mu\nu} \, \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}} \,, \tag{2.51}$$

where the higher derivative Lagrangian $\mathcal{L}_3$ is considered without the $1/\kappa^2$ factor. Applying this formula to NMG yields

$$c_{R/L} = \frac{3L}{2\,G_3} \left(\sigma \pm \frac{\Lambda}{2m^2} \right) \,. \tag{2.52}$$

In order to have a unitary CFT on the boundary, we need the central charges to be positive. Therefore, we have to find a region in the parameter space where this condition is satisfied that is compatible with the constraints coming from the bulk modes. As we will shortly see, this is not an easy task.

2.4.3 The Bulk-Boundary Clash

We will now try to find a region in the parameter space where all the constraints are satisfied. The parameters σ and m^2 can be chosen to be independently either positive or negative, we thus divide our search into four cases. The summary of the analysis will be displayed in Table 2.

The requirements are to have an AdS (thus $\Lambda < 0$) vacuum without tachyons or ghosts, and having positive central charges. Thus, for every choice of σ and m^2 we will use the constraints derived in sections 2.4.1 and 2.4.2 to restrict the range of the other NMG parameter λ.

- $\sigma = -1$ and $m^2 > 0$
  The condition (2.44) yields $\Lambda > -2m^2$ and, using the equation (2.31) relating the NMG parameters, we find that the no-ghost and no-tachyon conditions require
  $$0 < \lambda < 3 \,. \tag{2.53}$$
  However, the positivity of the central charges requires $\Lambda < -2m^2$ and thus we can't find a region where we the bulk graviton and the BTZ black hole are both well-behaved.

- $\sigma = -1$ and $m^2 < 0$
  It is convenient to define the new mass parameter by $\tilde{m}^2 = -m^2$. In this case, the BF bound (2.45) reads $-2\tilde{m}^2 \geqslant \Lambda$. If we use the explicit expression for Λ we obtain
  $$\pm\sqrt{1+\lambda} \leqslant 0 \,, \tag{2.54}$$
  which is true if we choose the lower branch. However, this choice of parameter leads to the manifest negativity of the central charges.

- $\sigma = 1$ and $m^2 > 0$
  This choice makes impossible to satisfy (2.44), thus allowing the presence of ghost modes. On the other hand, both central charges are manifestly positive.

- $\sigma = 1$ and $m^2 < 0$
  The no-ghost condition implies $2\tilde{m}^2 + \Lambda < 0$ and, again in combination with the explicit expression of Λ, we have $\mp\sqrt{1+\lambda} > 2$. The AdS vacua are then obtained by choosing the lower branch and the condition is thus

$$\lambda > 3\,, \tag{2.55}$$

  while all the other constraints coming from the bulk physics are automatically satisfied. On the other hand, the positivity of the central charges imposes that $\sqrt{1+\lambda} < 2$, which is true if

$$0 < \lambda < 3\,. \tag{2.56}$$

To summarize, we are not able to find a region in the parameter space where we have unitary positive-energy modes in the bulk and a unitary dual CFT at the boundary simultaneously. The problem is then analogous to the one we encountered for TMG and can potentially be solved by looking at the chiral limit $\lambda = 3$. In this limit, the central charges vanish and the bulk modes become unitary massive spin-1 excitations. For further readings on the topic, the reader can refer to [38, 51, 52].

		Bulk	Boundary
$\sigma = -1$	$m^2 > 0$	$0 < \lambda < 3$	$\lambda > 3$
	$m^2 < 0$	Stable solution	$c_{R,L} < 0$
$\sigma = 1$	$m^2 > 0$	Ghosts	$c_{R,L} > 0$
	$m^2 < 0$	$\lambda > 3$	$0 < \lambda < 3$

Table 2
A summary of the bulk-boundary clash. The conditions for the unitarity of the bulk and boundary modes are displayed and the clash is thus visible: there is no region of the parameter space where all the constraints are simultaneously satisfied. The only exception would be to consider the chiral limit $\lambda = 3$.

3

Massive $\mathcal{N} = 2$ Supergravity in three dimensions

In this chapter, we will construct a supersymmetric extension of General Massive Gravity, a model that combines NMG and TMG. Following [53], we will see that we have two choices to do so.

The two supersymmetric models are constructed by using the so-called superconformal method. Here, we will see explicit examples of the construction whose strategy was introduced in Chapter 1.

Both cases will be first introduced by a recap of the steps needed, followed by an exhaustive display of the details of the specific construction.

In order to facilitate the reading of the technical details, we will emphasize the key steps by putting them in a box.

In the previous chapter, we have seen how useful it can be to consider a theory of massive gravity in three dimensions. We have also encountered some obstacles that emerge regardless the use of a simplified model. In particular, it was shown in [54] that if we take a higher derivative theory with special relations between the coefficients, such as NMG, we observe a worsening in the behavior of the graviton's propagator. On the other hand, we have seen in Chapter 1 that supersymmetry helps by softening the divergences, thus improving the renormalizability properties of a theory.

The general principle that more symmetries give us more control over the theory motivates us to construct higher-derivative supergravity theories with extended supersymmetry in three dimensions. The method that will be used can be implemented in multiple ways. The results will be two theories with two different sets of symmetries.

If we consider supergravity theories that admit an anti-de Sitter spacetime as a vacuum solution, the underlying supersymmetry algebra is $OSp(p, q)$ whose bosonic part is $O(2, 2) \oplus SO(p) \times SO(q)$ [55–57]. We refer to these theories as $\mathcal{N} = (p, q)$ supergravities. Our aim is to generalize the construction of higher-derivative supergravity invariants to those with underlying $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ supersymmetries and to look for their ghost-free combinations.

Conformal $\mathcal{N} = 2$ supergravity and the two-derivative invariants were considered in [58–61]. Off-shell matter-coupled supergravity theories were investigated in the superspace framework in [62–65]. The on-shell construction and the matter couplings of the three dimensional $\mathcal{N} = 2$ supergravity were studied in [66–68].

Here, we will first define our strategy and then present all the details of the calculation. Since we recognize the very technical character of this exposition, the key steps will be emphasized by very convenient boxes in order to keep track of the path leading to the invariant terms we are looking for.

Taking into account the new invariants we construct here, we end up with seven parameter action with $\mathcal{N} = (1, 1)$ supersymmetry (see section 3.2.4) and a six parameter action with $\mathcal{N} = (2, 0)$ supersymmetry (see section 3.3.4). We find that the former, after choosing a four parameter subfamily, admits an AdS vacuum solution around which the spectrum of small fluctuations is ghost-free. In the latter case, however, we find that a ghost-free scenario does not exist. This turns out to be due to the fact that a particular type of invariant that exists for the $\mathcal{N} = (1, 1)$ model does not seem to exist for the $\mathcal{N} = (2, 0)$ model.

3.1 Strategy and Main Ingredients

The strategy to achieve the supersymmetric version of a theory passes through the construction of a model that respects a bigger symmetry. More concretely, we will first construct a superconformal theory, which is governed by a combination of supersymmetry and conformal symmetry, and then eliminate the extra symmetries by applying the appropriate constraints.

The superconformal algebra is generated by elements with both commuting and anticommuting components. In general, we can express these elements in terms of matrices of the form [12]

$$\begin{bmatrix} \text{conformal algebra} & Q, S \\ Q, S & \text{R-symmetry} \end{bmatrix}$$

As we have seen in section 1.1.2, the conformal algebra extends the Poincaré symmetry with dilatation and special conformal transformations. When combined with supersymmetry, it requires the introduction of a new fermionic generator, namely the special supersymmetry charge S_α. The R-symmetry rotates the supercharges and its presence is essential in order to achieve the closure of the algebra.

Our goal is to construct the $\mathcal{N} = 2$ extension of a gravity model that contains terms up to four derivatives in the metric such as R^2. To achieve this goal, we will first construct the superconformal extension of the theory and then perform a gauge-fixing procedure to eliminate the undesired symmetries. Therefore, we want to build actions that, after fixing the gauge, will reproduce the terms originally present in the gravity model plus other terms coming from the introduction of two local supersymmetries.

The main ingredient is the Weyl multiplet, which contains all the gauge fields associated to the symmetries of the theory. In order to obtain the (superconformal) actions, we will couple the Weyl multiplet with the appropriate compensating multiplet: by choosing a scalar multiplet we will construct the $\mathcal{N} = (1, 1)$ extension, while the $\mathcal{N} = (2, 0)$ will be obtained by using a vector compensating multiplet. At last, we will impose conditions on the compensating multiplet, thus gauge-fixing the extra symmetries.

As we have seen in Chapter 1, the driving principle to construct any multiplet is that it must contain the same number of bosonic and fermionic degrees of freedom both on-shell and off-shell. Moreover, the presence of other symmetries, such as dilatation and R-symmetry, will guide us in the choice of the appropriate multiplet to obtain a specific term. For example, if we need a component with a specific dilatation weight and R-symmetry charge, we can use the transformation rules to trace down the multiplet from which it belongs and thus construct it.

In Table 3, the reader will find a summary of all the multiplets that will play a role in this chapter, with their behavior under the two symmetries we mentioned. For both constructions, $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$, we will need a method to build multiplets with different weights. This will be done in detail, thus introducing every supermultiplet we will need before using it. These will be simply particular cases of the summary presented in Table 3.

Multiplet	Field	Type	Off-shell	w	q
Weyl	$e_\mu{}^a$	dreibein	2	-1	0
	ψ_μ	gravitino	4	$-\frac{1}{2}$	1
	V_μ	$U(1)_R$ gauge field	2	0	0
Scalar	A	complex scalar	2	w_A	$-w_A$
	χ	Dirac spinor	4	$w_A+\frac{1}{2}$	$-w_A+1$
	$\mathcal{F}$	complex auxiliary	2	w_A+1	$-w_A+2$
Vector	ρ	real scalar	1	1	0
	C_μ	gauge field	2	0	0
	λ	Dirac spinor	4	$\frac{3}{2}$	1
	D	real auxiliary	1	2	0

TABLE 3
Properties of the $3D,\mathcal{N}=2$ Weyl and compensating multiplets where (w,q) label the dilatation weight and the $U(1)_R$ charge, respectively.

3.1.1 The Weyl Multiplet.

The $\mathcal{N} = 2$ Weyl multiplet in three dimensions is based on the conformal superalgebra $OSp(4|2)$ and consists of the fields

$$(e_\mu{}^a\,,\,\psi_\mu\,,\,V_\mu\,,\,b_\mu\,,\,\omega_\mu{}^{ab}\,,\,f_\mu{}^a\,,\,\phi_\mu)\ , \tag{3.1}$$

where $e_\mu{}^a$ is the dreibein, ψ_μ is the gravitino represented by a Dirac vector-spinor, V_μ is the $U(1)$ R-symmetry gauge field, b_μ is the dilatation gauge field, $\omega_\mu{}^{ab}$ is the spin connection, $f_\mu{}^a$ is the conformal boost gauge field and ϕ_μ is the special supersymmetry gauge field represented by a Dirac vector-spinor. The corresponding gauge parameters are

$$(\xi^a, \epsilon\,, \Lambda, \Lambda_D\,, \Lambda^{ab}, \Lambda_K^a\,, \eta)\ . \tag{3.2}$$

The gauge fields $\omega_\mu{}^{ab}, \phi_\mu, f_\mu{}^a$ can be expressed in terms of the remaining fields by imposing the constraints [58]

$$\widehat{R}^a_{\mu\nu}(P) = 0\ , \qquad \widehat{R}_{\mu\nu}{}^{ab}(M) = 0\ , \qquad \widehat{R}_{\mu\nu}(Q) = 0\ , \tag{3.3}$$

where the supercovariant curvatures associated with translations, Lorentz rotations, and supersymmetry are defined as

$$\begin{aligned}
\widehat{R}_{\mu\nu}{}^a(P) &= 2(\partial_{[\mu} + b_{[\mu})\, e_{\nu]}{}^a + 2\,\omega_{[\mu}{}^{ab} e_{\nu]b} - \frac{1}{2}(\bar{\psi}_{[\mu}\gamma^a\psi_{\nu]} + h.c.)\ ,\\
\widehat{R}_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^b + 8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]}\\
&\qquad -\frac{1}{2}\bar{\psi}_\mu\gamma^{ab}\phi_\nu - \frac{1}{2}\bar{\phi}_\mu\gamma^{ab}\psi_\nu + h.c.\,,\\
\widehat{R}_{\mu v}(Q) &= 2\partial_{[\mu}\psi_{\nu]} + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\,\psi_{\nu]} + b_{[\mu}\psi_{\nu]}\\
&\qquad -2\,\gamma_{[\mu}\phi_{\nu]} - 2\mathrm{i}V_{[\mu}\psi_{\nu]}\ .
\end{aligned} \tag{3.4}$$

These constraints together with the Bianchi identity for $\widehat{R}_{\mu\nu}(P)$ also imply that the curvature associated with dilatation vanishes, *viz.* $\widehat{R}_{\mu\nu}(D) = 0$. Solving the constraints (3.3) gives

$$\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\,\partial_\nu e_\sigma{}^c + 2e_\mu{}^{[a}b^{b]} + \frac{1}{2}\bar{\psi}_\mu\gamma^{[a}\psi^{b]} \\
&\quad + \frac{1}{2}\bar{\psi}^{[a}\gamma^{b]}\psi_\mu + \frac{1}{2}\bar{\psi}^{[a}\gamma_\mu\psi^{b]}\ , \\
\phi_\mu &= -\gamma^a\widehat{R}'_{\mu a}(Q) + \frac{1}{4}\gamma_\mu\gamma^{ab}\widehat{R}'_{ab}(Q)\ , \\
f_\mu^a &= -\frac{1}{2}\widehat{R}'_\mu{}^a(M) + \frac{1}{8}e_\mu{}^a\widehat{R}'(M)\ ,
\end{aligned} \tag{3.5}$$

where the prime in the curvatures used in (3.5) means that the term including the field we are solving for is excluded. The transformation rules for the independent fields are given by

$$\begin{aligned}
\delta e_\mu{}^a &= -\Lambda^a{}_b\,e_\mu{}^b - \Lambda_D e_\mu{}^a + \frac{1}{2}\bar{\epsilon}\,\gamma^a\psi_\mu + h.c.\ , \\
\delta\psi_\mu &= -\frac{1}{4}\Lambda^{ab}\gamma_{ab}\psi_\mu - \frac{1}{2}\Lambda_D\psi_\mu + \mathcal{D}_\mu\epsilon - \gamma_\mu\eta + \mathrm{i}\Lambda\psi_\mu\ , \\
\delta b_\mu &= \partial_\mu\Lambda_D + 2\Lambda_{K\mu} + \frac{1}{2}\bar{\epsilon}\,\phi_\mu - \tfrac{1}{2}\bar{\eta}\,\psi_\mu + h.c\ , \\
\delta V_\mu &= \partial_\mu\Lambda + \frac{\mathrm{i}}{2}\bar{\epsilon}\,\phi_\mu + \frac{\mathrm{i}}{2}\bar{\eta}\,\psi_\mu + h.c.\ ,
\end{aligned} \tag{3.6}$$

where

$$\mathcal{D}_\mu\epsilon = \Big(\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}V_\mu\Big)\epsilon\ . \tag{3.7}$$

Finally, we give the transformation rule for ϕ_μ for later convenience

$$\delta\phi_\mu = \cdots + \mathrm{i}\gamma^\nu\widehat{F}_{\mu\nu}\epsilon - \frac{\mathrm{i}}{4}\gamma_\mu\gamma\cdot\widehat{F}_{\mu\nu}\epsilon\ , \tag{3.8}$$

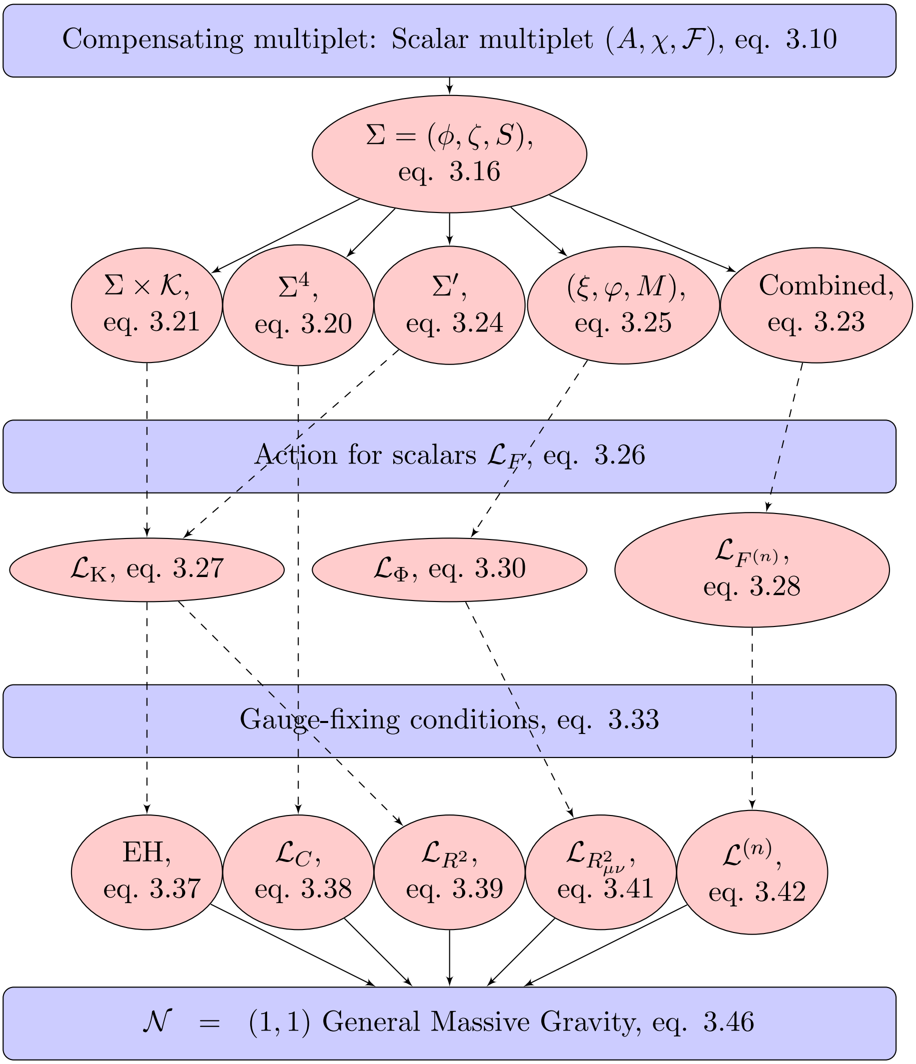
where we have displayed the supercovariant terms and the ellipses refer to the remaining terms implied by the $OSp(4|2)$ algebra. $\widehat{F}_{\mu\nu}$ is given by

$$\widehat{F}_{\mu\nu} = 2\partial_{[\mu}V_{\nu]} - \mathrm{i}\bar{\psi}_{[\mu}\phi_{\nu]} - \mathrm{i}\bar{\phi}_{[\mu}\psi_{\nu]}\ . \tag{3.9}$$

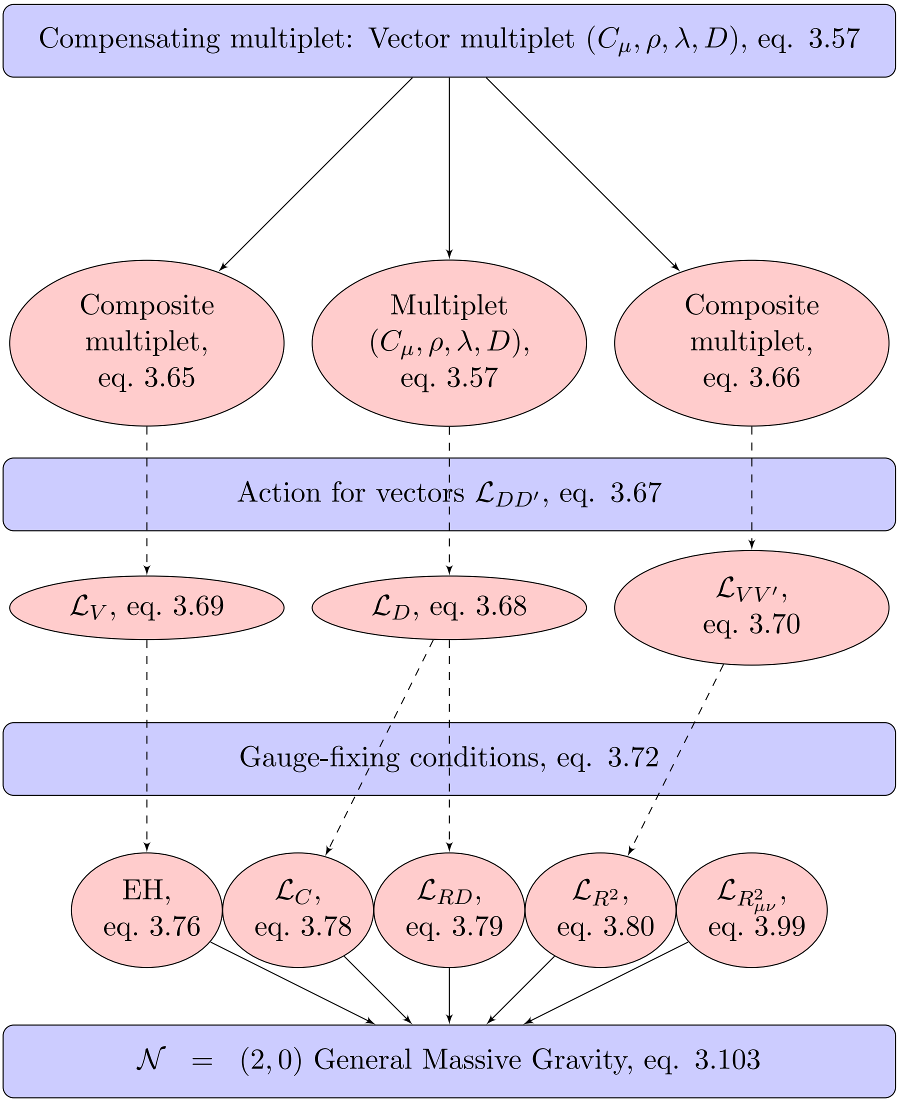
To summarize, the Weyl multiplet contains all the gauge fields associated with the symmetries of the theory. By imposing the appropriate constraints (3.3) we can reduce the number of independent fields.

3.1.2 The Blueprints

This subsection is dedicated to summarize the steps and the equations to construct the supersymmetric extensions.



In this chart, we have omitted the Chern-Simons term, which will be taken from the literature.



In this case, not only the Chern-Simons term is omitted, but also the $\mathcal{L}_{R^2_{\mu\nu}}$ term is obtained with a different method.

3.2 $\mathcal{N} = (1,1)$ Supergravity

The $\mathcal{N} = (1,1)$ supersymmetric extension of the gravity model with terms up to four derivatives is obtained by using a compensating scalar multiplet. For each term in the Lagrangian, the goal will be to find the appropriate multiplet such that its action reproduces those terms after gauge-fixing the extra symmetries.

As mentioned before, the charges under dilatation and $U(1)_R$-symmetry, i.e. (w, q), will play a central role to guide us in the construction. In fact, for symmetry reasons that will be clear later on, we want to use a scalar multiplet whose third component has weights $(3, 0)$.

We will first introduce the structure of the scalar multiplet that has to be thought as the blueprint for all the others. Indeed, we will then use this blueprint to construct scalar multiplets with different weights. At this stage, we will present a technique to build all these objects and try to keep track of where we will need each one of them (section 3.2.2).

Once that we have all the fundamental objects, we will construct a superconformal action in section 3.2.3 and then perform the gauge-fixing procedure leading to the final result in section 3.2.4.

3.2.1 The Scalar Multiplet

The off-shell $\mathcal{N} = 2$ scalar multiplet with 4+4 degrees of freedom consists of a physical complex scalar A, a Dirac fermion χ and an auxiliary complex scalar $\mathcal{F}$ with the following transformation rules [1]

$$
\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}\,\chi + w\Lambda_D A - \mathrm{i}w\Lambda A \ , \\
\delta\chi &= \not{D}A\,\epsilon - \frac{1}{2}\mathcal{F}\,(B\epsilon)^* + 2wA\,\eta + \left(w + \frac{1}{2}\right)\Lambda_D\chi + \mathrm{i}(-w+1)\Lambda\,\chi \ , \\
\delta\mathcal{F} &= -\tilde{\epsilon}\,\not{D}\chi + 2\left(w - \frac{1}{2}\right)\tilde{\eta}\,\chi + (w+1)\Lambda_D\mathcal{F} + \mathrm{i}(-w+2)\Lambda\,\mathcal{F} \ , \qquad (3.10)
\end{aligned}
$$

[1] See Appendix A for the definition of $\widetilde{\eta}$ and the constant matrix B.

where the supercovariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_\mu A &= (\partial_\mu - w\, b_\mu + \mathrm{i}wV_\mu)A - \frac{1}{2}\bar{\psi}_\mu\chi \ , \\
\mathcal{D}_\mu\chi &= \left[\partial_\mu - \left(w+\frac{1}{2}\right) b_\mu + \frac{1}{4}\,\omega_\mu{}^{ab}\,\gamma_{ab} + \mathrm{i}(w-1)V_\mu\right]\chi - \not{\mathcal{D}}A\psi_\mu \\
&\quad + \frac{1}{2}\mathcal{F}(B\psi_\mu)^* - 2wA\,\phi_\mu \ . \qquad (3.11)
\end{aligned}$$

Note that the lowest component has Weyl weight w and $U(1)_R$ weight $-w$. Another multiplet with its lowest component having Weyl weight w and $U(1)_R$ weight w can be obtained by charge conjugation

$$\begin{aligned}
\delta A^* &= \frac{1}{2}\tilde{\epsilon}\,(B\chi)^* + w\Lambda_D A^* + iw\Lambda A^* \ , \\
\delta\,(B\chi)^* &= \not{\mathcal{D}}A^*\,(B\epsilon)^* - \frac{1}{2}F^*\epsilon + 2wA^*\,(B\eta)^* + \left(w+\frac{1}{2}\right)\Lambda_D\,(B\chi)^* \\
&\quad + \mathrm{i}\,(w-1)\,\Lambda\,(B\chi)^* \ , \\
\delta\mathcal{F}^* &= -\bar{\epsilon}\not{\mathcal{D}}\,(B\chi)^* + 2\left(w-\frac{1}{2}\right)\bar{\eta}(B\chi)^* + (w+1)\Lambda_D\mathcal{F}^* \\
&\quad + \mathrm{i}(w-2)\Lambda\,\mathcal{F}^* \ , \qquad (3.12)
\end{aligned}$$

where the supercovariant derivatives are

$$\begin{aligned}
\mathcal{D}_\mu A^* &= (\partial_\mu - wb_\mu - \mathrm{i}wV_\mu)A^* - \frac{1}{2}\tilde{\psi}_\mu(B\chi)^* \ , \\
\mathcal{D}_\mu(B\chi)^* &= \left[\partial_\mu - \left(w+\frac{1}{2}\right) b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}(w-1)V_\mu\right](B\chi)^* \\
&\quad - \not{\mathcal{D}}A^*(B\psi_\mu)^* + \frac{1}{2}\mathcal{F}^*\psi_\mu - 2wA^*(B\phi_\mu)^* \ , \\
\mathcal{D}_\mu P^* &= \left[\partial_\mu - \frac{1}{2}b_\mu - (w-2)\mathrm{i}\,V_\mu\right]P^* + \bar{\psi}_\mu\not{\mathcal{D}}(B\chi)^* \\
&\quad - 2\left(w-\frac{1}{2}\right)\bar{\phi}_\mu(B\chi)^* \ . \qquad (3.13)
\end{aligned}$$

In equation (3.10), we have constructed a scalar multiplet $\big(A,\chi,\mathcal{F}\big)$ with $4+4$ degrees of freedom and charges $(w,q)=(w,-w)$. To obtain a multiplet with opposite $U(1)_R$ charge, i.e. $(w,q)=(w,w)$, we need to take the conjugated multiplet $\big(A^*,(B\chi)^*,\mathcal{F}^*\big)$.

3.2.2 How to Obtain Other Multiplets

We will construct composite scalar multiplets using the multiplication rules for scalar multiplets. The results of this section are summarized in Table 4. One can start with two scalar multiplets (A_i, χ_i, F_i), $i = 1, 2$ and obtain a multiplet whose lowest component has Weyl weight $w = w_1 + w_2$ and $U(1)_R$ weight $q = q_1 + q_2$ as follows

$$\begin{aligned} A &= A_1 A_2 \ , \\ \chi &= A_1 \chi_2 + A_2 \chi_1 , \\ F &= A_1 F_2 + A_2 F_1 + \tilde{\chi}_1 \chi_2 \ . \end{aligned} \tag{3.14}$$

It is also possible to use the inverse of the multiplication rule (3.14) to obtain a multiplet with Weyl weight $w = w_1 - w_2$ and $U(1)_R$ weight $q = q_1 - q_2$

$$\begin{aligned} A &= A_1 A_2^{-1} \ , \\ \chi &= A_2^{-1} \chi_1 - A_1 A_2^{-2} \chi_2 \ , \\ F &= A_2^{-1} F_1 - A_1 A_2^{-2} F_2 - A_2^{-2} \tilde{\chi}_2 \chi_1 + A_1 A_2^{-3} \tilde{\chi}_2 \chi_2 \ . \end{aligned} \tag{3.15}$$

> Using the multiplication rules for scalar multiplets we can obtain new multiplets whose lowest component has weights $(w_1 + w_2, q_1 + q_2)$. Analogously, we can reverse those rules to obtain a lowest component with weights $(w_1 - w_2, q_1 - q_2)$.

Given the scalar multiplet (see Table 3),

$$\Sigma = (\phi,\, \zeta,\, S) \ , \tag{3.16}$$

the associated inverse multiplet has the components

$$\Sigma^{-1} \equiv (\Phi, \Psi, P) = \left(\phi^{-1},\, -\phi^{-2}\zeta,\, -\phi^{-2} S + \phi^{-3} \tilde{\zeta}\, \zeta\right) \ . \tag{3.17}$$

as can be seen by considering the multiplication of the unit multiplet $(A_1, \chi_1, F_1) = (1, 0, 0)$, which has weights $(\omega, c) = (0, 0)$, with the multiplet $\Sigma = (A_2, \chi_2, F_2)$, by means of the formula (3.15).

