# Circle formation for anonymous mobile robots with order preservation

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# Circle formation for anonymous mobile robots with order preservation

Chen Wang, Guangming Xie, Ming Cao and Long Wang

***Abstract*— This paper proposes a distributed control law for a group of mobile robots to form any given formation on a circle of a prescribed radius. The robots are modeled by point masses with the constraint that all of them can move only on the circle. In particular, robots are oblivious, anonymous, and unable to communicate directly; they share the common knowledge of the orientation of the circle, and can only measure the relative angular positions of their neighbors. A sampled-data control law is proposed as well. For both cases, theoretical analysis and numerical simulations are given to show the effectiveness of the proposed formation control strategies with the desired property of order preservation.**

## I. Introduction

Teams of mobile robots have been utilized more and more often for a growing variety of coordination tasks, such as environment monitoring [1], [2], surveillance [3], exploration [4], pursuit and evasion [5], search and rescue [6], and transportation [7], [8]. Among these applications, the pattern formation problem has attracted considerable attention since positioning the robots to form certain patterns can be useful for various tasks [1], [9].

Pattern formation is observed in various kinds of animal groups in nature, such as flocks of birds and schools of fish [10], [11], [12]. The pattern formation problem, which requires robots to form and maintain a specific geometric pattern, has been investigated intensively in the literatures [13], [14]. Solutions to the pattern forming problem can be classified into centralized and distributed approaches [14]. The latter has been studied more extensively partly due to the emergence and wide application of classes of autonomous systems, such as sensor networks and multi-robot systems [10].

Significant research efforts have been made on the development of distributed protocols that allow the robots or sensors to form a circle from initial random configurations [15]. Most of them have focused on the uniform circle formation problem for which a group of mobile robots or sensors are required to form a circle with equal distances between them.

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Suzuki and Yamashita [15] have first proposed a computational model, called the *semi-synchronous* model, which has been used in [16], [17], [18], [19], etc. Using this model, Défago and Konagaya [16] have proposed an algorithm to solve the circle formation problem which has been decomposed into two subproblems. The first is to form a circle in finite time, and the second is to guide the robots to the configuration where all of them are positioned evenly on the circle. The algorithm they proposed is a composition of two independent algorithms that solve the two subproblems separately. Their algorithm operates under the assumptions that the robots are oblivious (i.e., have no memory of past actions and observations), anonymous (i.e., cannot be distinguished from each others), unable to communicate directly, share no common sense of direction, and can only interact by observing each other's position.

Under Défago and Konagaya's model, Chatzigiannakis et al. [17] have proposed a distributed algorithm to solve the circle formation problem and have tried to simplify the algorithm of [16]. Later, Défago and Souissi [18] have solved both subproblems in [16] using a single algorithm. The assumptions in [16] are also used in [17] and [18]. Flocchini et al. [19] have then stipulated a constraint, which is also used in this paper, that all the robots can move only on a circle. Under this constraint, they have investigated under what conditions the circle formation problem is solvable by a collection of identical sensors without a global coordinate system.

There have also been research efforts focusing on developing control theoretic ideas. For example, Marshall et al. [20] have studied a control law under which the multivehicle system's equilibrium formations are generalized circular pursuit patterns in the plane. We also refer the interested reader to [1] for a more complete list of references.

In this paper, we consider the problem of the realization of circle formations for anonymous mobile robots with local information. For such groups of robots, we define the notion of neighbor relationships and design a distributed control law to realize any given circle formation on a circle. We aim at developing a general control law that works for any given circle formation tasks. And we pay special attention to the property of order preservation, which makes the control strategy easier to be implemented in real-robot systems.

To be more specific, we consider a system in which a group of robots are modeled by point masses with the constraint that all the robots can move only on a circle of a prescribed radius. The robots are oblivious, anonymous, and unable to communicate directly; they share the common knowledge of the orientation of the circle. We assume that their neighbor relationships can be described by a cycle and thus the robots can only get relative positions of their neighboring robots. Then we propose a control law to realize any given circle formation on the circle. The angular position is the only variable to specify the circle formation, since the radius of the circle is fixed.

