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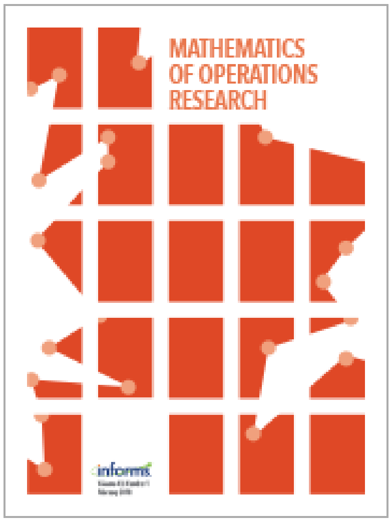

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Genericity Results in Linear Conic Programming—A Tour d'Horizon

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Abstract. This paper is concerned with so-called generic properties of general linear conic programs. Many results have been obtained on this subject during the last two decades. For example, it is known that uniqueness, strict complementarity, and nondegeneracy of optimal solutions hold for almost all problem instances. Strong duality holds generically in a stronger sense, i.e., it holds for a generic subset of problem instances.

In this paper, we survey known results and present new ones. In particular we give an easy proof of the fact that Slater's condition holds generically in linear conic programming. We further discuss the problem of stability of uniqueness, nondegeneracy, and strict complementarity. We also comment on the fact that in general, a conic program cannot be treated as a smooth problem and that techniques from nonsmooth geometric measure theory are needed.

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Keywords: conic optimization • generic properties • Slater's condition • uniqueness and nondegeneracy of optimal solutions • strict complementarity • stability

1. Introduction

Linear conic programs (CPs) can be given in different equivalent forms. In this paper, we consider the pair of primal-dual linear conic programs

$$\max c^T x \quad \text{s.t. } B - Ax \in \mathcal{K}, \tag{P}$$

$$\min\langle B, Y\rangle \quad \text{s.t. } A^T Y = c, \;\; Y \in \mathcal{K}^*, \tag{D}$$

with given vectors $c \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$, a matrix $A \in \mathbb{R}^{m\times n}$, and variables $x \in \mathbb{R}^n and Y \in \mathbb{R}^m$. We assume that $\mathcal{K} \subseteq \mathbb{R}^m$ is a pointed full-dimensional closed convex cone and $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^m$; i.e., $\mathcal{K}^* := \{Y \in \mathbb{R}^m \mid \langle Y, X\rangle \geq 0 \text{ for all } X \in \mathcal{K}\}$.

Often, for example in semidefinite and copositive programming, the elements Y, B and the columns A_i of A are matrices from the set $\mathcal{S}_k$ of real symmetric $k\times k$ matrices. We therefore write the vectors $x, c \in \mathbb{R}^n$ in lower case but the vectors (matrices) $Y, B, A_i \in \mathbb{R}^m$ in capital letters. Note that we can simply identify $\mathcal{S}_k \equiv \mathbb{R}^m$, where $m := \frac{1}{2}k(k+1)$.

Linear conic programming represents an important class of convex problems with a multitude of applications. It contains linear programming (LP), semidefinite programming (SDP), and copositive programming as special cases. We refer, e.g., to Nemirovski [25], Shapiro [35], and Pataki [27] for surveys on this topic.

In this paper, we study genericity results for such programs; i.e., we wish to show that certain "nice" regularity conditions hold generically. Let $\mathcal{P}$ be (a subset of) a Euclidean space $\mathbb{R}^N$. In what follows, we say that a subset $\mathcal{P}_r \subseteq \mathcal{P}$ is a *weakly generic subset* of $\mathcal{P}$ if $\mathcal{P}\backslash\mathcal{P}_r$ has Lebesgue measure zero. We call $\mathcal{P}_r$ a *generic subset* of $\mathcal{P}$ if $\mathcal{P}_r$ is open in $\mathcal{P}$ and $\mathcal{P}\backslash\mathcal{P}_r$ has Lebesgue measure zero. So the weakly generic sets $\mathcal{P}_r$ need not be open. A property is said to be (*weakly*) *generic in the problem set* $\mathcal{P}$ if it holds for a (weakly) generic subset $\mathcal{P}_r$ of $\mathcal{P}$. Hence, genericity implies both density and stability of the nice problem instances, whereas weak genericity only assures density. Note that from a numerical viewpoint, stability (i.e., openness of $\mathcal{P}_r$) is crucial, so genericity is the desirable property.

Remark 1. Genericity can be defined in different ways. In Alizadeh et al. [2] and Bolte et al. [4] (weak) genericity results have been formulated with respect to the Lebesque measure, in Pataki and Tunçel [28] with respect to the Hausdorff measure, and in Borwein and Moors [5] in terms of σ-porosity (cf. Lemma 2). It is well known that in $\mathbb{R}^N$ the N-dimensional Lebesgue and Hausdorff measures coincide (see, e.g., Morgan [24, Corollary 2.8]) and that a σ-porous set has Lebesque measure zero (the converse does not hold). In Schurr et al. [33], weak genericity is called metric genericity and some genericity results are given in terms of open and dense sets (see Schurr et al. [33, Theorems 4.6 and 4.7]). Note, however, that openness and density of a set $\mathcal{A} \subset \mathbb{R}^N$ does not imply that the complement $\mathbb{R}^N \backslash \mathcal{A}$ has Lebesque measure zero. So our concept of genericity is stronger and we think that for our purpose (i.e., for problem sets in $\mathbb{R}^N$), our definition of genericity is appropriate and meaningful.

Genericity of properties like strong duality, nondegeneracy, strict complementarity, and uniqueness of solutions of linear conic programs have been discussed before. Alizadeh et al. [2] and Shapiro [34] specifically discuss generic properties of semidefinite programs. Pataki and Tunçel [28] derive weak genericity results on strict complementarity, uniqueness, and nondegeneracy for general linear conic programs. Note, however, that the results in Alizadeh et al. [2] have been proven under the assumption that the Slater condition is satisfied, and in Pataki and Tunçel [28], the genericity results are restricted to so-called gap-free problems (i.e., problems with finite optimal value and zero duality gap). The possibility that these assumptions generically fail has not been excluded, so strictly speaking these genericity results were lacking some foundation. Merely for the SDP case, it is indicated in Shapiro [34, p. 310] that the Slater condition (Mangasarian-Fromovitz condition) is generic. Recently Bolte et al. [4] gave special full genericity results with respect to uniqueness of solutions under the extra assumption that the cone $\mathcal{K}$ is a semialgebraic set.

While we were working on an earlier version of this article, other results were brought to our attention, e.g., the paper by Schurr et al. [33] and the one by Borwein and Moors [5]. This led to a complete revision of our earlier paper and resulted in the present article, which has the following aims: to survey the known genericity results, to add new ones, and to discuss the relations between the different genericity statements.

We start with some general remarks. Usually, genericity results in smooth optimization are proven by applying transversality theory from differential topology. We refer to Jongen et al. [22] for such genericity results in smooth nonlinear finite programming and to Alizadeh et al. [2] for results in SDP. We also refer the reader to Section 5 for the special case of LP and SDP.

However, a general conic program is not a completely smooth problem. Indeed, a part of the problem is given by the specific cone $\mathcal{K}$ (or its dual $\mathcal{K}^*$), and boundaries of convex cones are generally described by convex and hence Lipschitz-continuous functions rather than by smooth functions. So, to obtain genericity results in general linear conic programming, we have to use techniques from nonsmooth convex analysis. Fortunately, in the field of geometric measure theory many results of differential geometry for C^1-functions have been generalized to similar results for Lipschitz functions. Founding work for this theory goes back to Federer and others (see Federer [11], Morgan [24], Schneider [31] for an overview). The results in Pataki and Tunçel [28], Schurr et al. [33], and Bolte et al. [4] are based on this theory, and we also will use techniques from geometric measure theory.

In this paper, we try to prove our genericity results with techniques that are as basic as possible. Genericity of strong duality will be proven (based on Lemma 1) by purely topological arguments. More structure is needed for weak genericity of uniqueness. As we shall see, the classical result that Lipschitz functions (convex functions) are differentiable almost everywhere will do the job. For weak genericity of nondegeneracy and strict complementarity, more sophisticated techniques from geometric measure theory are needed (see Pataki and Tunçel [28]).

The paper is organized as follows. Section 2 introduces some notation and presents two equivalent formulations for the conic programs (P) and (D). In Section 3 we show that the Slater condition holds generically in conic programming. By using well-known techniques, this leads to genericity results for strong duality similar to the results in Schurr et al. [33]. We compare the statements in Schurr et al. [33] with our result and discuss related work. Section 4 deals with weak genericity results concerning uniqueness, nondegeneracy, and strict complementarity in CP. In Section 4.1, we give an independent proof of the fact that uniqueness is weakly generic. This approach was brought to our attention by Alexander Shapiro (personal communication). The proof does not rely on deeper results from geometric measure theory as used in Pataki and Tunçel [28, Theorem 3]. Section 4.2 summarizes the weak genericity results for nondegeneracy and strict complementarity from Pataki and Tunçel [28]. Section 4.3 comments on the fact that nondegeneracy implies Slater's condition. It further explains why most genericity results from linear semi-infinite optimization (SIP) cannot be directly applied to CP.

In Section 5 we discuss stability of properties like uniqueness, nondegeneracy, and strict complementarity. For some special classes of CP, such as LP and SDP, full genericity can be shown. For general conic programs it is still open whether the stability for these properties holds generically.

2. Preliminaries

We next discuss two other formulations for CP. Many authors (e.g., Pataki and Tunçel [28]) consider conic programs in

Self-dual formulation:

$$\max\{\langle C,B\rangle - \langle C,X\rangle\} \quad \text{s.t. } X \in (B+\mathcal{L})\cap\mathcal{K} \tag{P_0}$$

$$\min\langle B,Y\rangle \quad \text{s.t. } Y \in (\mathcal{L}^\perp + C)\cap\mathcal{K}^*, \tag{D_0}$$

where $C,B\in\mathbb{R}^m$, $\mathcal{L}=\operatorname{span}\{A_1,\dots,A_n\}\subset\mathbb{R}^m$ is the linear subspace spanned by $A_i\in\mathbb{R}^m$, $i=1,\dots,n$, and $\mathcal{K}$ is a cone in $\mathbb{R}^m$, as above.

It is easy to see that the problems (P_0) and (D_0) are equivalent to (P) and (D), respectively. Indeed, let A_i denote the columns of A and choose some $C\in\mathbb{R}^m$ satisfying $\langle A_i,C\rangle = c_i$ for $i=1,\dots,n$. Then the feasible sets of (P_0) and (P) are directly related via the affine mapping $X = B - Ax$. If the A_is are linearly independent, the map is bijective. Also their objective function values are the same, since for $X = B-\sum_{i=1}^n x_iA_i$ we obtain

$$\langle C,B\rangle - \langle C,X\rangle = \langle C, B-X\rangle = \left\langle C,\sum_{i=1}^n x_iA_i\right\rangle = \sum_{i=1}^n x_i\langle C,A_i\rangle = c^Tx.$$

The dual problems (D_0) and (D) have the same objective function, and in view of the relation

$$Y - C\in\mathcal{L}^\perp \quad\Leftrightarrow\quad \langle Y-C,A_i\rangle = 0 \quad \text{for all } i \quad\Leftrightarrow\quad \langle Y,A_i\rangle = c_i \quad \text{for all } i \quad\Leftrightarrow\quad A^TY = c$$

the feasible sets coincide, so (D_0) and (D) are equivalent as well.