Next, we note that a scalar multiplet (ϕ, ζ, S) with weights $(w, q) = (\frac{1}{2}, -\frac{1}{2})$ has the corresponding kinetic multiplet with weights $(w, q) = (\frac{3}{2}, -\frac{3}{2})$ given by

$$\mathcal{K} = \left(S^*,\, -2\not{D}(B\zeta)^*,\, 4\Box^C \phi^*\right) \ , \tag{3.18}$$

Components	w	q
(ξ, φ, M)	$\frac{5}{2}$	$-\frac{5}{2}$
(Z, Ω, F)	2	-2
(ϕ, ζ, S)	$\frac{1}{2}$	$-\frac{1}{2}$
(σ, ψ, N)	0	0
(Φ, Ψ, P)	$-\frac{1}{2}$	$\frac{1}{2}$

Table 4
The compensating scalar multiplets constructed in this section.
(w, q) denotes the Weyl weight and the $U(1)_R$ charge of the lowest component scalar field.

where

$$\begin{aligned} \Box^C \phi^* \quad &= \quad \left(\partial^a - \frac{3}{2}b^a - \frac{\mathrm{i}}{2}V^a\right)\mathcal{D}_a\phi^* + \omega_a{}^{ab}\mathcal{D}_b\,\phi^* + f_a^a\,\phi^* \\ &\quad + \frac{1}{2}\tilde{\phi}_a\gamma^a(B\zeta)^* - \frac{1}{2}\tilde{\psi}^a\mathcal{D}_a(B\zeta)^* \;. \end{aligned} \tag{3.19}$$

Using the above multiplets as building blocks and using the product formula (3.14) we can construct a number of multiplets that will be useful in building actions. First, we consider the four-fold product of Σ:

$$\Sigma^4: \qquad (Z, \Omega, F) = \left(\phi^4, \;\; 4\phi^3\zeta, \;\; 4\phi^3 S + 6\phi^2\tilde{\zeta}\zeta\right) \;. \tag{3.20}$$

Note that the complex scalar Z has the weights $(w, q) = (2, -2)$ and will be useful to construct a cosmological constant invariant. Another multiplet with the same weights $(2, -2)$ can be obtained by multiplying Σ with the kinetic multiplet $\mathcal{K}$

$$\begin{aligned} \Sigma\times\mathcal{K} \quad : \quad Z' \quad &= \quad \phi S^* \;, \\ \Omega' \quad &= \quad \zeta\, S^* - 2\phi\, \not{\mathcal{D}}(B\zeta) \;, \\ F' \quad &= \quad 4\phi\Box^C\phi^* + |S|^2 - 2\,\tilde{\zeta}\,\not{\mathcal{D}}(B\zeta)^* \;. \end{aligned} \tag{3.21}$$

We will use this multiplet to construct the Einstein-Hilbert action.

We build two multiplets, see eqs. (3.20)–(3.21), where the lowest component has weights $(2,2)$ and thus, as Table 3 shows, its highest component has weights $(3,0)$.
This is a condition to have an invariant action.

A composite neutral multiplet with $(w,q)=(0,0)$ can be obtained as follows

$$\begin{aligned}
&\mathcal{K}\times\Sigma^{-3} \quad : \\
&\qquad \sigma = \phi^{-1}S^* \,, \\
&\qquad \psi = -2\,\phi^{-3}\not{D}\,(B\zeta)^* - 3\,\phi^{-4}S^*\zeta \,, \\
&\qquad N = 4\,\phi^{-3}\Box^C\phi - 3\,\phi^{-4}\,|S|^2 + 6\phi^{-4}\tilde{\zeta}\,\not{D}\,(B\zeta)^* + 6\,\phi^{-5}S^*\tilde{\lambda}\,\zeta,
\end{aligned} \tag{3.22}$$

which can be used to produce new scalar multiplets without changing the weights of the original multiplets

$$\begin{aligned}
&(\sigma,\psi,N)^n\times(Z,\Omega,F) \quad : \\
&\qquad Z^{(n)} = \sigma^n Z \,, \\
&\qquad \Omega^{(n)} = n\,\sigma^{n-1}Z\,\psi + \sigma^n\,\Omega \,, \\
&\qquad F^{(n)} = \sigma^n F + n\,\sigma^{n-1}ZN + n(n-1)\,\sigma^{n-2}Z\tilde{\psi}\,\psi \\
&\qquad\qquad + n\,\sigma^{n-1}\tilde{\psi}\,\Omega \,.
\end{aligned} \tag{3.23}$$

$$(\sigma,\psi,N)\times\Sigma : \qquad (\phi',\zeta',S') = \Big(\sigma\phi \;,\;\; \sigma\zeta+\phi\psi \;,\;\; \sigma S + \phi\,N + \tilde{\zeta}\,\psi\Big) \;. \tag{3.24}$$

Finally, we construct the multiplet (ξ,φ,M), with weights $(\frac{5}{2},-\frac{5}{2})$, in terms of the elements of the multiplet (Φ,Ψ,P) as follows:

$$\begin{aligned}
\xi &= \Box^c P^* \,, \\
\varphi &= -2\,\Box^c\not{D}\,(B\Psi)^* - 2\,\mathrm{i}\gamma^\nu\mathcal{D}^\mu\,\widehat{F}_{\mu\nu}(B\lambda)^* + 2\,\mathrm{i}\gamma^\nu\widehat{F}_{\mu\nu}\,\mathcal{D}^\nu(B\lambda)^* \\
&\quad + \mathrm{i}\gamma^{\mu\nu}\not{D}\,\widehat{F}_{\mu\nu}(B\lambda)^* + \frac{5}{2}\,\mathrm{i}\gamma^{\mu\nu}\widehat{F}_{\mu\nu}\not{D}(B\lambda)^* \,, \\
M &= 4\Box^c\Box^c\Phi^* - 8\,\mathrm{i}\mathcal{D}^a\widehat{F}_{ab}\,\mathcal{D}^b\Phi^* - 2\widehat{F}_{ab}\,\widehat{F}^{ab}\Phi^* + \text{fermions} \,.
\end{aligned} \tag{3.25}$$

Here we have omitted the complicated fermionic expressions in the composite formula for M as we shall be interested in the bosonic part of an action formula for which this multiplet will be used. With this multiplet we will produce a Ricci tensor squared invariant.

With the help of the neutral multiplet (3.23) we can increase the level of complexity of the other multiplets without changing their weights. We thus obtain the building blocks for the higher-derivative invariants.

3.2.3 Superconformal Actions

We start with the action for a scalar multiplet (Z, Ω, F)

$$e^{-1}\mathcal{L}_F = Re\left(F - \tilde{\psi}_\mu\gamma^\mu\Omega - Z\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right) , \tag{3.26}$$

which is invariant under dilatations and $U(1)_R$ transformations since the highest component field F has the weight $(w, q) = (3, 0)$.

Next, we use the components of the composite scalar multiplet (3.21) in the action formula (3.26) which yields the following action that will be used to construct the supersymmetric completions of the Einstein-Hilbert term as well as the R^2 term

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{K}} &= 4\phi\,\Box^C\phi^* + |S|^2 - 2\,\tilde{\zeta}\,\not{\mathcal{D}}\,(B\zeta)^* + 2\,\phi\,\tilde{\psi}_\mu\gamma^\mu\not{\mathcal{D}}\,(B\zeta)^* \\ &\quad -S^*\tilde{\psi}_\mu\gamma^\mu\zeta - \phi\,S^*\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\ . \end{aligned} \tag{3.27}$$

The composite multiplet given in (3.23) can be used in the action formula (3.26) to produce

$$\begin{aligned} e^{-1}\mathcal{L}_{F(n)} &= \mathrm{Re}\Big(\sigma^n F + n\,\sigma^{n-1}ZN + n(n-1)\sigma^{n-2}Z\,\tilde{\psi}\,\psi + n\,\sigma^{n-1}\tilde{\psi}\Omega \\ &\quad -n\,\sigma^{n-1}Z\,\tilde{\psi}_\mu\gamma^\mu\psi - \sigma^n\tilde{\psi}_\mu\gamma^\mu\,\Omega - \sigma^n Z\,\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\Big) , \end{aligned} \tag{3.28}$$

which we shall use below to obtain an action providing a supersymmetric completion of RS^n.

We next consider the scalar multiplets (ξ, φ, M) and (Φ, Ψ, P). Using the multiplication rule (3.14), the action describing the coupling of these multiplets is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{\xi\Phi} &= \mathrm{Re}\Big(\xi P + \Phi M + \tilde{\Psi}\varphi - \Phi\,\tilde{\psi}_\mu\gamma^\mu\varphi \\ &\quad - \xi\tilde{\psi}_\mu\gamma^\mu\Psi - \Phi\,\xi\,\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\Big) . \end{aligned} \tag{3.29}$$

Using the composite expressions (3.25), the bosonic part of the action that will give rise to the $R^2_{\mu\nu}$ invariant is given by

$$e^{-1}\mathcal{L}_\Phi = \mathrm{Re}\Big(4\Phi\,\Box^C\Box^C\Phi^* + P\Box^C P^* - 8\mathrm{i}\Phi\,\mathcal{D}^a\widehat{F}_{ab}\mathcal{D}^b\Phi^* - 2\widehat{F}_{ab}\,\widehat{F}^{ab}\Phi\Phi^*\Big)\ . \quad (3.30)$$

Finally, there also exists an action that constitutes the superconformal completion of the Lorentz Chern-Simons term. It is given by [58]

$$\begin{aligned}\mathcal{L}_{\mathrm{CS}} \quad = \quad & -\frac{1}{4}\,\varepsilon^{\mu\nu\rho}\left[R_{\mu\nu}{}^{ab}(\omega)\,\omega_{\rho ab} + \frac{2}{3}\omega_\mu{}^{ab}\,\omega_{\nu b}{}^c\,\omega_{\rho ca}\right] \\ & + \varepsilon^{\mu\nu\rho}F_{\mu\nu}V_\rho - \bar{R}^\mu\,\gamma_\nu\gamma_\mu R^\nu\ ,\end{aligned} \quad (3.31)$$

where the Hodge dual of the gravitino curvature is defined by

$$R^\mu = \varepsilon^{\mu\nu\rho}(D_\nu(\omega) - \mathrm{i}V_\nu)\psi_\rho\ . \quad (3.32)$$

The supersymmetric Chern-Simons action is invariant under the Weyl multiplet transformation rules (3.6). Therefore, it can be used for both $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravities.

To summarize, we use the action for a generic scalar multiplet as a blueprint. We then use different multiplets to obtain different invariants. In particular

- $\mathcal{L}_{\mathrm{K}}$, in eq. (3.27), will give us the EH and the R^2 terms.
- $\mathcal{L}_{F^{(n)}}$, in eq. (3.28), will give us the RS^n term.
- $\mathcal{L}_\Phi$, in eq. (3.30), will give us the $R^2_{\mu\nu}$ term.
- $\mathcal{L}_{\mathrm{CS}}$, in eq. (3.31), is the supersymmetric Chern-Simons action.

3.2.4 Gauge Fixing and Invariant Terms

The off-shell Poincaré supergravity action is readily obtained from the action formula (3.27) by fixing the dilatation, conformal boost and special supersymmetry transformation by imposing

$$\phi = 1 \,, \qquad \zeta = 0 \,, \qquad b_\mu = 0 \,. \tag{3.33}$$

The first one fixes dilatation and $U(1)_R$ transformation, the second fixes the S-supersymmetry and the last one fixes the special conformal transformations. Maintaining these gauge conditions requires that

$$\begin{aligned}
\Lambda_D &= \mathrm{i}\Lambda = 0 \,, \\
\Lambda_{K\mu} &= \frac{1}{4}\bar{\eta}\,\psi_\mu - \frac{1}{4}\bar{\epsilon}\,\phi_\mu + h.c. \,, \\
\eta &= -\frac{\mathrm{i}}{2}\gamma^\nu V_\nu \epsilon + \frac{1}{2} S\,(B\epsilon)^* \,.
\end{aligned} \tag{3.34}$$

These constraints on the transformation parameters imply the supersymmetry transformation rules

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu + h.c. \,, \\
\delta\psi_\mu &= D_\mu(\omega)\,\epsilon - \frac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \frac{1}{2}S\gamma_\mu\,(B\epsilon)^* \,, \\
\delta V_\mu &= \frac{\mathrm{i}}{8}\bar{\epsilon}\,\gamma^{\nu\rho}\gamma_\mu\Big(\psi_{\nu\rho} - \mathrm{i}V_\sigma\gamma^\sigma\gamma_\nu\,\psi_\rho - S\gamma_\nu\left(B\psi_\rho\right)^*\Big) + h.c. \,, \\
\delta S &= -\frac{1}{4}\tilde{\epsilon}\,\gamma^{\mu\nu}\Big(\psi_{\mu\nu} - \mathrm{i}V_\sigma\,\gamma^\sigma\gamma_\mu\psi_\nu - S\gamma_\mu\,(B\psi_\nu)^*\Big) \,,
\end{aligned} \tag{3.35}$$

where

$$D_\mu(\omega)\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab}\right)\epsilon \,, \qquad \psi_{\mu\nu} = 2D_{[\mu}(\omega)\psi_{\nu]} \,. \tag{3.36}$$

> We gauge-fixed the dilatation, conformal boost and special supersymmetry transformations. Due to the fact that ϕ is complex, the constraint we impose is actually translated into two conditions: one that fixes dilatations and one that fixes the $U(1)_R$ symmetry. We can now see the effects of this fixing on the superconformal actions constructed before.

$\mathcal{N} = (1,1)$ Cosmological Poincaré Supergravity

Using the gauge-fixing conditions (3.33) in the action (3.27) gives the action of Poincaré supergravity

$$e^{-1}\mathcal{L}_{EH} = R + 2V^2 - 2\,|S|^2 - \left(\bar{\psi}_\mu\, \gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho + h.c.\right) \ , \tag{3.37}$$

where $V^2 := V_\mu V^\mu$. Next, we construct the supersymmetric cosmological term by using the multiplet (Z, Ω, F) given in (3.20) in the action formula (3.26), imposing the gauge fixing conditions (3.33), and multiplying the action by 1/2. The result is

$$e^{-1}\mathcal{L}_C = S - \frac{1}{4}\tilde{\psi}_\mu\, \gamma^{\mu\nu}\, \psi_\nu + h.c. \tag{3.38}$$

$\mathcal{N} = (1,1)$ Higher-Dimensional Invariants

We begin with the construction of the R^2 invariant. To this end, we employ the composite scalar multiplet Σ' from (3.24) in the action formula (3.27). In the resulting action we use the composite neutral multiplet from (3.23). Subsequently, we fix the extra gauge symmetries as in (3.33). These are straightforward manipulations which give the full R^2 invariant whose bosonic part is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2} \;\; = \;\; & R^2 + 16\,|S|^4 + 4(V^2)^2 + 6R\,|S|^2 + 4RV^2 + 12\,|S|^2\, V^2 \\ & -16\partial_\mu S\, \partial^\mu S^* - 8\mathrm{i}V^\mu S^* \overleftrightarrow{\partial_\mu} S + 16\,(\nabla_\mu V^\mu)^2 \ , \end{aligned} \tag{3.39}$$

where $S^* \overleftrightarrow{\partial_\mu} S = S^*\partial_\mu S - S\partial_\mu S^*$.

To construct the supersymmetric $R^2_{\mu\nu}$ invariant, we employ the action formula (3.30). Substituting for the components of the multiplet (Φ, Ψ, P), given in (3.17), and imposing gauge-fixing conditions (3.33), give the supersymmetric completion of the Ricci tensor squared as follows

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2_{\mu\nu}+R^2} \;\; = \;\; & R_{\mu\nu}R^{\mu\nu} - \frac{23}{64}R^2 - \frac{1}{32}R\,|S|^2 - R_{\mu\nu}V^\mu V^\nu + \frac{5}{16}RV^2 \\ & + \frac{1}{16}(V^2)^2 - \frac{25}{16}V^2\,|S|^2 - \frac{1}{4}\partial_\mu\, S\partial^\mu S^* - \frac{5}{8}\mathrm{i}V^\mu S^* \overleftrightarrow{\partial_\mu} S \\ & + \frac{1}{4}\,(\nabla_\mu V^\mu)^2 - F_{\mu\nu}F^{\mu\nu} \ , \end{aligned} \tag{3.40}$$

where we have exhibited the bosonic part of the Lagrangian. The R^2 dependent part can be removed by adding $\frac{23}{64}\mathcal{L}_{R^2}$ to this Lagrangian, obtaining

$$
\begin{aligned}
e^{-1}\mathcal{L}_{R^2_{\mu\nu}} &= R_{\mu\nu}R^{\mu\nu} - R_{\mu\nu}V^\mu V^\nu + \frac{7}{4}RV^2 + \frac{17}{8}R\,|S|^2 + \frac{23}{4}|S|^4 \\
&\quad -F_{\mu\nu}F^{\mu\nu} + 6\,(\nabla_\mu V^\mu)^2 + \frac{3}{2}(V^2)^2 + \frac{11}{4}V^2\,|S|^2 \\
&\quad -6\partial_\mu S\partial^\mu S^* - \frac{7}{2}\mathrm{i}V^\mu S^*\overleftrightarrow{\partial_\mu} S\ . \qquad (3.41)
\end{aligned}
$$

Next, we construct the supersymmetric completion of the RS^n term. To this end, we employ the action formula (3.28), in which we substitute for the components of the multiplets (σ, ψ, N) and (Z', Ω', F') given in (3.23) and (3.21), respectively. Imposing the gauge-fixing conditions (3.33) in the resulting Lagrangian, and dividing by an overall constant factor of $-(n+1)$, we obtain

$$
e^{-1}\mathcal{L}^{(n)} = \frac{1}{2}\left[R + \frac{2(3n-1)}{n+1}|S|^2 + 2V^2 - 4i\nabla_\mu V^\mu\right]S^n + h.c.\ , \qquad (3.42)
$$

where we have given the bosonic part of the result. Note that the $n = 0$ case agrees with the Poincaré supergravity action (3.37) which we obtained by an alternative procedure.

We now have all the ingredients, namely the invariant terms up to four derivatives, and therefore we can combine them to obtain the desired supersymmetric extension.

$\mathcal{N}=(1,1)$ Generalized Massive Supergravity

We consider a combination of the invariants up to dimension four, namely,

$$\begin{aligned} I &= \frac{1}{\kappa^2}\int d^3x \left[\frac{1}{2}M\mathcal{L}_C + \sigma\mathcal{L}_{EH} + \frac{1}{\mu}\mathcal{L}_{CS} \right.\\ &\qquad\qquad \left. + \frac{1}{\nu}\mathcal{L}_{RS} + \frac{1}{m^2}\mathcal{L}_{R^2_{\mu\nu}} + c_1\mathcal{L}_{R^2} + c_2\mathcal{L}_{RS^2}\right] , \end{aligned} \tag{3.43}$$

where $(\sigma, M, \mu, \nu, m^2, c_1, c_2)$ are arbitrary real constants. Defining

$$S = A + \mathrm{i}B \ , \tag{3.44}$$

where A and B are real scalar fields, the $\mathcal{N}=(1,0)$ supersymmetric truncation is achieved by setting $V_\mu = 0$ and $B=0$. In that case, the so-called Generalized Massive Gravity (GMG) model is defined by setting

$$\nu = \infty \ , \qquad c_1 = -\frac{3}{8m^2} \ , \qquad c_2 = \frac{1}{8m^2} \ . \tag{3.45}$$

With these choices of the coupling constants the model expanded around a supersymmetric AdS_3 vacuum propagates only helicity ± 2 and $\pm 3/2$ states with energies that respect perturbative unitarity. We shall define the $\mathcal{N}=(1,1)$ supersymmetric version of the GMG model by choosing the coupling constants as in (3.45) as well, since the quadratic action obtained by expanding around the supersymmetric AdS_3 vacuum contains the $\mathcal{N}=(1,0)$ sector as an independent subsector. In this case, the total Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{GMG} &= \sigma(R + 2V^2 - 2|S|^2) + MA \\ &\quad -\frac{1}{4\mu}\left[\epsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\,\omega_{\rho ab} + \frac{2}{3}\omega_\mu{}^{ab}\,\omega_{\nu b}{}^{c}\,\omega_{\rho ca}\right) - 8\epsilon^{\mu\nu\rho}V_\mu\partial_\nu V_\rho\right] \\ &\quad +\frac{1}{m^2}\left[R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 - R_{\mu\nu}V^\mu V^\nu - F_{\mu\nu}F^{\mu\nu}\right. \\ &\qquad\qquad + \frac{1}{4}R(V^2 - B^2) + \frac{1}{6}|S|^2(A^2 - 4B^2) \\ &\qquad\qquad \left. -\frac{1}{2}V^2(3A^2 + 4B^2) - 2V^\mu B\partial_\mu A\right] . \end{aligned} \tag{3.46}$$

Remarkably, all terms proportional to $|\partial S|^2, RA^2, (\nabla_\mu V^\mu)^2$ and $(V_\mu V^\mu)^2$ have cancelled. The cancellation of the $|\partial S|^2$ and RA^2 require c_1 and c_2 to have the values given in (3.45), and it is crucial for having ghost-free propagation of massive modes, as we shall see below.

Notwithstanding that the fields A and B do not propagate, their elimination yields highly nonlinear interactions, including those which take the form of an infinite power series in the Ricci curvature scalar R. In that sense, the notion of a supersymmetric GMG model is extended here, compared to the case of the $\mathcal{N}=(1,0)$ supersymmetric version where the single auxiliary field, a real scalar, can be eliminated from the action by means of its algebraic equation of motion, yielding the standard bosonic GMG action. Nonetheless, in both cases the action contains the combination $(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2)$, and if we take this feature to be the defining one for an extended definition of super GMG models, it is clear that such an extension is not unique. In such models, there is no need for eliminating the auxiliary fields, even when they are non-propagating, unless their field equations are algebraic ones [53].

We now turn to the model with parameters chosen as in (3.45), focus on the maximally supersymmetric AdS vacuum, and determine the spectrum of fluctuations around it. In view of the results of [69], the following background is maximally supersymmetric

$$\bar{R}_{\mu\nu}=-\frac{2}{\ell^2}\bar{g}_{\mu\nu}\ , \qquad \bar{A}=-\frac{1}{\ell}\ , \qquad \bar{V}_\mu=0\ , \qquad \bar{B}=0\ , \tag{3.47}$$

where $\bar{g}_{\mu\nu}$ is the AdS_3 metric, and ℓ is the AdS_3 radius which must obey the equation

$$4\sigma+\ell M+\frac{2}{3\ell^2m^2}=0\ . \tag{3.48}$$

Let us define the fluctuation fields around this vacuum as

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu}\left(1+\frac{1}{3}h\right)+H_{\mu\nu}\ , \qquad \bar{g}^{\mu\nu}H_{\mu\nu}=0\ , \\ A &= \bar{A}+a\ , \qquad B=\bar{B}+b\ , \qquad V_\mu=\bar{V}_\mu+v_\mu\ , \end{aligned} \tag{3.49}$$

and choose the gauge condition

$$\bar{\nabla}^\mu H_{\mu\nu}=0\ . \tag{3.50}$$

The linearized field equations then take the form

$$
\begin{aligned}
&[\mathcal{D}(1)\,\mathcal{D}(-1)\,\mathcal{D}(\eta_+)\,\mathcal{D}(\eta_-)\,H]_{\mu\nu} = -\frac{1}{3\ell^2}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu - \frac{1}{3}\bar{g}_{\mu\nu}\bar{\Box}\right)h\ , \\
&\frac{\Omega}{m^2}\left(\ell^2\bar{\Box} - 3\right)h = 0\ , \qquad \frac{\Omega}{m^2}\,a = 0\ , \qquad \frac{\Omega}{m^2}\,b = 0\ , \\
&\frac{\Omega}{m^2}\left[\mathcal{D}(\eta_+)\,\mathcal{D}(\eta_-)v\right]_\mu = 0\ ,
\end{aligned} \tag{3.51}
$$

where

$$
\eta_\pm = \Omega^{-1}\left(-\frac{\ell m^2}{2\mu} \pm \sqrt{\frac{\ell^2 m^4}{4\mu^2} - \Omega}\right)\ , \qquad \Omega \equiv \sigma\,\ell^2 m^2 - \frac{1}{2}\ , \tag{3.52}
$$

and $\mathcal{D}(\eta)$ is a first-order linear differential operator, parametrized by a dimensionless constant η, that acts on a rank-$s \geq 1$ totally symmetric, traceless and divergence-free tensor as follows

$$
\begin{aligned}
\left[\mathcal{D}(\eta)\,\varphi^{(s)}\right]_{\mu_1\cdots\mu_s} &= [\mathcal{D}(\eta)]_{\mu_1}{}^{\rho}\,\varphi^{(s)}_{\rho\mu_2\cdots\mu_s}\ , \\
[\mathcal{D}\,(\eta)]_\mu{}^\nu &= \ell^{-1}\,\delta_\mu^\nu + \frac{\eta}{\sqrt{|\bar{g}|}}\,\varepsilon_\mu{}^{\tau\nu}\bar{\nabla}_\tau\ .
\end{aligned} \tag{3.53}
$$

The equations for $H_{\mu\nu}$ and h agree precisely with those arising in the $\mathcal{N} = (1,0)$ GMG model [70, 71] whose spectrum was studied in detail in [72], extending earlier results of [73] for the bosonic model. For "non-critical" values of the couplings summarized by the condition $m^{-2}\Omega(\eta_+ - \eta_-)(|\eta_+| - 1)(|\eta_-| - 1) \neq 0$, these equations describe the UIRs of $SO(2,2)$ with lowest weight (E_0, s), and where $\ell^{-1}E_0$ is the lowest energy, and s is the helicity, their values given by

$$
(E_0, s):\quad (2,2),\ (2,-2),\quad \left(1 + \frac{1}{|\eta_+|}, \frac{2\eta_+}{|\eta_+|}\right),\ \left(1 + \frac{1}{|\eta_-|}, \frac{2\eta_-}{|\eta_-|}\right). \tag{3.54}
$$

The new degrees of freedom arising here are generated by the field v_μ. From (3.51) it follows that the propagating modes have the representation content

$$
(E_0, s):\quad \left(1 + \frac{1}{|\eta_+|}, \frac{\eta_+}{|\eta_+|}\right)\ , \quad \left(1 + \frac{1}{|\eta_-|}, \frac{\eta_-}{|\eta_-|}\right)\ . \tag{3.55}
$$

Together with the spin-2 modes displayed in (3.54), these form the bosonic content of a massive spin-2 supermultiplet of $\mathcal{N} = (1,1)$ supersymmetry in three dimensions. The structure of this multiplet is similar to the one studied in [74]. The critical versions of our $\mathcal{N} = (1,1)$ GMG model arises for

$$m^{-2}\Omega(\eta_+ - \eta_-)(|\eta_+| - 1)(|\eta_-| - 1) = 0 \ . \tag{3.56}$$

We shall not examine these points here but we note that the spin-2 sector at critical points has been analyzed in considerable detail in [70]. As for the spin-1 sector, it follows a pattern similar to the one discussed in [74], in the context of a parent supergravity theory whose off-shell degrees of freedom coincide with those of $\mathcal{N} = (1,1)$ supergravity in three dimensions upon a circle reduction.

3.3 $\mathcal{N} = (2,0)$ supergravity

In this section, we will follow the same procedure as before. The main difference is that the compensating multiplet will be a vector multiplet instead of a scalar one.

The construction of the invariants will follow the same strategy delineated before with one difference: one of the invariants, the $R^2_{\mu\nu}$ term, will require a different technique. This will be done in section 3.3.4 by presenting the new method and its results.

3.3.1 The Vector Multiplet

The off-shell $\mathcal{N} = 2$ vector multiplet with 4 + 4 degrees of freedom consists of a gauge field C_μ, a real scalar ρ, a spinor λ and an auxiliary scalar D. Their transformation rules are given by

$$\begin{aligned}
\delta C_\mu &= \frac{1}{2}\,\bar\epsilon\,\gamma_\mu\lambda - \frac{\mathrm{i}}{4}\,\rho\,\bar\epsilon\,\psi_\mu + h.c.\ , \\
\delta\rho &= (\mathrm{i}\,\bar\epsilon\,\lambda + h.c.) + \Lambda_D\rho\ , \\
\delta\lambda &= -\frac{1}{4}\gamma^{\mu\nu}\widehat{G}_{\mu\nu}\,\epsilon + \frac{\mathrm{i}}{2}D\,\epsilon - \frac{\mathrm{i}}{4}\not{\mathcal{D}}\rho\,\epsilon - \frac{\mathrm{i}}{2}\rho\,\eta + \mathrm{i}\Lambda\,\lambda + \frac{3}{2}\Lambda_D\lambda\ , \\
\delta D &= \left(-\frac{\mathrm{i}}{2}\bar\epsilon\not{\mathcal{D}}\lambda + \frac{\mathrm{i}}{2}\bar\eta\,\lambda + h.c.\right) + 2\,\Lambda_D\,D\ ,
\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}
\mathcal{D}_\mu \rho &= (\partial_\mu - b_\mu)\,\rho + \left(-\mathrm{i}\bar{\psi}_\mu \lambda + h.c.\right)\,, \\
\mathcal{D}_\mu \lambda &= \left(\partial_\mu - \frac{3}{2} b_\mu + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}V_\mu\right)\lambda + \frac{1}{4}\gamma^{\rho\sigma}\widehat{G}_{\rho\sigma}\,\psi_\mu \\
&\quad -\frac{\mathrm{i}}{2} D\psi_\mu + \frac{\mathrm{i}}{4}\not{\mathcal{D}}\rho\,\psi_\mu + \frac{\mathrm{i}}{2}\rho\,\phi_\mu\,, \\
\widehat{G}_{\mu\nu} &= 2\,\partial_{[\mu} C_{\nu]} + \left(-\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda + \frac{\mathrm{i}}{2}\rho\,\bar{\psi}_\mu \psi_\nu + h.c.\right)\,.
\end{aligned} \tag{3.58}$$

As we shall discuss in subsection 3.3.4, the nonabelian versions of (3.57) and (3.58) can be obtained by taking the fields of the vector multiplet in the adjoint representation of a Lie group G, and imposing the closure of the algebra accordingly.