We present theoretical analysis to show the effectiveness of our proposed control law. We also prove that the group of robots can preserve their angular-position ordering under our proposed control law. We further propose a sampled-data control law that can be easier to be used in real-robot systems. We prove its performance as well. We perform numerical simulations for both continuous-time and sampled-data cases to show the effectiveness of our proposed control laws.

The main contribution of this paper is the general control law under which a group of robots can form any given circle formation on a circle of a prescribed radius. Our control laws allow the robots to preserve their ordering on the circle, and its effectiveness can be proved theoretically. On the other hand, using angles as the positions make the method easy to be used in different robots installed with different sensors. And the use of angular position make the control law work for any prescribed radius of the circle as long as the robots are within the sensing ranges of its neighboring peers.

The outline of this paper is as follows. In Section II, we present the model for the mobile robots and formulate the circle formation problem we study. Then we propose a control law and give its theoretical analysis in Section III. In Section IV, a sampled-data based control law is proposed, and theoretical analysis is given as well. Numerical simulations for both continuous-time and sampled-data cases are given in Section V.

Before presenting the main body of the paper, we first define some notations. $\mathbf{1}_n$ is the column vector $[1,\ldots,1]^T$ with dimension $n$. $I_n$ is the $n$-by-$n$ identity matrix. $\mathcal{M}_n(\mathbb{R})$ denotes the set of all $n$-by-$n$ real matrices. Given a matrix $A=[a_{ij}]\in\mathcal{M}_n(\mathbb{R})$, we say that $A\geq 0$ ($A$ is nonnegative) if all its entries $a_{ij}$ are nonnegative. We say that $A>0$ ($A$ is positive) if all its entries $a_{ij}$ are positive. $\rho(A)$ is the spectral radius of $A$. The directed graph of $A$, denoted by $\Gamma(A)$, is the directed graph on $n$ nodes $v_i$, $i\in\{1,2,\ldots,n\}$, such that there is an edge in $\Gamma(A)$ from $v_j$ to $v_i$ if and only if $a_{ij}\neq 0$ [21]. We say that $x\in\mathbb{R}^n$ is positive if all its entries $x_i$ are positive, denoted by $x>0$. For $n\geq 2$ and $a,b,c\in\mathbb{R}$, $\mathrm{Circ}_n(a,b,c)$ defines the $n$-by-$n$ matrix

$$\mathrm{Circ}_n(a,b,c)=\begin{bmatrix} b & c & 0 & \ldots & \ldots & a\\ a & b & c & 0 & \ldots & 0\\ 0 & & & & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & & & a & b & c\\ c & 0 & \ldots & 0 & a & b \end{bmatrix}.$$

## II. Model and Problem formulation

We consider a group of $N$, $N\geq 2$, robots that can only gather local information (i.e., the relative angular positions of its neighbors) and share the common knowledge of the orientation of the circle (i.e., with common knowledge about which way is clockwise or counterclockwise). The robots are initially located on the circle randomly with distinct positions and are labeled counterclockwise, such that

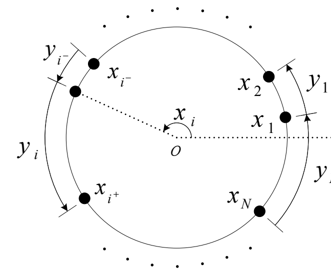

Fig. 1. Cycle Topology

$$0\leq x_1(0)<\ldots<x_i(0)<x_{i+1}(0)\ldots<x_N(0)<2\pi, \quad (1)$$

where $x_i(t)$ denotes the angular position of agent $i$ on the circle at time $t$.

According to their labels, the robots can only get local information of relative angular positions of their neighbors described by the undirected ring graph $\mathbb{G}=(\mathcal{V},\mathcal{E})$, with $\mathcal{V}=\{1,2,\ldots,N\}$ and $\mathcal{E}=\{(1,2),\ldots,(i,i+1),\ldots,(N-1,N),(N,1)\}$. In other words, each agent has only two neighbors. We define the neighbor of agent $i$ counterclockwise (resp. clockwise) to be its previous neighbor (resp. next neighbor), which is denoted by $i^+$ (resp. $i^-$). Then we have

$$i^+=\begin{cases} i+1 & \text{when } i=1,2,\ldots,N-1\\ 1 & \text{when } i=N \end{cases},$$

and

$$i^-=\begin{cases} N & \text{when } i=1\\ i-1 & \text{when } i=2,3,\ldots,N \end{cases}. \quad (2)$$

The problem addressed in this paper is to form a given circle formation by the set of robots we introduced above with the kinematic continuous-time model

$$\dot{x}_i(t)=u_i(t), \qquad i=1,2,\ldots,N, \quad (3)$$

where $u_i(t)\in\mathbb{R}$ represents the control input of agent $i$ at time $t$.