Remark 2. Important special cases of CP are linear programs, where $\mathcal{K}=\mathcal{K}^*=\mathbb{R}^m_+$, and semidefinite programs, where the columns A_i of A (i.e., the basis of $\mathcal{L}$) as well as B and C are elements of the space $\mathcal{S}_k$ of symmetric $k\times k$ matrices and $\mathcal{K}=\mathcal{K}^*$ equals the set $\mathcal{S}_k^+$ of positive semidefinite matrices in $\mathcal{S}_k$. Note that we can identify $\mathcal{S}_k\equiv\mathbb{R}^m$ with $m=\frac{1}{2}k(k+1)$.

Another example is given by the class of copositive programs (COP), where $\mathcal{K}$ is the cone of copositive matrices with dual $\mathcal{K}^*$, the cone of completely positive matrices (see, e.g., Burer [6] for details).

In the sequel, the feasible sets and optimal values of these conic programs will be denoted by $\mathcal{F}_{P_0}$ and $\mathcal{F}_{D_0}$, and v_{P_0} and v_{D_0}, respectively. As usual, we say that strong duality holds for a pair of primal, dual programs (P_0), (D_0) if the relation $v_{P_0}=v_{D_0}$ holds.

SIP formulation: Linear conic programs can also be seen as a special case of linear semi-infinite programs (SIP) of the form

$$\max_{x\in\mathbb{R}^n} c^Tx \quad \text{s.t. } b(Y) - a(Y)^Tx \ge 0 \ \text{ for all } Y\in Z, \tag{SIP_P}$$

with a possibly infinite index set $Z\subset\mathbb{R}^m$ and functions $a\colon Z\to\mathbb{R}^n$ and $b\colon Z\to\mathbb{R}$. The (Haar-) dual reads

$$\min \sum_{Y_j\in Z} y_j b(Y_j) \quad \text{s.t. } \sum_{Y_j\in Z} y_j a(Y_j) = c, \quad y_j\ge 0, \tag{SIP_D}$$

where the min is taken over all finite sums. For an introduction to (linear) SIP, we refer, e.g., to Goberna and López [13]. Note that the condition $X\in\mathcal{K}$ can be equivalently expressed as

$$\langle X,Y\rangle \ge 0 \quad \text{for all } Y\in\mathcal{K}^*.$$

In view of this relation the primal program (P) can be written as (SIP_P), with

$$a(Y) := A^TY, \qquad b(Y) := \langle B,Y\rangle, \qquad \text{and} \qquad Z := \mathcal{K}^*. \tag{1}$$

The feasibility condition for (SIP_D) then becomes

$$c = \sum_j y_jA^TY_j, \quad y_j\ge 0$$

and by putting $Y := \sum_j y_jY_j\in\mathcal{K}^*$, this coincides with the feasibility condition $c = A^TY$ of (D). Moreover, in view of $\sum_j y_jb(Y_j) = \sum_j y_j\langle Y_j,B\rangle = \langle Y,B\rangle$, the dual ($SIP_D$) is equivalent to (D), and we simply denote both versions by (D).

For the genericity results in this article, we always assume that the cone $\mathcal{K}$ (and thus $\mathcal{K}^*$) and n and m are arbitrarily fixed. Then the set of problem instances of (P) and (D) is given by

$$\mathcal{P} := \{(A,B,c)\in\mathbb{R}^{m\times n}\times\mathbb{R}^m\times\mathbb{R}^n\}\equiv\mathbb{R}^{m\cdot n+m+n}$$

endowed with some norm.

Often we prove results of the sort that for arbitrarily fixed $A \in \mathbb{R}^{m\times n}$ a property holds for all (B, c) from a generic set $S = S(A) \subset \mathbb{R}^{m+n}$. We emphasize that this implies that the property holds for almost all problem instances in the whole space $\mathcal{P} = \{(A, B, c)\}$. Indeed, under this assumption, for any fixed $A \in \mathbb{R}^{m\times n}$ the property holds on the whole $\mathbb{R}^{m+n}$, except for the set $S(A)^c := \mathbb{R}^{m+n} \backslash S(A)$ of Lebesgue measure $\mu(S(A)^c) = 0$ in $\mathbb{R}^{m+n}$. But then by Fubini's theorem the property holds for $(A, B, c) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n$, except for a set of measure $\int_{\mathbb{R}^{m\times n}} \mu(S(A)^c)\, dA = 0$.

Concerning openness, however, we have to be careful: If for any fixed A a property holds for any (B, c) from an open set $S \subset \mathbb{R}^{m+n}$, then this property need not hold for an open set in $\mathcal{P}$. A counterexample is given in Example 1.

Throughout the paper we assume that $n \le m$ holds. For the case $n > m$ the genericity results can be summarized by the following statement: Generically for the case $n > m$

- the dual (D) is infeasible and
- the primal program (P) is unbounded.

So in this case strong duality with $v_P = v_D = +\infty$ holds generically. To prove this, we use the well-known fact that (see, e.g., Jongen et al. [22, Example 7.3.23])

$$\text{a matrix } U \in \mathbb{R}^{N\times M} \text{ with } N \ge M \text{ generically has full rank } M. \tag{2}$$

We first show that generically with respect to (A, c), the system

$$c = A^T Y \quad \text{has no solution } Y \in \mathbb{R}^m. \tag{3}$$

Indeed, by (2) the matrix $U := [A^T c] \in \mathbb{R}^{n\times(m+1)}$ generically has rank $m + 1$ which is why $Uz = 0$ does not allow a nonzero solution. This means that generically the system in (3) and hence problem (D) are infeasible.

To show that (P) is generically unbounded, we consider the system $Ax = B$, $c^T x = \tau$, any solution of which yields a primal feasible x with objective value τ. Again, using (2), generically, the matrix $U := \left[\begin{smallmatrix} A \\ c^T \end{smallmatrix}\right] \in \mathbb{R}^{(m+1)\times n}$ has full rank $m + 1$, so $Ax = B$, $c^T x = \tau$ is solvable for any τ and B.

3. Genericity of Slater's Condition and Strong Duality

It is well known that strong duality always holds in linear programming (unless both programs are infeasible), but strong duality need not hold in general conic programming. However, as we shall see, strong duality is a generic property.

In this section we give an independent easy proof of the fact that in conic programming the Slater condition holds generically. To do so we only make use of the result that the boundary of a convex set has measure zero. By applying well-known duality theorems, this leads to an alternative proof of the genericity result for strong duality in Schurr et al. [33]. We also summarize other related results from Borwein and Moors [5] and Schurr et al. [33].

3.1. Genericity of Slater's Condition

In this section we provide a purely topological proof of the genericity of Slater's condition.

Definition 1. We say that Slater's condition holds for (P) if there exists a feasible x such that $X := B - Ax \in \operatorname{int} \mathcal{K}$. Analogously, we say that Slater's condition holds for (D) if there exists a feasible Y, i.e., $A^T Y = c$, such that $Y \in \operatorname{int} \mathcal{K}^*$.

Roughly speaking, Slater's condition says that the feasible set of the problem is not entirely contained in the boundary of the convex cone. For this reason, it is intuitive that the proof of a genericity result should be based on properties of this boundary. More specifically, we will use the fact that the boundary of a convex set has measure zero.

Lemma 1. *Let $\mathcal{T}$ be a full-dimensional closed convex set in $\mathbb{R}^s$. Then the boundary of $\mathcal{T}$ has s-dimensional Lebesgue measure zero.*

Proof. For the sake of completeness we repeat here the elegant proof of Lang [23]. Consider an open ball $\mathcal{B}_\varepsilon(p)$ with center $p \in \operatorname{bd} \mathcal{T}$ and radius $\varepsilon > 0$. Since there exists a hyperplane supporting the convex set $\mathcal{T}$ at p, at least half of the ball does not contain points of $\mathcal{T}$. Therefore,

$$\limsup_{\varepsilon \downarrow 0} \frac{\mu(\mathcal{T} \cap \mathcal{B}_\varepsilon(p))}{\mu(\mathcal{B}_\varepsilon(p))} \le \frac{1}{2}.$$

On the other hand, Lebesgue's density theorem (see, e.g., Faure [10]) says that for almost all points p of the Lebesgue measurable set $\mathcal{T}$ we have that

$$\lim_{\varepsilon\downarrow 0}\frac{\mu(\mathcal{T}\cap\mathcal{B}_\varepsilon(p))}{\mu(\mathcal{B}_\varepsilon(p))}=1.$$

This immediately implies that $\operatorname{bd}\mathcal{T}$ has measure zero. □

The next theorem shows that Slater's condition is indeed generic.

Theorem 1. *Let $A\in\mathbb{R}^{m\times n}$ be given arbitrarily. Then there exists a generic subset $S_1\subset\mathbb{R}^n$ (open with complement of measure zero) such that for any $c\in S_1$ precisely one of the following alternatives holds for the corresponding problem instance of* (D):

(1) *either the feasible set of* (D) *is empty, i.e., $\{Y\in\mathcal{K}^*\mid A^TY=c\}=\varnothing$, or*
(2) *Slater's condition holds for* (D), *i.e., $\{Y\in\operatorname{int}\mathcal{K}^*\mid A^TY=c\}\neq\varnothing$.*

An analogous result holds for the primal program (P); *i.e., there is a generic subset $\tilde{S}_1$ of $\mathbb{R}^m$ such that for any $B\in\tilde{S}_1$ either the corresponding program* (P) *is infeasible or* (P) *satisfies the Slater condition.*

Proof. For the case of program (D), note that the set $S:=\{c=A^TY\mid Y\in\mathcal{K}^*\}\subset\mathbb{R}^n$ is a convex set with $\dim S=: k\le n$. We define $S_1:=\operatorname{int}S\cup(\mathbb{R}^n\setminus\operatorname{cl}S)$ and show that this is the generic set we are looking for. As a union of two open sets, S_1 is clearly open. Note that for $c\in\mathbb{R}^n\setminus\operatorname{cl}S$ the alternative (1) holds; i.e., the feasible set of (D) is empty. If $k<n$, i.e., A does not have full rank n, then the statement is true with $\operatorname{int}S=\varnothing$. So we can assume $\dim S=n$, and since by Lemma 1 the set $\operatorname{bd}S=\mathbb{R}^n\setminus S_1$ has measure zero, it is sufficient to show that for $c\in\operatorname{int}S$ the Slater condition holds (alternative (2)).

So let $c\in\operatorname{int}S$ be given. By assumption, there exists some $Y_0\in\mathcal{K}^*$ for which $A^TY_0=c$ holds. Consider the affine space $Y_0+\ker A^T$. If $Y_0+\ker A^T\cap\operatorname{int}\mathcal{K}^*\neq\varnothing$, then Slater's condition holds and we are done.