> In equation (3.57), we have constructed a vector multiplet $\left(C_\mu\,, \rho, \lambda, D\right)$ with $4+4$ degrees of freedom.

3.3.2 How to Obtain Other Multiplets

For the construction of the n vector multiplet action, we first introduce a real function $C_{IJ}(\rho)$, which is a function of the vector multiplet scalars ρ^I, and the n vector multiplets are labeled by $I, J, \ldots = 1, 2, \ldots, n$. The lowest component of a vector multiplet can then be composed as

$$\rho_I = C_{IJ} D^J + C_{IJK}\,\bar{\lambda}^J \lambda^K\,. \tag{3.59}$$

The label I is fixed, and it differs from the indices that are being summed over. We also define

$$C_{IJK} = \frac{\partial C_{IJ}}{\partial \rho^K}\,, \quad C_{IJKL} = \frac{\partial^2 C_{IJ}}{\partial \rho^K\,\partial \rho^L}\,, \qquad C_{IJKLM} = \frac{\partial^3 C_{IJ}}{\partial \rho^K\,\partial \rho^L\,\partial \rho^M}\,. \tag{3.60}$$

In order to ensure that the ρ_I is the scalar of a superconformal vector multiplet, we impose that the conformal weight of C_{IJ} is $\omega(C_{IJ}) = -1$, and the following constraints are satisfied

$$C_{IJK} = C_{I(JK)}\,, \quad C_{IJK}\,\rho^K = -C_{IJ}\,. \tag{3.61}$$

Furthermore, additional constraints are needed to ensure that λ_I, D_I and $\widehat{G}_{\mu\nu I}$ are also the elements of a superconformal vector multiplet

$$C_{IJKL}\,\rho^L = -2\,C_{IJK}\,, \quad C_{IJKLM}\,\rho^M = -3\,C_{IJKL}\,. \tag{3.62}$$

Applying a sequence of Q- and S-transformations, we find the elements of the composite vector multiplet as

$$\begin{aligned}
\rho_I &= C_{IJ}\,D^J + C_{IJK}\,\bar\lambda^J\lambda^K\,,\\
\lambda_I &= \frac{1}{2}\,C_{IJK}\,D^J\lambda^K - \frac{1}{2}\,C_{IJ}\,\not{\mathcal{D}}\lambda^J - \frac{\mathrm{i}}{4}\,C_{IJK}\,\gamma^{\mu\nu}\widehat{G}^J_{\mu\nu}\lambda^K\\
&\quad -\frac{1}{4}\,C_{IJK}\,\not{\mathcal{D}}\rho^J\lambda^K + C_{IJKL}\,\lambda^L\bar\lambda^J\lambda^K,\\
D_I &= \frac{1}{2}\,C_{IJK}\,D^JD^K + \frac{1}{4}\,C_{IJ}\,\Box^C\rho^J - \frac{1}{4}\,C_{IJK}\,\widehat{G}^J_{\mu\nu}\,\widehat{G}^{\mu\nu K}\\
&\quad +\frac{1}{8}\,C_{IJK}\,\mathcal{D}_\mu\rho^J\mathcal{D}^\mu\rho^K - \frac{1}{2}\,C_{IJK}\,\bar\lambda^J\not{\mathcal{D}}\lambda^K + \frac{1}{2}\,C_{IJK}\,\overline{\mathcal{D}_\mu\lambda^J}\gamma^\mu\lambda^K\,,\\
&\quad -\frac{\mathrm{i}}{2}\,C_{IJKL}\,\bar\lambda^L\gamma^{\mu\nu}\widehat{G}^J_{\mu\nu}\lambda^K + C_{IJKL}\,D^J\bar\lambda^K\lambda^L + C_{IJKLM}\,\bar\lambda^J\lambda^K\bar\lambda^L\lambda^M\,,\\
\widehat{G}_{\mu\nu I} &= \frac{1}{2}\mathcal{D}_\sigma\left(\epsilon_{\lambda\mu\nu}\,C_{IJ}\,\widehat{G}^{\sigma\lambda J}\right) + 2\,\mathrm{i}\,\mathcal{D}_{[\mu}\Big(C_{IJK}\,\bar\lambda^J\gamma_{\nu]}\lambda^K\Big) - \frac{1}{4}\,C_{IJ}\,\rho^J\,\widehat{F}_{\mu\nu},
\end{aligned} \tag{3.63}$$

where the superconformal d'Alambertian for ρ^I is given by

$$\begin{aligned}
\Box^C\rho^I &= \left(\partial^a - 2b^a + \omega_b{}^{ba}\right)\mathcal{D}_a\rho^I + 2f^a_a\rho^I\\
&\qquad + \left(-\mathrm{i}\bar\psi^a\mathcal{D}_a\lambda^I + i\bar\phi_a\gamma^a\lambda^I + h.c.\right)\,.
\end{aligned} \tag{3.64}$$

Note that $\widehat{G}_{\mu\nu I}$ satisfies the Bianchi identity due to the constraints (3.61).

The composition formula (3.64) can be truncated to a map between two vector multiplets by choosing $C_{21} = \rho^{-1}$, in which case one obtains, for the bosonic fields,

$$\begin{aligned}
\rho' &= \rho^{-1}D - \rho^{-2}\bar\lambda\lambda\,,\\
D' &= -\frac{1}{2}\rho^{-2}D^2 + \frac{1}{4}\rho^{-1}\Box^C\rho + \frac{1}{4}\rho^{-2}\widehat{G}_{\mu\nu}\,\widehat{G}^{\mu\nu} - \frac{1}{8}\rho^{-2}\mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho\\
\widehat{G}'_{\mu\nu} &= \frac{1}{2}\mathcal{D}_\sigma\left(\epsilon_{\lambda\mu\nu}\rho^{-1}\widehat{G}^{\sigma\lambda}\right) - \frac{1}{4}\widehat{F}_{\mu\nu}\,,
\end{aligned} \tag{3.65}$$

where 1 labels the multiplet $(\rho, C_\mu, \lambda, D)$, and 2 labels the multiplet $(\rho', C'_\mu, \lambda', D')$. Another composite multiplet is obtained by choosing $C_{31} = -\rho^{-2}\rho'$ and $C_{32} =$

ρ^{-1} in the composition formula (3.64). The bosonic components of the composite multiplet $(\rho'', \lambda'', C''_\mu, D'')$, labeled by 3, can then be written as

$$\begin{aligned}
\rho'' &= -\rho^{-2}\rho' D + \rho^{-1} D' \ , \\
D'' &= \rho^{-3}\rho' D^2 - \rho^{-2} D D' - \frac{1}{4}\rho^{-2}\rho' \Box^C \rho + \frac{1}{4}\rho^{-1}\Box^C \rho' \\
&\quad -\frac{1}{2}\rho^{-3}\rho' \widehat{G}_{\mu\nu}\,\widehat{G}^{\mu\nu} + \frac{1}{2}\rho^{-2}\widehat{G}'_{\mu\nu}\widehat{G}^{\mu\nu} + \frac{1}{4}\rho^{-3}\rho' \mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho \\
&\quad -\frac{1}{4}\rho^{-2}\mathcal{D}_\mu\rho'\mathcal{D}^\mu\rho \ , \\
\hat{G}''_{\mu\nu} &= \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{D}_\sigma\left(-\rho^{-2}\rho'\widehat{G}^{\sigma\lambda} + \rho^{-1}\widehat{G}'^{\sigma\lambda}\right) \ .
\end{aligned} \tag{3.66}$$

3.3.3 Superconformal Actions

Supersymmetric Lagrangians for the vector multiplet can be constructed starting from an action formula which describes the coupling of two vector multiplets as

$$\begin{aligned}
e^{-1}\mathcal{L}_{DD'} &= \rho D' + \rho' D + 2\left(\bar\lambda\lambda' + h.c.\right) - 2\epsilon^{\mu\nu\rho}C_\mu\partial_\nu C'_\rho \\
&\quad -\frac{\mathrm{i}}{2}\left(\rho\,\bar\psi_\mu\gamma^\mu\lambda' + \rho'\bar\psi_\mu\gamma^\mu\lambda + h.c.\right) \\
&\quad -\frac{1}{8}\left(\rho\rho'\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu + h.c.\right) \ .
\end{aligned} \tag{3.67}$$

As a special case, one can set the primed and the un-primed multiplet equal to each other, obtaining [64]

$$\begin{aligned}
e^{-1}\mathcal{L}_D &= 2\rho D - \epsilon^{\mu\nu\rho}C_\mu\,G_{\nu\rho} + 4\bar\lambda\,\lambda - \mathrm{i}\left(\rho\,\bar\psi_\mu\gamma^\mu\lambda + h.c.\right) \\
&\quad -\frac{1}{4}(\rho^2\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu + h.c) \ .
\end{aligned} \tag{3.68}$$

Using the composite multiplets (3.65) in this action formula, we also obtain the conformal vector multiplet action

$$\begin{aligned}
e^{-1}\mathcal{L}_V &= \frac{1}{4}\Box^C\rho + \frac{1}{2}\rho^{-1}D^2 - \frac{1}{8}\rho^{-1}\partial_\mu\rho\,\partial^\mu\rho \\
&\quad -\frac{1}{4}\rho^{-1}G_{\mu\nu}\,G^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho}\,C_\mu\,\partial_\nu V_\rho + \dots \ ,
\end{aligned} \tag{3.69}$$

where the ellipses refer to the fermionic terms.

Considering the coupling of a primed and double-primed multiplets in accordance with the action formula (3.67), and employing the composite expressions (3.66) result into an action that will be used in the construction of a supersymmetric completion of the R^2 term,

$$\begin{aligned} e^{-1}\mathcal{L}_{VV'} &= \rho^{-3}(\rho')^2 D^2 - 2\rho^{-2}\rho' D D' + \rho^{-1}(D')^2 + \frac{1}{4}\rho^{-1}\rho'\Box^c\rho' \\ &\quad -\frac{1}{4}\rho^{-2}\rho'^2\Box^c\rho + \frac{1}{4}\rho^{-3}\rho'^2\mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho - \frac{1}{4}\rho'\rho^{-2}\mathcal{D}_\mu\rho'\mathcal{D}^\mu\rho \\ &\quad -\frac{1}{2}\rho^{-3}(\rho')^2\widehat{G}_{\mu\nu}\,\widehat{G}^{\mu\nu} + \rho'\rho^{-2}\widehat{G}'_{\mu\nu}\,\widehat{G}^{\mu\nu} - \frac{1}{2}\rho^{-1}\widehat{G}'_{\mu\nu}\,\widehat{G}'^{\mu\nu} \end{aligned} \tag{3.70}$$

Here we have provided the terms that contributes to the bosonic part of the action. More generally, we obtain the most general 2−derivative vector multiplet coupling, by using the action formula (3.67), as follows

$$\begin{aligned} e^{-1}\mathcal{L}_{V_I} &= \frac{1}{4}C_{IJ}\,\rho^I\Box^c\rho^J + \frac{1}{8}C_{IJK}\,\rho^I\mathcal{D}_\mu\rho^J\mathcal{D}^\mu\rho^K - \frac{1}{2}C_{IJ}\,\widehat{G}^I_{\mu\nu}\widehat{G}^{\mu\nu J} \\ &\quad -\frac{1}{4}C_{IJK}\,\rho^I\widehat{G}^J_{\mu\nu}\widehat{G}^{\mu\nu K} + C_{IJ}\,D^I D^J + \frac{1}{2}C_{IJK}\,\rho^I D^J D^K \\ &\quad +\frac{1}{4}C_{IJ}\,\rho^J\epsilon^{\mu\nu\rho}C^I_\mu\,F_{\nu\rho}\,. \end{aligned} \tag{3.71}$$

Note that the index I is fixed to represent a certain multiplet by construction due to (3.64), and summing over I indices correspond to summing different off-shell invariants.

To summarize, we use the action (3.67) describing the coupling of two vector multiplets as a blueprint. We then use different multiplets to obtain different invariants. In particular

- $\mathcal{L}_V$, in eq. (3.69), will give us the EH term.
- $\mathcal{L}_D$, in eq. (3.68), will give us the RD term.
- $\mathcal{L}_{VV'}$, in eq. (3.70), will give us the R^2 term.
- The $R^2_{\mu\nu}$ term is going to be obtained later with a different technique.
- $\mathcal{L}_{\mathrm{CS}}$, in eq. (3.31), is the supersymmetric Chern-Simons action.

3.3.4 Gauge Fixing and Invariant Terms

The off-shell Poincaré supergravity is obtained from the action formula (3.69) and gauge-fixing the superconformal transformations by imposing the following gauge conditions

$$\rho = 1, \qquad \lambda = 0, \qquad b_\mu = 0\ , \tag{3.72}$$

where the first choice fixes dilatations, the second fixes the S-supersymmetry and the third fixes the special conformal symmetry. These gauge choices are maintained provided that

$$\begin{aligned} \Lambda_D &= 0\ , \\ \Lambda_{K\mu} &= -\frac{1}{4}\bar{\epsilon}\,\phi_\mu + \frac{1}{4}\bar{\eta}\,\psi_\mu + h.c.\ , \\ \eta &= \frac{\mathrm{i}}{2}\gamma\cdot\widehat{G}\,\epsilon + D\epsilon\ . \end{aligned} \tag{3.73}$$

We therefore end up with the new minimal Poincaré multiplet consisting of a dreibein e_μ^a, a gravitino ψ_μ, a $U(1)_R$ symmetry gauge field V_μ, a vector gauge field C_μ, and an auxiliary scalar D. The resulting local supersymmetry transformation rules are

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}\,\gamma^a\,\psi_\mu + h.c.\,, \\ \delta\psi_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} - \mathrm{i}V_\mu\right)\epsilon - \frac{1}{2}\mathrm{i}\,\gamma_\mu\gamma\cdot\widehat{G}\epsilon - \gamma_\mu\,D\epsilon\,, \\ \delta C_\mu &= -\frac{\mathrm{i}}{4}\,\bar{\epsilon}\,\psi_\mu + h.c.\,, \\ \delta V_\mu &= -\frac{\mathrm{i}}{2}\bar{\epsilon}\,\gamma^\nu\widehat{\psi}_{\mu\nu} + \frac{1}{8}\mathrm{i}\,\bar{\epsilon}\,\gamma_\mu\gamma\cdot\widehat{\psi} - \frac{1}{2}\,\bar{\epsilon}\,\gamma\cdot\widehat{G}\,\psi_\mu + \mathrm{i}D\,\bar{\epsilon}\,\psi_\mu + h.c.\,, \\ \delta D &= -\frac{1}{16}\bar{\epsilon}\,\gamma\cdot\widehat{\psi} + h.c.\,, \end{aligned} \tag{3.74}$$

where the $U(1)_R$ covariant gravitino field strength is given by

$$\widehat{\psi}_{\mu\nu} = 2\left(\partial_{[\mu} + \frac{1}{4}\omega_{[\mu|}{}^{ab}\,\gamma_{ab} - \mathrm{i}V_{[\mu}\right)\psi_{\nu]} - \mathrm{i}\,\gamma_{[\mu}\gamma\cdot\widehat{G}\psi_{\nu]} - 2D\,\gamma_{[\mu}\psi_{\nu]}\ . \tag{3.75}$$

Similarly to the $\mathcal{N} = (1,1)$ case, we gauge-fix the dilatation, conformal boos and special supersymmetry transformations. The conditions are not the same because the compensating multiplet is different, thus we do not "accidentally" fix the $U(1)_R$ gauge symmetry.

$\mathcal{N} = (2,0)$ Cosmological Poincaré Supergravity

Substituting the gauge-fixing conditions (3.72) into the Lagrangian (3.69), and rescaling with a factor of -16, we obtain the following Poincaré supergravity Lagrangian

$$e^{-1}\mathcal{L}_{EH} = R - 2G^2 - 8D^2 - 8\epsilon^{\mu\nu\rho}\, C_\mu\, \partial_\nu V_\rho \ , \tag{3.76}$$

where we have defined

$$G_\mu := \epsilon_{\mu\nu\rho} G^{\nu\rho} \ , \qquad G^2 := G_\mu G^\mu \, . \tag{3.77}$$

Consequently, G_μ is a covariantly conserved tensor $\nabla^\mu G_\mu = 0$. A supersymmetric cosmological constant can be added to the Poincaré supergravity Lagrangian (3.76), which can be obtained from the action formula (3.68), imposing the gauge-fixing choices (3.72), and thus obtaining

$$e^{-1}\mathcal{L}_C = 2D - \epsilon^{\mu\nu\rho}\, C_\mu\, G_{\nu\rho} - \left(\frac{1}{8}\bar{\psi}_\mu\, \gamma^{\mu\nu}\, \psi_\nu + h.c.\right) \ . \tag{3.78}$$

The $\mathcal{N} = (2,0)$ RD and R^2 Invariants

For the construction of the RD invariant, we consider the vector multiplet action (3.68) for the primed vector multiplet $(\rho', C'_\mu, \lambda', D')$. Using the composite expressions given in (3.65) and fixing the redundant superconformal symmetries by using the gauge-fixing choices (3.72), give the supersymmetric completion of the RD action

$$e^{-1}\mathcal{L}_{RD} = RD + 8D^3 - 2G^{\mu\nu}\left(F_{\mu\nu} + \nabla_\mu G_\nu + 2DG_{\mu\nu}\right) + \frac{1}{2}\epsilon^{\mu\nu\rho}\, V_\mu\, F_{\nu\rho} \ , \tag{3.79}$$

where we have rescaled the Lagrangian with an overall factor of -8. Note that although the RD invariant and the Lorentz-Chern-Simons invariant (3.31) have the same conformal $\epsilon^{\mu\nu\rho}V_\mu F_{\nu\rho}$ term, the RD invariant is not conformally invariant as can be understood from the presence of the Ricci scalar. Such non-conformal invariants are studied in detail in the context of Chern-Simons contact terms in three dimensions [75–77].

Next, we construct the supersymmetric completion of R^2. Using the composition formula (3.65) and employing the gauge-fixing conditions (3.72) in the action formula (3.70), we obtain

$$e^{-1}\mathcal{L}_{R^2} = (R+24D^2+2G^2)^2 - 8\left(F_{\mu\nu} + 2\nabla_{[\mu}G_{\nu]} + 4DG_{\mu\nu}\right)^2 + 64D\Box D\,. \quad (3.80)$$

The $\mathcal{N}=(2,0)$ $R^2_{\mu\nu}$ Invariant

The supersymmetric completion of the Ricci tensor-squared term is most conveniently obtained by establishing a map between Yang-Mills and supergravity multiplets. To do so, we begin by gauge fixing the nonabelian version of the transformation rules (3.57) in accordance with (3.72), obtaining

$$\begin{aligned}
\delta C^I_\mu &= \frac{1}{2}\bar\epsilon\gamma_\mu\lambda^I - \frac{1}{4}\mathrm{i}\rho^I\bar\epsilon\,\psi_\mu + h.c.\,,\\
\delta\rho^I &= \mathrm{i}\bar\epsilon\lambda^I + h.c.\,,\\
\delta\lambda^I &= -\frac{1}{4}\gamma^{\mu\nu}\widehat{G}^I_{\mu\nu}\epsilon + \frac{1}{2}\mathrm{i}D^I\epsilon - \frac{1}{4}\mathrm{i}\widehat{\not{D}}\rho^I\epsilon - \frac{1}{2}\mathrm{i}\rho^I D\epsilon + \frac{1}{4}\rho^I\gamma\cdot\widehat{G}\epsilon\,,\\
\delta D^I &= -\frac{\mathrm{i}}{2}\bar\epsilon\,\widehat{\not{D}}\lambda^I + \frac{\mathrm{i}}{2}D\bar\epsilon\lambda^I - \frac{1}{4}\bar\epsilon\,\gamma\cdot\widehat{G}\lambda^I + \frac{1}{4}g\,\bar\epsilon\,f_{JK}{}^I\,\rho^J\lambda^K + h.c.
\end{aligned} \quad (3.81)$$

where

$$\begin{aligned}
\widehat{D}_\mu\rho^I &= \partial_\mu\rho^I + \left(-\mathrm{i}\,\bar\psi_\mu\,\lambda^I + h.c.\right) + g\,f_{JK}{}^I\,C^J_\mu\,\rho^K\,,\\
\widehat{D}_\mu\lambda^I &= (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} - \mathrm{i}V_\mu)\lambda^I + \frac{1}{4}\gamma^{\rho\sigma}\widehat{G}^I_{\rho\sigma}\,\psi_\mu - \frac{\mathrm{i}}{2}D^I\psi_\mu + \frac{\mathrm{i}}{4}\widehat{\not{D}}\rho^I\psi_\mu\\
&\quad + \frac{\mathrm{i}}{2}\rho^I D\,\psi_\mu - \frac{1}{4}\rho^I\gamma\cdot\widehat{G}\psi_\mu + g\,f_{JK}{}^I C^J_\mu\,\lambda^K\,,\\
\widehat{G}^I_{\mu\nu} &= 2\partial_{[\mu}C^I_{\nu]} - \left(\bar\psi_{[\mu}\gamma_{\nu]}\lambda^I - \frac{\mathrm{i}}{2}\rho^I\bar\psi_\mu\,\psi_\nu + h.c.\right) + g\,f_{JK}{}^I\,C^J_\mu\,C^K_\nu\,.
\end{aligned} \quad (3.82)$$

We will next show that the following set of fields

$$(\Omega_\mu{}^{-ab}\,,\widehat{G}^{ab}\,,\widehat{\psi}^{ab}\,,\widehat{F}^{ab}(V_+,\omega,\Omega^-)) \quad (3.83)$$

transforms as a Yang-Mills multiplet $(C^I_\mu\,,\rho^I\,,\lambda^I\,,D^I)$, where the ab index pair plays the role of Yang-Mills index. The definitions of the torsionful spin connection $\Omega_\mu{}^{-ab}$, the gravitino field strength $\widehat{\psi}_{ab}$, and the modified $U(1)_R$ gauge field are given by

$$\begin{aligned}
\Omega_\mu{}^{ab\pm} &= \omega_\mu{}^{ab} \pm 2\,\varepsilon_\mu{}^{ab} D\ , && (3.84)\\
\widehat{\psi}_{ab} &= 2\nabla_{[a}(\omega,\Omega^+,V)\psi_{b]} - \mathrm{i}\,\gamma_{[a}\gamma\cdot\widehat{G}\psi_{b]}\ , && (3.85)\\
V_{a+} &= V_a + \frac{1}{2}\epsilon_a{}^{bc}\widehat{G}_{bc}\ , && (3.86)
\end{aligned}$$

where in the definition of $\widehat{\psi}^{ab}$, the connection ω rotates the Lorentz vector index while the connection Ω^+ rotates the Lorentz spinor index.

First, we calculate the transformation rules for $\omega_\mu{}^{ab}$, D and $\widehat{G}_{ab}$

$$\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\frac{1}{4}\,\bar{\epsilon}\,\gamma_\mu\widehat{\psi}_{ab} + \frac{1}{2}\,\bar{\epsilon}\,\gamma^{[a}\widehat{\psi}^{b]}{}_\mu + D\,\bar{\epsilon}\,\gamma_{ab}\,\psi_\mu - \mathrm{i}\,\bar{\epsilon}\,\psi_\mu\widehat{G}^{ab} + h.c.\ , && (3.87)\\
\delta D &= -\frac{1}{16}\bar{\epsilon}\,\gamma\cdot\widehat{\psi} + h.c.\ , && (3.88)\\
\delta\widehat{G}_{ab} &= -\frac{\mathrm{i}}{4}\bar{\epsilon}\,\widehat{\psi}_{ab} + h.c.\ . && (3.89)
\end{aligned}$$

From the first two transformation rules, we observe that

$$\delta\Omega_\mu{}^{-ab} = -\frac{1}{2}\bar{\epsilon}\,\gamma_\mu\widehat{\psi}^{ab} - \mathrm{i}\,\bar{\epsilon}\,\psi_\mu\widehat{G}^{ab} + h.c.\ . \qquad (3.90)$$

Next, we compute the transformation rule for the gravitino curvature

$$\begin{aligned}
\delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{abcd}(\Omega^+)\epsilon - \mathrm{i}\widehat{F}_{ab}(V)\epsilon - 2\,\mathrm{i}\,\nabla_{[a}(\omega)\widehat{G}_{b]c}\gamma^c\epsilon\\
&\quad -\mathrm{i}\nabla_{[a}(\omega)\,\widehat{G}^{cd}\varepsilon_{b]cd} + 2\,\mathrm{i}\,D\,\widehat{G}_{ab}\epsilon - \widehat{G}_{ab}\gamma\cdot\widehat{G}\epsilon\\
&\quad +\mathrm{i}\,\widehat{G}\gamma_{ab}\,\gamma\cdot\widehat{G}\epsilon\,, && (3.91)
\end{aligned}$$

where $\widehat{R}_{abcd}(\Omega^+)$ represents a torsionful supercovariant Riemann tensor. Using the definition of V_{a+} given in (3.86), the Bianchi identity $\nabla_{[a}\widehat{G}_{bc]} = 0$ and $\widehat{R}_{abcd}(\Omega^+) = \widehat{R}_{cdab}(\Omega^-)$, we rewrite the transformation rule for the gravitino curvature as

$$\begin{aligned}
\delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{cdab}(\Omega^-)\epsilon - \mathrm{i}\widehat{F}_{ab}(V_+)\epsilon\\
&\quad +\mathrm{i}\not{\nabla}(\Omega^-)\widehat{G}_{ab}\,\epsilon - \widehat{G}_{ab}\,\gamma\cdot\widehat{G}\epsilon\ , && (3.92)
\end{aligned}$$

where in $\nabla_\mu(\Omega^-)\widehat{G}_{ab}$, the connection Ω^- rotates both a and b indices. Finally, defining $\widehat{F}_{ab}(V_+,\omega,\Omega^-)$ where ω rotates the Lorentz vector index b, whereas the connection Ω^- rotates the index c in the covariant derivative acting on $\widehat{G}_{bc}$, we have

$$\begin{aligned} \delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{cdab}(\Omega^-)\epsilon - \mathrm{i}\,\widehat{F}_{ab}(V_+,\omega,\Omega^-)\epsilon + \mathrm{i}\not{\nabla}(\Omega^-)\widehat{G}_{ab}\,\epsilon \\ & -\widehat{G}_{ab}\,\gamma\cdot\widehat{G}\,\epsilon + 2\,\mathrm{i}\,DG_{ab}\,\epsilon\,. \end{aligned} \tag{3.93}$$

Finally, we consider the transformation rule for $\widehat{F}_{ab}(V_+,\omega,\Omega^-)$,

$$\begin{aligned} \delta\widehat{F}_{ab}(V_+,\omega,\Omega^-) &= \frac{\mathrm{i}}{4}\bar{\epsilon}\not{\nabla}(\omega,\Omega^-)\widehat{\psi}_{ab} - \frac{\mathrm{i}}{4}D\,\bar{\epsilon}\,\widehat{\psi}_{ab} \\ & +\frac{1}{8}\,\bar{\epsilon}\gamma\cdot\widehat{G}\,\widehat{\psi}_{ab} - \mathrm{i}\,\bar{\epsilon}\,\widehat{G}_{c[a}\widehat{\psi}_{b]}{}^{c} + h.c.\,, \end{aligned} \tag{3.94}$$

where in $\nabla_c(\omega,\Omega^-)\widehat{\psi}_{ab}$ the connection ω acts on the spinor index, whereas Ω^- acts on both a and b indices.

Comparing the transformation rules (3.89), (3.90), (3.93) and (3.94) with those of the nonabelian vector multiplet, we find the following correspondence

$$\begin{aligned} \Omega_\mu{}^{-ab} &\leftrightarrow C_\mu^I\,, & 4\,\widehat{G}^{ab} &\leftrightarrow \rho^I\,, \\ -\widehat{\psi}^{ab} &\leftrightarrow \lambda^I\,, & 2\widehat{F}^{ab}(V_+,\omega,\Omega^-) &\leftrightarrow D^I\,. \end{aligned} \tag{3.95}$$

We now turn to the supersymmetric completion of the Ricci squared term. To this end, we first construct the following Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{YM} &= \frac{1}{4}\Big(G^I_{\mu\nu} - \rho^I G_{\mu\nu}\Big)\Big(G^{\mu\nu I} - \rho^I G^{\mu\nu}\Big) \\ & -\frac{1}{2}(D^I - \rho^I D)^2 + \frac{1}{8}D_\mu\rho^I D^\mu\rho^I\,, \end{aligned} \tag{3.96}$$

describing the bosonic sector of a Yang-Mills multiplet coupled to supergravity. This is obtained by generalizing the superconformal invariant action (3.70) and then gauge-fixing according to (3.72). It is now straightforward to use the map (3.95) which gives the bosonic part of the supersymmetric completion of the Riemann squared action

$$\begin{aligned} e^{-1}\mathcal{L}_{Riem^2} &= \frac{1}{4}\Big(R_{\mu\nu ab}(\Omega^-) - 4G_{ab}\,G_{\mu\nu}\Big)\Big(R^{\mu\nu ab}(\Omega^-) - 4G^{ab}\,G^{\mu\nu}\Big) \\ & -2\Big(F_{ab}(V_+,\omega,\Omega^-) - 2DG_{ab}\Big)\Big(F^{ab}(V_+,\omega,\Omega^-) - 2DG^{ab}\Big) \\ & +2\nabla_\mu(\Omega^-)G_{ab}\,\nabla^\mu(\Omega^-)G^{ab}\,. \end{aligned} \tag{3.97}$$

Finally, expanding the torsion terms and using the definition of three-dimensional Riemann tensor

$$R_{\mu\nu ab} = \varepsilon_{\mu\nu\rho}\,\varepsilon_{abc}\left(R^{\rho c} - \frac{1}{2}e^{\rho c}R\right)\,, \tag{3.98}$$

we obtain the supersymmetric completion of the Ricci squared action

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2_{\mu\nu}} \;=\;& R_{\mu\nu}\,R^{\mu\nu} - \frac{1}{4}R^2 + 4RD^2 + RG^2 - 2R_{\mu\nu}\,G^\mu\,G^\nu + 48D^4 \\ & +8D\Box D + 8D^2G^2 + (G^2)^2 - 2(F_{\mu\nu} + \nabla_{[\mu}G_{\nu]})^2 \\ & -(\nabla_\mu G_\nu + 4DG_{\mu\nu})^2\;, \end{aligned} \tag{3.99}$$

where we recall that $G_\mu := \epsilon_{\mu\nu\rho}\,G^{\nu\rho}$. If desired, a term proportional to $\mathcal{L}_{R^2}$ from (3.80) can be added to this result to obtain the invariant in which the only curvature squared term is that of the Ricci tensor.