A circle formation is denoted by $d=[d_1,d_2,\ldots,d_N]^T\in\mathbb{R}^N$ satisfying $d>0$ and $\sum_{i=1}^N d_i=2\pi$, where $d_i$ describes the angular distance between agent $i$ and its previous neighbor $i^+$.

To clarify the problem, we introduce the variables:

$$y_i=\begin{cases} x_{i^+}-x_i & \text{when } i=1,2,\ldots,N-1\\ x_{i^+}-x_i+2\pi & \text{when } i=N \end{cases}, \quad (4)$$

where $y_i$ is the angular distance between agent $i$ and its previous neighbor $i^+$. The cycle topology described in (2) ensures that $\sum_{i=1}^N y_i=2\pi$ all the time.

Then we formulate the circle formation problem as follows.

*Definition 1 (Circle Formation Problem):* Given a circle formation described by $d$. Consider a group of $N$ robots with neighbor relationship described by $\mathbb{G}$, design distributed control laws $u_i(t) = u_i(y_i, y_{i-}, d_i, d_{i-})$, $i = 1, 2, \ldots, N$, such that under any initial condition (1) the solution to system (3) converges to some equilibrium point $x^*$ (dependent on $x(0)$) satisfying $y^* = d$.

*Remark 1:* The Circle Formation Problem becomes a Uniform Circle Formation Problem when $d = \frac{2\pi}{N}\mathbf{1}_N$.

We further define a useful property as follows.

*Definition 2 (Order preservation):* Consider the group of $N$ robots in Definition 1. We say it has the property of order preservation if under initial condition (1) the solution to system (3) under control laws $u_i(t)$ satisfies $y(t) > 0$ for all $t \geq 0$.

In the next section, we propose a control law to solve the Circle Formation Problem with the requirement of order preservation.

## III. Way-point control law for the Circle Formation Problem

### A. Control law

The proposed way-point control law takes the following form:

$$u_i(t) = \frac{d_{i-}}{d_i + d_{i-}} y_i(t) - \frac{d_i}{d_i + d_{i-}} y_{i-}(t), \quad i = 1, 2, \ldots, N. \tag{5}$$

Thus, the resulting closed-loop system is

$$\begin{cases} \dot{x}_1 = -x_1 + \frac{d_1}{d_1+d_N} x_N + \frac{d_N}{d_1+d_N} x_2 - \frac{2\pi d_1}{d_1+d_N} \\ \dot{x}_i = -x_i + \frac{d_i}{d_i+d_{i-1}} x_{i-1} + \frac{d_{i-1}}{d_i+d_{i-1}} x_{i+1} \qquad i = 2, \ldots, N-1. \\ \dot{x}_N = -x_N + \frac{d_N}{d_N+d_{N-1}} x_{N-1} + \frac{d_{N-1}}{d_N+d_{N-1}} x_1 + \frac{2\pi d_{N-1}}{d_N+d_{N-1}} \end{cases} \tag{6}$$

Then the overall system described by $y_i$'s is

$$\dot{y}_i = \left(-\frac{d_{i+}}{d_{i+}+d_i} - \frac{d_{i-}}{d_i+d_{i-}}\right) y_i + \frac{d_i}{d_{i+}+d_i} y_{i+} + \frac{d_i}{d_i+d_{i-}} y_{i-},$$

$$i = 1, 2, \ldots, N. \tag{7}$$

We rewrite the system into a compact form

$$\dot{y}(t) = -L(d) y(t), \tag{8}$$

where $y(t) = [y_1(t), y_2(t), \ldots, y_N(t)]^T$ and $L(d)$ is given by (9).

### B. Analysis

In this section, we analyze the closed-loop system (8). Before doing this, we list some existing results that will become useful in our analysis.