So assume by contradiction that $Y_0+\ker A^T\cap\operatorname{int}\mathcal{K}^*=\varnothing$. This implies in particular that $Y_0\in\operatorname{bd}\mathcal{K}^*$, and since $\operatorname{int}\mathcal{K}^*\neq\varnothing$, there exists a separating hyperplane with normal vector N such that (see Rockafellar [29, Theorem 11.2])

$$\langle N,Y\rangle\ge\langle N,Y_0\rangle\quad\text{for all } Y\in\mathcal{K}^*\ \text{ and }\ N\perp\ker A^T. \tag{4}$$

Since $c\in\operatorname{int}S$, there exists an open neighborhood $\varnothing\neq\mathcal{N}_\varepsilon(c)\subset\operatorname{int}S$ of c, and by continuity of the mapping A^TY there exists an open neighborhood $\varnothing\neq\mathcal{N}_\delta(Y_0)$ of Y_0 such that $A^T\mathcal{N}_\delta(Y_0)\subset\mathcal{N}_\varepsilon(c)$. The separating hyperplane divides $\mathcal{N}_\delta(Y_0)$ into two parts. Take a point $Y_1\in\mathcal{N}_\delta(Y_0)$ such that $\langle N,Y_1\rangle<\langle N,Y_0\rangle$. By construction, $c_1:=A^TY_1\in\mathcal{N}_\varepsilon(c)\subset\operatorname{int}S$. So there must exist a pre-image $\tilde{Y}_1\in\mathcal{K}^*$ with $A^T\tilde{Y}_1=c_1$, i.e., $\tilde{Y}_1=Y_1+\tilde{Y}_0$ with $\tilde{Y}_0\in\ker A^T$. Putting it all together using $\langle N,\tilde{Y}_0\rangle=0$ and (4), we obtain

$$\langle N,Y_0\rangle\le\langle N,\tilde{Y}_1\rangle=\langle N,Y_1+\tilde{Y}_0\rangle=\langle N,Y_1\rangle<\langle N,Y_0\rangle,$$

a contradiction. So the assumption $Y_0+\ker A^T\cap\operatorname{int}\mathcal{K}^*=\varnothing$ must be false. This concludes the proof for problem (D).

For the primal program we proceed as follows. We note that $\mathbb{R}^m$ allows an orthogonal decomposition

$$\mathbb{R}^m=\operatorname{im}A\oplus\ker A^T,\qquad B=B_1\oplus B_2\quad\text{for } B\in\mathbb{R}^m,$$

where B_2 is the projection $\operatorname{proj}_{\ker A^T}B$ of $B\in\mathbb{R}^m$ onto the linear space $\ker A^T$. Let $Q\in\mathbb{R}^{m\times m}$ be the matrix representation of this projection; i.e., $B_2=\operatorname{proj}_{\ker A^T}B=QB$. We now consider the convex cone $R:=Q\mathcal{K}$. As before, we have

$$QB\in\ker A^T\setminus\operatorname{cl}R\quad\Rightarrow\quad\{B-Ax\mid x\in\mathbb{R}^n\}\cap\mathcal{K}=\varnothing$$

and we can show (with $\operatorname{int}R$ relative to $\ker A^T$)

$$QB\in\operatorname{int}R\quad\Rightarrow\quad\{B-Ax\mid x\in\mathbb{R}^n\}\cap\operatorname{int}\mathcal{K}\neq\varnothing.$$

Here again $\operatorname{bd}R$ has measure zero and thus $R_1:=\operatorname{int}R\cup(\ker A^T\setminus\operatorname{cl}R)$ is relatively open in $\ker A^T$ with $\ker A^T\setminus R_1$ of measure zero in $\ker A^T$. Consequently, the set $\tilde{S}_1:=\operatorname{im}A\oplus R_1$ is open in $\mathbb{R}^m$ with $\mathbb{R}^m\setminus\tilde{S}_1$ of measure zero in $\mathbb{R}^m$. By construction, for $B\in\tilde{S}_1$, precisely one of the two alternatives holds. □

Remark 3. The Slater conditions for (P) and (P_0) are clearly equivalent. The genericity result for (D) in Theorem 1 with respect to parameter c can also be translated to the following corresponding result for (D_0): Let $\mathcal{L}$ be given. Then there exists a generic subset $Q_1\subset\mathbb{R}^m$ such that for any $C\in Q_1$ precisely one of the following alternatives holds for the corresponding problem instance of (D_0):

(1′) either the feasible set of (D_0) is empty, or
(2′) Slater's condition holds for (D_0), i.e., $\{Y\mid Y\in(\mathcal{L}^\perp+C)\cap\operatorname{int}\mathcal{K}^*\}\neq\varnothing$.

To see this, we proceed as in the second part of the proof of Theorem 1: consider the orthogonal decomposition

$$\mathbb{R}^m = \mathcal{L}^\perp \oplus \mathcal{L}, \qquad C = C_1 \oplus C_2 \quad \text{for } C \in \mathbb{R}^m.$$

Let $P \in \mathbb{R}^{m\times m}$ be the matrix representation of the projection $\operatorname{proj}_{\mathcal{L}}$ onto $\mathcal{L}$, and let $C_2 = PC = \operatorname{proj}_{\mathcal{L}} C$. Then as in the proof of Theorem 1 above, we consider the convex cone $S := P\mathcal{K}^* \subset \mathcal{L}$ and the set

$$S_1 = \operatorname{int} S \cup (\mathcal{L} \setminus \operatorname{cl} S),$$

which is relatively open in $\mathcal{L}$ and for which $\mathcal{L}\setminus S_1$ has measure zero. Note that for $PC \in \operatorname{int} S$ the alternative (2′) holds, whereas for $PC \in \mathcal{L}\setminus \operatorname{cl} S$ the condition (1′) is true. So the set $Q_1 = \mathcal{L}^\perp \oplus S_1$ is the required generic set in $\mathbb{R}^m$.

It is well known (see Rockafellar [30], Schurr et al. [33, Lemma 3.2], Goberna and López [13, Theorem 8.1]) that if for some A, B the problem (P) satisfies the Slater condition, then for all c the strong duality relation $v_P = v_D$ holds and, in case $v_P = v_D$ is finite, the optimal value of (D) is attained. So the genericity of Slater's condition in Theorem 1 leads to the following genericity result for strong duality (similar to Schurr et al. [33]):

Corollary 1. *Let $A \in \mathbb{R}^{m\times n}$ be given arbitrarily. Then with the generic subset $\tilde{S}_1 \subset \mathbb{R}^m$ from Theorem 1 one of the following holds for $B \in \tilde{S}_1$:*

- either the feasible set of (P) is empty,
- or (P) is strictly feasible and for any $c \in \mathbb{R}^n$ we have $v_P = v_D$, meaning that if (D) is infeasible, then $v_P = v_D = +\infty$, and if (D) is feasible, then $v_P = v_D$ is finite and the minimum value of (D) is attained.

An analogous result holds for the dual program (D) with respect to $c \in S_1 \subset \mathbb{R}^n$ (with S_1 from Theorem 1).

By combining the results for the primal and dual programs we obtain:

Corollary 2. *Let $A \in \mathbb{R}^{m\times n}$ be given arbitrarily. Then with the generic subsets $S_1 \subset \mathbb{R}^n$, $\tilde{S}_1 \subset \mathbb{R}^m$ from Theorem 1, for any $(B, c) \in \tilde{S}_1 \times S_1$ precisely one of the following alternatives holds:*

(1) *Both feasible sets of (P) and (D) are empty.*
(2) *Precisely one of the feasible sets of (P) or (D) is empty and $v_P = v_D = \pm\infty$.*
(3) *Both (P) and (D) are feasible and for both problems the optimal value is attained with $v_P = v_D$.*

A corresponding result holds for (P_0), (D_0) with respect to to a generic set $\tilde{S}_1 \times Q_1 \subset \mathbb{R}^m \times \mathbb{R}^m$ of parameters (B, C) (cf. Remark 3).

The statement in Corollary 2 could be called genericity of universal strong duality with respect to parameters (B, c) for any fixed A.

We next compare our result with that in Schurr et al. [33], where the authors take A as a parameter, and they define: For given A, universal duality is said to hold (with respect to A), if for any (B, c) the equality $v_P = v_D$ holds for (P) and (D) (see also Section 3.2). They prove the following:

Theorem 2 (see Schurr et al. [33, Theorem 4.5, Theorem 4.7]). *There is a generic subset $S \subset \mathbb{R}^{m\times n}$ such that for any $A \in S$ universal duality holds.*

The main difference between this statement and ours above is that by taking A as a parameter in the generic set S of Theorem 2, the case that both primal and dual are infeasible is excluded. In our approach, for fixed A we cannot exclude generically in (B, c) the infeasibility of both programs (P) and (D) simultaneously. We illustrate this difference between our result in Corollary 2 and the result from Schurr et al. [33] as stated in Theorem 2 by an example.

Example 1. Consider the LP:

$$\text{(P)} \quad \max c^T x \quad \text{s.t. } B - Ax \geq 0 \quad \text{with } c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$\text{(D)} \quad \min B^T Y \quad \text{s.t. } A^T Y = c, \quad Y = (y_1, y_2, y_3) \geq 0.$$

The primal (resp. dual) feasibility conditions are

$$x_2 \leq 0, \quad x_2 \geq 1, \quad x_1 \leq 0 \quad \text{resp.} \quad y_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad y_i \geq 0.$$

Both programs are infeasible, and for fixed A this property is stable with respect to small perturbations of (B, c). So in Corollary 2, the alternative (1) cannot be excluded generically. Recall, however, that according to the genericity concept in Theorem 2 (where A is the parameter) a generic perturbation of the matrix A above makes either (P) or (D) feasible.

Moreover, note that in Corollary 2 (in contrast to Theorem 2) also the existence of solutions is assured in case (3). We also emphasize that the proof of our genericity statement is more elementary than the proof in Schurr et al. [33], which is based on a deep result (see Schurr et al. [33, Lemma A.1]) from geometric measure theory.

The notion of universal duality goes back to Duffin et al. [9]. His results allow another approach to genericity of strong duality, which is briefly discussed in the next section.

3.2. Genericity Results Based on Generic Closedness of the Image $M\mathcal{K}$

It was brought to our attention by Warren Moors that an approach from Borwein and Moors [5] allows another way to prove genericity of strong duality for conic programs. We briefly outline this alternative: It is well known that for $M \in \mathbb{R}^{k\times m}$ the linear image $M\mathcal{K} := \{MY \mid Y \in \mathcal{K}\}$ of a polyhedral closed convex cone $\mathcal{K} \subset \mathbb{R}^m$ is closed. This is not generally true for nonpolyhedral cones (see, e.g., Rockafellar [29, pp. 73–74] for a counterexample). In Borwein and Moors [5] the following genericity statement has been shown.

Lemma 2 (see Borwein and Moors [5, Theorem 2]). *Let $k \in \mathbb{N}$ and let $\mathcal{K} \subset \mathbb{R}^m$ be a closed convex cone. Then the set*

$$S_1 := \mathbb{R}^{k\times m} \setminus \operatorname{int}\{M \in \mathbb{R}^{k\times m} \mid M\mathcal{K} \text{ is closed}\}$$

is σ-porous.

Note that σ-porosity of S_1 implies that S_1 has Lebesgue measure zero and is the countable union of nowhere dense sets (see Borwein and Moors [5]).

The following result for SIP, by Duffin et al. [9], provides the connection between strong duality and closedness of images $M\mathcal{K}$. We formulate these statements in terms of our problems (P) and (D).

Under the assumption that (P) is feasible, in Duffin et al. [9] the data $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ are said to yield primal uniform LP duality for (P) and (D), if for any $c \in \mathbb{R}^n$ either $\mathcal{F}_D = \emptyset$ and $v_P = v_D = \infty$ or $v_P = v_D < \infty$ and a solution of (D) exists.

Lemma 3 (See Duffin et al. [9, Theorem 3.2] and Hettich and Kortanek [19, Theorem 6.14]). *Let $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ be such that (P) is feasible. Then the data (A, B) yield primal uniform LP duality if and only if the cone*

$$\mathcal{C} := \operatorname{cone}\left(\left\{\begin{pmatrix} A^T \\ B^T \end{pmatrix} Y \,\middle|\, Y \in \mathcal{K}^*\right\} \cup e_{m+1}\right)$$

is closed. Here, $e_{m+1} := (0, \ldots, 0, 1)^T$ is a unit vector in $\mathbb{R}^{m+1}$.