We conclude this subsection with a few comments on the existence of an off-shell RD^2 invariant. Considering the vector multiplet action (3.71) and the composite formulæ (3.65)–(3.66), we find the following choices for C_{IJ} to obtain a supersymmetric completion for the RD^2 term:

1. The supersymmetric completion of the RD^2 term can be obtained by supersymmetrizing the $\Box^c(\rho^{-3}D^2)$ term. In order to do so, we can consider two vector multiplets: $(\rho, C_\mu, \lambda, D)$ labeled by 1, and $(\rho'', C''_\mu, \lambda'', D'')$ labeled by 3, and set $C_{13} = \rho^{-1}$. Making this choice, we find that all the terms in the Lagrangian (3.71) cancel each other out, thus not giving rise to an RD^2 invariant.

2. Alternatively, one can consider the supersymmetric completion of the term $\rho^{-2}D^2\Box^c\rho$ which gives rise to an RD^2 term after fixing the gauge. Such a model can be obtained by considering two vector multiplets: $(\rho, C_\mu, \lambda, D)$ labeled by 1, and $(\rho', C'_\mu, \lambda', D')$ labeled by 2, and set $C_{22} = \rho^{-1}$. With this choice, however, we find that the resulting action is the R^2 action given in (3.80).

3. Another alternative is the supersymmetric completion of $\rho^{-2}D\Box^c(\rho^{-1}D)$. This construction also corresponds to the choice $C_{22} = \rho^{-1}$, and coincides with the R^2 action given in (3.80)

In view of these arguments, it is not clear to us how the supersymmetric completion of RD^2 as an off-shell invariant independent of the R^2 and $R^2_{\mu\nu}$ invariants can be obtained within the tensor calculus framework presented in section 3.3.3.

$\mathcal{N} = (2,0)$ Generalized Massive Supergravity

We now consider a combination of the invariants up to dimension four, namely,

$$I = \frac{1}{\kappa^2}\int d^3x \left[M\mathcal{L}_C + \sigma\mathcal{L}_{EH} + \frac{1}{\mu}\mathcal{L}_{CS} + \frac{1}{\nu}\mathcal{L}_{RD} + \frac{1}{m^2}\mathcal{L}_{R^2_{\mu\nu}} + c\,\mathcal{L}_{R^2}\right], \tag{3.100}$$

where $(\sigma, M, \mu, \nu, m^2, c)$ are arbitrary real constants. This action is invariant under the off-shell supersymmetry transformation rules given in (3.74). If we consider the defining feature of a super GMG model to be that it contains the term $R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2$, such an extension is clearly not unique, as discussed earlier. Focusing on maximally supersymmetric backgrounds and ghost-free fluctuations around it, we begin by noting that the metric for such backgrounds is AdS or Minkowski. In the former case, D must be non-vanishing, and this is problematic for ghost-freedom due the presence of the RD^2 term in the action. Such a term is akin to the RA^2 term in the $\mathcal{N} = (1,1)$ model which we were able to eliminate. In the case of a Minkowski background, the presence of the RD^2 term is harmless. Thus, to obtain a maximally supersymmetric Minkowski background, we are led to consider the model with the following choice of parameters

$$M = 0\,, \qquad \nu = \infty\,, \qquad c = -\frac{1}{8m^2}\,. \tag{3.101}$$

In this case, the total Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{GMG} = {}& \sigma\left(R - 2G_\mu G^\mu - 8D^2 - 4G^\mu V_\mu\right) \\ & -\frac{1}{4\mu}\epsilon^{\mu\nu\rho}\left[R_{\mu\nu}{}^{ab}\,\omega_{\rho ab} + \frac{2}{3}\omega_\mu{}^{ab}\,\omega_{\nu b}{}^{c}\,\omega_{\rho ca} - 8V_\mu\partial_\nu V_\rho\right] \\ & +\frac{1}{m^2}\Big[R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 - 2RD^2 - R_{\mu\nu}\,G^\mu G^\nu + \frac{1}{2}RG^2 \\ & -24D^4 + \frac{1}{2}(G^2)^2 - 4D^2G^2 - F_{\mu\nu}F^{\mu\nu} + 8DG^{\mu\nu}\left(F_{\mu\nu} + \nabla_\mu G_\nu\right)\Big]. \end{aligned} \tag{3.102}$$

For the maximally supersymmetric Minkowski background, the fields (D, V_μ, C_μ) are vanishing. Therefore, the analysis of the linearized fluctuations for spin-2 modes around this background is the same as that of standard GMG model, amounting to the purely gravitational part of the action above. Thus, we know that the system describes two massive helicity ± 2 modes with masses [78]

$$m_\pm^2 = -\sigma m^2 + \frac{m^4}{2\mu^2}\left[1 \pm \sqrt{1 - \frac{4\sigma\mu^2}{m^2}}\right] . \tag{3.103}$$

Ghosts are absent for $m^2 > 0$ and $\sigma \leq 0$ [44, 71, 78]. We note that the linearized fluctuation of the field D vanishes. Denoting the linearized vector fluctuations of (V_μ, C_μ) by the same symbols and choosing the Lorentz gauges $\partial_\mu V^\mu = 0$ and $\partial_\mu C^\mu = 0$, one finds that their linearized field equations are

$$\frac{1}{m^2}\Box V^\mu - \frac{1}{\mu}\epsilon^{\mu\nu\rho}\,\partial_\nu V_\rho + \sigma G^\mu = 0 \ , \qquad F_{\mu\nu} + 2\partial_{[\mu}G_{\nu]} = 0 \ . \tag{3.104}$$

A simple manipulation of these equations gives

$$\left[(\Box + \sigma m^2)\delta_\mu^\rho\,\delta_\nu^\sigma - \frac{\sigma m^2}{\mu}\epsilon_{[\mu}{}^{\rho\sigma}\partial_{\nu]}\right]\begin{pmatrix} F_{\rho\sigma} \\ G_{\rho\sigma}\end{pmatrix} = 0 \ , \tag{3.105}$$

Diagonalizing the mass matrix one finds that the masses for V^μ and C^μ are given by the formula (3.103). Thus, we have found the bosonic sector of two massive spin-2 multiplets of $\mathcal{N} = (2, 0)$ supersymmetry.

3.4 Summary

In this chapter, we have introduced and extensively applied the superconformal tensor calculus as the technique that better suits our goal of constructing a supersymmetric extension of a gravity model. We followed the same strategy to achieve an explicit expression of the necessary supersymmetric invariants. The only exception was the case of the supersymmetric completion of the Ricci tensor squared invariant (3.99) with $\mathcal{N} = (2, 0)$ supersymmetry, where we have employed a map between the Yang-Mills multiplet and the Poincaré multiplet.

Moreover, we have determined the relation between the parameters of the resulting Lagrangians so that the spectrum of fluctuations about a maximally symmetric vacuum solution is ghost-free. For ghost-free fluctuations about AdS_3 vacuum, certain type of off-diagonal invariants with mass dimension four, namely RS^2 for $\mathcal{N} = (1, 1)$ supersymmetry and RD^2 for $\mathcal{N} = (2, 0)$ supersymmetry, play a crucial role. We have constructed the former, but surprisingly we have found that the latter does not seem to exist. Consequently, the $\mathcal{N} = (2, 0)$ model does not seem to have a supersymmetric AdS_3 vacuum with a ghost-free spectrum, even though it does admit a supersymmetric Minkowski vacuum that gives a ghost-free massive spin-2 multiplet.

4

Supersymmetric Solutions and Black Holes

The construction performed in the previous chapter gave us two supersymmetric extensions of cosmological General Massive Gravity, a model that combines NMG and TMG.

It is then natural to wonder which solutions of such theories preserve supersymmetry.

Following [79], we will study the supersymmetric backgrounds and black holes solutions of the $\mathcal{N} = (1, 1)$ cosmological NMG.

We shall provide a brief, non-technical, introduction to the methods hereby implemented, followed by a detailed exposition of the results. As in the previous chapter, the key steps will be empathized by a box.

In general, we find the solutions of a theory by solving its equations of motion, which we derive from the action. When supersymmetry is involved, one may also wonder if a given solution preserves the supersymmetries of the theory or not.

Every background solution has vanishing value of fermions and can be fully described by the values of the bosonic fields included in the theory. If the solution is preserving a certain number of supersymmetries, the background is invariant under a subset of local supersymmetries of the supergravity theory.

Schematically [12], we can write the local supersymmetric transformations as

$$\begin{aligned} \delta B(x) &= \bar{\epsilon}(x)\, f_1[B(x)]\, F(x) + \mathcal{O}(F^3), \\ \delta F(x) &= f_2[B(x)]\, \epsilon(x) \; + \mathcal{O}(F^2), \end{aligned} \tag{4.1}$$

where B denotes collectively the bosonic fields and F the fermionic ones. If we require our solution to be invariant, we want both of these variations to vanish. The first condition is automatically satisfied because the fermionic fields are vanishing classically, while the second one will provide us a set of constraints on the spinor $\epsilon(x)$. Moreover, in order to obtain this set of constraints, we only need to solve the equation at the linear level in the fermionic fields, i.e.

$$\delta F(x)|_{\text{lin}} = f_2[B(x)]\, \epsilon(x) \; = 0\,, \tag{4.2}$$

since any other term will vanish due to the vanishing value of the fermionic fields.

The spinors that solve the last equation are called *Killing spinors* and they are useful for three reasons. First, in determining which non-trivial configuration is preserving supersymmetry we also determine how many supersymmetries are preserved. Secondly, finding the Killing spinors also provides a lot of information about the field configuration $B(x)$, such that it gives us a solution for the full set of equations of motion. Finally, often there is an interesting relation between Killing spinors and Killing vectors.

The merit of the $\mathcal{N} = (1,1)$ cosmological New Massive Gravity (CNMG), as we have seen in the previous chapter, is that its spinors are Dirac instead of Majorana. Therefore, we shall observe a larger variety of supersymmetric solutions with respect to the $\mathcal{N} = 1$ case.

The analysis presented in this chapter is performed for the off-shell supersymmetric configurations. The method is particularly powerful because, once the conditions of the possible field configurations are obtained by using the off-shell transformation rules, one can use them to study the solution of any model that respects the same set of transformation rules.

In section 4.1, we will introduce the Killing spinor equation and study its solutions. The existence of a Killing spinor will impose a series of algebraic and differential identities on the metric and the other bosonic fields. These identities are the backbone of our analysis.

An interesting consequence of the Killing spinor equation is that the background solution can be categorized according to the nature of the Killing vector that is formed out of the Killing spinors. We will have two possibilities: a null Killing vector (section 4.2) or a timelike one (section 4.3).

We will conclude the chapter with a study of the supersymmetric Black Holes with AdS_3 and Lifshitz backgrounds in section 4.4.

4.1 $\mathcal{N} = (1,1)$ Cosmological NMG and Killing Spinors

As we have seen in section 3.2.4, the field content of the $\mathcal{N} = (1,1)$ supergravity theory consists of the dreibein $e_\mu{}^a$, the gravitino ψ_μ, a complex scalar S, and a vector V_μ. For convenience, we shall re-write the Lagrangian of our interest. Focusing on the bosonic part of the supersymmetric CNMG Lagrangian, we obtain

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{CNMG}} &= \sigma(R + 2V^2 - 2|S|^2) + 4MA \\ &\quad + \frac{1}{m^2}\bigg[R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 - R_{\mu\nu}V^\mu V^\nu - F_{\mu\nu}F^{\mu\nu} \\ &\qquad + \frac{1}{4}R(V^2 - B^2) + \frac{1}{6}|S|^2(A^2 - 4B^2) \\ &\qquad - \frac{1}{2}V^2(3A^2 + 4B^2) - 2V^\mu B\partial_\mu A\bigg]\,, \end{aligned} \tag{4.3}$$

where (σ, M, m^2) are arbitrary real constants and we have defined $S = A + \mathrm{i}B$. The action corresponding to this Lagrangian is invariant under the following off-shell supersymmetry transformation rules [1]

$$
\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu + h.c.\,, \\
\delta\psi_\mu &= D_\mu(\widehat{\omega})\,\epsilon - \frac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \frac{1}{2}S\gamma_\mu\epsilon^\star\,, \\
\delta V_\mu &= \frac{1}{8}\mathrm{i}\bar{\epsilon}\,\gamma^{\nu\rho}\gamma_\mu\left(\psi_{\nu\rho} - \mathrm{i}V_\sigma\gamma^\sigma\gamma_\nu\,\psi_\rho - S\gamma_\nu\psi^\star_\rho\right) + h.c.\,, \\
\delta S &= -\frac{1}{4}\tilde{\epsilon}\,\gamma^{\mu\nu}\left(\psi_{\mu\nu} - \mathrm{i}V_\sigma\,\gamma^\sigma\gamma_\mu\psi_\nu - S\gamma_\mu\psi^\star_\nu\right)\,,
\end{aligned}
\tag{4.4}
$$

where $\tilde{\epsilon} = \overline{\epsilon^\star}$, $\widehat{\omega}$ is the super-covariant spin-connection (3.5) and

$$
D_\mu(\widehat{\omega})\epsilon = \left(\partial_\mu + \frac{1}{4}\widehat{\omega}_\mu{}^{ab}\,\gamma_{ab}\right)\epsilon\,, \qquad \psi_{\mu\nu} = 2D_{[\mu}(\widehat{\omega})\psi_{\nu]}\,. \tag{4.5}
$$

The transformation rules (4.4) are off-shell as the algebra closes on these fields without imposing the field equations corresponding to the Lagrangian (4.3).

In order to determine the supersymmetric backgrounds allowed by a model with the transformation rules (4.4), one considers the Killing spinor equation

$$
\mathcal{D}_\mu\,\epsilon = \partial_\mu\,\epsilon + \frac{1}{4}\widehat{\omega}_\mu{}^{ab}\,\gamma_{ab}\,\epsilon - \frac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \frac{1}{2}S\gamma_\mu\epsilon^\star = 0\,. \tag{4.6}
$$

Any Killing spinor ϵ satisfying this equation must also satisfy the integrability condition

$$
\begin{aligned}
[\mathcal{D}_\mu,\mathcal{D}_\nu]\epsilon &= \frac{1}{4}\Big(R_{\mu\nu}{}^{\rho\sigma} + 2\delta^\rho_\mu\delta^\sigma_\nu(A^2+B^2) + 2\delta^\rho_\mu\delta^\sigma_\nu V^2 \\
&\qquad\qquad -4\mathrm{i}\delta^\sigma_{[\nu}\nabla_{\mu]}V^\rho - 4\delta^\sigma_{[\nu}V_{\mu]}V^\rho\Big)\gamma_{\rho\sigma}\epsilon \\
&\quad -\delta^\sigma_{[\nu}\Big(\partial_{\mu]}A + BV_{\mu]}\Big)\gamma_\sigma\epsilon^* - \mathrm{i}\delta^\sigma_{[\nu}\Big(\partial_{\mu]}B - AV_{\mu]}\Big)\gamma_\sigma\epsilon^* \\
&\quad -\frac{1}{2}\mathrm{i}F_{\mu\nu}\epsilon + \mathrm{i}\epsilon_{\mu\nu\rho}V^\rho(A+\mathrm{i}B)\epsilon^* = 0\,.
\end{aligned}
\tag{4.7}
$$

[1]Here, we follow the conventions of [69], with the only difference being that the S we are using here is replaced by $S \to -Z$.

Considering the field equations for A, B, V_μ and $g_{\mu\nu}$,

$$
\begin{aligned}
0 &= 4M - 4\sigma A + \frac{1}{m^2}\left[\frac{2}{3}A^3 - B^2 A - 3V^2 A + 2\left(\nabla\cdot V\right)B + 2V^\mu\partial_\mu B\right]\,,\\
0 &= 4\sigma B + \frac{1}{m^2}\left[\frac{1}{2}RB + A^2 B + \frac{8}{3}B^3 + 4V^2 B + 2V^\mu\partial_\mu A\right]\,,\\
0 &= 4\sigma V_\mu - \frac{1}{m^2}\left[2R_{\mu\nu}V^\nu + 4\nabla_\nu F_\mu{}^\nu + V_\mu\left(3A^2 + 4B^2 - \frac{R}{2}\right) + 2B\partial_\mu A\right]\,,\\
0 &= \sigma\Big(R_{\mu\nu} + 2V_\mu V_\nu - \frac{1}{2}g_{\mu\nu}[R + 2V^2 - 2(A^2+B^2)]\Big) - 2g_{\mu\nu}MA\\
&\quad+\frac{1}{m^2}\Bigg[\Box R_{\mu\nu} - \frac{1}{4}\nabla_\mu\nabla_\nu R + \frac{9}{4}RR_{\mu\nu} - 4R^\rho_\mu R_{\nu\rho} - 2F_\mu{}^\rho F_{\nu\rho}\\
&\quad+\frac{1}{4}RV_\mu V_\nu - 2R^\rho_{(\mu}V_{\nu)}V_\rho - \frac{1}{2}\Box(V_\mu V_\nu) + \nabla_\rho\nabla_{(\mu}(V_{\nu)}V^\rho)\\
&\quad+\frac{1}{4}R_{\mu\nu}(V^2 - B^2) - \frac{1}{4}\nabla_\mu\nabla_\nu(V^2 - B^2) - \frac{1}{2}V_\mu V_\nu(3A^2 + 4B^2)\\
&\quad-2BV_{(\mu}\partial_{\nu)}A - \frac{1}{2}g_{\mu\nu}\Big(\frac{13}{8}R^2 + \frac{1}{2}\Box R - 3R^2_{\rho\sigma} - R_{\rho\sigma}V^\rho V^\sigma\\
&\quad+\nabla_\rho\nabla_\sigma(V^\rho V^\sigma) - F^2_{\rho\sigma} + \frac{1}{4}R(V^2 - B^2) - \frac{1}{2}\Box(V^2 - B^2)\\
&\quad+\frac{1}{6}(A^2+B^2)(A^2 - 4B^2) - \frac{1}{2}V^2(3A^2 + 4B^2) - 2BV^\rho\partial_\rho A\Big)\Bigg]\,,
\end{aligned}
\tag{4.8}
$$

it can be seen that, for cosmological Poincaré supergravity (i.e. $m \to \infty$), A, B and V_μ can be eliminated algebraically. In this case, the integrability condition (4.7) reduces to

$$
\Big(R_{\mu\nu}{}^{\rho\sigma} + 2\,\delta^\rho_\mu\,\delta^\sigma_\nu\,M^2\Big)\gamma_{\rho\sigma}\,\epsilon = 0\,,
\tag{4.9}
$$

which implies that the maximally supersymmetric background is either Minkowski with $M = 0$, or AdS_3 with radius $1/M^2$. More solutions, with less supersymmetry, can be obtained by imposing projection conditions on ϵ.

Note that, even with the higher-derivative contributions, the maximally supersymmetric solution is still given by the same background solution with a shifted value of the cosmological constant. The reason for this is can be seen from the integrability conditions (4.7) and the field equations (4.8): the expectation value of A receives a contribution from the higher-derivative corrections whereas B and V_μ do not and, therefore, can still be set to zero.

In the case of cosmological Poincaré supergravity, the auxiliary fields can be eliminated from the theory, resulting in an on-shell supergravity theory with the field content $(e_\mu{}^a, \psi_\mu)$. However, with the higher-derivative contributions added, the massive vector and the real scalars become dynamical and hence cannot be solved algebraically. These 'auxiliary' fields play a crucial role in determining the supersymmetric backgrounds allowed by the CNMG Lagrangian (4.3).

> The Killing spinor equation (4.6) and the integrability condition (4.7) allow us to find the maximally supersymmetric background solutions. More solutions can be obtained by preserving less supersymmetries thanks to the higher-derivative contribution and the consequent role of the auxiliary fields.

Now that we have clarified the maximally supersymmetric backgrounds, let us proceed to the case where we have at least one unbroken supersymmetry. In order to do so, we will briefly review the implications of an off-shell Killing spinor following the discussion of [69]. From the symmetries of the gamma matrices, one finds the following identities for a commuting Killing spinor ϵ

$$\bar\epsilon\,\epsilon^\star = \tilde\epsilon\,\epsilon = 0\,. \tag{4.10}$$

Thus, non-vanishing spinor bilinears can be defined as follows

$$\bar\epsilon\,\epsilon = -\tilde\epsilon\,\epsilon^\star \equiv \mathrm{i}f\,, \quad \bar\epsilon\,\gamma_\mu\,\epsilon = \tilde\epsilon\,\gamma_\mu\,\epsilon^\star \equiv K_\mu\,, \quad \bar\epsilon\,\gamma_\mu\,\epsilon^\star \equiv L_\mu = S_\mu + \mathrm{i}T_\mu\,, \tag{4.11}$$

where f is a real function and $K_\mu\,(L_\mu)$ is a real (complex) vector. Using the Fierz identities for commuting spinors, one can show that

$$K_\mu K^\mu = -f^2\,, \qquad\qquad K_\mu\gamma^\mu\epsilon = \mathrm{i}f\epsilon\,. \tag{4.12}$$

The first equation implies that the vector is either null or timelike. Using the Killing spinor equation (4.6) one finds that

$$\nabla_{(\mu}K_{\nu)} = 0\,, \tag{4.13}$$

showing that K_μ is a Killing vector. Finally, we may derive the following differential identities following from the Killing spinor equation (4.6)

$$\partial_{[\mu}K_{\nu]} = \epsilon_{\mu\nu\rho}\Big(- fV^\rho - \frac{1}{2}\left(SL^\rho + S^\star (L^\star)^\rho\right)\Big) \ , \tag{4.14}$$

$$\partial_\mu f = -\epsilon_{\mu\nu\rho}V^\nu K^\rho - \frac{1}{2}\mathrm{i}\left(SL_\mu - S^\star L^\star_\mu\right) \ . \tag{4.15}$$

We refer to [69] for the readers interested in the derivation of these Killing spinor identities and of other implications of the existence of a Killing spinor.

> We are able to construct spinor bilinears out of the Killing spinor, i.e. $\bar\epsilon\,\gamma_\mu\,\epsilon = \tilde\epsilon\,\gamma_\mu\,\epsilon^\star \equiv K_\mu$, which turns out to be a Killing vector that can be either null or timelike. We will explore these two possibilities in the next two sections.

4.2 The Null Killing Vector

We first consider the case that the function f introduced in eq. (4.11) is zero everywhere, i.e. $f = 0$. This implies that K_μ is a null Killing vector. In our conventions, a Majorana spinor field has all real components.

The first spinor bilinear equation in (4.11) leads to a Dirac spinor ϵ that is proportional to a real spinor ϵ_0 up to a phase factor characterized by an angle θ [69],

$$\epsilon = e^{-\mathrm{i}\frac{\theta}{2}}\,\epsilon_0\,, \tag{4.16}$$

which implies that $L_\mu = e^{\mathrm{i}\theta}K_\mu$. Taking this into account, the differential equation (4.14) reads

$$\partial_{[\mu}K_{\nu]} = -\mathrm{Re}(\mathrm{S}e^{\mathrm{i}\theta})\,\epsilon_{\mu\nu\rho}\,\mathrm{K}^\rho \ . \tag{4.17}$$

Contracting this equation with K^μ we find that

$$K^\mu\,\nabla_\mu\,K_\nu = 0\,. \tag{4.18}$$

The same equation also implies that $K\wedge dK = 0$, i.e. K is hypersurface orthogonal. Thus, there exist functions u and P of the three-dimensional spacetime coordinates such that

$$K_\mu \, dx^\mu = P du \, . \tag{4.19}$$

Eq. (4.18) implies that that K is tangent to affinely parameterized geodesics in the surface of constant P. One can, then, choose coordinates (u, v, x) such that v is an affine parameter along these geodesics, i.e.

$$K^\mu \, \partial_\mu = \frac{\partial}{\partial v} \, . \tag{4.20}$$

By virtue of our choice for K_μ the metric components further simplify to

$$g_{uv} = P(u, x), \qquad g_{vv} = g_{xv} = 0 \, , \tag{4.21}$$

where $P = P(u, x)$ since we demand the null direction to be along the v direction. All these choices yield a metric of the following generic form

$$ds^2 = h_{ij}(x, u) \, dx^i \, dx^j + 2P(x, u) \, du \, dv \, , \tag{4.22}$$

where $x^i = (x, u)$. Without loss of generality, this metric can be cast in the following form by a coordinate transformation [71, 80]

$$ds^2 = dx^2 + 2P(x, u) \, du \, dv + Q(x, u) \, du^2 \, , \tag{4.23}$$

with $\sqrt{|g|} = P$. With these results in hand, the auxiliary fields of the theory should satisfy the following constraints [69]

$$\begin{aligned} V_\mu &= -\frac{1}{2} \partial_\mu \, \theta(x, u) \, , \\ S e^{\mathrm{i}\theta} + S^\star e^{-\mathrm{i}\theta} &= \partial_x \log P(x, u) \, . \end{aligned} \tag{4.24}$$

Setting $f = 0$ leads to a Dirac spinor of the form (4.16). As a consequence, the vector K_μ is hypersuface orthogonal, a result that allows us to find the constraints (4.24) on the auxiliary fields.

In the next subsection we will investigate the solutions of CNMG under the assumption that $f = 0$.

4.2.1 The General Solution

To find the general solution with $f = 0$, we set S to be a constant. To be precise we set $A = -\frac{1}{l}$, and $B = 0$. Using (4.15) we obtain

$$\epsilon_{\mu\nu\rho} V^\nu K^\rho = -\frac{1}{l} K_\mu \sin\theta(u,x) \,. \tag{4.25}$$

The u component of this equation reads

$$\frac{1}{l} K_u \sin\theta(u,x) = P(u,x) V_x \,, \tag{4.26}$$

where we have used that $\varepsilon_{xuv} = 1$. Provided that the function $P(u,x)$ is nowhere vanishing, it is straightforward to integrate the first (vector) equation in (4.24) and obtain

$$\theta(u,x) = \arctan\left(\frac{2\,c(u)\,e^{-2x/l}}{1 - c^2(u)\,e^{-4x/l}}\right) , \tag{4.27}$$

for arbitrary $c(u)$. From the second (scalar) equation in (4.24) we deduce that

$$-\frac{2}{l}\cos\theta(u,x) = \partial_x \log P(u,x) \,, \tag{4.28}$$

which, upon using eq. (4.27), yields

$$P(x,u) = P(u)[\,e^{2x/l} + e^{-2x/l} c^2(u)] \,, \tag{4.29}$$

where $P(u)$ is an arbitrary function of u. We may set $P(u)$ to unity without loss of generality [71]. Using eqs. (4.28)–(4.29) in the vector field equation (4.27), we deduce that $c(u) = 0$ and $\theta(u,x) = n\pi$. In order to fix n we use the trace of the gravity equation and find that $\theta(u,x) = \pi$. We thus find that the metric (4.23) takes the following final form

$$ds^2 = dx^2 + 2\,e^{2x/l}\,du\,dv + Q(x,u)\,du^2 \,. \tag{4.30}$$

This is the general form of a pp-wave metric. Taking the limit $l \to \infty$ gives rise to the pp-wave in a Minkowski background. Setting $l = 1$ and substituting $A = -1$, $B = 0$, $V_x = V_u = V_v = 0$ into the metric field equation, we find that $Q(x,u)$ satisfies the following differential equation

$$(2 + 4\sigma m^2)\,Q' - (9 + 2\sigma m^2)\,Q'' + 8Q''' - 2Q'''' = 0 \,, \tag{4.31}$$

where the prime denotes a derivative with respect to x. The most general solution of this differential equation is given by

$$\begin{aligned} Q(x,u) &= e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}\, C_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}\, C_2(u) \\ &\quad + e^{2x}\, C_3(u) + C_4(u)\,, \end{aligned} \tag{4.32}$$

where the functions $C_i(u)\,, i=1,\cdots,4$, are arbitrary functions of u. We note that this expression for $Q(x,u)$ matches with that of [70,81]. It differs, however, from the expression given in [71]. This is due to the fact that the off-diagonal coupling of gravity to the scalar A was included in the supersymmetric New Massive Gravity model studied in [71], whereas such a term is absent in our case, see eq. (4.3).

The solution for $Q(x,u)$ given in (4.32) has a redundancy represented by the functions $C_3(u)$ and $C_4(u)$ [80]. To make this redundancy manifest we consider the following coordinate transformation

$$x = \widetilde{x} - \tfrac{1}{2}\log a'\,, \qquad u = a(\widetilde{u})\,, \qquad v = \widetilde{v} - \tfrac{1}{4} e^{-2\widetilde{x}}\,\frac{a''}{a'} + b(\widetilde{u})\,, \tag{4.33}$$

where $a(\widetilde{u})$ and $b(\widetilde{u})$ are arbitrary functions of $\widetilde{u}$ and the prime denotes a derivative with respect to $\widetilde{u}$. By choosing the function $a(\widetilde{u})$ and $b(\widetilde{u})$ such that the differential equations

$$\left(\frac{a''}{a'}\right)' - \frac{1}{2}\left(\frac{a''}{a'}\right)^2 - 2(a')^2\, \widetilde{C}_4(\widetilde{u}) = 0\,, \qquad b' + \frac{1}{2} a'\, \widetilde{C}_3(\widetilde{u}) = 0\,, \tag{4.34}$$

are satisfied, the functions $\widetilde{C}_3$ and $\widetilde{C}_4$ can be set to zero. This implies that, without loss of generality, we may set $C_3 = C_4 = 0$. In addition to this, we obtain

$$\begin{aligned} \widetilde{C}_1(\widetilde{u}) &= C_1(a(\widetilde{u}))\, [a'(\widetilde{u})]^{\frac{1}{2}(3+\sqrt{\frac{1}{2}-\sigma m^2})}\,, \\ \widetilde{C}_2(\widetilde{u}) &= C_2(a(\widetilde{u}))\, [a'(\widetilde{u})]^{\frac{1}{2}(3-\sqrt{\frac{1}{2}-\sigma m^2})}\,. \end{aligned} \tag{4.35}$$

There are two special values of parameters which must be handled separately. These are the cases $\sigma m^2 = \pm\frac{1}{2}$. The reason is that, for the $\sigma m^2 = \frac{1}{2}$ case, the function C_1 degenerates with C_2, whereas, for the $\sigma m^2 = -\frac{1}{2}$ case, the function C_1 degenerates with C_4 while the function C_2 degenerates with C_3.

Therefore, we solve the field equation (4.31) for these special cases and display the solutions $Q(x,u)$ for these special values of the parameters explicitly:

$$\begin{aligned} \sigma m^2 = \tfrac{1}{2}: &\quad Q(x,u) = e^x\, D_1(u) + x\, e^x\, D_2(u) + e^{2x}\, D_3(u) + D_4(u)\,, \\ \sigma m^2 = -\tfrac{1}{2}: &\quad Q(x,u) = x\, e^{2x}\, D_1(u) + x\, D_2(u) + e^{2x}\, D_3(u) + D_4(u)\,. \end{aligned} \tag{4.36}$$

Here, $D_i(u)\,,\; i = 1, \ldots, 4$, are arbitrary functions of u. Setting $D_3 = D_4 = 0$, we are led to the following cases:

$$\begin{aligned} \sigma m^2 \neq \pm\tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\, e^{2x}\, du\, dv \\ &\qquad\quad + \Big(e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}\, D_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}\, D_2(u) \Big)\, du^2\,, \\ \sigma m^2 = \tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\, e^{2x}\, du\, dv + \Big(e^x\, D_1(u) + x\, e^x\, D_2(u) \Big)\, du^2\,, \\ \sigma m^2 = -\tfrac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\, e^{2x}\, du\, dv + \Big(x\, e^{2x}\, D_1(u) + x\, D_2(u) \Big)\, du^2. \end{aligned} \tag{4.37}$$

The pp-wave solutions (4.37) coincide with the solutions of $\mathcal{N} = 1$ CNMG [70].

Having found the most general solutions for the null case, we will continue in the next subsection with determining the amount of supersymmetry that these solutions preserve by working out the Killing spinor equation (4.6).

4.2.2 Killing Spinor Analysis

In order to construct the Killing spinors for the pp-wave metric (4.30) we introduce the following orthonormal frame [80]

$$e^0 = e^{\frac{2x}{l}-\beta}\, dv, \qquad e^1 = e^{\beta} du + e^{\frac{2x}{l}-\beta}\, dv, \qquad e^2 = dx\,, \tag{4.38}$$

where $Q(u,x) = e^{2\beta(u,x)}$. It follows that the components of the spin connection are given by

$$\begin{aligned} \omega_{01} &= -\dot\beta\, du - \Big(\beta' - \frac{1}{l}\Big) dx\,, \\ \omega_{02} &= -\Big(\beta' - \frac{1}{l}\Big) e^{\beta}\, du - \frac{1}{l}\, e^{\frac{2x}{l}-\beta}\, dv\,, \\ \omega_{12} &= \beta'\, e^{\beta}\, du + \frac{1}{l}\, e^{\frac{2x}{l}-\beta}\, dv\,, \end{aligned} \tag{4.39}$$

where

$$\dot\beta \equiv \frac{\partial\beta}{\partial u}\,, \qquad \beta' \equiv \frac{\partial\beta}{\partial x}\,. \tag{4.40}$$

For the null case, the Killing spinor equation (4.6) reads

$$0 = d\epsilon + \frac{1}{4}\,\omega_{ab}\,\gamma^{ab}\epsilon + \frac{1}{2l}\,\gamma_a\, e^a\, \epsilon^\star\,. \tag{4.41}$$

We make the following choice of the γ matrices

$$\gamma_0 = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \sigma_1\,, \qquad \gamma_2 = \sigma_3\,, \tag{4.42}$$

where σ_i's are the standard Pauli matrices. With this choice the Killing spinor equation reads

$$\begin{aligned} 0 = d\epsilon \;&+\; \tfrac{1}{2}\Big(\dot\beta\,\sigma_3\,\epsilon - e^{\beta}\beta'(\sigma_1 + \mathrm{i}\sigma_2)\,\epsilon + \frac{1}{l} e^{\beta}\sigma_1\,(\epsilon + \epsilon^\star)\Big) du \\ &-\; \frac{1}{2l}\, e^{\frac{2x}{l}-\beta}\,(\sigma_1 + \mathrm{i}\sigma_2)\,(\epsilon - \epsilon^\star)\, dv \\ &+\; \frac{1}{2}\,\Big(\beta'\sigma_3\,\epsilon - \frac{1}{l}\,\sigma_3\,(\epsilon - \epsilon^\star)\Big)\, dx\,. \end{aligned} \tag{4.43}$$

Decomposing a Dirac spinor into two Majorana spinors as $\epsilon = \xi + \mathrm{i}\zeta$, i.e.

$$\epsilon = \begin{pmatrix} \xi_1 + \mathrm{i}\zeta_1 \\ \xi_2 + \mathrm{i}\zeta_2 \end{pmatrix} , \tag{4.44}$$

we find the following equations for the components

$$\begin{aligned}
0 &= d\xi_1 + \frac{1}{2}\dot\beta\,\xi_1\,du - e^{\beta}(\beta' - \frac{1}{l})\,\xi_2\,du + \frac{1}{2}\,\xi_1\,\beta'\,dx\,, \\
0 &= d\xi_2 + \frac{1}{l}\,e^{\beta}\,\xi_1\,du - \frac{1}{2}\,\dot\beta\,\xi_2\,du - \frac{1}{2}\,\beta'\,\xi_2\,dx\,, \\
0 &= d\zeta_1 + \frac{1}{2}\,\dot\beta\,\zeta_1\,du - e^{\beta}\,\beta'\,\zeta_2\,du - \frac{2}{l}\,e^{\frac{2x}{l}-\beta}\,\zeta_2\,dv + \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_1\,dx\,, \\
0 &= d\zeta_2 - \frac{1}{2}\,\dot\beta\,\zeta_2\,du - \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_2\,dx\,.
\end{aligned} \tag{4.45}$$

The first two equations are uniquely solved by $\xi_1 = \xi_2 = 0$. For the last two equations, the solution for a generic function $\beta(u, x)$ is given by

$$\zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,, \qquad \zeta_2 = 0\,. \tag{4.46}$$

There is an additional solution for the special case that $\beta = x$. It is given by

$$\zeta_1 = (u + 2v)e^{\frac{1}{2}x}\,, \qquad \zeta_2 = e^{-\frac{1}{2}x}\,. \tag{4.47}$$

This solution corresponds to the first case given in eq. (4.37) with $D_1(u) = 0$ and $D_2(u) = 1$. There is, however, a problem with this solution. One must choose $\sigma m^2 = -\frac{1}{2}$ and this conflicts with the condition imposed on this pp-wave solution when we classified the different solutions in the previous subsection. Therefore, we conclude that the pp-wave Killing spinor equation is uniquely solved by

$$\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,. \tag{4.48}$$

> The pp-wave solutions all preserve 1/4 of the supersymmetries. Note that in the Minkowski limit $l \to \infty$, the equations for ξ and ζ degenerate. Thus, the number of Killing spinors are the same for both AdS and Minkowski pp-wave solutions.

We conclude this section by noting that when $D_1 = D_2 = 0$, the metric reduces to

$$ds^2 = dx^2 + 2e^{2x/l}\, du\, dv = dx^2 + e^{2x/l}\,(-dt^2 + d\phi^2)\,, \tag{4.49}$$

which is the AdS_3 metric in a Poincaré patch. In this case, we have

$$e^0 = e^{x/l}\, dt\,, \qquad e^1 = e^{x/l}\, d\phi\,, \qquad e^2 = dx\,, \tag{4.50}$$

which implies that

$$\omega_{02} = -\frac{1}{l}\, e^{x/l}\, dt\,, \qquad \omega_{12} = \frac{1}{l}\, e^{x/l}\, d\phi\,. \tag{4.51}$$

The Killing spinor equation then turns into

$$d\epsilon - \frac{1}{2l}\, e^{x/l}\Big(\sigma_1\epsilon - \mathrm{i}\sigma_2\epsilon^\star\Big)dt - \frac{1}{2l}\, e^{x/l}\Big(\mathrm{i}\sigma_2\epsilon - \sigma_1\epsilon^\star\Big)d\phi + \frac{1}{2l}\,\sigma_3\,\epsilon^\star dx = 0\,. \tag{4.52}$$

Decomposing the Dirac spinor into two Majorana spinors as $\epsilon = \xi + \mathrm{i}\zeta$, see eq. (4.44), the Killing spinor equation gives rise to the following equations

$$\begin{aligned}
0 &= d\xi_1 + \tfrac{1}{2l}\,\xi_1 dx\,,\\
0 &= d\xi_2 - \tfrac{1}{l}\, e^{x/l}\,\xi_1 dt + \tfrac{1}{l}\, e^{x/l}\,\xi_1 d\phi - \tfrac{1}{2l}\,\xi_2 dx\,,\\
0 &= d\zeta_1 - \tfrac{1}{l}\, e^{x/l}\,\zeta_2\, dt - \tfrac{1}{l}\, e^{x/l}\,\zeta_2\, d\phi - \tfrac{1}{2l}\,\zeta_1 dx\,,\\
0 &= d\zeta_2 + \tfrac{1}{2l}\,\zeta_2\, dx\,.
\end{aligned} \tag{4.53}$$

Making use of the fact that that the ξ and ζ equations are decoupled from each other, we find the following four independent solutions:

1. $\xi_1 = 0, \quad \xi_2 = e^{\frac{x}{2l}}, \quad \zeta_1 = \zeta_2 = 0,$
2. $\xi_1 = e^{-\frac{x}{2l}}, \quad \xi_2 = \frac{1}{l}\, e^{\frac{x}{2l}}\,(t-\phi), \quad \zeta_1 = \zeta_2 = 0,$
3. $\xi_1 = \xi_2 = 0, \quad \zeta_1 = e^{\frac{x}{2l}}, \quad \zeta_2 = 0,$
4. $\xi_1 = \xi_2 = 0, \quad \zeta_1 = \frac{1}{l}\, e^{\frac{x}{2l}}\,(t+\phi), \quad \zeta_2 = e^{-\frac{x}{2l}},$

Therefore, the AdS_3 solution has a supersymmetry enhancement with four Killing spinors.

4.3 The Timelike Killing Vector

In this section, we shall consider the case that $f \neq 0$ and hence that K is a timelike Killing vector field. Introducing a coordinate t such that $K^\mu \partial_\mu = \partial_t$, the metric can be written as [69]

$$ds^2 = -e^{2\varphi(x,y)}\left(dt + B_\alpha(x,y)\, dx^\alpha\right)^2 + e^{2\lambda(x,y)}(dx^2 + dy^2)\ , \tag{4.54}$$

where $\lambda(x,y)$ and $\varphi(x,y)$ are arbitrary functions and B_α $(\alpha = x, y)$ is a vector with two components. The dreibein corresponding to this metric is naturally written as

$$e^t{}_0 = f^{-1}\,, \qquad e^t{}_i = -f^2 W_i\,, \qquad e^\alpha{}_0 = 0\,, \qquad e^\alpha{}_i = e^{-\lambda}\delta_i^\alpha\ , \tag{4.55}$$

where we have defined $f \equiv e^\varphi$ and $W_\alpha = e^{2\varphi-\lambda}B_\alpha$. We write $\mu = (t, \alpha)$ for the curved indices and $a = (0, i)$ for the flat ones, respectively. We also require that all functions occurring in the metric (4.55) are independent of the coordinate t. Taking everything into account, the components of the spin connection ω_{abc} in the flat basis read as follows,

$$\begin{aligned}
\omega_{00i} &= -e^{-\lambda} f^{-1}\partial_i f\ ,\\
\omega_{0ij} &= -\omega_{ij0} = f\, e^{-2\lambda}\,\partial_{[i}\left(W_{j]}e^{\sigma} f^{-2}\right)\ ,\\
\omega_{ijk} &= 2e^{-\lambda}\,\delta_{i[j}\partial_{k]}\lambda\ .
\end{aligned} \tag{4.56}$$

Following [69], it can be shown that the existence of a timelike Killing spinor leads to the following relations between the auxiliary fields V_μ, S and the metric functions

$$V_0 = \frac{1}{2}\epsilon^{ij}\,\omega_{ij0}\ , \tag{4.57}$$

$$V_1 - iV_2 = \mathrm{i}e^{-\lambda}\,\partial_z\left(\varphi - \lambda + ic\right)\ , \tag{4.58}$$

$$S = \mathrm{i}e^{-\lambda - ic}\,\partial_z\left(\varphi + \lambda - ic\right)\ , \tag{4.59}$$

$$\epsilon^{ij}\partial_i B_j = -2V_0\, e^{2\lambda-\varphi}\ , \tag{4.60}$$

where $c(x,y)$ is an arbitrary time-independent real function and $z = x + iy$ denotes the complex coordinates.

At this stage we have paved the way for constructing supersymmetric background solutions by exploiting the Killing spinor identities. Making an ansatz for the vector field V_μ, we can now solve eqs. (4.57)–(4.60) and determine the metric functions λ and φ. Following the same logic in [69], we now look for solutions with the following field configuration

$$S = \Lambda\,, \qquad V_a = \text{const}\,, \qquad V_2 = 0\,, \qquad c = 0\,. \tag{4.61}$$

With these choices, the non-vanishing components of the spin connection given in eq. (4.56) in a flat basis read as follows

$$\begin{aligned} \omega_{002} &= -(\Lambda + V_1)\,, \qquad \omega_{112} = \Lambda - V_1\,, \\ \omega_{120} &= \omega_{201} = -\,\omega_{012} = V_0\,. \end{aligned} \tag{4.62}$$

Note that, by setting $V_2 = c = 0$, we can solve for λ and φ using eqs. (4.58)–(4.59) and their integrability conditions (4.7). Furthermore, B_y can be set to zero by a gauge choice. As a result, we obtain the following differential equations for the functions φ, λ and B_x

$$e^{-\lambda}\partial_y\varphi = V_1 + \Lambda, \tag{4.63}$$

$$e^{-\lambda}\partial_y\lambda = \Lambda - V_1, \tag{4.64}$$

$$\partial_y B_x = 2V_0\, e^{2\lambda-\varphi}, \tag{4.65}$$

with $\partial_x\varphi = \partial_x\lambda = 0$.

So far we have not used the equations of motion, we have only considered the constraints that follow from supersymmetry. The solutions of eqs. (4.63)–(4.65) will bifurcate depending on the value of the vector component V_1.

In the next subsection we will classify the supersymmetric solutions of the CNMG Lagrangian (4.3) with respect to the value of this vector field component by imposing the field equations.

4.3.1 Classification of Supersymmetric Background Solutions

In this subsection, we first integrate the differential equations (4.63)–(4.65) depending on the different values of the vector field components V_a, which yields the metric functions λ and φ. Next, we impose the field equations and determine the couplings. The results for the different cases are given in three subsubsections. For the convenience of the reader, we have summarized all supersymmetric background solutions allowed by the theory described by the Lagrangian (4.3) in Table 5.

	V^2	V_0	V_1	Equation	Sol. of STMG?
Round AdS_3	0	0	0	4.68	✓
$AdS_2 \times \mathbb{R}$	>0	0	Λ	4.71	✗
Null-Warped AdS_3	0	$\pm\Lambda$	Λ	4.74	✓
Spacelike Squashed AdS_3	>0	$<\Lambda$	Λ	4.78	✓
Timelike Streched AdS_3	<0	$>\Lambda$	Λ	4.80	✓
AdS_3 pp-wave	0	V_0	εV_0	4.87	✓
Lifshitz	>0	0	$\neq 0,\ \Lambda$	4.92	✗

TABLE 5
Classification of the supersymmetric background solutions of $\mathcal{N} = (1, 1)$ CNMG. The solutions are classified with respect to the values of the components of the auxiliary vector V_a, and compared with the solutions of the $\mathcal{N} = (1, 1)$ TMG theory (STMG).

The case $V_1 = 0$.

We start with the simplest case, i.e. $V_1 = 0$. The supersymmetry constraint equations (4.63)–(4.65) yield

$$\lambda = -\log(-\Lambda y), \qquad \varphi = \log(-\frac{1}{\Lambda y}), \qquad B_x = -\frac{2V_0}{\Lambda}\log(-\Lambda y). \quad (4.66)$$

The vector equation (4.8) then implies $V_0 = 0$ for $\Lambda \neq 0$. Finally, from the scalar equation we fix M to be

$$M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma. \quad (4.67)$$

Thus, the metric becomes

$$ds^2 = \frac{l^2}{y^2}\left(-dt^2 + dx^2 + dy^2\right), \tag{4.68}$$

which describes a **round $\mathbf{AdS_3}$ spacetime** with $l = -\frac{1}{\Lambda}$, see Table 5.

The case $V_1 = \Lambda \neq 0$.

For $V_1 = \Lambda$, we obtain

$$\lambda = 0, \qquad \varphi = 2\Lambda y, \qquad B_x = -\frac{V_0}{\Lambda} e^{-2\Lambda y}. \tag{4.69}$$

The vector and the scalar field equation lead to the following subclasses **A, B** and **C** which we describe below.

A. $V_0 = 0, \quad \Lambda = -2\sqrt{\frac{m^2\sigma}{7}}, \quad M = \frac{7\Lambda^3}{12m^2} + \Lambda\sigma.$

With this choice of parameters the metric reads

$$ds^2 = -e^{4\Lambda y} dt^2 + dx^2 + dy^2. \tag{4.70}$$

After a simple coordinate transformation $y = \frac{\log r}{2\Lambda}$, $x = \frac{x'}{2\Lambda}$ the metric is brought into the following form

$$ds^2 = \frac{l^2}{4}\left(-r^2 dt^2 + \frac{dr^2}{r^2} + dx^2\right), \tag{4.71}$$

which is $\mathbf{AdS_2} \times \mathbb{R}$. This background also appeared in the bosonic version of NMG, although given in different coordinates [45, 82].

B. $V_0 = \pm\Lambda, \quad \Lambda = -\sqrt{\frac{-2m^2\sigma}{7}}, \quad M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma.$

This choice of parameters leads to the metric

$$ds^2 = -e^{4\Lambda y} dt^2 \pm 2e^{2\Lambda y} dt dx + dy^2. \tag{4.72}$$

Performing a coordinate transformation

$$y = l \log u, \qquad t = l x^{-}, \qquad x = \pm \frac{l x^{+}}{2}, \tag{4.73}$$

the metric (4.72) can be put into the more familiar form [83]

$$ds^2 = l^2 \left[\frac{du^2 + dx^+ dx^-}{u^2} - \left(\frac{dx^-}{u^2} \right)^2 \right], \tag{4.74}$$

which is **null warped AdS$_3$**.

C. $V_0 = \pm\sqrt{\frac{7\Lambda^2 - 4m^2\sigma}{21}}$**,** $\quad M = -\frac{\Lambda^3}{3m^2} + \frac{8\Lambda\sigma}{7}$**.**

Using these values for the parameters and fixing the value of V_0 we deduce from the vector equation that

$$ds^2 = \frac{V^2}{\Lambda^2} \Big(dx + \frac{V_0 \Lambda}{V^2}\, e^{2\Lambda y}\, dt \Big)^2 - \frac{\Lambda^2}{V^2}\, e^{4\Lambda y}\, dt^2 + dy^2\, . \tag{4.75}$$

After making a coordinate transformation $\frac{V_0 \Lambda}{V^2} e^{2\Lambda y} = \frac{1}{z}$, the metric reads

$$ds^2 = \frac{V^2}{\Lambda^2} \Big(dx + \frac{dt}{z} \Big)^2 - \frac{1}{z^2} \frac{V^2}{\Lambda^2} \frac{dt^2}{\nu^2} + \frac{dy^2}{4\Lambda^2 z^2}\, , \tag{4.76}$$

where $\nu^2 = 1 - \frac{V^2}{\Lambda^2} < 1$.

This is not yet the end of the story for this subclass: provided that $V^2 > 0$, which implies $7\Lambda^2 + 2m^2\sigma > 0$, we have $1 > \nu^2 > 0$. By making a coordinate transformation

$$x = \frac{x' \nu}{2V}\, , \qquad t = \frac{t' \nu}{2V}\, , \tag{4.77}$$

the metric (4.75) can be cast into the following form

$$ds^2 \;=\; \frac{l^2}{4} \Big[\frac{-dt^2 + dz^2}{z^2} + \nu^2 \Big(dx + \frac{dt}{z} \Big)^2 \Big]\, , \tag{4.78}$$

which is the metric of **spacelike squashed AdS$_3$** with squashing parameter ν^2.

For $V^2 < 0$, i.e. $7\Lambda^2 + 2m^2\sigma < 0$, we perform a coordinate transformation

$$x = \frac{x'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \qquad t = \frac{t'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \tag{4.79}$$

after which the metric (4.75) can be written in the following form

$$ds^2 = \frac{l^2}{4}\left[\frac{dt^2 + dz^2}{z^2} - \nu^2\left(dx + \frac{dt}{z}\right)^2\right], \tag{4.80}$$

where $\nu^2 > 1$. The metric (4.80) is one of the incarnations of the **timelike stretched AdS$_3$** background.

The case $V_1 \neq \Lambda$ and $V_1 \neq 0$.

This class of solutions has $V_1 \neq \Lambda$ and $V_1 \neq 0$. The calculation of the metric functions follows the computations performed in the previous subsubsections with the extra definitions

$$\sigma = -\log(z), \qquad \varphi = \log(z^\alpha), \qquad B_x = -\frac{V_0}{V_1} z^{-(1+\alpha)}, \tag{4.81}$$

where

$$z \equiv (V_1 - \Lambda)y, \qquad \alpha \equiv \frac{V_1 + \Lambda}{V_1 - \Lambda}\,. \tag{4.82}$$

Using the components of the vector equation, we find

$$V_0(V_0^2 - V_1^2)(V_1 - \Lambda) = 0\,. \tag{4.83}$$

From eq. (4.83) it is straightforward to see that this subclass has two different branches, i.e. $V_0 = 0$ and $V_1 = \varepsilon V_0$ with $\varepsilon^2 = 1$. We will discuss these two branches as separate cases **A** and **B** below.

A. $V_1 = \varepsilon V_0\,, \varepsilon = \pm 1\,, \quad V_0 = -\varepsilon\Lambda \pm \sqrt{\frac{\Lambda^2 - 2m^2\sigma}{2}}$**.**

With this choice of parameters the vector equation gives rise to

$$2V_0^2 + 4\varepsilon V_0 + \Lambda^2 + 2m^2\sigma = 0\,. \tag{4.84}$$

The parameter M can be solved by using the field equation for A (4.8) as follows,

$$M = \frac{-\Lambda^3}{6m^2} + \Lambda\sigma \,. \tag{4.85}$$

Plugging in the metric functions, we obtain the following expression for the metric

$$ds^2 = -z^{2\alpha}(-dt + 2\varepsilon z^{-1-\alpha}dx)dt + \frac{1}{(V_1 - \Lambda)^2}\frac{dz^2}{z^2} \,.$$

Performing the coordinate transformation [69]

$$z = u^{\frac{(\Lambda - V_1)}{\Lambda}} \,, \qquad t = lx^- \,, \qquad x = \frac{\varepsilon l x^+}{2} \,, \tag{4.86}$$

this metric can be written as follows

$$ds^2 = l^2 \left[\frac{du^2 + dx^+dx^-}{u^2} - u^{2(\frac{\Lambda - V_1}{\Lambda})} \left(\frac{dx^-}{u^2} \right)^2 \right] \,. \tag{4.87}$$

This is the metric of an **$\mathbf{AdS_3}$ pp-wave**. Note that the limit $V_1 \to \Lambda$ is well defined and gives rise to the *minus branch* of null warped AdS_3 metric in eq. (4.74), as expected.

B. $V_0 = 0$**,** $\quad V_1 = \dfrac{\alpha+1}{\alpha-1}$**,** $\quad M = \frac{\Lambda(9V_1^2 - 2\Lambda^2)}{12m^2} + \Lambda\sigma$**.**

Finally, we consider the case in which $V_0 = 0$. Rather than solving the vector equation for V_1, as we did in the previous cases, we set $V_1 = \dfrac{\alpha+1}{\alpha-1}$ using eq. (4.82). The field equations (4.8) further imply that

$$(1 - 14\alpha - 7\alpha^2)\Lambda^2 + 4m^2(-1+\alpha)^2\sigma = 0, \tag{4.88}$$

whose solution is given by

$$\Lambda = -\sqrt{\frac{4m^2\sigma(\alpha-1)^2}{(1 - 14\alpha - 7\alpha^2)}} \,. \tag{4.89}$$

Here, we would like to restrict our attention to $\alpha < 0$, as α will be minus the Lifshitz exponent, thus giving rise to spacetimes with positive Lifshitz exponent

(1) $\alpha < \frac{1}{7}(-7-2\sqrt{14})$, then $m^2\sigma > 0$,

(2) $\frac{1}{7}(-7-2\sqrt{14}) < \alpha < 0$, then $m^2\sigma < 0$.

Provided that the vector field components are chosen as discussed, we obtain the **Lifshitz** metric

$$ds^2 = l_L^2\Big[-y^{2\alpha}dt^2 + \frac{1}{y^2}(dx^2+dy^2)\Big], \tag{4.90}$$

where l_L is the Lifshitz radius and it is defined as

$$l_L^2 = \frac{1}{(V_1-\Lambda)^2}. \tag{4.91}$$

We have redefined t as $t \to (V_1-\Lambda)^{2\alpha+2}t$. Note that, in the limit $V_1 \to 0$, one obtains the round AdS_3 metric given in eq. (4.68). Taking $y=\frac{1}{r}$ gives the metric in the standard form

$$ds^2 = l_L^2\Big(-r^{-2\alpha}dt^2 + r^2dx^2 + \frac{1}{r^2}dr^2\Big), \tag{4.92}$$

where l_L^2 and V_1 are given in terms of α and Λ as [2]

$$l_L^2 = \Big(\frac{\alpha-1}{2\Lambda}\Big)^2. \tag{4.93}$$

As shown in [69], all the supersymmetric backgrounds that we have found in this section except the AdS_3 metric preserve 1/4 of the supersymmetries.

> In this section, we have classified the solutions summarized in Table 5. With the exception of the AdS_3 metric, all the solutions preserve 1/4 of the supersymmetries.

[2] Note that the standard Lifshitz exponent z in the literature is given by $z=-\alpha$.

4.4 Supersymmetric Black Holes

In this section, we discuss the supersymmetry aspects of black hole solutions of CNMG in an AdS$_3$ or Lifshitz background. The existence of a Killing spinor is highly restricted due to the global requirement that the angular coordinate ϕ should be periodic. As shown in [69], the spacelike squashed AdS_3 solution can be interpreted as an extremal black hole upon making a coordinate transformation. In this section, we will discuss three specific cases of black hole solutions. We start our discussion in subsection 4.4.1 with a generalization of the BTZ black hole, and show that the periodicity condition implies the extremality of the black hole. In the subsection 4.4.2 we investigate the 'logarithmic' black hole, given in [84], and show that it is supersymmetric. Finally, in a third subsection we investigate the possible black holes in a Lifshitz background.

4.4.1 The Rotating Hairy BTZ Black Hole

The CNMG Lagrangian (4.3) admits the following rotating black hole solution [85]

$$ds^2 = -N^2 F^2 dt^2 + \frac{dr^2}{F^2} + r^2 \left(d\phi + N^\phi dt\right)^2 \ , \tag{4.94}$$

where N, N^ϕ and F are functions of the radial coordinate r, given by

$$\begin{aligned} N^2 &= \left[1 + \frac{b}{4H}\left(1 - \Xi^{\frac{1}{2}}\right)\right]^2 \ , \\ N^\phi &= -\frac{\mathcal{J}}{2\mathcal{M}r^2}\left(\mathcal{M} - bH\right) \ , \\ F^2 &= \frac{H^2}{r^2}\left[H^2 + \frac{b}{2}\left(1 + \Xi^{\frac{1}{2}}\right)H + \frac{b^2}{16}\left(1 - \Xi^{\frac{1}{2}}\right)^2 - \mathcal{M}\ \Xi^{\frac{1}{2}}\right] \ , \end{aligned} \tag{4.95}$$

and

$$H = \left[r^2 - \frac{1}{2}\mathcal{M}\left(1 - \Xi^{\frac{1}{2}}\right) - \frac{b^2}{16}\left(1 - \Xi^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \ , \tag{4.96}$$

where we have set the AdS_3 radius $l = 1$. Here $\Xi := 1 - \mathcal{J}^2/\mathcal{M}^2$, and the rotation parameter $\mathcal{J}/\mathcal{M}$ is bounded in terms of the AdS radius according to

$$-1 \leq \mathcal{J}/\mathcal{M} \leq 1 \, . \tag{4.97}$$

The parameter b is the gravitational hair and, for $b = 0$, one recovers the BTZ black hole [28]. Since we impose the global requirement that ϕ should be periodic, i.e. $0 \leq \phi \leq 2\pi$, the vacuum of the BTZ black hole with gravitational hair, defined by $\mathcal{M} = \mathcal{J} = b = 0$, admits only two Killing spinors. In order to see that, we consider the Killing spinor equations (4.53). Since the equations for ξ_1 and ζ_2 enforce exponential solutions for ξ_1 and ζ_2, we cannot find a solution for ξ_2 and ζ_1 that is periodic in ϕ. Therefore, finding a periodic solution requires setting $\xi_1 = \zeta_2 = 0$. This implies that only two of the solutions of equations (4.53) are valid.