Under the assumption that (P) is feasible, it is easy to show that

$$\text{if the cone } \mathcal{C}_1 := \left\{\begin{pmatrix} A^T \\ B^T \end{pmatrix} Y \,\middle|\, Y \in \mathcal{K}^*\right\} \text{ is closed, then } \mathcal{C} \text{ is closed.} \tag{5}$$

To see this, we note that the cone $\mathcal{C}_2 := \operatorname{cone}(e_{m+1})$ is closed and apply a well-known result, e.g., in the form Goberna and López [13, Theorem A4]:

$$\text{Let } \mathcal{C}_1, \mathcal{C}_2 \text{ be closed cones with } \mathcal{C}_1 \cap -\mathcal{C}_2 = \{0\}. \text{ Then } \mathcal{C}_1 + \mathcal{C}_2 \text{ is closed.}$$

To show that under the assumption $\mathcal{F}_P \neq \emptyset$ the relation $\mathcal{C}_1 \cap -\mathcal{C}_2 = \{0\}$ holds, let us assume to the contrary that there exists an element $0 \neq Z \in \mathcal{C}_1 \cap -\mathcal{C}_2$. This means that there exists some $\tilde{Y} \in \mathcal{K}^*$ such that $Z := \binom{A^T}{B^T}\tilde{Y} = -\alpha e_{m+1}$ with $\alpha > 0$; i.e., $A^T\tilde{Y} = 0$ and $B^T\tilde{Y} = -\alpha < 0$. But for any $\bar{x} \in \mathbb{R}^n$ we then obtain $(B - A\bar{x})^T\tilde{Y} = B^T\tilde{Y} < 0$, i.e., (P) is not feasible, a contradiction.

By combining Lemma 3 and (5) with Lemma 2 we obtain:

Theorem 3. *The set of parameters $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ with nonempty primal feasible set $\mathcal{F}_P$ where the (primal) uniform LP-duality fails is σ-porous in $\mathbb{R}^{m\times n} \times \mathbb{R}^m$. So, in particular, uniform LP duality as defined above is weakly generic in the space of parameters $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$.*

A corresponding dual genericity result holds with respect to parameters (A^T, c).

4. Genericity Analysis for Other Properties

In this section we analyze the generic behavior of conic programs with respect to the uniqueness, nondegeneracy, and strict complementarity of solutions. Note that even for linear programs, these properties are not always fulfilled. But it appears that these properties hold for almost all instances of conic programs. We emphasize that the stability (openness of the "set of nice instances") cannot be answered generally without extra assumptions on the cone. This aspect will be addressed in Section 5.

Trying to derive the genericity results using techniques that are as basic as possible, we show in the next section how the analysis of uniqueness can be based on the classical result that a convex function is differentiable almost everywhere. The weak genericity results for nondegeneracy and strict complementarity in Section 4.2 require deeper results from geometric measure theory.

4.1. Analysis of Uniqueness of Solutions

We now study the uniqueness of solutions of conic programs. Weak genericity of uniqueness can be proven, as in Pataki and Tunçel [28], by using a result from geometric measure theory for convex bodies (Schneider [31, Theorem 2.2.9]). Alternatively, we will derive this result by using the fact that convex functions are differentiable almost everywhere. This approach was brought to our attention by Alexander Shapiro (personal communication). It is based on a duality theory developed in Rockafellar [30]. Similar results have been proven for SIP programs in Goberna and López [12]. We will make use of these results, directly formulated in terms of conic programs.

In this section we consider for fixed A, B our primal problem $\mathrm{P} = \mathrm{P}(c)$, as the linear SIP (see Section 2)

$$(\mathrm{P}(c)) \quad \max c^T x \quad \text{s.t. } (B - Ax)^T Y \geq 0 \quad \text{for all } Y \in Z := \mathcal{K}^*,$$

depending on c as a parameter, with optimal value function $v_{\mathrm{P}(c)}$, feasible set $\mathcal{F}_{\mathrm{P}(c)}$ (not depending on c), and the set $\mathcal{S}^*_{\mathrm{P}(c)} := \{x \in \mathcal{F}_{\mathrm{P}(c)} \mid c^T x = v_{\mathrm{P}(c)}\}$ of optimal solutions. The dual $(\mathrm{D}(c))$ has corresponding optimal value $v_{\mathrm{D}(c)}$ and feasible set $\mathcal{F}_{\mathrm{D}(c)}$, etc.

We introduce the cone $\mathcal{M} := \{a(Y) = A^T Y \mid Y \in \mathcal{K}^*\}$, which will play a crucial role.

Remark 4. In semi-infinite optimization, the condition $c \in \operatorname{int} \mathcal{M}$ is just the standard Slater condition for $(\mathrm{SIP}_\mathrm{D})$, and it is not difficult to see that this condition is equivalent to the Slater condition for (D) in Definition 1 (see Ahmed et al. [1, Lemma 3.1]).

In the following, $\mathcal{D}_\mathrm{P} := \{c \in \mathbb{R}^n \mid v_{\mathrm{P}(c)} < \infty\}$ denotes the effective domain of the function $v_{\mathrm{P}(c)}$ and $\partial v_{\mathrm{P}(c)}$ its subdifferential with respect to c.

Lemma 4 (see Goberna and López [12]). *Let B and A be such that $\mathcal{F}_{\mathrm{P}(c)} \neq \emptyset$. Then the following holds:*

(1) *$v_{\mathrm{P}(c)}$ is a proper closed convex function of c on its effective domain $\mathcal{D}_\mathrm{P}$.*
(2) *$\partial v_{\mathrm{P}(c)} = \mathcal{S}^*_{\mathrm{P}(c)}$.*
(3) *$\mathcal{S}^*_{\mathrm{P}(c)}$ is nonempty and compact if and only if $c \in \operatorname{int} \mathcal{M}$.*

Proof. See Goberna and López [12, p. 262] for (1) and Goberna and López [12, Theorem 2.1] for (2) and (3). □

Using Lemma 4 and Rademacher's theorem for convex functions we can now prove the weak genericity of uniqueness in CP and obtain at the same time an alternative proof for the genericity of the Slater condition.

Theorem 4. *Let A and B be such that $\mathcal{F}_{\mathrm{P}(c)} \neq \emptyset$. Then for almost all $c \in \mathbb{R}^n$, one of the following alternatives holds:*

- either $\mathcal{F}_{\mathrm{D}(c)} = \emptyset$,
- or the Slater condition holds for $(D(c))$ and the optimal solution of $(\mathrm{P}(c))$ is unique.

A corresponding dual result holds with respect to parameter B (for fixed A, c).

Proof. Let A, B be such that $\mathcal{F}_{\mathrm{P}(c)} \neq \emptyset$. Let $\mathcal{D}_\mathrm{P}$, with boundary $\operatorname{bd} \mathcal{D}_\mathrm{P}$, be the convex effective domain of the convex function $v_{\mathrm{P}(c)}$ from Lemma 4. We distinguish the following three cases for $c \in \mathbb{R}^n$:

$$\text{(i) } c \in \operatorname{bd} \mathcal{D}_\mathrm{P}, \qquad \text{(ii) } c \notin \operatorname{cl} \mathcal{D}_\mathrm{P}, \qquad \text{(iii) } c \in \operatorname{int} \mathcal{D}_\mathrm{P}.$$

By Lemma 1, case (i) occurs on a set of measure zero in $\mathbb{R}^n$. To prove the theorem, we show that in case (ii) (resp. (iii)), the first (resp. second) alternative of the theorem holds. Indeed for case (ii), in view of the relation

$$\mathcal{F}_{\mathrm{D}(c)} \neq \emptyset \quad \Rightarrow \quad c \in \mathcal{D}_\mathrm{P},$$

we get $\mathcal{F}_{\mathrm{D}(c)} = \emptyset$ and the first alternative holds.

In case (iii), we use the fact that the convex function $v_{P(c)}$ defined on the open set $\operatorname{int}\mathcal{D}_P$ is differentiable for almost all $c \in \operatorname{int}\mathcal{D}_P$ (see, e.g., Rockafellar [29, Theorem 25.5]). Using Lemma 4(2), this means that for these values of c the subgradient $\partial v_{P(c)} = \mathcal{S}^*_{P(c)} = \{\nabla v_{P(c)}\}$ is a singleton. Moreover, in this case, by Lemma 4(3) the Slater condition holds for $\mathcal{F}_{D(c)}$ (cf. Remark 4).

The proof of the dual statement is similar. □

A uniqueness result similar to the uniqueness statement in Theorem 4 can also be found in Bolte et al. [4], even for more general convex programs.

By combining the statements of Theorem 4 for the primal and dual we obtain:

Corollary 3. *Let $A \in \mathbb{R}^{m\times n}$ be given arbitrarily. Then for almost all $(B, c) \in \mathbb{R}^m \times \mathbb{R}^n$ the following holds: If both* (P) *and* (D) *are feasible, then both satisfy the Slater condition and both have unique optimal solutions $\bar{X}$ and $\bar{Y}$.*

A corresponding result holds for (P_0), (D_0) *with respect to almost all $(B, C) \in \mathbb{R}^m \times \mathbb{R}^m$.*

4.2. Nondegeneracy and Strict Complementarity

We now discuss nondegeneracy and strict complementarity of optimal solutions in conic programming. It has been shown by Pataki and Tunçel [28] that both properties hold for almost all problem instances. For completeness we summarize their results, which are formulated in terms of the conic programs in self-dual form (P_0), (D_0), as described in Section 2. Note that in Pataki and Tunçel [28] these results have been proven under the assumption that the problems are gap free. We emphasize that their arguments are completed by the results in Section 3, which assure (weak) genericity of gap freeness.

We now introduce some notation. Let us denote the minimal face of $\mathcal{K}$ containing X and the minimal face of $\mathcal{K}^*$ containing Y, respectively, by

$$\mathrm{J}(X) = \operatorname{face}(X, \mathcal{K}) \quad \text{and} \quad \mathrm{G}(Y) = \operatorname{face}(Y, \mathcal{K}^*).$$

Observe that for each feasible X, we have $X \in \operatorname{rint}\mathrm{J}(X)$. For a face F of $\mathcal{K}$, we define the complementary face as $F^\triangle := \{Q \in \mathcal{K}^* \mid \langle Q, S\rangle = 0 \text{ for all } S \in F\}$. Clearly, $F^\triangle$ is a closed convex cone. Moreover, it is not difficult to see that if $X \in \operatorname{rint} F$, then $F^\triangle = \{Q \in \mathcal{K}^* \mid \langle Q, X\rangle = 0\}$. This immediately implies that the complementary face of $\mathrm{J}(X)$ is equivalently given by

$$\mathrm{J}^\triangle(X) = \{Q \in \mathcal{K}^* \mid \langle Q, X\rangle = 0\}. \tag{6}$$

Analogous definitions and results apply to $\mathrm{G}^\triangle(Y)$, the complementary face of $\mathrm{G}(Y)$.

Definition 2. The extreme points of $\mathcal{F}_{P_0}$ (resp. $\mathcal{F}_{D_0}$) are called *primal* (resp. *dual*) *basic feasible solutions*.

The following characterization of basic solutions is given in Pataki and Tunçel [28, Theorem 1]:

Lemma 5. *Let X be feasible for* (P_0). *Then X is a basic feasible solution if and only if*

$$\operatorname{span}(\mathrm{J}(X)) \cap \mathcal{L} = \{0\}.$$

A similar condition for the dual program leads to the concept of (primal) nondegeneracy:

Definition 3. A primal feasible solution X is called *nondegenerate* if

$$\operatorname{span}(\mathrm{J}^\triangle(X)) \cap \mathcal{L}^\perp = \{0\}. \tag{7}$$

Nondegeneracy of a dual feasible solution Y is defined analogously.