Introducing the following orthonormal frame for the metric

$$e^0 = NFdt\,, \qquad e^1 = rd\phi + rN^\phi dt\,, \qquad e^2 = F^{-1}dr\,, \tag{4.98}$$

the spin-connection components are given by

$$\begin{aligned} \omega_{01} &= \frac{1}{2}\frac{rN^{\phi\prime}}{FN}dr\,, \qquad \omega_{02} = \left(-FNF' + \frac{r^2 N^\phi N^{\phi\prime}}{2N} - F^2 N'\right)dt + \frac{r^2 N^{\phi\prime}}{2N}d\phi\,, \\ \omega_{12} &= \frac{1}{2}F(2N^\phi + rN^{\phi\prime})dt + Fd\phi\,, \end{aligned} \tag{4.99}$$

and hence the Killing spinor equation reads

$$\begin{aligned} 0 = d\epsilon &+ \frac{1}{2}\Big(-\frac{rN^{\phi\prime}}{2FN}\sigma_3\epsilon + \frac{1}{F}\sigma_3\epsilon^\star\Big)dr + \frac{1}{2}\Big(\frac{r^2 N^{\phi\prime}}{2N}\sigma_1\epsilon - \mathrm{i}F\sigma_2\epsilon + r\sigma_1\epsilon^\star\Big)d\phi \\ &+ \frac{1}{2}\Bigg[\Big(-FNF' + \frac{r^2 N^\phi N^{\phi\prime}}{2N} - F^2 N'\Big)\sigma_1\epsilon - \mathrm{i}\Big(FN^\phi + \frac{1}{2}rFN'\Big)\sigma_2\epsilon \\ &\qquad + \mathrm{i}NF\sigma_2\epsilon^\star + rN^\phi\sigma_1\epsilon^\star\Bigg]dt\,. \end{aligned} \tag{4.100}$$

Decomposing the Dirac spinor into two Majorana spinors like in eq. (4.44), we obtain the following equations

$$
\begin{aligned}
0 &= d\xi_1 + \frac{1}{4N}\Big(N^\phi\left[2N(r-F)+r^2N^{\phi\prime}\right] \\
&\qquad -FN(-2N+2rF'+rN^{\phi\prime}+2FN')\Big)\xi_2\,dt \\
&\quad +\frac{1}{4N}\Big(2N(r-F)+r^2N^{\phi\prime}\Big)\xi_2\,d\phi+\frac{1}{4FN}\Big(2N-rN^{\phi\prime}\Big)\xi_1\,dr\,, \\
0 &= d\xi_2 + \frac{1}{4N}\Big(N^\phi\left[2N(r+F)+r^2N^{\phi\prime}\right] \\
&\qquad +FN(-2N-2rF'+rN^{\phi\prime}-2FN')\Big)\xi_1\,dt \\
&\quad +\frac{1}{4N}\Big(2N(r+F)+r^2N^{\phi\prime}\Big)\xi_1\,d\phi+\frac{1}{4FN}\Big(-2N+rN^{\phi\prime}\Big)\xi_2\,dr\,, \\
0 &= d\zeta_1 + \frac{1}{4N}\Big(N^\phi\left[-2N(r+F)+r^2N^{\phi\prime}\right] \\
&\qquad -FN(2N+2rF'+rN^{\phi\prime}+2FN')\Big)\zeta_2\,dt \\
&\quad +\frac{1}{4N}\Big(-2N(r+F)+r^2N^{\phi\prime}\Big)\zeta_2\,d\phi-\frac{1}{4FN}\Big(2N+rN^{\phi\prime}\Big)\zeta_1\,dr\,, \\
0 &= d\zeta_2 + \frac{1}{4N}\Big(N^\phi\left[-2N(r-F)+r^2N^{\phi\prime}\right] \\
&\qquad +FN(2N-2rF'+rN^{\phi\prime}-2FN')\Big)\zeta_1\,dt \qquad (4.101) \\
&\quad +\frac{1}{4N}\Big(-2N(r-F)+r^2N^{\phi\prime}\Big)\zeta_1\,d\phi+\frac{1}{4FN}\Big(2N+rN^{\phi\prime}\Big)\zeta_2\,dr\,.
\end{aligned}
$$

From these equations it follows that for the generic case not all the $d\phi$ components can be set to zero, which is the requirement for finding a periodic Killing spinor. Therefore, we turn our attention to the extremal solutions with $\mathcal{M}=|\mathcal{J}|$. For this case we find the following Killing spinors that are periodic in ϕ

(1) $\underline{\mathcal{M}=-\mathcal{J}}$

$$
\xi_1=\zeta_1=\zeta_2=0\,, \qquad \xi_2=\frac{b+\sqrt{-b^2+8J+16r^2}}{\sqrt{r}}\,, \tag{4.102}
$$

(2) $\underline{\mathcal{M} = \mathcal{J}}$

$$\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = \frac{b + \sqrt{-b^2 - 8J + 16r^2}}{\sqrt{r}}\,. \tag{4.103}$$

Note that for zero hair, i.e. $b \to 0$, one re-obtains the Killing spinors for a BTZ black hole.

4.4.2 The 'Logarithmic' Black Hole

The supersymmetric CNMG Lagrangian (4.3) also admits the following so-called 'logarithmic' black hole solution [84]

$$ds^2 = -\frac{4\rho^2}{l^2 f^2(\rho)}\, dt^2 + f^2(\rho)\Big(d\phi - \varepsilon\, \frac{q\, l \ln[\frac{\rho}{\rho_0}]}{f^2(\rho)}\, dt\Big)^2 + \frac{l^2}{4\rho^2}\, d\rho^2\,, \tag{4.104}$$

where $q \le 0$ and $0 < \phi < 2\pi$. The function $f^2(\rho)$ is defined by

$$f^2(\rho) = 2\rho + q\, l^2 \ln[\frac{\rho}{\rho_0}]\,, \tag{4.105}$$

and the parameter $\varepsilon = \pm 1$ determines the direction of the rotation since

$$M = 2q\,, \qquad J = 2\,\varepsilon\, lq\,. \tag{4.106}$$

Setting $q = 0$ and making the coordinate transformation $\rho = r^2/2$, we obtain an AdS_3 background with ϕ being periodic. This implies that the background of the 'logarithmic' black hole preserves only half of the supersymmetries like in the case of the rotating hairy BTZ black hole in the previous subsection.

We now determine the explicit expressions for the Killing spinors. Introducing the following orthonormal frame for the metric

$$e^0 = \frac{2\rho}{l f(\rho)}\, dt\,, \qquad e^1 = f(\rho)\, d\phi - \frac{lq\,\varepsilon \ln[\frac{\rho}{\rho_0}]}{f(\rho)}\, dt\,, \qquad e^2 = \frac{l}{2\rho} d\rho\,, \tag{4.107}$$

we find the following expressions for the spin connection components

$$
\begin{aligned}
\omega_{01} &= -\frac{l^2 q\,\varepsilon}{4\rho^2 f(\rho)}\Big[f(\rho) - 2\rho\, f'(\rho)\ln[\frac{\rho}{\rho_0}]\Big]\, d\rho\,,\\
\omega_{12} &= -\frac{q\,\varepsilon}{f(\rho)}\, dt + \frac{2\rho\, f'(\rho)}{l}\, d\phi\,,\\
\omega_{02} &= -\frac{1}{2l^2\,\rho\, f^2(\rho)}\Bigg(f(\rho)\Big[8\rho^2 - l^4 q^2\ln[\frac{\rho}{\rho_0}]\Big]\\
&\qquad\qquad +2\rho f'(\rho)\Big[-4\rho^2 + l^4 q^2\ln[\frac{\rho}{\rho_0}]\Big]\Bigg)dt\\
&\qquad\qquad -lq\,\epsilon\Big(\frac{f(\rho)}{2\rho} + \ln[\frac{\rho}{\rho_0}]\, f'(\rho)\Big)\, d\phi\,.
\end{aligned}
\tag{4.108}
$$

Using these expressions in the Killing spinor equation (4.6), we find that the Killing spinors of the logarithmic black hole are given by

i. $\underline{\varepsilon = 1}$

$$
\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = \sqrt{\frac{\rho}{\rho_0}}\Big(\frac{1}{2r + l^2 q\ln[\frac{\rho}{\rho_0}]}\Big)^{1/4}\,, \tag{4.109}
$$

ii. $\underline{\varepsilon = -1}$

$$
\xi_1 = \zeta_1 = \zeta_2 = 0\,, \qquad \xi_2 = \sqrt{\frac{\rho}{\rho_0}}\Big(\frac{1}{2r + l^2 q\ln[\frac{\rho}{\rho_0}]}\Big)^{1/4}\,. \tag{4.110}
$$

The result may be somewhat surprising considering the expectation that the only existing supersymmetric black hole in an AdS_3 background is an extremal BTZ black hole [70]. However, unlinke the rotating BTZ black hole, the 'logarithmic' black hole does not have a non-extremal limit $J \neq M$. Thus, one cannot recover a static, non-supersymmetric black hole from the $J \to 0$ limit of the 'logarithmic' black hole. Therefore, this particular case evades the argument presented in [70].[3]

[3]We thank Paul Townsend for a clarifying discussion on this exceptional case.

4.4.3 Searching For a Supersymmetric Lifshitz Black Hole

In this section, we briefly present our attempts to find a supersymmetric Lifshitz black hole. Following [86], we first try to saturate the BPS bound using the vector field V_μ, since it can, in principle, contribute as a massive vector hair. In order to do so, we consider the following metric ansatz

$$ds^2 = l_L^2\Big(- adt^2 + r^2dx^2 + \frac{1}{f}dr^2\Big)\,, \tag{4.111}$$

where the functions a and f depend on the coordinate r only. With this ansatz for the metric, one can show that the Killing spinor equation imposes the following constraint on these functions

$$\frac{a'\sqrt{f}}{a} + \frac{2\sqrt{f}}{r} + 2(\alpha - 1) = 0\,. \tag{4.112}$$

Having obtained this constraint, we next turn to the vector equation (4.8). Using the metric ansatz (4.111), the V_0 and V_2 components of the vector equation are automatically satisfied, while the V_1 component reads

$$\begin{aligned} 0 \;\; = \;\; & (1+\alpha)\Big[r^2 f a' + 2a^2\Big(- 8f + r[2r(-1+5\alpha+\alpha^2) + 5f']\Big) \\ & -ra\Big(ra'f' + 2f(-5a' + ra'')\Big)\Big]. \end{aligned} \tag{4.113}$$

Imposing the Killing spinor constraint (4.112) to simplify the vector equation, we obtain

$$-7r\sqrt{f}(\alpha - 1) - 11f + r\Big(r(7\alpha - 2) + 3f'\Big) = 0\,. \tag{4.114}$$

Since we wish to find a solution for f that has a double root at $r = r_0$, which is a necessary condition for an extremal black hole, we need to be able to eliminate the f terms in the vector equation. Using the fact that the Killing spinor constraint (4.112) can be cast into the following form

$$\sqrt{f}(1-\alpha) = -\frac{1}{2}\Big(\frac{a'}{a} + \frac{2}{r}\Big)f\,, \tag{4.115}$$

the vector equation can be written as

$$\frac{7}{2}r\Big(\frac{a'}{a} - \frac{8}{7r}\Big)f + r\Big(r(7\alpha - 2) + 3f'\Big) = 0\,, \tag{4.116}$$

which has the following solution

$$a = r^{8/7} , \qquad f = r^2 - r_0^2 . \tag{4.117}$$

However, using this equation in the Killing spinor constraint (4.112), we find that $r_0 = 0$. A further check with the metric equation also imposes $r_0 = 0$. Therefore, although the Killing spinor equation allows the existence of a supersymmetric black hole, we find that the vector and metric equations are incompatible with that possibility.

Alternatively, one may try to start with a rotating Lifshitz black hole using the following metric ansatz

$$ds^2 = l_L^2 \left[-r^{-2\alpha} F(r) dt^2 + \Big(r dx + r^{-\alpha} G(r) dt \Big)^2 + \frac{1}{r^2 F(r)} dr^2 \right] , \tag{4.118}$$

where $F(r)$ and $G(r)$ are arbitrary functions that depend on the coordinate r only. In this case, the Killing spinor equation constrains the function $F(r)$ to be of the form

$$F(r) = 1 + a r^{-2+2\alpha} , \tag{4.119}$$

where a is a constant. Furthermore, the vector equation constraints the function $G(r)$ via the following differential equation

$$21 r^4 G'^2 - 42 r^3 (\alpha + 1) G G' + 21 r^2 (1 + \alpha)^2 G^2 + 4 a r^{2\alpha} (6\alpha - 11) = 0 . \tag{4.120}$$

Using the solutions of this differential equation, along with the expression (4.119) in the gravity equation, we find that it takes us back to the Lifshitz background, not allowing a rotating black hole solution.

The result of this subsection is somewhat expected, considering the fact that for the rotating Lifshitz solution known to us [87], the couplings are determined by using a stationary Lifshitz spacetime which has a rotation term. This is not allowed by the given matter configuration of the $\mathcal{N} = (1, 1)$ CNMG theory.

Finally, we would like to comment that, although our attempts to find a supersymmetric Lifshitz black hole were not successful with the parity-even theory under our consideration (4.3), one may consider to modify the CNMG by adding a parity violating Lorentz-Chern-Simons term, which gives rise to the so-called $\mathcal{N} = (1, 1)$ Generaized Massive Gravity [53]. In that case, we found that the vector equation is modified in such a way that the Lifshitz background is no longer a solution with the field configuration given in (4.61).

4.5 Summary

In this chapter, we have used the off-shell Killing spinor analysis to investigate the supersymmetric backgrounds of the $\mathcal{N} = (1,1)$ CNMG model given by the Lagrangian (4.3). The background solutions are classified according to the norm of the Killing vector constructed out of Killing spinors. There are only two cases.

The first case provides for the possibility of having a null Killing vector, see section 4.2. Here, the $\mathcal{N} = (1,1)$ analysis reduces to that of the $\mathcal{N} = 1$ CNMG model since the null choice forces the auxiliary massive vector V_μ and the auxiliary pseudo-scalar B to vanish. Therefore, the solution is of the pp-wave type which preserves 1/4 of the supersymmetries. In the AdS_3 limit, there is a supersymmetry enhancement, and the AdS_3 solution is maximally supersymmetric.

As a second case, in section 4.3 we have investigated the case where the Killing vector is taken to be timelike. In particular, we did consider a special class of solutions in which the pseudo-scalar B vanishes. In that case, all the supersymmetric solutions can be classified in terms of the components V_a of the massive vector in the flat basis. A subclass of these solutions, with different parameters, are also solutions of the supersymmetric TMG model, see Table 5. In addition to these solutions, we have found that the $\mathcal{N} = (1,1)$ CNMG model possesses Lifshitz and $AdS_2 \times \mathbb{R}$ solutions. All these background solutions preserve 1/4 of the supersymmetries.

We then dedicated our efforts to the investigation of three cases of black hole solutions in a AdS_3 or Lifshitz background. In the case of AdS_3, we studied the rotating hairy BTZ black hole and the logarithmic black hole. We have found that in general the rotating hairy BTZ black hole is not supersymmetric due to the fact that the periodicity condition on the ϕ coordinate and the periodic Killing spinors only arise when the black hole is extremal. In the case of the logarithmic black hole, we found that only the extremal black hole solution exists, which is supersymmetric by its own nature. Finally, we analyzed the conditions for the existence of a supersymmetric Lifshitz black hole and showed that it does not exist given the field configuration of the $\mathcal{N} = (1,1)$ CNMG model.

5

Holographic Entanglement Entropy

This chapter will be dedicated to explore a different merit of New Massive Gravity. As we have seen in the previous chapter, despite the simplifications introduced by working in only three dimensions, the theory admits a rich set of solutions.

Following [88], we will use a holographic method to compute Entanglement Entropy in the context of a higher-derivative theory of gravity such as NMG.

We will thus observe that the presence of the higher-derivative terms determines a change in the nature of the holographic surface describing this quantity.

This chapter will give us a chance to see a simple application of the proposal introduced in Chapter 1. We will then be able to focus on richer geometries.

We have seen in section 1.2 how, despite the simplicity of the definition of Entanglement Entropy (EE), we are often forced to use tricks and techniques to practically perform the calculation. In particular, we have seen that the computation of EE in quantum field theory often requires the use of the so-called *replica trick* [9]. Here, we can compute the Rényi entropy of n copies of the system and then take the limit $n \to 1$ to obtain the EE. However, for higher dimensional conformal field theories, the replica method can be applicable only for certain topologies of the entangling region. It is also important to note that, although the presence of infinitely many degrees of freedom in field theory makes this quantity divergent, it can be regularized by introducing a UV cut-off [9,10]. Finally, see section 1.2.2, these obstacles have been overcome by considering a holographic realization of Entanglement Entropy (HEE) originally proposed by Ryu and Takayanagi (RT) [15, 16]. To simplify the reading, let's briefly summarize the proposal.

The entanglement entropy S_A of a region A in a d dimensional boundary theory holographically corresponds to the area of a codimension-2 spacelike minimal surface γ_A in the $(d+1)$-dimensional dual gravity theory. The minimal surface is anchored to the boundary in such a way that it satisfies the homology constraint $\partial\gamma_A = \partial A$. The EE is then computed by

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}}\,, \tag{5.1}$$

where $G_N^{(d+1)}$ is the $(d+1)$-dimensional Newton constant.

The holographic proof of the RT proposal, presented by Lewkowycz and Maldacena (LM) in [21], is based on the implementation of the n-copy replica trick in the dual bulk geometry. In this way, the replicated bulk geometry acquires a conical singularity on the hypersurface, as explained in section 1.2.2. Once we impose the limit $n \to 1$, we notice that the hypersurface converges to the usual minimal surface and, by collecting the leading divergences, we obtain the desired EE.

It is a very appropriate question to ask whether the LM formalism remains valid beyond the Einstein-Hilbert theory. The effective description of the UV limit of the Einstein-Hilbert theory comprises higher-derivative terms and turns out to be one of the most natural arenas for this investigation. The LM formalism encourages the generalized formulations of holographic entanglement entropy for various higher-derivative theories [89–91]. These generalizations allow contributions coming from the Wald's entropy [92] as well as from the extrinsic curvature evaluated on the entangling surface. In particular, for our present analysis we follow the prescription of [91] for the holographic computation.

The philosophy of the generalization proposed in [91] goes as follows. When we take the $n \to 1$ limit, we are interested only in the $\mathcal{O}(n-1)$ terms because they will give the EE. In the case of Einstein gravity, the variation of the action is purely a boundary term at this order due to the equations of motion. This is not necessarily true for higher-derivative gravity. In other words, the number of terms that can give a linear contribution is enhanced by the presence of higher derivatives, thus modifying the functional describing the entanglement entropy.

We will investigate this prescription in the context of New Massive Gravity (NMG) [44] due to its wide class of background solutions. In addition to that, since such a theory lives in only three dimensions, the co-dimension 2 surface is simply a line and thus the technical obstacles to carry out the actual holographic computations are drastically reduced. However, due to the presence of higher-derivative terms, NMG provides a structure complex enough to observe nontrivial changes in the behavior of EE. For instance, we will show the existence of new minimal surfaces.

Motivated along this line of research, we compute the holographic entanglement entropy for Lifshitz and Warped AdS backgrounds in New Massive Gravity. Similarly to [93], we adopt the perturbative approach for our analysis. We observe that a suitable perturbative ansatz for the entangling surface significantly reduces the technical complexities of extremizing the higher-derivative entropy functional. In particular, we study the nontrivial modification of the AdS_3 geodesic by introducing a suitable perturbation resulting from the higher-derivative contribution in the NMG theory. The modified entangling surface satisfies the equation of motion derived from the entropy functional order by order with a certain set of appropriate boundary conditions. We also give an interpretation of our holographic analysis consistent with the corresponding boundary theory.

This chapter is organized as follows. In section 5.1, we briefly introduce the necessary ingredients to determine the geometry of the entangling surface and compute the entanglement entropy. In section 5.2, we reproduce the analysis presented in [94] in order to review the procedure in the simple example provided by the Anti-de Sitter spacetime. We then apply the same technique to the Lifshitz spacetime in section 5.3. In this background, we will prove the existence of a new entangling surface by deforming the geodesic. Consequently, we establish that the existent analysis in this regard [95] requires further attention. As a final example, in section 5.4 we investigate the deformation of the entangling surface for the Warped AdS_3 spacetime. Previous works on this particular case can be found in [96–98], but our analysis will focus specifically on the higher-derivative contribution.

5.1 NMG and Holographic Entanglement Entropy

We have seen in Chapter 2 how New Massive Gravity [44] is a modification of the Einstein-Hilbert gravity theory realized by adding a combination of higher-derivative terms. In particular, the action, in the Euclidean signature, takes the following form

$$S = -\frac{1}{16\pi G}\int d^3x\sqrt{g}\left[R + \frac{2}{L^2} + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right]. \tag{5.2}$$

One of the merits of this theory is that, although it lives in three dimensions, it presents a richer dynamics compared to Einstein gravity without introducing many of the well-known pathologies that we can find in general four dimensional higher-derivative theories.

Since New Massive Gravity is a higher derivative theory with curvature squared terms, we need a reassessment of the Ryu-Takayanagi prescription. It is known that the finite part of the holographic entanglement entropy evaluated in a black hole background in Einstein gravity corresponds to the Bekenstein-Hawking thermal entropy [17,99]. Therefore it is natural to expect the same in gravity theories with higher derivatives, with the understanding that the thermal entropy is now realized as Wald's entropy [92]. Recently, motivated by the analysis in [21], a general prescription for computing the holographic entanglement entropy for a higher-derivative theory was proposed in [91]. When applied to New Massive Gravity, the prescription [91] yields the entropy functional [100]

$$S_{EE} = \frac{1}{4G}\int_\Sigma dz\,\sqrt{h}\left[1 + \frac{1}{m^2}\left(R_{||} - \frac{1}{2}K^2 - \frac{3}{4}R\right)\right], \tag{5.3}$$

where h is the induced metric on the entangling surface Σ. Such surface is taken to be co-dimension two (thus it is just a line), anchored to the boundary and propagating deep in the bulk. The projected Ricci tensor $R_{||}$ is given by

$$R_{||} = \eta^{\alpha\beta}(n_{(\alpha)})^{\mu}(n_{(\beta)})^{\nu} R_{\mu\nu}\,, \tag{5.4}$$

while $(n_{(\alpha)})^{\mu}$ are the orthogonal vectors defined on Σ. The extrinsic curvature is given by

$$(K_{(\alpha)})_{\mu\nu} = h^{\lambda}_{\mu} h^{\rho}_{\nu} \nabla_{\rho}(n_{(\alpha)})_{\lambda}\,, \tag{5.5}$$

where ∇ is the covariant derivative with respect to the bulk metric. Consequently, the contracted form of the extrinsic curvature entering the functional can be written as

$$K^2 = \eta^{\alpha\beta}(K_{(\alpha)})^{\mu}_{\mu}(K_{(\beta)})^{\nu}_{\nu}\,. \tag{5.6}$$

All these quantities are required to be evaluated on the appropriate entangling surface, determined by minimizing the entropy functional (5.3) itself. In three dimensional pure gravity this problem is easily solved by taking the geodesic as entangling surface since it is, by definition, the curve with minimal length. However, as we will see in the next sections, this is not necessarily the case for a more general theory of gravity. Intuitively, due to the presence of higher-derivative terms, the entropy functional given in (5.3) fails to be interpreted as a length anymore [101]. From a technical point of view, minimizing such functionals leads to a higher-order differential equation that opens up the possibility of finding different entangling surfaces as opposed to the one in the context of Einstein-Hilbert gravity. In the next section, we elaborate upon this issue further with a concrete example, i.e. the AdS_3 spacetime as a background geometry in New Massive Gravity.

5.2 New Entangling Surfaces

In this section, we elaborate on the geometry of the entangling surface embedded in a three dimensional background describing a more general theory of gravity, in particular New Massive Gravity. Our principal aim is to achieve an entangling surface by minimizing the entropy functional prescribed in [91] and to compute the corresponding holographic entanglement entropy. The simplest example to realize the richer behaviour of the geometry of the entangling surface we are interested in is the AdS_3 spacetime. Such a background is described by the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\widetilde{L}^2}{z^2}(dt^2 + dz^2 + dx^2)\,, \tag{5.7}$$

where $\widetilde{L}$ is the AdS_3 radius and is related to the cosmological parameter L present in (5.2) by

$$L^2 = F\widetilde{L}^2\,, \qquad\qquad F^2 - 4m^2L^2F + 4m^2L^2 = 0\,. \tag{5.8}$$

With this background metric, we systematically reproduce the results presented in [94] in order to give a clear overview of the general procedure.

The entangling region in the dual field theory is a one-dimensional line located at the boundary of AdS_3 ($z = 0$). Correspondingly, to obtain the co-dimension 2 extremal hypersurface embedded in the constant time slice of the AdS_3 geometry, we choose the following ansatz consistent with the so-called boundary parametrization

$$t = 0\,, \qquad\qquad x = f(z)\,. \tag{5.9}$$

With this AdS_3 background metric and the prescribed profile ansatz for the entangling surface, the computation of the entropy functional (5.3) (we refer to [94] for the details of the calculation) leads to

$$\begin{aligned} S_{EE} \;=\; & \frac{2\pi}{\ell_p}\int dz\,\frac{\widetilde{L}}{z}\,\sqrt{A}\left[1 + 2\,\frac{F-1}{F}\right. \\ & \left.\left[1 - \frac{1}{A^3}\left(f'(z)^3 + f'(z) - zf''(z)\right)^2\right]\right]\,, \end{aligned} \tag{5.10}$$

where $A = f'(z)^2 + 1$. In order to minimize this functional, we consider the expression (5.10) as a one-dimensional action, so that the corresponding equation of motion is

$$\frac{\partial^2}{\partial z^2}\left(\frac{\delta\mathcal{L}}{\delta f''}\right)-\frac{\partial}{\partial z}\left(\frac{\delta\mathcal{L}}{\delta f'}\right)+\frac{\delta\mathcal{L}}{\delta f}=0\,. \tag{5.11}$$

Since (5.10) is independent of $f(z)$, the last term in (5.11) identically vanishes. The resultant fourth-order differential equation is of a highly nonlinear nature. However, as mentioned in [100], there exists a very simple analytic solution of (5.11), namely

$$f_1(z)=\sqrt{z_0^2-z^2}\,. \tag{5.12}$$

where $z_0 = f(z=0)$ is a tunable parameter, expressing the length of the region of interest. The turning point, that is how deep the surface goes into the bulk, is located at $z_t = z_0$. Moreover, the corresponding extrinsic curvature vanishes. It is very interesting to note that the same solution can be obtained even without the higher-derivative terms, being a geodesic anchored to the boundary region of interest. As mentioned before, the profile (5.12) is the only possible solution in Einstein gravity, because it is the trajectory that minimizes the length. However, there is no a priori reason to expect that no solution other than (5.12) exists for a higher-derivative theory of gravity.

Indeed, the authors of [94] present a different entangling surface. Making the ansatz $f(z) = \sqrt{z_0^2 - z^2 + az}$, and requiring that it solves the differential equation (5.11), we obtain

$$f_2(z)=\sqrt{z_0^2-z^2+2z_0qz}\,,\qquad\qquad q=\sqrt{\frac{F-2}{F}}\,. \tag{5.13}$$

In this case, the turning point goes deeper into the bulk and it is located at

$$z_t=z_0\left[q+\sqrt{q^2+1}\right]\,. \tag{5.14}$$

Moreover, unlike the case for $f_1(z)$, the contracted form of the extrinsic curvature evaluated on this new extremal surface

$$K^2|_{f_2(z)}=\frac{q^2}{\widetilde{L}^2\left(q^2+1\right)}=\frac{F-2}{2\widetilde{L}^2\left(F-1\right)}\,, \tag{5.15}$$

is non-vanishing.

For both solutions we have the respective universal terms

$$
\begin{aligned}
S_{EE}^{(1)} &= \frac{c_1}{3}\log\left(\frac{z_0}{\epsilon}\right), & \frac{c_1}{3} &= \frac{L}{4G}\,\frac{3F-2}{F^{\frac{3}{2}}}, \\
S_{EE}^{(2)} &= \frac{c_2}{3}\log\left(\frac{z_0}{\epsilon}\right), & \frac{c_2}{3} &= \frac{L}{4G}\sqrt{8\frac{F-1}{F^2}}\,.
\end{aligned} \tag{5.16}
$$

It is easy to verify that as long as $F > 2$, the coefficients c_1 and c_2 follow the inequalities $c_1 \geq c_2 > 0$. Therefore, in contrast with the results previously presented in the literature, the solution that minimizes the entropy is $f_2(z)$. Both solutions coincide for $F = 2$, as can be verified from equation (5.13). There exists another range of parameters ($2 \geq F \geq \frac{2}{3}$) where only the first solution is real valued and correspondingly c_1 is a positive number. It is important to notice that c_1 can be interpreted as the central charge from the boundary theory prospective [100]. Therefore, the boundary EE is expected to be reproduced by the holographic computation performed by taking $f_1(z)$ as entangling surface. Moreover, in [102] the authors used the field-redefinition invariance to restrict the admissible entangling surfaces to those with vanishing trace of extrinsic curvature, thus giving an argument in favor of the first type of solution.

We dedicate the next two sections to investigate what happens in the more complex backgrounds that NMG provides us.

5.3 HEE for Lifshitz spacetime in NMG

The next solution of New Massive Gravity that we want to consider is the Lifshitz background. The isometry of the Lifshitz spacetime can be holographically mapped to the symmetry of the dual non-Lorentz invariant boundary theories [103, 104]. It thus offers a substantial understanding of strongly coupled nonrelativistic conformal field theories characterizing a large class of condensed matter systems [105–107]. Moreover, for the purposes of our analysis, this case presents a lot of technical similarities to the AdS spacetime. Therefore, it seems that the Lifshitz spacetime is the natural choice to show the existence of new entangling surfaces in a more general context.

It is important to notice that, being an asymptotically non-AdS background, the Lifshitz spacetime needs special attentions for holographic computations. In the context of the Einstein-Hilbert theory coupled to matter, the authors of [108] have constructed the bulk-to-boundary dictionary for the Lifshitz spacetime by treating it as a deformation over AdS. In particular, the authors have considered a perturbative expansion with respect to the Lifshitz exponent around unity. In this scenario, the dual boundary theory is a deformed conformal field theory consistent with the Lifshitz symmetry. In the following analysis, with the similar spirit of [108], we perform the bulk analysis of the entanglement entropy for the Lifshitz spacetime in the context of NMG. Another example of obtaining the holographic entanglement entropy for Lifshitz spacetime in Lovelock gravity can be found in [109].