Definition 4. Optimal solutions $\bar{X}$ of (P_0) and $\bar{Y}$ of (D_0) are called *complementary* if $\langle \bar{X}, \bar{Y}\rangle = 0$, i.e., if $\bar{Y} \in \mathrm{J}^\triangle(\bar{X})$. The solutions $\bar{X}$ and $\bar{Y}$ are called *strictly complementary*, if we have

$$\bar{Y} \in \operatorname{rint}\mathrm{J}^\triangle(\bar{X}). \tag{8}$$

Recall that $\bar{X} \in \operatorname{rint}\mathrm{J}(\bar{X})$ holds by definition.

The following lemma shows some relations between nondegeneracy, strict complementarity, basic solutions and uniqueness.

Lemma 6 (see Pataki [27], Pataki and Tunçel [28, Theorem 2]). *Let X be an optimal solution of* (P_0). *Then the following hold.*

(a) *If X is a unique optimal solution, then X is a basic solution.*

(b) *If X is nondegenerate, then any complementary solution Y of* (D_0) *must be basic. Moreover, if there is a complementary solution Y, it must be unique.*

(c) *Suppose that Y is a dual feasible solution and X and Y are strictly complementary. Then Y is basic if and only if X is nondegenerate.*

Remark 5. In Pataki and Tunçel [28], a slightly different definition of strict complementarity is given: the optimal solutions $\bar{X}$ and $\bar{Y}$ are called strictly complementary if

$$\bar{X} \in \operatorname{rint} F \quad \text{and} \quad \bar{Y} \in \operatorname{rint} F^{\triangle} \qquad \text{holds for some face } F \text{ of } \mathcal{K}. \tag{9}$$

It is clear that (8) implies (9). Conversely, let (9) be satisfied. We always have $\bar{X} \in \operatorname{rint} \mathrm{J}(\bar{X})$. So $\bar{X} \in \operatorname{rint} F$ implies $F^{\triangle} = \mathrm{J}^{\triangle}(\bar{X})$ by (6). Therefore, (8) and (9) are equivalent.

In Pataki [27], strict complementarity for $\bar{X}, \bar{Y}$ is defined by $\mathrm{J}^{\triangle}(\bar{X}) = \mathrm{G}(\bar{Y})$. It can be shown that this condition and (8) are equivalent; see the proof of Pataki and Tunçel [28, Theorem 2]. By considering the dual problem, strict complementarity can similarly be defined as (again, $\bar{Y} \in \operatorname{rint} \mathrm{G}(\bar{Y})$ holds by definition)

$$\bar{X} \in \operatorname{rint} \mathrm{G}^{\triangle}(\bar{Y}). \tag{10}$$

Neither condition (8) nor (10) implies the other unless $\mathcal{K}$ or $\mathcal{K}^*$ is facially exposed, as noted in Pataki [27, Remark 3.3.2]. For an example of these "asymmetric" definitions of strict complementarity, see Davi and Jarre [7, Example 1].

Note that not all cones appearing in conic programming are facially exposed: it is well known that the cone of semidefinite matrices is facially exposed, but the cone of copositive matrices is not (see Dickinson [8, Theorem 8.22]).

We now sketch the weak genericity result for nondegeneracy and strict complementarity of Pataki and Tunçel [28]. To prove their result, they consider for fixed $\mathcal{L}$ the sets (see Pataki and Tunçel [28, p. 455 and Proposition 1])

$$\bar{\mathcal{D}}(\mathcal{L}) := \{(B, C) \mid \text{the corresponding problems } (\mathrm{P}_0) \text{ and } (\mathrm{D}_0) \text{ are feasible with } v_{\mathrm{P}_0} = v_{\mathrm{D}_0}\}$$

and

$$\mathcal{D}(\mathcal{L}) := \{(B, C) \in \bar{\mathcal{D}}(\mathcal{L}) \mid \text{ some optimal solutions } \bar{X}, \bar{Y} \text{ of } (\mathrm{P}_0), (\mathrm{D}_0) \text{ are strictly complementary}\}.$$

Using a deep result from geometric measure theory, Pataki and Tunçel [28, Theorem 3] derive the following result.

Lemma 7 (see Pataki and Tunçel [28, Proposition 2]). *For fixed $\mathcal{L}$, the set $\bar{\mathcal{D}}(\mathcal{L}) \backslash \mathcal{D}(\mathcal{L})$ has $\dim(\bar{\mathcal{D}}(\mathcal{L}))$-dimensional Hausdorff measure zero.*

Combining Pataki and Tunçel [28, Theorem 4] with Corollary 2, the result of Pataki and Tunçel can be formulated as follows:

Theorem 5. *Let $\mathcal{L}$ be given arbitrarily. Then for almost all $(B, C) \in \mathbb{R}^{2m}$ the following is true: If the corresponding programs (P_0) and (D_0) are both feasible, then there exist unique optimal solutions $\bar{X}$ of (P_0) and $\bar{Y}$ of (D_0). These solutions are nondegenerate and satisfy the strict complementarity condition.*

Proof. Similar to the arguments in Pataki and Tunçel [28, p. 456], we combine several results. For fixed $\mathcal{L}$ we consider the set $\mathcal{P}^0$ of instances (B, C) such that the primal and dual are feasible. Note that this set $\mathcal{P}^0$ is of full dimension by Corollary 2. Corollary 3 and Lemma 7 guarantee that for almost all instances in $\mathcal{P}^0$ the primal and dual optimal solutions are unique and strictly complementary (as defined in (8)). Let $\mathcal{P}^0_{sc}$ denote this weakly generic subset of $\mathcal{P}^0$. In view of Lemma 6(a), which is also valid for the optimal solution Y of (D_0), the dual optimal solutions of instances in $\mathcal{P}^0_{sc}$ are basic, and by Lemma 6(c) the primal maximizers X are nondegenerate.

Note that Lemma 6(c) does not hold for X and Y interchanged unless $\mathcal{K}$ is facially exposed (cf. Remark 5). However, if we define strict complementarity as in (10), then Lemma 6(c) holds for X and Y interchanged. Analogous to (8) following Pataki and Tunçel [28], one can show that (10) is a weakly generic property. Thus, using the same arguments, weakly generically at optimal solutions of (D_0) the nondegeneracy condition holds. □

Remark 6. With the same projection trick as in Remark 3, the genericity result of Theorem 5 for (P_0) and (D_0) can directly be translated to the following statement for the programs in the form (P), (D): Let $A \in \mathbb{R}^{m \times n}$ be arbitrary. Then for almost all $(B, c) \in \mathbb{R}^n \times \mathbb{R}^m$ we have that if (P) and (D) are both feasible, then there exist unique optimal solutions $\bar{X}$ of (P) and $\bar{Y}$ of (D). Moreover, $\bar{X}$ and $\bar{Y}$ are both nondegenerate and satisfy the strict complementarity condition.

Note that to assure uniqueness of the solution of (P) in terms of the variable $x \in \mathbb{R}^n$, we have to assume that A has full rank n. However, recall from (2) that for $m \geq n$ a matrix $A \in \mathbb{R}^{m\times n}$ generically has full rank n.

4.3. Connection Between Nondegeneracy and Slater's Condition

We briefly comment on the fact that nondegeneracy implies the Slater condition. We again analyze this for conic programs of the form (P_0). The following is true.

Theorem 6. *Let X be a nondegenerate feasible solution of* (P_0)*. Then Slater's condition holds for* (P_0)*. An analogous result is true for the problems* (D_0), (P), *and* (D).

Proof. To prove the statement, we will construct an element $L \in \mathcal{L}$ such that with small $\alpha > 0$ we have that $X + \alpha L \in \operatorname{int}\mathcal{K}$ and $X + \alpha L$ is feasible for (P_0). To do so, we first note that the nondegeneracy condition $\mathcal{L}^\perp \cap \operatorname{span}(\mathrm{J}^\triangle(X)) = \{0\}$ is equivalent to $\mathcal{L} + [\operatorname{span}(\mathrm{J}^\triangle(X))]^\perp = \mathbb{R}^m$. So for $X_0 \in \operatorname{int}\mathcal{K}$ there is a representation

$$X_0 = L + Z \quad \text{with } L \in \mathcal{L} \text{ and } Z \in [\operatorname{span}(\mathrm{J}^\triangle(X))]^\perp.$$

Fix such an element $X_0 = L + Z \in \operatorname{int}\mathcal{K}$ and let $S \in \mathrm{J}^\triangle(X)\setminus\{0\}$. Then $\langle S, Z\rangle = 0$, and since $X_0 \in \operatorname{int}\mathcal{K}$, we get

$$0 < \langle S, X_0\rangle = \langle S, X_0\rangle - \langle S, Z\rangle = \langle S, L\rangle.$$

Let $\mathcal{B}_1 := \{S \mid \|S\| = 1\}$ be the unit sphere in $\mathbb{R}^m$. By compactness of $\mathcal{B}_1$ and continuity of the linear function $\langle L, \cdot\rangle$, there exists some $\varepsilon > 0$ such that L satisfies

$$\langle L, S\rangle \geq 2\varepsilon \quad \text{for all } S \in \mathrm{J}^\triangle(X) \cap \mathcal{B}_1. \tag{11}$$

We will show now that for $\alpha > 0$ small enough we have $(X + \alpha L) \in (B + \mathcal{L}) \cap \operatorname{int}\mathcal{K}$; i.e., Slater's condition holds for (P_0). Clearly $(X + \alpha L) \in B + \mathcal{L}$ since $X \in B + \mathcal{L}$ and $L \in \mathcal{L}$. To prove $(X + \alpha L) \in \operatorname{int}\mathcal{K}$, we have to show that

$$\langle X + \alpha L, S\rangle > 0 \quad \text{for all } S \in \mathcal{K}^* \cap \mathcal{B}_1. \tag{12}$$

To do so, in view of (11), a continuity argument shows that there exists some $\delta > 0$ such that

$$\langle L, S\rangle \geq \varepsilon \quad \text{for all } S \in \mathrm{J}^\triangle_\delta(X) \cap \mathcal{B}_1, \tag{13}$$

where $\mathrm{J}^\triangle_\delta(X) := \{S \in \mathcal{K}^* \mid \|S - \bar{S}\| < \delta \text{ for some } \bar{S} \in \mathrm{J}^\triangle(X)\}$. Since $X \in \mathcal{K}$, we have $\langle X, S\rangle \geq 0$ for all $S \in \mathcal{K}^*$, and by the definition of $\mathrm{J}^\triangle(X)$ in (6) we have that $\langle X, S\rangle > 0$ for all $S \in (\mathcal{K}^* \setminus \mathrm{J}^\triangle_\delta(X)) \cap \mathcal{B}_1$. By compactness of this set, there exists some τ such that

$$\langle X, S\rangle \geq \tau > 0 \quad \text{for all } S \in (\mathcal{K}^* \setminus \mathrm{J}^\triangle_\delta(X)) \cap \mathcal{B}_1.$$

Let $\mu := \min\{\langle L, S\rangle \mid S \in (\mathcal{K}^* \setminus \mathrm{J}^\triangle_\delta(X)) \cap \mathcal{B}_1\}$. We claim that $X + \alpha L \in \operatorname{int}\mathcal{K}$ for all $0 < \alpha < \tau/|\mu|$. We have the following two cases:

If $S \in (\mathcal{K}^* \setminus \mathrm{J}^\triangle_\delta(X)) \cap \mathcal{B}_1$: then $\langle X + \alpha L, S\rangle = \langle X, S\rangle + \langle \alpha L, S\rangle \geq \tau + \alpha\mu > 0$.