The metric of the Lifshitz background is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\widetilde{L}^{2\nu}}{z^{2\nu}}dt^2 + \frac{\widetilde{L}^2}{z^2}(dz^2 + dx^2)\,, \tag{5.17}$$

where $\widetilde{L}$ is the Lifshitz radius and ν is the Lifshitz exponent. Note that the limit $\nu \to 1$ leads to the AdS spacetime discussed in the previous section. As explained in [110], the exponent ν and the NMG parameters are related by

$$m^2\widetilde{L}^2 = \frac{1}{2}\left(\nu^2 - 3\nu + 1\right)\,, \qquad \frac{\widetilde{L}^2}{L^2} = \frac{1}{2}\left(\nu^2 + \nu + 1\right)\,. \tag{5.18}$$

The extremal surface, parametrized by the following relations

$$t = 0\,, \qquad x = f(z)\,, \tag{5.19}$$

leads to the induced metric

$$ds_h^2 = h_{\mu\nu}dx^\mu dx^\nu = \frac{\widetilde{L}^2}{z^2}\left(f'(z)^2+1\right)dz^2\,. \tag{5.20}$$

We note that the induced metric in the present context is structurally identical to the one we find in the AdS_3 spacetime. Although the timelike orthogonal vectors, defined on the co-dimensional two entangling surface, posses an explicit dependence on the Lifshitz exponent ν

$$n_{\alpha\,1} = \left(0, -\frac{\widetilde{L}f'(z)}{z\sqrt{A}}, \frac{\widetilde{L}}{z\sqrt{A}}\right)\,, \qquad n_{\alpha\,2} = \left(\frac{\widetilde{L}^\nu}{z^\nu}, 0, 0\right)\,, \tag{5.21}$$

where $A = f'(z)^2+1$, the components of the extrinsic curvature are the same as before. As the metric is diagonal, we are only interested in the diagonal terms of the extrinsic curvature. Since we have a Killing vector in the time direction, the component of the extrinsic curvature in that direction, $K^2_{\alpha\alpha}$, vanishes. On the other hand, once we compute $K^1_{\alpha\alpha}$, it is easy to verify that $K^1_{tt}=0$ and $K^1_{rr} = f'(z)^2K^1_{zz}$. So the component we need to know is

$$K^1_{zz} = \frac{\widetilde{L}}{z^2A^{5/2}}\left[f'(z)^3+f'(z)-zf''(z)\right]\,, \tag{5.22}$$

leading to

$$K^2 = \frac{1}{\widetilde{L}A^3}\left[f'(z)^3+f'(z)-zf''(z)\right]^2\,. \tag{5.23}$$

Since the Lifshitz and the Anti-de Sitter spacetimes differ only in the g_{tt} component and we are working on a time slice, there is no difference in the induced metric and in the extrinsic curvature of the two cases. However the intrinsic curvature is a quantity that does not depend on the embedding, therefore it presents differences with respect to the AdS_3 case, namely

$$R = -\frac{2\left(\nu^2+\nu+1\right)}{\widetilde{L}^2}\,, \qquad R_{||} = -\left[\frac{\nu\left(\nu f'(z)^2+1\right)}{\widetilde{L}^2\,A}+\frac{\left(\nu^2+\nu+1\right)}{\widetilde{L}^2}\right]\,. \tag{5.24}$$

Collecting all these results together and plugging them back into (5.3), we obtain the entropy functional for the Lifshitz spacetime

$$S_{EE} = \frac{1}{4G}\int dz\frac{\widetilde{L}}{z}\sqrt{A}\Bigg[1+\frac{2}{\nu^2-3\nu+1}\Bigg[-\frac{\nu\left(\nu f'(z)^2+1\right)}{A}+\frac{\left(\nu^2+\nu+1\right)}{2} - \frac{1}{2A}\left(f'(z)^3+f'(z)-zf''(z)\right)^2\Bigg]\Bigg] \tag{5.25}$$

Upon minimizing the functional (5.25) in the same way as described in section 5.2, it leads to the following highly nonlinear differential equation

$$\begin{aligned}
&(2\nu-1)f'(z)^9+(2\nu(\nu+3)-3)f'(z)^7+(6\nu(\nu+1)-3)f'(z)^5\\
&+\left(4\nu^2-6\nu+3\right)zf'(z)^6f''(z)+z\Big[-5z^2f''(z)^3-2\nu^2f''(z)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +2z\left(zf^{(4)}(z)+2f^{(3)}(z)\right)\Big]\\
&+f'(z)^3\left(-5z^2f''(z)\left(4zf^{(3)}(z)+3f''(z)\right)+6\nu^2+2\nu-1\right)\\
&+f'(z)\left(2\nu^2-5z^2f''(z)\left(4zf^{(3)}(z)+3f''(z)\right)\right)\\
&+zf'(z)^2\left(30z^2f''(z)^3+(3-6\nu)f''(z)+4z\left(zf^{(4)}(z)+2f^{(3)}(z)\right)\right)\\
&+2zf'(z)^4\left(3(\nu-1)^2f''(z)+z\left(zf^{(4)}(z)+2f^{(3)}(z)\right)\right)=0\,.
\end{aligned}\tag{5.26}$$

In [95], it is stated that this equation is solved by the geodesic, specifically

$$f(z)=\sqrt{z_0^2-z^2}\,,\tag{5.27}$$

constrained by a causal boundary condition [111]. However, if we insert the ansatz (5.27) in this nonlinear equation of motion (5.26), we obtain

$$-\frac{4z_0^6(\nu-1)\nu z^3}{(z_0^2-z^2)^{9/2}}=0\,.\tag{5.28}$$

It is clear that for a generic non-zero ν, (5.27) is not a solution of (5.26) except for the case $\nu=1$. For $\nu=1$ we recover the AdS_3 spacetime as a special limit of the Lifshitz spacetime (in Appendix C we reproduce the same result using the notation of [95]).

Since obtaining an exact solution for a nonlinear fourth order differential equation like (5.26) is generically difficult, we rather aim to solve the equation using a perturbative technique. We already know that for $\nu=1$ we have an exact solution (5.27) of the differential equation (5.26). We consider $\nu=1+\delta$, where δ is a tiny positive deformation around unity and by following [93][1] we introduce the ansatz function

$$\nu=1+\delta\,,\qquad\qquad f'(z)=h'(z)\Big(1+\delta g(z)+\delta^2n(z)+\mathcal{O}(\delta^3)\Big)\,,\tag{5.29}$$

[1] Another application of this method can be found in [112]

where $h(z) = \sqrt{z_t^2 - z^2}$ is the geodesic and z_t is the turning point. We choose the solution (5.12) for the AdS spacetime because, by taking the limit from Lifshitz to AdS (i.e by setting $\delta = 0$), we end up in a region of the parameter space where the second solution (5.13) proposed in section 5.2 is not well defined ($F = 2/3$ in the language of section 5.2). The downside of our choice is that the AdS central charge and the extrinsic curvature (at 0th-order in δ) are vanishing. After imposing our ansatz, we also notice that $g(z)$ starts to appear in the expansion of the differential equation at second order, while $n(z)$ starts to appear at third order, and so on. Consequently, by expanding the entropy functional, the first nontrivial contribution to the entanglement entropy is coming at second order.

This ansatz will simplify the differential equation whose solution is the desired entangling surface. Following [93], we can easily determine the surface by imposing a few boundary conditions. In particular, we require z_0 to be finite and real. Moreover, we require that our surface is anchored to the region of our interest. To do so, the condition we employ is $z_0 = \int_0^{z_t} f'(z)$. Finally, we also require that the turning point approaches to 0 when the size of the entangling region $z_0 \to 0$.

Thanks to this approach, the differential equation becomes linear and is solved by

$$\begin{aligned} f'(z) &= -\frac{z}{\sqrt{z_t^2 - z^2}} \left[1 + \delta \left(\frac{z_t^2 (1 - 2\log z_t)}{2(z_t^2 - z^2)} + \frac{z_t^2 (2 \log z - 1)}{2(z_t^2 - z^2)} \right) \right. \\ &\quad \left. + \delta^2 \, n(z) + \mathcal{O}(\delta^3) \right], \end{aligned} \tag{5.30}$$

where the contribution at the second order is given by,

$$n(z) = \frac{z_t^2 \left(\left(z_t^2 - z^2 \right)^2 + z^2 \left(\left(z_t^2 + 2z^2 \right) (\log(z_t) - \log(z)) - z_t^2 + z^2 \right) (\log(z_t) - \log(z)) \right)}{2 \left(z^3 - z \, z_t^2 \right)^2} \,. \tag{5.31}$$

However, since (5.31) is not contributing at the leading order, we will not include it in the present analysis. By integrating our result, we can obtain the form of the entangling surface,

$$\begin{aligned} f(z) &= \sqrt{z_t^2 - z^2} \\ &\quad + 2 z_t \delta \left[\log z - \frac{z_t \log(z/z_t)}{\sqrt{z_t^2 - z^2}} - \log \left(z_t + \sqrt{z_t^2 - z^2} \right) \right] + \mathcal{O}(\delta^2) \,. \end{aligned} \tag{5.32}$$

Now we just need to determine our turning point z_t as a function of z_0, i.e. the size of the entangling region of our interest. By requiring that $z_0 = \int_0^{z_t} f'(z)$, we obtain

$$z_t = z_0(1 + 2\delta \log 2\,) + \mathcal{O}(\delta^2)\,. \tag{5.33}$$

It is then clear that the turning point z_t is located deeper in the bulk with respect to the one reached by the geodesic (see Fig. 3). Notice also that, as it is expected, if we take the limit $z_0 \to 0$, the turning point $z_t \to 0$.

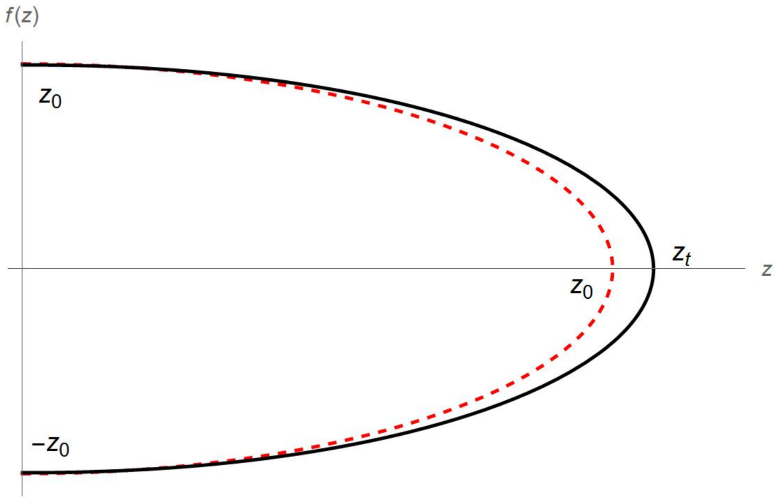


FIGURE 3
The entangling surface (in black) is going deeper in the bulk with respect to the geodesic (in dashed red)

Substituting these results back into the functional (5.25), we obtain the universal term of EE for the Lifshitz spacetime

$$S_{EE}^{Lif} = \frac{c_{Lif}}{3}\log\left(\frac{z_0}{\epsilon}\right)\,, \qquad \frac{c_{Lif}}{3} = \left[0_{AdS} - \delta^2\sqrt{\frac{3}{2}}\frac{L}{2G} + \mathcal{O}(\delta^3)\right]\,. \tag{5.34}$$

Since our expansion takes place around the chiral point of New Massive Gravity, the leading contribution (indicated as 0_{AdS} to keep track of it) is vanishing and the equations are extremely simplified. However, we can see how the entangling surface is necessarily deformed around the geodesic in order to extremize the entropy functional.

From the boundary field theory point of view, the difficulties in computing the central charge for the asymptotic Lifshitz symmetry and arbitrary Lifshitz exponent have been addressed in [113]. The technical reason behind this obstacle is the appearance of infinities while integrating over the non-compact x direction to obtain the conserved charge of the symmetry algebra[2]. It is beyond the scope of our present analysis to check our holographic results for Lifshitz spacetime from the point of view of the boundary theory. However, from our holographic analysis, we are able to propose an approximate result of central charge as a coefficient of the leading UV logarithmic divergent term in the HEE.

Interestingly, although the entangling surface reported in [95] does not extremize the entropy functional (see Appendix C), the expansion around $\nu = 1$ of their result matches the one here derived. The reason is that in proximity of the boundary (where the main contribution to EE is coming from) the two entangling surfaces do not present relevant differences, as already commented in [115].

Our analysis requires the existence of an exact solution at the zeroth order in δ, that is only known for $\nu = 1$, i.e. at the chiral point. In order to explore a more complicated case, we now turn our attention to the Warped AdS spacetime as a solution of New Massive Gravity.

5.4 HEE for Warped AdS$_3$ Spacetime in NMG

We dedicate this section to study the effect of higher-derivative contributions on the holographic entanglement entropy by exploring a WAdS geometry. Such spacetime has received great attention in the context of a non-AdS extension of holography that allows squashed and stretched deformations of the AdS geometry as a dual gravity spacetime. In the present discussion we are particularly interested in the timelike Warped AdS$_3$ background having an asymptotic symmetry as $SL(2, R) \times U(1)$. The boundary theory of such a background has been proposed to be a warped conformal field theory [116,117] describing a particular class of non-Lorentzian physical systems.

The metric of the timelike WAdS$_3$ can be characterized by the 3-dimensional

[2] A recent attempt can be found in [114].

version of the Gödel spacetime

$$ds^2 = -dt^2 - 4\omega r dt d\phi + 2(r(\ell^{-2} - \omega^2)r^2)d\phi^2 + \frac{dr^2}{2(r(\ell^{-2} + \omega^2)r^2)}\,, \tag{5.35}$$

that solves the NMG equations of motion, provided that

$$m^2\ell^2 = -\frac{19\omega^2\ell^2 - 2}{2}\,, \qquad \frac{\ell^2}{L^2} = \frac{11\omega^4\ell^4 + 28\omega^2\ell^2 - 4}{2(19\omega^2\ell^2 - 2)}\,. \tag{5.36}$$

In the particular case that $\omega^2\ell^2 = 1$, we find again the AdS$_3$ spacetime. In order to simplify the discussion, following [118] and setting $\ell = 1$, we perform the following change of coordinates,

$$\theta = t - \phi\,, \qquad \rho^2 = 2r\,. \tag{5.37}$$

With this change of coordinates, the metric takes the following form

$$\begin{aligned} ds^2 \;=\; & -\left(1 + (2\omega - 1)\rho^2 - (1 - \omega^2)\frac{\rho^4}{2}\right) dt^2 \\ & +2\left((\omega - 1)\rho^2 - (1 - \omega^2)\frac{\rho^4}{2}\right) dt d\theta \\ & +\left(\rho^2 + (1 - \omega^2)\frac{\rho^4}{2}\right) d\theta^2 + \frac{d\rho^2}{1 + (1 + \omega^2)\frac{\rho^2}{2}}\,. \end{aligned} \tag{5.38}$$

We choose again the boundary parametrization to describe the entangling surface

$$t = 0\,, \qquad \theta = f(\rho)\,. \tag{5.39}$$

The curious reader will find the details of the calculation in Appendix D. Intuitively, one can imagine that the complexity of the equations is forcing us to look again only for approximate solutions. It is important to notice that our choice of coordinates is made in order to recover the AdS results in the limit $\omega \to 1$ at any given step of the calculation. Here we consider a particular ansatz signifying a deformation of the entangling surface for pure AdS$_3$ spacetime in the global coordinate system, namely

$$\omega = 1 + \delta\,, \qquad f(\rho) = h(\rho) + \delta g(\rho) + \delta^2 n(\rho) + \mathcal{O}(\delta^3)\,, \tag{5.40}$$

where δ is a positive small deformation and

$$h(\rho) = \tan^{-1}\left(\frac{\sqrt{\rho^2-\rho_t^2}}{\rho_t\sqrt{\rho^2+1}}\right), \tag{5.41}$$

is the geodesic in the AdS_3 background (in global coordinates), i.e the entangling surface if we set $\omega = 1$. Also in this case, we choose the geodesic of the AdS_3 spacetime since, by taking $\omega = 1$, we fall down in a region of the NMG parameter space where the second solution proposed in [94] is not well-defined.

Unlike the previous example, the function $g(\rho)$ is contributing already at the linear order in δ to the Entanglement Entropy and we are going to ignore higher order contributions. However, in order to determine the profile of $g(\rho)$, we need to solve the differential equation at order δ^2. The reason is again that our ansatz, with the AdS_3 geodesic at leading order, is simplifying a lot the differential equation, forcing us to expand up to second order to find the equation that constraints the profile $g(\rho)$.

By imposing the same boundary conditions of the previous two sections, we obtain

$$\begin{aligned} g(\rho) \;=\; & \frac{1}{2}\Bigg(\frac{\left(\rho^2+1\right)\left(\frac{\left(\rho_t^2+1\right)^3}{\rho_t}-\rho_t\left(4\rho_t^2+5\right)\rho_\infty^2\right)}{\left(\rho_t^2+1\right)^2\rho_\infty^2} \\ & +\frac{\rho_t\left(4\rho_t^2+5\right)\rho_\infty^2+2\rho_t\left(\rho_t^2+1\right)\rho_\infty^2\log\left(\rho_t^2+1\right)-\frac{\left(\rho_t^2+1\right)^3}{\rho_t}}{\left(\rho_t^2+1\right)\rho_\infty^2} \\ & +\frac{\rho_t\sqrt{\rho^2-\rho_t^2}\left(\rho_t^2\left(4\rho^2+3\right)+5\rho^2+4\right)}{\left(\rho_t^2+1\right)^2\sqrt{\rho^2+1}} \\ & -4\rho_t\log\left(\sqrt{\rho^2-\rho_t^2}+\sqrt{\rho^2+1}\right)\Bigg), \end{aligned} \tag{5.42}$$

where the turning point ρ_t is determined by solving the transcendental equation

$$\frac{\Delta\theta}{2} = \tan^{-1}\frac{1}{\rho_t} - 2\delta\rho_t\log\rho_\infty\,. \tag{5.43}$$

Here $\Delta\theta$ is the size of the entangling region of our interest at the UV cut-off ρ_∞ (in other words, it is the analogue of the z_0 of previous sections).

Now that we have all the ingredients, we can finally write down the universal contribution of the warping parameter to the entanglement entropy. We find

$$S_{EE}^{W} = \frac{c_W}{3}\log\left(\frac{\rho_\infty}{\Delta\theta}\right)\,, \qquad \frac{c_W}{3} = \frac{4}{17G} + \delta\,\frac{52}{289G} + \mathcal{O}(\delta^2)\,. \tag{5.44}$$

In [119], by exploring the asymptotic analysis of the $WAdS_3$ spacetime in NMG, the authors have conjectured a dual WCFT_2. It is interesting to notice that, by expanding their result around $\omega = 1$, we can consistently reproduce the central charge we obtain in eq. (5.44).

In this case, the leading term (the order δ^0 in the expansion of the central charge, corresponding to the AdS spacetime) is non-vanishing and we can appreciate the contribution coming from the warping parameter already at first order. The nature of the deformation of the entangling surface is different from the one showed in section 5.3. Although in both cases we are studying a small deformation around the exact solution of the AdS spacetime, in this case the leading order is not coming from the geodesic of the spacetime under examination, therefore there is no reason to compare the turning points of the two solutions.

5.5 Summary

In this chapter, we have explored the role of higher derivatives in the context of a holographic computation of entanglement entropy (EE). In particular, we showed the deformation in the geometry of the entangling surface due to the presence of higher derivatives in the gravity theory. Within the perturbative approximation, we proved *the existence of new entangling surfaces* for the Lifshitz and the timelike WAdS_3 backgrounds in the NMG theory.

The main purpose of this holographic study in the Lifshitz background is to show that, unlike the AdS_3 case, the Lifshitz geodesic is not the correct entangling surface that extremizes the entropy functional. Moreover, we have applied the similar holographic technique to the case of timelike WAdS_3 spacetime and found an entangling surface that extremizes the entropy functional. Consequently, we compute the leading logarithmic term of holographic entanglement entropy and show that our result is consistent with the expectation of the boundary Warped CFT_2. In both analyses, we constructed new entangling surfaces as perturbative deformations over the AdS_3 geodesic.

Conclusions

We began this thesis by asking ourselves what it would mean to attempt to gain a better understanding of the universe. We decided to tackle this problem from the angle of a theoretical physicist who is trying to explore the nature of the gravitational interaction. As briefly reviewed in Chapter 2, the study of gravity has been characterized by several twists and turns, and it is still not free of mystery. One of these twists was the formulation of the holographic principle, born within the context of the study of the properties of black holes and nowadays being applied in many different fields.

In our search for a better understanding of this principle, we have ended up focusing on a fascinating correspondence between different theories. Our approach was to study the effects of adding a specific set of (higher derivative) terms, thus modifying the theory that lies on one side of the correspondence. In doing so, it was inevitable that technical obstacles, as well as a richer range of behaviors, would be encountered. Therefore, the main goals were to observe any new feature our model may have introduced, to overcome any technical obstacle to the best of our abilities, and, consequentially, provide potential hints for a deeper understanding of this mystery.

In a way, we were performing *experiments* on a theoretical model by extending it and studying its new properties, or by using it as a playground to study a specific physical quantity, thus eventually obtaining interesting hints about the existence of new features characterizing the behavior of this quantity. Our laboratory, our playground, was a theory called New Massive Gravity (NMG) and the choice matched our needs because NMG maintains a certain richness despite its simplifications, as we have seen in Chapter 2 and, more generally, in the results presented in this thesis.

Naturally, one should not think for a moment that this is the only path to explore gravity supplemented with higher derivative terms. In the introduction, for example, we saw how in the late 1970s these theories proved to be useful in the study of quantum gravity, thus arousing the interest of the scientific community. In more recent years, it has been suggested that infinite derivative gravity theories might provide a solution to the cosmological singularity problem, thus contributing to achieving a better understanding of the inflationary phase in the early universe [120, 121] [3].

In Chapter 3, we followed [53] in the construction of all off-shell Poincaré supergravity invariants up to mass dimension four and with $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ supersymmetry. To achieve this result, we applied the superconformal tensor calculus. This technique is the backbone of the strategy that we have employed to obtain the supersymmetric extensions mentioned earlier. The merit of this construction is that it can be performed almost mechanically by following the prescribed steps. In addition to finding all the invariants needed to supersymmetrically extend our model, we studied which combinations of parameters would allow a ghost-free spectrum about a maximally supersymmetric vacuum. Due to the non-existence of a specific supersymmetric invariant, we found that the $\mathcal{N} = (2, 0)$ model does not allow a supersymmetric AdS_3 vacuum with a ghost-free spectrum.

While the supersymmetric AdS vacuum solutions have been examined here, it would be instructive to study the non-supersymmetric AdS vacuum solutions as well. Moreover, a systematic study of the ghost-free vacua and their stability under quantum corrections would shed light on the role of extended supersymmetry and the differences between the two versions of the off-shell $\mathcal{N} = 2$ theory at the quantum level [53].

In our construction of the vector multiplet actions, we used an arbitrary function of vector multiplet scalars, as given in (3.71). However, we did not consider such constructions for the scalar multiplet in this chapter and it would be interesting to consider the coupling of an arbitrary number of scalar multiplets and vector multiplets since that would enable us to construct a large class of supergravity Lagrangians [123]. Moreover, as already pointed out in [53], the composite expression we derived for both the scalar and vector multiplet can also be used to construct matter-coupled higher derivative supergravity

[3] A more complete review of the topic can be found in [122]

models. Such theories have attracted a considerable amount of attention in the context of rigid supersymmetric theories on three-manifolds [75, 124]. The compensating multiplet used in the construction of $\mathcal{N} = (1,1)$ theory includes a complex scalar and we have seen how the R-symmetry is fixed once we fix the dilatations. However, in the case of the $\mathcal{N} = (2,0)$ theory, we gauge fix dilatations with a real scalar, thus not accidentally fixing the R-symmetry too. Therefore, one can use this setup to obtain an Einstein-Maxwell theory where the R-symmetry is dynamically gauged.

In Chapter 4, we have followed [79] and used the Killing spinor analysis to classify the supersymmetric solutions of the $\mathcal{N} = (1,1)$ extension of New Massive Gravity with a cosmological constant (CNMG), a particular case of the model constructed in Chapter 3.

An intriguing problem is to find a supersymmetric Lifshitz black hole. Indeed, we provided a number of motivations to prove that no field configuration allows such a solution for our model. However, one could attempt a different approach. For instance, one could saturate the BPS bound with a $U(1)$ charge by coupling the $\mathcal{N} = (1,1)$ CNMG model to an off-shell vector multiplet and repeat the analysis presented in [79].

Moreover, the same procedure presented in this chapter could be applied to the $\mathcal{N} = (2,0)$ CNMG model introduced in Chapter 3. We have seen how the model is characterized by a different field content consisting of two auxiliary vectors and a real scalar as well as the graviton and the gravitino. Given that the $\mathcal{N} = (2,0)$ theory with matter couplings has new supersymmetric solutions [57], we would expect that the $\mathcal{N} = (2,0)$ CNMG model will exhibit different supersymmetric solutions as well. Therefore, it would be interesting to see what the consequences of the different field content are for the supersymmetric solutions of the model.

In Chapter 5, we followed [88] and studied the effects of the presence of higher derivative terms in a holographic computation. In particular, we computed the entanglement entropy (EE) by using a holographic technique and observed that, in the context of a higher derivative theory such as NMG, there is a substantial change in the nature of the geometrical object encoding this particular information. While in the case of pure gravity one must always study a particular geodesic propagating into the bulk, when higher derivative terms are involved other surfaces appear as solutions to the problem.

One may be tempted to view the results presented in [88] as an indication that the EE is increased or decreased as one turns on the higher derivative contributions. We hereby present an argument for not making such comparison in the context of our present holographic analysis.

- As pointed out at the end of section 5.2, the correct result is the one obtained by the geodesic, further explanation of why the second result, which was providing a lower entanglement entropy, should be discharged can be found in [102].

- Regarding the Lifshitz case, it is not appropriate to interpret the result as a reduction in the EE, since the 0th order, i.e. the geodesic (both of the Lifshitz and the AdS spacetime), is not an admissible surface for computing the EE in Lifshitz spacetime (see equation 5.28) [4].

- For the $WAdS_3$ spacetime, we can interpret our result as a perturbative enhancement of the holographic entanglement entropy due to warping of the AdS spacetime. However, the comparison is not very meaningful since the entangling at the zeroth order does not represent any particular surface for the WAdS_3 spacetime (being simply the AdS geodesic).

In [115], the authors showed how the presence of higher derivative terms in the gravity theory does not change the structure of the divergences in the entanglement entropy with respect to the Einstein gravity case. However, since the backgrounds under examination are not solutions to pure Einstein gravity, we find it more appropriate to show the shifts in the central charges explicitly. These changes in the shape of the entangling surface, as well as in the coefficient of the leading divergence in the entanglement entropy, are present exclusively because we are taking into account the higher derivative terms. For this purpose, the Lifshitz case, presented in section 5.3, is a perfect example, since *the surface determined by ignoring the higher derivative terms (i.e. the geodesic) is not extremizing the entropy functional.*

As suggested in [101], the technical reason behind the emergence of multiple entangling surfaces is that we are unable to impose a sufficient number of boundary conditions to solve our differential equations. In [111], the authors

[4] Further studies on this case can be found in [114], where the authors approach the problem from the field theory side

proposed the so-called *free-kick condition* in order to solve the problem in the context of hairy black holes as solutions of NMG. However, such a condition constrains the entangling surface to be the geodesic at its turning point and we can find counter examples both in [94] and in the analysis presented here.

We believe that the central question to be addressed in the near future is: how can we give a physical reason that solves such a technical problem? We are now providing explicit (although approximate) solutions and we hope, as a result, that this work will pave the way for a discussion to find a rigorous method to solve such problems.

In [125], the computation of the holographic entanglement entropy for WAdS spacetime has been investigated in the light of non-AdS holography. It would be very interesting to check whether the non-AdS correction considered in [125] can be consistently implemented together with higher derivative contribution of the NMG theory. Another interesting study on this topic is presented in [126]. Moreover, the same kind of analysis can be performed for geometries with a horizon (i.e. black holes), which will impose a limit on how deep we can push the entangling surface into the bulk, thus providing more hints on the nature of this geometric object.

The question may now arise: have we arrived at a better understanding of the universe after 150 pages of hypotheses, calculations, and discussions? Understanding the universe is not a match against a problem that you may win or lose. It is a relentless drive that has to push every step, every abstraction, and every assumption. It is a quest that may take you far from everyday reality but that has to remain true to it. Here, we provided a new theory and clarified an effect of the presence of higher derivatives. Our results can be used in multiple directions, as we have just seen, and thus will hopefully fit in with the quest for understanding, or at least help to clarify, some deeper questions.

List of Publications

- Chapter 3,
  G. Alkaç, L. Basanisi, E. A. Bergshoeff, M. Ozkan and E. Sezgin,
  Massive $\mathcal{N} = 2$ supergravity in three dimensions,
  JHEP **1502** (2015) 125
  [arXiv:1412.3118].

- Chapter 4,
  G. Alkaç, L. Basanisi, E. A. Bergshoeff, D. O. Deveciolu and M. Ozkan,
  Supersymmetric backgrounds and black holes in $\mathcal{N} = (1, 1)$ cosmological new massive supergravity,
  JHEP **1510** (2015) 141
  [arXiv:1507.06928].

- Chapter 5,
  L. Basanisi and S. Chakrabortty,
  Holographic Entanglement Entropy in NMG,
  JHEP **1609** (2016) 144
  [arXiv:1606.01920].

- G. Alkaç, L. Basanisi, E. Kilicarslan and B. Tekin,
  Unitarity Problems in 3D Gravity Theories,
  To appear on Physics. Rev. D
  arXiv:1703.03630.

List of Abbreviations

AdS	Anti-de Sitter	A maximally symmetric solution of Einstein gravity with a negative cosmological constant.
BF	Breitenlohner-Freedman	A bound to ensure the positivity of energy in AdS spacetime.
BH	Bekenstein-Hawking	Entropy associated to a black hole.
BPS	Bogomol'nyi, Prasad Sommerfield	Bound saturated when half of the supersymmetries is unbroken.
BTZ	Bañados, Teitelboim, Zanelli	Black hole solution in 3D.
EE	Entanglement Entropy	Measure for a quantum state.
EH	Einstein-Hilbert	Action describing a theory of pure gravity.
FP	Fierz-Pauli	Theory of massive gravity with explicit mass term.
GMG	General Massive Gravity	Model of massive gravity combining TMG and NMG
GR	General Relativity	Modern theory of gravity.
HEE	Holographic EE	Method to compute EE.
LM	Lewkowicz-Maldacena	Proof of the RT proposal.
NMG	New Massive Gravity	Theory of massive gravity with higher derivatives.
RT	Ryu-Takayanagi	Method to compute HEE.
TMG	Topologically Massive Gravity	Theory of massive gravity with higher derivatives.
WAdS	Warped AdS	Deformation of the AdS spacetime.

Samenvatting

De afgelopen eeuw werd gekenmerkt door twee belangrijke periodes: de eerste helft, waarin revolutionaire theorieën onze kijk op de natuur compleet veranderden, en de tweede helft, waarin men diezelfde theorieën gebruikte om de verborgen elegantie van de fundamentele interacties bloot te leggen. Dit is natuurlijk een enigszins gesimplificeerde weergave van één van de meest (zo niet dé meest) vruchtbare periodes in de geschiedenis van de natuurkunde. Echter, men kan het als volgt beschrijven: uitzonderlijke geesten ontwikkelden allereerst een reeks aan concepten en theorieën die uiteindelijk als hoeksteen voor de moderne natuurkunde gingen fungeren (namelijk zogenaamde kwantumveldentheorieën), waarna andere evenzo briljante geesten diezelfde hoeksteen gebruikten om een breed scala aan fysische fenomenen op een eendrachtige manier te beschrijven. Zo kon men uiteindelijk drie fundamentele krachten (namelijk de elektromagnetische kracht, en de zwakke- en sterke kernkrachten) in dezelfde taal beschrijven.

Genspireerd door de mogelijkheid om steeds meer fysische fenomenen op een al maar eenvoudigere wijze te beschrijven, is men gaan proberen hier de nog als enige overgebleven kracht aan toe te voegen: de zwaartekracht.

Een belangrijke rol in de zoektocht naar zo'n allesomvattende theorie is weggelegd voor *zwarte gaten*: regio's in de ruimte waar de zwaartekracht zo sterk is dat zelfs licht er niet aan kan ontsnappen. Inderdaad, dankzij het werk van Stephen Hawking [1, 2], wordt geloofd dat in die extreme situaties men zowel zwaartekracht- als kwantumeffecten dient te beschouwen, waardoor de vervaardiging van een onderliggende alomvattende theorie noodzakelijk is.

Ook al loopt met bij elke tot nu toe ondernomen poging voor het formuleren van zo'n theorie tegen grote theoretische en technische obstakels aan, heeft het onderzoek naar zwarte gaten geleid tot een van de grootste conceptuele revoluties van de laatste jaren: de formulering van het holografische principe.

Dit principe werd als eerst geopperd door Gerard 't Hooft [3] en vervolgens genterpreteerd als een eigenschap van de snaartheorie door Leonard Susskind [4]. Het principe stelt dat men een volume in de ruimte kan beschrijven door slechts de informatie die bevat is in zijn rand te gebruiken. Zoals een hologram op papiergeld een tweedimensionale weergave is maar niettemin driedimensionaal lijkt, kan met een driedimensionaal volume in de ruimte beschrijven door een geometrisch object in een twee dimensies te beschouwen.

De meest in het oog springende verwezenlijking van het holografische principe is een vermoeden dat bekend staat als de AdS/CFT correspondentie [5]. Dit vermoeden wordt ook wel ijk/zwaartekracht dualiteit genoemd en poneert een relatie tussen twee zeer verschillende theorieën. Aan de ene kant van de correspondentie staat een zwaartekrachttheorie in N dimensies, met aan de andere kant een hoekgetrouwe veldentheorie in $N-1$ dimensies. De mogelijkheid om een theorie van de zwaartekracht aan een veel beter begrepen kwantumveldentheorie te koppelen is fascinerend: het stelt ons in staat om de n uit te drukken in de ander, volledig gebruikmakend van het eerdergenoemde holografische principe.

Ondanks het feit dat er nog geen formeel bewijs van de correspondentie is, is hij de afgelopen 20 jaar uitvoerig bestudeerd, met als gevolg dat [5] het op n na vaakst geciteerde artikel is in de geschiedenis van de hoge energie fysica. De voornaamste reden hiervoor is dat het een krachtig hulpmiddel kan zijn bij het bestuderen van sterk gekoppelde systemen. Dergelijke systemen vormen normaal gesproken een uitdaging omdat men de standaard technieken als storingsrekening er niet op kan toepassen. Het opmerkelijke is dat de correspondentie zegt dat een sterk gekoppelde theorie duaal is aan een zwak gekoppelde theorie die dus wiskundig gezien makkelijker te hanteren is.

De correspondentie is uitvoerig toegepast op allerlei systemen, maar men kan zich afvragen wat precies de grenzen zijn van de correspondentie, welke beperkingen men kan tegenkomen en wat men hier vervolgens uit kan opmaken. Dit proefschrift houdt zich hier mee bezig aangezien we gebruik zullen maken van holografie in de context van een gemodificeerde zwaartekrachttheorie (om precies te zijn, een zwaartekrachttheorie met hogere orde afgeleiden), waarmee we dus buiten de directe reikwijdte van de AdS/CFT correspondentie treden.

Zwaartekrachttheorieën met hogere afgeleiden is men gaan bestuderen omdat ze de mogelijkheid geven om Einstein's theorie uit te breiden door een beschrijving te geven van meer algemene fysische effecten, zoals bijvoorbeeld de voortplanting van gravitonen met massa. Daarbij komt dat de zoektocht naar een renormaliseerbare zwaartekrachttheorie heeft geleid tot de gedachte dat Einstein's theorie opgevat dient te worden als een effectieve theorie en daarmee gevoelig is voor hogere orde correcties. Dergelijke correcties zullen steeds belangrijker worden naarmate the energieschaal toeneemt [6]. De interesse in deze theorieën nam toe nadat men had laten zien dat men een renormaliseerbare theorie kan verkrijgen door alle mogelijke kwadratische krommingstermen toe te voegen aan Einstein's theorie [7]. Echter, de prijs die men hier voor betaalt is dat men tegelijkertijd spookdeeltjes toevoegt.

Met de komst van snaartheorie als een mogelijke consistente kwantumzwaartekrachttheorie, hebben dergelijke theorieën met hogere afgeleiden nog meer in de theoretische natuurkunde. Eén interessante uitkomst is dat perturbatieve snaartheorie niet alleen de Einstein-Hilbert actie bevat, maar ook termen van hogere orde. Zo is bijvoorbeeld laten zien dat dergelijke termen een mechanisme voor de breking van supersymmetrie leveren [8].

Supersymmetrie is één van de steunpilaren van snaartheorie en geeft een relatie tussen bosonen en fermionen die ons, zoals we in Hoofdstuk 1 zullen zien, in staat stelt interne symmetrieën met die van de ruimte-tijd te combineren. Het levert ook een interessante theoretische (en experimentele) uitdaging omdat de symmetrie gebroken dient te zijn, aangezien men deze niet heeft geobserveerd door middel van experimenten. Het bestuderen van mechanismes voor de breking van supersymmetrie door middel van termen met hogere afgeleiden kan licht werpen op het belang van deze theorieën.

In de context van holografie kunnen zwaartekrachttheorieën met hogere afgeleiden een nuttig instrument zijn om de grenzen van de AdS/CFT correspondentie mee te onderzoeken. Door aan de ene kant van de dualiteit de dynamica en andere eigenschappen te veranderen, hoopt men te kunnen bepalen hoe die verandering doorwerken aan de andere kant. Aangezien het langetermijndoel is de aard van de connectie tussen theorieën te begrijpen, zou een meer algemene kijk hierop een grondiger inzicht kunnen leveren.

Een ander hulpmiddel waar we in dit proefschrift gebruik van zullen maken om mogelijke verborgen eigenschappen van de dualiteit te ontdekken, is de zogenaamde verstrengelingsentropie (VE). Het is welbekend dat als een verzameling deeltjes met elkaar interacteert we hun kwantumstaten niet afzonderlijk kunnen beschrijven maar alleen als geheel samengenomen, ongeacht hoever de deeltje van elkaar verwijderd zijn. Dit fenomeen noemt met *kwantumverstrengeling* en de VE geeft de mate van verstrengeling weer daar het een uitdrukking is van de hoeveelheid informatie die verloren gaat wanneer een deel van het kwantumsysteem niet meer toegankelijk is.

VE is een interessante grootheid die in verscheidene velden optreedt. Zo speelt het bijvoorbeeld een cruciale rol als ordeparameter bij kwantumfasetransities in allerlei fysische situaties [9]. Vanuit een meer praktisch oogpunt is duidelijk geworden dat een begrip van de mate van correlatie nuttig is om een kwantumgrondtoestand efficiënnt weer te geven, wat onze klassieke computers helpt om kwantumsystemen te beschrijven. Omgekeerd kan men zelfs een kwantumfase identificeren door te kijken naar hoe complex het is deze computationeel te beschrijven.

Zoals we in Hoofdstuk 1 zullen zien, zijn er nogal wat uitdagingen wat betreft het daadwerkelijk berekenen van de VE. Juist hier helpt holografie ons door deze kwantumeigenschap van materie te koppelen aan een geometrisch object in een zwaartekrachtstheorie die significant eenvoudiger te berekenen is. Omgekeerd kan VE het idee van holografie helpen omdat VE een universele grootheid is die in elk kwantumsysteem gedefinieerd kan worden en daarmee dus een interessante grootheid is om holografie in verschillende situaties te bestuderen.

De mogelijk geometrische grootheden te relateren aan de kwantumeigenschappen van materie is slechts n van de fascinerende aspecten van het bestuderen van de VE van een systeem. We kunnen ook proberen de grenzen hiervan beter te bepalen door het toe te passen in meer algemene situaties zoals gegeven door zwaartekrachttheorieën met hogere afgeleiden. We zullen zien dat de aanwezigheid van hogere afgeleiden niet alleen een effect heeft op numerieke waarden (zoals natuurlijk te verwachten is wanneer men een aanpassing doet), maar daarnaast de gehele aard van de geometrische grootheid veranderd die men holografisch gezien associeert met de VE.

Hoofdstuk 1 introduceert de twee al eerder genoemde protagonisten van dit proefschrift: supersymmetrie en verstrengelingsentropie. Deze zullen gebruikt worden als hulpmiddelen om onze modellen te onderzoeken, uit te zoeken welke nieuwe mogelijkheden we wellicht hebben, en om inzicht te bieden in de grenzen die we mogelijk tegen kunnen komen. We zullen niet al te veel details geven maar meer de nadruk leggen op de onderliggende ideeën.

In Hoofdstuk 2, het laatste introductiehoofdstuk, wordt een specifieke zwaartekrachttheorie met hogere afgeleiden, namelijk New Massive Gravity (NMG), geïnntroduceerd die als speelveld zal dienen voor ons onderzoek. We zullen hierbij een pad volgen dat begint bij de basisprincipes van de Algemene Relativiteitstheorie en leidt tot de mogelijkheid om deze principes te realiseren in een simpeler model. Gaandeweg zullen we het probleem vereenvoudigen door het aantal dimensies te verlagen alvorens termen met hogere afgeleiden toe te voegen om, ondanks de simplificaties, een breder scala aan fysische effecten te beschrijven. Pas daarna zullen we de eigenschappen van NMG bestuderen en zijn we gereed om in het overige deel van het proefschrift binnen deze theorie te werken.

Het eerste originele werk kan men vinden in Hoofdstuk 3. Hier zullen we supersymmetrische uitbreidingen van New Massive Gravity construeren. Er zal een overzicht van de methode worden gegeven alsook de details die leiden tot het eindresultaat. Het lezen wordt vergemakkelijkt doordat we de blauwdruk van onze methode zullen volgen, waardoor duidelijk wordt hoe elk tussenresultaat bijdraagt aan het construeren van de theorie.

Zodra je een nieuwe theorie hebt geformuleerd ligt het voor de hand zijn oplossingen te bestuderen. Dit zullen we doen in Hoofdstuk 4, waar we een aantal voordelen van het werken met supersymmetrische theorieën aanstippen. In plaats van de bewegingsvergelijking op te lossen en wat vaak nogal een uitdaging is, kan men ook andere objecten bestuderen, namelijk de zogenaamde Killing spinoren, die helpen bij het classificeren van de oplossingen. Vanuit dat oogpunt zullen we eerst bekijken hoe men deze objecten kan vinden en daarmee dus ook de supersymmetrische oplossingen van de theorie zoals geconstrueerd in Hoofdstuk 3. Hier zal ook duidelijk worden hoe de termen met hogere afgeleiden een zekere rijkdom introduceren in de oplossingen.

Zoals verwacht kan men met de aanwezigheid van termen met hogere afgeleiden nog een interessant resultaat verkrijgen door de verstrengelingsentropie via een holografische benadering te bestuderen. Hoofdstuk 5 is gewijd aan het toepassen van de concepten zoals geïnntroduceerd in Hoofdstuk 1 op de ingewikkeldere context van NMG. We zullen holografie toepassen op theorieën met hogere afgeleiden, opmerken welke nieuwe moeilijkheden dit met zich meebrengt en tenslotte zien welke interessante resultaten verkregen kunnen worden door de mogelijkheid om met rijkere geometrieën te werken dan NMG als oplossingen toestaat.

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A

Complex Spinor Conventions

The metric signature is $(-,+,+)$. The gamma matrices satisfy the Clifford algebra

$$\left\{\gamma^a,\gamma^b\right\} = 2\eta^{ab}, \tag{A.1}$$

and the identities

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \qquad (\gamma^\mu)^T = -C\gamma^\mu C^{-1}, \qquad (\gamma^\mu)^* = B\gamma^\mu B^{-1}\ , \tag{A.2}$$

where C is the charge conjugation matrix and B is a unitary matrix with the following properties

$$CC^\dagger = 1\ , \qquad CC^* = -1\ , \qquad C^T = -C\ . \tag{A.3}$$

$$C = iB\gamma^0\ , \qquad BB^\dagger = 1\ , \qquad BB^* = 1, \qquad B^T = B\ . \tag{A.4}$$

For Dirac spinors, there are two different definitions of the conjugate which are given by [60]

$$\bar\epsilon = \mathrm{i}\epsilon^\dagger\gamma^0, \qquad \tilde\epsilon = \overline{(B\epsilon)^*}\ . \tag{A.5}$$

For Majorana spinors, we impose the reality condition $\epsilon^* = B\epsilon$ and Majorana conjugation $\bar\epsilon = \epsilon^T C$ is equivalent to Dirac conjugation $\bar\epsilon = \mathrm{i}\epsilon^\dagger\gamma^0$.

In order to obtain the flipping rules for bilinears formed by Dirac spinors, it is useful to decompose a Dirac spinor into two Majorana spinors as $\epsilon_D = \epsilon_{M1} + \mathrm{i}\epsilon_{M2}$. As a result, we have

$$(B\epsilon_D)^* \;=\; \epsilon_{M1} - \mathrm{i}\epsilon_{M2}\ , \qquad \bar\epsilon_D = \bar\epsilon_{M1} - \mathrm{i}\bar\epsilon_{M2}, \qquad \tilde\epsilon_D = \bar\epsilon_{M1} + \mathrm{i}\bar\epsilon_{M2}, \tag{A.6}$$

from which one can obtain

$$\bar{\epsilon}_1\Gamma\,(B\epsilon_2)^* = \alpha\,\bar{\epsilon}_2\Gamma\,(B\epsilon_1)^*\,, \qquad \tilde{\epsilon}_1\Gamma\epsilon_2 = \alpha\,\tilde{\epsilon}_2\Gamma\epsilon_1\ , \tag{A.7}$$

where Γ is any element of the Clifford algebra and α is the corresponding numerical factor in the Majorana flipping relations. Using the decomposition, one also gets

$$\tilde{\epsilon}_1\Gamma\,(B\epsilon_2)^* = \alpha\,\bar{\epsilon}_2\Gamma\epsilon_1\ . \tag{A.8}$$

Note that this time we get a different type of bilinear, which becomes an important issue in the closure of the algebra on the scalar multiplet. Namely, the commutator between two supersymmetries (with parameters ϵ_1 and ϵ_2) leads to a translation parameter

$$\xi_3^\mu = \frac{1}{2}\tilde{\epsilon}_2\gamma^\mu\,(B\epsilon_1)^* - \frac{1}{2}\tilde{\epsilon}_1\gamma^\mu\,(B\epsilon_2)^*\ , \tag{A.9}$$

which can be shown to be identical to the usual translation parameter

$$\xi_3^\mu = \frac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\ , \tag{A.10}$$

by using (A.8).

The charge conjugation of a spinor is defined by

$$\lambda^C = B^{-1}\lambda^* = (B\lambda)^*\,, \tag{A.11}$$

and the complex conjugation of bilinears are given by

$$(\bar{\chi}\Gamma\lambda)^* \ \equiv\ (\bar{\chi}\Gamma\lambda)^C = \overline{\chi^C}\Gamma^C\lambda^C = \tilde{\chi}\Gamma^C\,(B\lambda)^*\ , \tag{A.12}$$

$$(\tilde{\chi}\Gamma\lambda)^* \equiv (\tilde{\chi}\Gamma\lambda)^C = \widetilde{\chi^C}\Gamma^C\lambda^C = \bar{\chi}\Gamma^C\,(B\lambda)^*\ , \tag{A.13}$$

where the charge conjugation of matrices are determined by $(\Gamma_1\Gamma_2)^C = \Gamma_1^C\Gamma_2^C$ and $\gamma_\mu^C = \gamma_\mu$.

B

Fierz Identities

The elements of the Clifford algebra in $3D$ are $\left\{\Gamma^A = \mathbb{1}, \gamma^\mu\right\}$ with the orthogonality relation $Tr\left(\Gamma^A \Gamma_B\right) = 2\,\delta^A_B$. Therefore, any 2-dimensional matrix can be expanded in the basis $\left\{\Gamma^A\right\}$ as $M = \frac{1}{2}\sum_A \mathrm{Tr}\left(M\Gamma_A\right)\Gamma^A$. As a result, the Fierz identity in $3D$ is given by

$$\bar{\chi}_1\,\chi_2\,\epsilon = -\frac{1}{2}\left(\bar{\chi}_1\,\epsilon\,\chi_2 + \bar{\chi}_1\,\gamma^a\epsilon\,\gamma_a\chi_2\right)\ , \tag{B.1}$$

from which one can also obtain the relations

$$\begin{aligned} \bar{\chi}_1\,\gamma^a\chi_2\,\gamma_a\epsilon &= -\bar{\chi}_1\,\chi_2\,\epsilon - 2\bar{\chi}_1\epsilon\,\chi_2\ , && \text{(B.2)}\\ \bar{\chi}_1\,\gamma^{ab}\chi_2\,\gamma_{ab}\epsilon &= 2\bar{\chi}_1\,\chi_2\,\epsilon + 4\bar{\chi}_1\,\epsilon\,\chi_2\ . && \text{(B.3)} \end{aligned}$$

Whenever flipping relations are applicable, one can also obtain additional identities by antisymmetrizing eqs. (B.2)–(B.3) with respect to $1 \longleftrightarrow 2$

$$\begin{aligned} \tilde{\chi}_1\,\gamma^a\chi_2\,\gamma_a\epsilon &= -\tilde{\chi}_1\,\epsilon\,\chi_2 + \tilde{\chi}_2\,\epsilon\,\chi_1\ , && \text{(B.4)}\\ \tilde{\chi}_1\,\gamma^{ab}\chi_2\,\gamma_{ab}\epsilon &= 2\tilde{\chi}_1\,\epsilon\,\chi_2 - 2\tilde{\chi}_2\,\epsilon\,\chi_1\ , && \text{(B.5)} \end{aligned}$$

which are also true for bilinears of type $\bar{\epsilon}_1\Gamma\left(B\epsilon_2\right)^*$. Using (B.4) in (B.1) we obtain

$$\tilde{\chi}_1\,\chi_2\,\epsilon = -\tilde{\chi}_1\,\epsilon\,\chi_2 - \tilde{\chi}_2\,\epsilon\,\chi_1\ . \tag{B.6}$$

C

Details of the Lifshitz case

We dedicate this appendix to review the calculation presented in [95], using their notation to avoid confusion. The metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{r^{2\nu}}{\widetilde{L}^{2\nu}}dt^2 + \frac{\widetilde{L}^2}{r^2}dr^2 + r^2 d\phi^2 \tag{C.1}$$

where

$$m^2\widetilde{L}^2 = \frac{1}{2}\left(\nu^2 - 3\nu + 1\right) , \qquad \frac{\widetilde{L}^2}{L^2} = \frac{1}{2}\left(\nu^2 + \nu + 1\right) . \tag{C.2}$$

The determinant of the induced metric, the Ricci scalar and $R_{||}$ (defined in eq. (5.4)) are given by

$$\begin{aligned} h &= \frac{\widetilde{L}^2}{r^2} + r^2 f'(r)^2 , \\ R &= -\frac{2\left(\nu^2 + \nu + 1\right)}{\widetilde{L}^2} , \\ R_{||} &= \frac{(\nu - 1)\nu}{\widetilde{L}^2 + r^4 f'(r)^2} - \frac{2\nu^2 + \nu + 1}{\widetilde{L}^2} . \end{aligned} \tag{C.3}$$

The last term we need is the squared of the extrinsic curvature

$$K^2 = \frac{r^4\left(\widetilde{L}^2\left(rf''(r) + 3f'(r)\right) + r^4 f'(r)^3\right)^2}{\widetilde{L}^2\left(\widetilde{L}^2 + r^4 f'(r)^2\right)^3} . \tag{C.4}$$

With this notation, the differential equation to be solved is

$$\begin{aligned}
&r^{12}\widetilde{L}^2 f'(r)^6\Big(\left(-4\nu^2+6\nu-3\right) r f''(r) - 3(2(\nu-3)\nu+3)f'(r)\Big) \\
&-r^8\widetilde{L}^4 f'(r)^2\Big[30r^3 f''(r)^3 + (6(\nu-5)\nu+99)f'(r)^3 \\
&\qquad +5r^2 f'(r)f''(r)\left(15f''(r)-4rf^{(3)}(r)\right) \\
&\qquad +2rf'(r)^2\left(3((\nu-2)\nu+19)f''(r)+r\left(rf^{(4)}(r)-10f^{(3)}(r)\right)\right)\Big] \\
&+r^4\widetilde{L}^6\Big[5r^3 f''(r)^3 + (2\nu(3\nu+7)+165)f'(r)^3 \\
&\qquad +rf'(r)^2\left(3(2\nu+87)f''(r)-4r^2 f^{(4)}(r)\right) \\
&\qquad +5r^2 f'(r)f''(r)\left(4rf^{(3)}(r)+27f''(r)\right)\Big] \\
&+2\widetilde{L}^8\left[3\left(\nu^2-4\right)f'(r) - r\left(\left(24-\nu^2\right)f''(r)+r\left(rf^{(4)}(r)+10f^{(3)}(r)\right)\right)\right] \\
&+(2\nu-1)r^{16}f'(r)^9 = 0\,. \qquad \text{(C.5)}
\end{aligned}$$

If we take the entangling surface to be the geodesic

$$f(r) = \frac{\widetilde{L}\sqrt{r^2\widetilde{L}^2 - r_t^2}}{r\, r_t}\,, \qquad \text{(C.6)}$$

we obtain

$$\frac{4\,\widetilde{L}^{15} r_t^3\,(\nu-1)\,\nu\, r^4}{\left(\widetilde{L}^2 r^2 - r_t^2\right)^{9/2}} = 0\,. \qquad \text{(C.7)}$$

Therefore we can conclude that the geodesic cannot be taken as an entangling surface in a Lifshitz background, since it doesn't minimize the entropy functional, except for the case $\nu = 1$, which corresponds to the Anti-de Sitter spacetime.

D Details of the WAdS case

In this appendix, we present the details of our calculation of section 5.4. We choose the boundary parametrization of the entangling surface to be $t = 0$ and $\theta = f(\rho)$. Thus the induced metric is given by

$$h = \rho^2 \left(\frac{1}{2}\rho^2 \left(\omega^2 - 1\right) + 1 \right) f'(\rho)^2 + \frac{1}{\frac{1}{2}\rho^2 \left(\omega^2 + 1\right) + 1} \,. \tag{D.1}$$

We compute the orthogonal vectors

$$\begin{aligned} n_{\alpha\,1} &= C \left(-\frac{(\omega - 1)\left(\rho^2(\omega + 1) + 2\right)}{\rho^2 \left(\omega^2 - 1\right) - 2}, -f'(\rho), 1 \right) , \\ n_{\alpha\,2} &= \left(-\sqrt{\left| \frac{\left(\omega^2 + 1\right)\rho^2 + 2}{2 - \rho^2 \left(\omega^2 - 1\right)} \right|}, 0, 0 \right) , \\ C &= \frac{\sqrt{2}\,\rho}{\sqrt{\rho^2 \left(\rho^2 \left(\omega^2 + 1\right) + 2\right) f'(\rho)^2 + \frac{4}{2 - \rho^2\left(\omega^2 - 1\right)}}} , \end{aligned} \tag{D.2}$$

as well as the contributions of the intrinsic curvature (remember that we set $\ell = 1$)

$$R = -2\left(\omega^2 + 2\right) \,, \tag{D.3}$$

$$R_{||} = \frac{2\rho^2\left(\rho^2\left(\omega^2-1\right)^2+2\left(\omega^2+1\right)\right)\left(\rho^2\left(\omega^2+1\right)+2\right)f'(\rho)^2+8\left(\omega^2+1\right)}{\rho^2\left(\rho^4\omega^4-\left(\rho^2+2\right)^2\right)f'(\rho)^2-4} \,, \tag{D.4}$$

and the extrinsic curvature contraction

$$
\begin{aligned}
K^2 \;=\;& \frac{2}{\left(\rho^2\left(\rho^4\omega^4-(\rho^2+2)^2\right)f'(\rho)^2-4\right)^3} \\
&\Big[4\rho^2\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^2 f''(\rho)^2 \\
&+\rho^4\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2-1\right)-1\right)^2\left(\rho^2\left(\omega^2+1\right)+2\right)^4 f'(\rho)^6 \\
&-4\rho^2\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^2\left(\rho^4\left(3\omega^4+2\omega^2-5\right)\right. \\
&\left.+3r^2\left(\omega^2-3\right)-4\right)f'(\rho)^4+4\left(\rho^6\left(\omega^4-1\right)\left(17\omega^2+25\right)\right. \\
&\left.+\rho^4\left(38\omega^4-20\omega^2-90\right)-96\rho^2-32\right)f'(\rho)^2 \\
&+8\rho\left(\rho^2\left(\omega^2+1\right)+2\right)\left(5\rho^4\left(\omega^4-1\right)+2\rho^2\left(\omega^2-7\right)-8\right)f'(\rho)f''(\rho) \\
&-4\rho^3\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2-1\right)-1\right) \\
&\qquad\left(\rho^2\left(\omega^2+1\right)+2\right)^3 f'(\rho)^3 f''(\rho)\Big]\,. \qquad \text{(D.5)}
\end{aligned}
$$

The merit of our choice of coordinates is that all these quantities, in the limit $\omega \to 1$, are precisely and smoothly the one that one obtain for the AdS_3 case (in global coordinates). With these results we can write down the entropy functional given in the equation (5.3) on the next page

$$
\begin{aligned}
S_{EE} \;=\; & \frac{1}{4G}\int d\rho\,\sqrt{h}\Bigg[1-\frac{2}{19\omega^2-2}\Bigg[\frac{3}{2}\left(\omega^2+2\right) \\
& +\frac{1}{\left(\rho^2\left(\rho^4\omega^4-\left(\rho^2+2\right)^2\right)f'(\rho)^2-4\right)^3} \\
& \Big[-4\rho^2\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^2 f''(\rho)^2 \\
& +\rho^4\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^3\left(\rho^2\left(\rho^2\left(\omega^2-1\right)\right.\right. \\
& \quad \left.\left.\left(\rho^2\left(\omega^4-4\omega^2+3\right)+12\right)-5\omega^2-13\right)-2\right)f'(\rho)^6 \\
& +4\rho^2\left(\rho^2\left(\omega^2-1\right)-2\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^2 \\
& \quad \left(\rho^4\left(\omega^4+10\omega^2-11\right)-3\rho^2\left(3\omega^2+7\right)-4\right)f'(\rho)^4 \\
& -4\Big[\rho^6\left(\omega^4-1\right)\left(25\omega^2+49\right)+6\rho^4\left(\omega^4-14\omega^2-31\right) \\
& \quad -96\rho^2\left(\omega^2+2\right)-32\Big]f'(\rho)^2 \\
& -8\rho\left(\rho^2\left(\omega^2+1\right)+2\right) \\
& \quad \left(5\rho^4\left(\omega^4-1\right)+2\rho^2\left(\omega^2-7\right)-8\right)f'(\rho)f''(\rho) \\
& +4\rho^3\left(\rho^2\left(\omega^2-1\right)-2\right) \\
& \quad \left(\rho^2\left(\omega^2-1\right)-1\right)\left(\rho^2\left(\omega^2+1\right)+2\right)^3 f'(\rho)^3 f''(\rho) \\
& +128\left(\omega^2+1\right)\Big]\Bigg]\Bigg]\,. \qquad \text{(D.6)}
\end{aligned}
$$

To extremize the action, we need to solve the equation of motion derived from this functional. However, as in the Lifshitz case, the resulting differential equation is highly non-linear. Therefore, we solve such a complex equation with the perturbative techniques of section 5.4.

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