If $S \in \mathrm{J}^\triangle_\delta(X) \cap \mathcal{B}_1$: using $\langle X, S\rangle \geq 0$ and (13), we have $\langle X + \alpha L, S\rangle = \langle X, S\rangle + \langle \alpha L, S\rangle \geq \alpha\varepsilon > 0$.

By combining these two cases, we have shown that (12) holds, and the result follows. □

For the case of semidefinite programming, it has been shown implicitly in Alizadeh et al. [2, Proof of Theorem 14] that, given $\mathcal{L}$, for almost all B all feasible points of $\mathcal{F}_P$ are nondegenerate. Note that in Alizadeh et al. [2] a definition of nondegeneracy is used that is different but equivalent to (7): nondegeneracy is defined in terms of tranversality conditions for certain tangent spaces. Hence, by applying Theorem 6, it follows for the SDP case that, given $\mathcal{L}$, for almost all B we have: if $\mathcal{F}_P \neq \emptyset$, then $\mathcal{F}_P$ has Slater points. This was also established in Shapiro [34, p. 310].

We wish to mention that in geometric measure theory, transversality results have been proven that—roughly speaking—assert that weakly generically all intersection points of two convex sets are nondegenerate. For example, the following result has been shown in Hug and Schätzle [20].

Lemma 8 (see Hug and Schätzle [20, Lemma 3.1]). *Let $K, L \subset \mathbb{R}^m$ be compact convex sets with nonempty interiors. Then for almost all $B \in \mathbb{R}^m$ (with respect to the Hausdorff measure) the sets K and $L_B := B + L$ intersect almost transversally; i.e., for all $X \in \operatorname{bd} K \cap \operatorname{bd} L_B$ we have*

$$N(K, X) \cap N(L_B, X) = \{0\} \quad \text{and} \quad N(K, X) \cap -N(L_B, X) = \{0\}$$

where $N(K, X)$ denotes the normal cone of K at X.

A similar result is given in Schneider [32, Theorem 2]. In combination with Theorem 6, these results could also be used to show that nondegeneracy and Slater's condition hold weakly generically in CP.

4.4. Genericity Results in Linear Semi-Infinite Optimization

In the preceding discussions we have made use of the fact that a conic program can be seen as a special case of a linear SIP (cf. Sections 3.2 and 4.1). There are many papers dealing with generic properties (in the sense of density and stability) of semi-infinite problems in the form ($\mathrm{SIP_P}$), ($\mathrm{SIP_D}$). We refer to Jongen and Zwier [21], Goberna et al. [17], Goberna and Todorov [15, 16], Goberna et al. [18], and Ochoa and de Serio [26]. In Goberna and López [14, Chapter 5] readers can find an overview of stability and genericity results for linear semi-infinite problems.

One might expect that these genericity results for SIP can directly be transferred to CPs, but unfortunately this is not the case for the following reason.

In the above articles, SIP programs are considered in the form ($\mathrm{SIP_P}$) with an infinite, compact index set $Z \subset \mathbb{R}^m$. In Jongen and Zwier [21] the problem data $(a(Y), b(Y), c)$ are elements of the space $C^2(Z)^n \times C^2(Z) \times \mathbb{R}^n$. In Goberna et al. [17], [18] and Goberna and Todorov [15, 16], the data $(a(Y), b(Y), c)$ are taken from $C(Z)^n \times C(Z) \times \mathbb{R}^n$ endowed with the norm of uniform convergence

$$\|(a, b, c)\| = \max\left\{\max_{Y \in Z} \|(a(Y), b(Y))\|_\infty, \|c\|_\infty\right\}.$$

But if we write CP in the form ($\mathrm{SIP_P}$) using (1), then the data $(a(Y), b(Y))$ are of the special form

$$a(Y) = A^T Y, \qquad b(Y) = \langle B, Y \rangle,$$

which is linear in Y. So the set of conic programs represents only a small subset of the set of all SIP instances, which is given, e.g., by $(a(Y), b(Y), c) \in C(Z)^n \times C(Z) \times \mathbb{R}^n$. This subset of conic problems allows much less freedom for perturbations, so roughly speaking we can say:

- Density results cannot be transferred from the general SIP theory to the special case of CPs.
- Openness results remain valid in the following sense: the sufficient conditions for stability in SIP remain valid for CPs, but the necessary conditions do not. Typically, the conditions for stability in SIP are too strong in CPs.

We just note that Goberna and Todorov [16, Theorem 1] gives genericity results (density and openness) for the special case of finite linear programs.

5. Stability Issues

The results so far do not present full genericity statements; i.e., the results so far do not guarantee stability with respect to perturbation of the whole set of parameters (A, B, c). As we will show, in general CP, the Slater condition and strong duality are fully generic properties (in the sense of density and openness). For the other desirable properties—namely, uniqueness, nondegeneracy, and strict complementarity of solutions—only weak genericity results (density without openness) have been established.

In smooth finite optimization (see Jongen et al. [22]), the stability of such properties is typically proven by applying the (smooth) Implicit Function Theorem to an appropriate system of optimality conditions. As we shall see, this approach can be applied to the special case of LP and SDP. For the latter, we make use of the fact (shown in Alizadeh et al. [2]) that the set of positive semidefinite matrices of a given rank can locally be described by smooth manifolds. Similar techniques can be used if the cones $\mathcal{K}$ and $\mathcal{K}^*$ are so-called semialgebraic sets: it is well known that a semialgebraic set allows a complete partition (stratification) of the set into smooth manifolds (see, e.g., Benedetti and Risler [3, Proposition 2.5.1]). For the sake of completeness we recall that a set $\mathcal{A} \subset \mathbb{R}^N$ is called semialgebraic if $\mathcal{A}$ is given as a finite union of sets of the form

$$\{x \in \mathbb{R}^N \mid p_i(x) = 0,\ i = 1, \ldots, k, \text{ and } q_j(x) > 0,\ j = 1, \ldots, s\}$$

with $k, s \in \mathbb{N}$, and polynomial functions $p_i, q_j \in \mathbb{R}[x_1, \ldots, x_N]$.

The theory of semialgebraic sets has been used in Bolte et al. [4] to prove a genericity result for primal uniqueness. However, the stability is shown only with respect to the objective vector c as parameter. We formulate one of their results in terms of our conic program (see Bolte et al. [4, Theorem 5.1]):

> Let $\mathcal{K}$ be a semialgebraic cone, and let A, B be given such that $\mathcal{F}_P$ is compact. Then there exists a generic set $S \subset \mathbb{R}^n$ such that for all $c \in S$ the corresponding program (P) has a unique maximizer.

It is not difficult to see that the cones of semidefinite, copositive and completely positive matrices are semialgebraic.

However, general cones $\mathcal{K}$ may have a much more complicated, nonsmooth structure. So whether in general CP the properties of uniqueness, nondegeneracy, and strict complementarity are stable (in a generic subset of the problem set) remains an open problem.

We now establish some full genericity results. By using the stability of Slater's condition we first prove that generically strong duality holds in general CP. To that end, we restrict ourselves to the following subset $\mathcal{P}^1$ of CP instances (with fixed $\mathcal{K}$, $m \geq n$)

$$\mathcal{P}^1 := \{(A, B, c) \mid \text{the corresponding programs (P), (D) are both feasible}\}.$$

Note that this set is of full dimension $m \cdot n + m + n$. Using results from Section 3 we can prove the following.

Theorem 7. *There is a generic subset $\mathcal{P}^1_{\text{reg}}$ of $\mathcal{P}^1$ such that for any $(A, B, c) \in \mathcal{P}^1_{\text{reg}}$ the Slater condition holds for* (P) and (D) *and both programs have optimal solutions with $v_P = v_D$.*

Proof. By Corollary 2 there is a weakly generic subset $\mathcal{P}^1_1$ of $\mathcal{P}^1$ such that for any $(A, B, c) \in \mathcal{P}^1_1$ the Slater condition holds for the corresponding programs (P) and (D). In view of (2), there also exists a generic subset $\mathcal{P}_A$ of $\mathbb{R}^{m\times n}$ such that for any $A \in \mathcal{P}_A$ we have $\operatorname{rank} A = n$ (recall $m \geq n$). We define the weakly generic subset $\mathcal{P}^1_{\text{reg}}$ of $\mathcal{P}^1$ by

$$\mathcal{P}^1_{\text{reg}} := \mathcal{P}^1_1 \cap (\mathcal{P}_A \times \mathbb{R}^m \times \mathbb{R}^n),$$

and we show that $\mathcal{P}^1_{\text{reg}}$ is open. To this end, suppose $(\bar{A}, \bar{b}, \bar{c}) \in \mathcal{P}^1_{\text{reg}}$. We show that if (A, b, c) is close to $(\bar{A}, \bar{b}, \bar{c})$, then $(A, b, c) \in \mathcal{P}^1_{\text{reg}}$. By definition, for any $(\bar{A}, \bar{B}, \bar{c}) \in \mathcal{P}^1_{\text{reg}}$ the Slater condition holds for the corresponding programs $(\bar{P})$ and $(\bar{D})$; i.e., there exist $\bar{x} \in \mathcal{F}_{\bar{P}}$ and $\bar{Y} \in \mathcal{F}_{\bar{D}}$, such that

$$\bar{B} - \bar{A}\bar{x} \in \operatorname{int} \mathcal{K}, \qquad \bar{A}^T \bar{Y} = \bar{c}, \quad \bar{Y} \in \operatorname{int} \mathcal{K}^*, \tag{14}$$

and $\bar{A}$ has full rank n. Both Slater conditions in (14) are stable with respect to small perturbations of $(\bar{A}, \bar{B}, \bar{c})$. Indeed, for (A, B, c) near $(\bar{A}, \bar{B}, \bar{c})$ the point $\bar{x}$ still satisfies the primal Slater condition. Moreover, if we define $Y = Y(A, c)$ as the unique solution of

$$\min \|Y - \bar{Y}\| \quad \text{s.t. } A^T Y = c,$$

then using $\operatorname{rank} \bar{A} = n$ it is not difficult to see that $Y(A, c)$ depends continuously on A and c and satisfies $Y(A, c) \to \bar{Y}$ for $(A, c) \to (\bar{A}, \bar{c})$. Thus, for (A, c) close to $(\bar{A}, \bar{c})$, the vector $Y(A, c)$ lies in the interior of $\mathcal{K}^*$. So the set $\mathcal{P}^1_{\text{reg}}$ is an (open) generic subset of $\mathcal{P}^1$.

Moreover, by the arguments before Corollary 1, for any $(A, B, c) \in \mathcal{P}^1_{\text{reg}}$ both programs (P) and (D) have optimal solutions and the strong duality relation $v_P = v_D$ holds. □

Before we give a full stability analysis for the case of SDP, we consider linear programs as an example.

Stability analysis for LP: Consider the pair of primal-dual LP's

$$\begin{aligned} &\text{(P)} \quad \max c^T x \quad \text{s.t. } X := B - Ax \in \mathbb{R}^m_+ \\ &\text{(D)} \quad \min \langle B, Y \rangle \quad \text{s.t. } A^T Y = c, \quad Y \in \mathbb{R}^m_+ \end{aligned}$$

for instances $Q := (A, B, c)$ with A of full rank n. Again, let $\mathcal{P}^1$ denote the set of LP instances Q such that the corresponding programs (P) and (D) are both feasible. In view of Theorem 5 and Remark 6, there exists a weakly generic subset $\mathcal{P}^1_{\text{reg}} \subset \mathcal{P}^1$ of instances Q such that the primal and dual optimal solutions $\bar{X}, \bar{Y}$ of (P), (D) are unique, nondegenerate, and strictly complementary. To show stability, i.e., openness of $\mathcal{P}^1_{\text{reg}}$, let $\bar{Q} := (\bar{A}, \bar{B}, \bar{c})$ be an element of $\mathcal{P}^1_{\text{reg}}$ with solutions $\bar{X}, \bar{Y}$. Let us denote the active index set $\bar{I} = \{i \in \{1, \ldots, m\} \mid \bar{X}_i = 0\}$, its complement $\bar{I}^C = \{i \in \{1, \ldots, m\} \mid \bar{X}_i > 0\}$, and $\bar{\mathcal{L}} := \operatorname{span}\{\bar{A}_j \mid j = 1, \ldots, n\}$, where $\bar{A}_j$ is the jth column of $\bar{A}$. It follows that

$$\mathrm{J}(\bar{X}) = \operatorname{cone}\{e_i \mid i \in \bar{I}^C\} = \mathrm{G}^{\triangle}(\bar{Y}), \qquad \mathrm{G}(\bar{Y}) = \operatorname{cone}\{e_i \mid i \in \bar{I}\} = \mathrm{J}^{\triangle}(\bar{X}).$$

The nondegeneracy condition for $\bar{X}$ resp. $\bar{Y}$ reads

$$\bar{\mathscr{L}}^{\perp} \cap \operatorname{lin} \mathrm{J}^{\triangle}(\bar{X}) = \{0\} \qquad \text{resp.} \qquad \bar{\mathscr{L}} \cap \operatorname{lin} \mathrm{G}^{\triangle}(\bar{Y}) = \{0\}. \tag{15}$$

The strict complementarity condition means that $\bar{Y}_i = 0$ holds if and only if $i \in \bar{I}^C$. From (15) we deduce $|\bar{I}| \le n$, resp. $|\bar{I}^C| \le m-n$ and thus, using $m = |\bar{I}| + |\bar{I}^C| \le m - n + n = m$, we find $|\bar{I}| = n$. Moreover, the condition $\bar{\mathscr{L}} \cap \operatorname{lin} \mathrm{G}^{\triangle}(\bar{Y}) = \bar{\mathscr{L}} \cap \operatorname{lin}\{e_i \mid i \in \bar{I}^C\} = \{0\}$ implies that the matrix

$$\begin{pmatrix} \bar{A}^T \\ e_i^T, i \in \bar{I}^C \end{pmatrix}$$

and thus the $n \times n$-matrix $\bar{A}_{\bar{I}} := ([\bar{A}_1]_{\bar{I}}, \ldots, [\bar{A}_n]_{\bar{I}})$ is nonsingular (where $[\bar{A}_1]_{\bar{I}} := ([\bar{A}_1]_j, j \in \bar{I})^T$). It finally follows that for (A, B, c) near $(\bar{A}, \bar{B}, \bar{c})$ the solutions x (resp. X) of (P) and Y of (D) are given as the solutions of the systems

$$B_{\bar{I}} - A_{\bar{I}} x = 0 \quad \text{and} \quad A_{\bar{I}}^T Y_{\bar{I}} - c = 0, \tag{16}$$

with Y defined by $Y_i = [Y_{\bar{I}}]_i$ for $i \in \bar{I}$, and $Y_i = 0$ otherwise. These solutions yield unique, nondegenerate, and strictly complementary optimal solutions X, Y of (P), (D). So we obtain the following (well-known) result.

Theorem 8. *There is a generic subset $\mathscr{P}^1_{\text{reg}} \subset \mathscr{P}^1$ such that for all instances $Q = (A, B, c) \in \mathscr{P}^1_{\text{reg}}$ the primal and dual optimal solutions are unique, nondegenerate, and strictly complementary. Moreover, for any $\bar{Q} \in \mathscr{P}^1_{\text{reg}}$ with primal solution $\bar{X}$ and corresponding active index set $\bar{I}$ (with $|\bar{I}| = n$), there exists a neighborhood $\mathscr{N}$ of $\bar{Q}$ such that for any $Q = (A, B, c) \in \mathscr{N}$ the optimal solutions X, Y of the corresponding LP are unique, nondegenerate, strictly complementary, and given as the solution of the system* (16).

Stability analysis for SDP: We now study the stability of uniqueness, nondegeneracy, and strict complementarity for SDP, i.e., for the case $\mathscr{K} = \mathscr{S}_k^+ = \{X \in \mathscr{S}_k \mid X \text{ is positive semidefinite}\}$ and $A_i \in \mathscr{S}_k \equiv \mathbb{R}^m$ with $m = \frac{1}{2} \cdot k(k+1)$. Since we will make use of results in Alizadeh et al. [2], we consider SDP in the form

$$\begin{aligned} &(\mathrm{P}_0) \quad \max \langle C, B\rangle - \langle C, X\rangle \quad \text{s.t. } X := B - \sum_{i=1}^n x_i A_i \in \mathscr{S}_k^+ \\ &(\mathrm{D}_0) \quad \min \langle B, Y\rangle \quad \text{s.t. } Y := \sum_{j=1}^{m-n} y_j A_j^{\perp} + C \in \mathscr{S}_k^+ \end{aligned} \tag{17}$$

as problems depending on the parameter $Q := (\{A_i\}_{i=1}^n, B, C) \in (\mathscr{S}_k)^{n+2}$ (with $m \ge n$). We can again assume that the matrices $A_i, i = 1, \ldots, n$, are linearly independent, which is a generic condition according to (2), and that $A_j^{\perp}, j = 1, \ldots, m-n$ is a basis of the orthogonal complement of $\operatorname{span}\{A_i\}_{i=1}^n$.

For completeness we sketch the proof of the weak genericity results in Alizadeh et al. [2]. However, we present the arguments in a more explicit form, which will enable us to apply the Implicit Function Theorem to establish stability, i.e., full genericity.

We start by collecting some well-known facts from differential geometry.

(1) Let f be a function in $C^1(\mathbb{R}^q, \mathbb{R}^s)$. Then $0 \in \mathbb{R}^s$ is called a regular value of f if

$$\nabla f(x) \text{ has (full) rank } s \text{ for all } x \text{ such that } f(x) = 0.$$

(2) (See, e.g., Jongen et al. [22, Remark 3.1.5].) A set $M \subset \mathbb{R}^s$ is called a C^r-manifold of codimension c_d (resp. dimension $s - c_d$) with $0 \le c_d \le s$, if for any $\bar{x} \in M$ there exist a neigborhood $\mathscr{N}_{\bar{x}}$ and a C^r vector function h: $\mathscr{N}_{\bar{x}} \to \mathbb{R}^{c_d}$ such that $\nabla h(x)$ has rank c_d for all $x \in \mathscr{N}_{\bar{x}}$ and

$$x \in \mathscr{N}_{\bar{x}} \text{ is in } M \text{ if and only if } h(x) = 0.$$

(3) Let f: $\mathbb{R}^q \to \mathbb{R}^s$ be a C^1 function and $M \subset \mathbb{R}^s$ a manifold of codimension c_d, locally (in $\mathscr{N} \subset \mathbb{R}^q$) defined by $h(y) = 0$ with a C^1-function h: $\mathscr{N} \to \mathbb{R}^{c_d}$. Then we say that f is transversal to M (cf. Jongen et al. [22, Theorem 7.3.4]) if

$$\nabla f(x)[\mathbb{R}^s] + T_{f(x)} M = \mathbb{R}^s \quad \text{holds for all } x \text{ with } f(x) \in M, \tag{18}$$

where $T_{f(x)}M$ is the tangent space to M at $f(x)$. By Jongen et al. [22, Remark 7.3.5] an equivalent formulation of (18) is (with the defining equations $h(y) = 0$ for M)

$$\nabla h(f(x)) \text{ has full rank } c_d \text{ for all } x \text{ with } f(x) \in M. \tag{19}$$

The following is a useful generalization of the Sard Theorem (see, e.g., Zeidler [36, Proposition 78.10] for a proof).

Theorem 9 (Parametric Sard Theorem). *Let $Q \subset \mathbb{R}^q$ and $P \subset \mathbb{R}^p$ be open sets and let h: $Q \times P \subset \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^s$ with $(x, y) \mapsto h(x, y)$ be a C^r mapping with $r > \max\{0, q-s\}$. If $0 \in \mathbb{R}^s$ is a regular value of h, then for almost all $y \in P$ the value 0 is a regular value of the function $h_x(x) := h(x, y)$.*

We now introduce the relevant functions and manifolds for the genericity results. It is well known (see, e.g., Jongen et al. [22, Example 7.3.24]) that for any s with $0 \le s \le k$, the set

$$W_s := \{X \in \mathcal{S}_k \mid \operatorname{rank} X = s\}$$

is a C^∞ manifold in $\mathcal{S}_k$ of codimension $c_d = \frac{1}{2}(k+1-s)(k-s)$. Let this manifold locally be defined by the system $K(X) = 0$.

In Alizadeh et al. [2, Lemma 22] it has been proven that for any r, s with $0 \le r, s$ and $0 \le r+s \le k$, the set

$$W_{r,s} := \{(X, Y) \in \mathcal{S}_k \times \mathcal{S}_k \mid \operatorname{rank} X = s, \operatorname{rank} Y = r, \langle X, Y\rangle = 0\}$$

is a smooth C^∞ submanifold of $\mathcal{S}_k \times \mathcal{S}_k$ with $\dim W_{r,s} = m - \frac{1}{2}(k+1-r-s)(k-r-s)$ and thus with codimension $c_d = m + \frac{1}{2}(k+1-r-s)(k-r-s)$ where, again, $m = \frac{1}{2}k(k+1)$. Consider a pair $(\bar{X}, \bar{Y}) \in W_{r,s}$ such that $\bar{X} \in \mathcal{S}_s^+$, $\bar{Y} \in \mathcal{S}_r^+$. By continuity of the eigenvalues for (X, Y) close to $(\bar{X}, \bar{Y})$, the pair (X, Y) is in $W_{r,s}$ if and only if $(X, Y) \in W_{r,s}^+$ where

$$W_{r,s}^+ := \{(X, Y) \in \mathcal{S}_k^+ \times \mathcal{S}_k^+ \mid \operatorname{rank} X = s, \operatorname{rank} Y = r, \langle X, Y\rangle = 0\}.$$

So the set $W_{r,s}^+$ is a manifold of the same codimension c_d. This means that with locally defined smooth functions H (with $H(X, Y) \in \mathbb{R}^{c_d}$) we have $(X, Y) \in W_{r,s}^+$ if and only if $H(X, Y) = 0$. Note also that for $(X, Y) \in W_{r,s}^+$ the relation $\langle X, Y\rangle = 0$ implies $X \cdot Y = 0$. So the condition $r + s \le k$ must hold.

Now for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{m-n}$ and an SDP instance $Q := (\{A_i\}_{i=1}^n, B, C)$ we define the following mappings, which appear in (17):

$$F(x, Q) := B - \sum_{i=1}^n x_i A_i, \qquad G(y, Q) := C + \sum_{j=1}^{m-n} y_j A_j^\perp. \tag{20}$$

For parameters $Q = (\{A_i\}_{i=1}^n, B, C)$ in a sufficiently small neighborhood of $\bar{Q} = (\{\bar{A}_i\}_{i=1}^n, \bar{B}, \bar{C})$ we can assume that the orthogonal complement $\{A_j^\perp\}_{j=1,\dots,m-n}$ depends at least C^1 smoothly on the parameters $\{A_i\}_{i=1}^n$. Indeed, we obtain the $\{A_j^\perp\}$'s by a smooth Gram-Schmidt orthogonalization process (to compute $\{\bar{A}_j^\perp\}_{j=1}^{m-n}$). So the functions $F(x, Q)$ and $G(y, Q)$ can be seen as smooth functions of all parameters.

With these preparations we can prove the following full genericity result for SDP.

Theorem 10. *There is a generic subset $\mathcal{P}_{\text{reg}}^1$ of the set*

$$\mathcal{P}^1 = \{(\{A_i\}_{i=1}^n, B, C) \mid \text{the corresponding problems } (P_0) \text{ and } (D_0) \text{ are both feasible}\} \subset (\mathcal{S}_k)^{n+2}$$

of SDP instances such that the following holds. For any $Q \in \mathcal{P}_{\text{reg}}^1$ there exist unique, nondegenerate, and strictly complementary solutions x, y (or X, Y) of (P_0), (D_0). Moreover, if $\bar{Q} \in \mathcal{P}_{\text{reg}}^1$ is such that the corresponding (unique, nondegenerate, strictly complementary) solutions $\bar{x}, \bar{y}$ (or $\bar{X}, \bar{Y}$) of (P_0), (D_0) have $\operatorname{rank} \bar{X} = s$, $\operatorname{rank} \bar{Y} = r$ with $r + s = k$, then there exists a nonempty open neighborhood $\mathcal{N}$ of $\bar{Q}$ such that for any $Q \in \mathcal{N}$ the corresponding SDP programs (P_0) and (D_0) have unique, nondegenerate, strictly complementary solutions $x(Q) \approx \bar{x}$, $y(Q) \approx \bar{y}$ (or $(X(Q), Y(Q)) \approx (\bar{X}, \bar{Y})$) with the same ranks; i.e., $\operatorname{rank} X(Q) = s$ and $\operatorname{rank} Y(Q) = r$.

Proof. We first sketch the proof of the weak genericity result as in Alizadeh et al. [2]. Let $\mathcal{P}_0^1$ denote the weakly generic subset of $\mathcal{P}^1$ such that for all $Q \in \mathcal{P}_0^1$ optimal solutions X, Y of (P_0), (D_0) exist with $\langle X, Y\rangle = 0$ (see Corollary 2).

For fixed r, s with $0 \le r + s \le k$, we now consider the system of c_d equations $H(X, Y) = 0$, which (locally near some solution $\bar{x}, \bar{y}$ of (P_0), (D_0)) define the manifold $W_{r,s}^+$ of codimension $c_d = m + \frac{1}{2}(k+1-r-s)(k-r-s)$. With F, G as in (20), we introduce the equations

$$\tilde{H}(x, y, Q) := H(F(x, Q), G(y, Q)) = 0.$$

Let in the sequel $\nabla_z f(z, y)$ denote the partial derivative of f with respect to the variable z. Since the derivative $\nabla_{B,C}(F(x, Q), G(y, Q))$ has full rank $2m$, the derivative

$$\nabla\tilde{H}(x, y, Q) = \nabla H(F(x, Q), G(y, Q)) \cdot \nabla(F(x, Q), G(y, Q))$$

has full rank c_d for all x, y, Q with $(F(x,Q), G(y,Q)) \in W^+_{r,s}$. By the parametric Sard Theorem, for almost all Q and for the function $\tilde{H}_{x,y}(x,y) := \tilde{H}(x,y,Q)$ we have that

$$\begin{aligned} &\nabla \tilde{H}_{x,y}(x,y) = \nabla_{(x,y)}[H(F(x,Q), G(y,Q))] \\ &\text{has full rank } c_d \text{ for all } x, y \text{ with } (F(x,Q), G(y,Q)) \in W^+_{r,s}. \end{aligned} \tag{21}$$

Note that $r+s<k$ implies $c_d > m$, and with $(x,y) \in \mathbb{R}^m$ the matrix $\nabla_{(x,y)}[H(F(x,Q), G(y,Q))]$ cannot have rank c_d. So for $r+s<k$, the condition (21) means that for almost all Q there is no $(x,y) \in \mathbb{R}^m$ such that $(F(x,Q), G(y,Q)) \in W^+_{r,s}$. For $r+s=k$, strict complementarity holds for all feasible pairs $(X,Y) \in W^+_{r,s}$. Taking into account all finitely many combinations r,s with $r+s \le k$ (recall that k is fixed), we have proven that there is a weakly generic subset $\mathcal{P}^1_1$ of $\mathcal{P}^1$ such that for all $Q \in \mathcal{P}^1_1$ any complementary solutions X, Y of (P_0), (D_0) are strictly complementary.

For the weak genericity of primal nondegeneracy, we also proceed similar to Alizadeh et al. [2, Proof of Theorem 14]. Given s with $0 \le s \le k$ we consider the set W_s above and instances Q with primal feasible $X = F(x,Q)$ in W_s. With the linear independent system of c_d equations $K(X)=0$, which locally define the manifold W_s of codimension $c_d = \frac{1}{2}(k+1-s)(k-s)$, we thus consider x, Q such that

$$\tilde{K}(x,Q) := K(F(x,Q)) = 0.$$

Again, since $\nabla_B F(x,Q)$ has full rank m (cf. (20)), the derivative $\nabla F(x,Q)$ has full rank m for all x,Q and thus (in view of the definition of a manifold) $\nabla K(X)$ has full rank $c_d = \frac{1}{2}(k+1-s)(k-s)$ for $X \in W_s$. We find

$$\nabla \tilde{K}(x,Q) = \nabla K(F(x,Q)) \cdot \nabla F(x,Q) \text{ has full rank } c_d \text{ for all } x, Q \text{ with } F(x,Q) \in W_s.$$

The parametric Sard Theorem implies that for almost all Q we have for the function $\tilde{K}_x(x) := \tilde{K}(x,Q)$ that

$$\nabla \tilde{K}_x(x) = \nabla_x[K(F(x,Q))] \text{ has full rank } c_d \text{ for all } x \text{ with } F(x,Q) \in W_s. \tag{22}$$

With (18) and (19), this means that for almost all Q the function $F(x,Q)$ is transversal to the manifold W_s, so that for almost all Q we have

$$\nabla_x F(x,Q)[R^n] + T_{F(x,Q)} W_s = \mathcal{S}_k \quad \text{for all } x \text{ with } F(x,Q) \in W_s. \tag{23}$$

Since $\nabla_x F(x,Q)[\mathbb{R}^n] = \operatorname{span}\{\{A_i\}_{i=1}^n\}$ this condition is just the primal nondegeneracy condition of Alizadeh et al. [2, (18)]. (Note that our primal is the dual in Alizadeh et al. [2] and the nondegeneracy condition in Alizadeh et al. [2] is different but equivalent to the nondegeneracy relation in our paper.) Again by considering all possible s with $0 \le s \le k$, we obtain a weakly generic subset $\mathcal{P}^1_2$ of SDP instances such that for all $Q \in \mathcal{P}^1_2$, all primal feasible solutions are nondegenerate. The same can be done for the dual to obtain a set $\mathcal{P}^1_3$ of SDP instances such that for all $Q \in \mathcal{P}^1_3$, all dual feasible solutions are nondegenerate. Note that if the primal and dual solutions are nondegenerate, by Lemma 6(b) the optimal solutions must be unique. So by intersecting the weakly generic sets, $\mathcal{P}^1_{\text{reg}} := \bigcap_{i=0,1,2,3} \mathcal{P}^1_i$, we have constructed a weakly generic subset $\mathcal{P}^1_{\text{reg}}$ of $\mathcal{P}^1$ such that for any $Q \in \mathcal{P}^1_{\text{reg}}$ there exist unique, nondegenerate, and strictly complementary solutions x, y (or X, Y) of (P_0), (D_0).

We now show the stability of these nice properties, i.e., openness of $\mathcal{P}^1_{\text{reg}}$. This will be done by applying the Implicit Function Theorem to an appropriate system of equations.

To do so, let $\bar{Q} := (\{\bar{A}_i\}_{i=1}^n, \bar{B}, \bar{C})$ be a given instance in $\mathcal{P}^1_{\text{reg}}$. So $\bar{x}, \bar{y}$ (or $\bar{X}, \bar{Y}$) are unique, nondegenerate, strictly complementary solutions of the corresponding SDP pair (P_0) and (D_0) with $\operatorname{rank} \bar{X} = s$, $\operatorname{rank} \bar{Y} = r$, $r+s=k$ and $(\bar{X}, \bar{Y}) \in W^+_{r,s}$, where $\bar{X} = F(\bar{x}, \bar{Q}) = \bar{B} - \sum_{i=1}^n \bar{x}_i \bar{A}_i$, and $\bar{Y} = G(\bar{y}, \bar{Q}) = \sum_{j=1}^{m-n} \bar{y}_j \bar{A}_j^{\perp} + \bar{C}$. By the discussion above (see (21)), the derivative

$$\nabla_{(x,y)}[H(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q}))] \text{ has full rank } c_d = m \tag{24}$$

at $(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q}))$ satisfying $H(F(\bar{x}, \bar{Q}), G(\bar{y}, \bar{Q})) = 0$, a system of m equations. Locally near $(\bar{x}, \bar{y}, \bar{Q})$ we consider again the system

$$\tilde{H}(x,y,Q) := H(F(x,Q), G(y,Q)) = 0 \tag{25}$$

in the variables (x,y,Q). By applying the Implicit Function Theorem to (25) and taking into account (24), we see that for $Q \approx \bar{Q}$ there exists a unique C^∞-solution function $x(Q), y(Q)$ of the system

$$\tilde{H}(x(Q), y(Q), Q) = 0.$$

By construction, these solutions $x(Q), y(Q)$ define strictly complementary optimal solutions of the programs (P_0) and (D_0) with respect to the data $Q \approx \bar{Q}$ with $\operatorname{rank} F(x(Q), Q) = s$ and $\operatorname{rank} G(y(Q), Q) = r$. So we have proven the stability of strict complementarity.

To see that nondegeneracy of the solutions is stable we take an instance $\bar{Q} \in \mathcal{P}^1_{\text{reg}}$ with primal solution $\bar{X} = X(\bar{Q})$ as above. By the previous discussion (see (22)) with the defining equation $K(X) = 0$ for the manifold W_s of codimension $c_d = \frac{1}{2}(k - s + 1)(k - s)$, we have that

$$\nabla_x [K(F(\bar{x}, \bar{Q}))] \text{ has full rank } c_d.$$

But then, by continuity, for $Q \approx \bar{Q}$ and $x(Q) \approx \bar{x}$ also $\nabla_x [K(F(x(Q), Q))]$ has full rank c_d and (see (22) and (23)) the primal maximizers $x(Q)$ ($X(Q)$) are nondegenerate.

The same can be done for the dual. So finally we have established the full genericity result for SDP. □

6. Conclusion

In this paper we survey and complete genericity results for general conic programs. The results show that Slater's condition and strong duality are fully generic properties of CP; i.e., they hold for almost all problem instances and are stable at these instances with respect to small perturbations of all problem data. Other nice properties such as uniqueness, nondegeneracy, and strict complementarity are weakly generic, i.e., they hold for almost all problem instances. For the special cases of SDP these properties are also stable at these weakly generic instances. Whether this stability holds in general CP is still an open question.

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