*Lemma 1 (Theorem 6.2.24 of [21]):* Let $A \in \mathcal{M}_n$. The following are equivalent:

(a) $A$ is irreducible;

(b) $\Gamma(A)$ is strongly connected.

*Lemma 2 (Theorem 8.1.22 of [21]):* Let $A \in \mathcal{M}_n$ and suppose $A \geq 0$. Then

$$\min_{1 \leq j \leq n} \sum_{i=1}^{n} a_{ij} \leq \rho(A) \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} a_{ij}. \tag{10}$$

A nonnegative matrix $A \in \mathcal{M}_n$ is said to be *primitive* if it is irreducible and has only one eigenvalue of maximum modulus [21]. Then we have the following lemmas.

*Lemma 3 (Theorem 8.5.2 of [21]):* If $A \in \mathcal{M}_n$ is nonnegative, then $A$ is primitive if and only if $A^m > 0$ for some $m \geq 1$.

*Lemma 4 (Lemma 8.5.5 of [21]):* If $A \in \mathcal{M}_n$ is nonnegative and irreducible, and if all the main diagonal entries of $A$ are positive, then $A^{n-1} > 0$.

*Lemma 5 (Lemma 8.5.6 of [21]):* Let $A \in \mathcal{M}_n$ be nonnegative and primitive. Then $A^k$ is nonnegative, irreducible, and primitive for all $k = 1, 2, \ldots$

Now we first analyze the eigenvalues of $L(d)$ that will be used later.

*Lemma 6:* All the eigenvalues of $L(d)$ are located in the set $\{z \in \mathbb{C} : |z| \leq 2, |z - 2| \leq 2\}$.

*Proof*: All the eigenvalues of $L(d)$ are located in the union of $N$ discs

$$G(L) = \bigcup_{i=1}^{N} \{z \in \mathbb{C} : |z - l_{ii}| \leq R_i(L)\}, \tag{11}$$

where

$$R_i(L) = \frac{d_i}{d_{i+} + d_i} + \frac{d_i}{d_i + d_{i-}} = 2 - l_{ii}, \qquad 1 \leq i \leq N. \tag{12}$$

And all the eigenvalues of $L(d)$ are also located in the union of $N$ discs

$$G(L^T) = \bigcup_{i=1}^{N} \{z \in \mathbb{C} : |z - l_{ii}| \leq R_i(L^T)\}, \tag{13}$$

where

$$R_i(L^T) = \frac{d_{i+}}{d_{i+} + d_i} + \frac{d_{i-}}{d_i + d_{i-}} = l_{ii}, \qquad 1 \leq i \leq N. \tag{14}$$

Since $0 < l_{ii} < 2$, one can check that $G(L) \subseteq \{z \in \mathbb{C} : |z| \leq 2\}$ and $G(L^T) \subseteq \{z \in \mathbb{C} : |z - 2| \leq 2\}$. Thus, $G(L) \cap G(L^T) \subseteq \{z \in \mathbb{C} : |z| \leq 2, |z - 2| \leq 2\}$. This ends the proof. ■

We are now able to prove the main result in this section.

*Theorem 1:* The Circle Formation Problem is solved with order preservation under the proposed control law (5).

*Proof*: Define the matrix $Q(d) \triangleq 2I_N - L(d)$. Since $Q(d) \geq 0$ and the column sums of $Q(d)$ are the constant 2, we have $\rho(Q) = 2$ in view of Lemma 2, and one can check that 2 is one of the eigenvalues of $Q(d)$. Since $\Gamma(Q)$ is strongly connected, we know that $Q(d)$ is irreducible from Lemma 1. Furthermore, all the main diagonal entries of $Q(d)$ are positive. Then we know that $Q^{N-1}(d) > 0$ from Lemma 4. From Lemma 3, $Q(d)$ is primitive, which means $Q(d)$ has only one eigenvalue of maximum modulus. More concretely, the largest eigenvalue of $Q(d)$, 2, is a single one. Thus, the eigenvalue $\lambda^* = 0$ of $L(d)$, corresponding to 2 of $Q(d)$, is a single eigenvalue. Together with Lemma 6, we know that all the other eigenvalues of $L(d)$ are located on the right-half plane. Denote the corresponding eigenvector of $\lambda^*$ by $\nu^*$. One can check that $\nu^* = cd$, where $c$ is a constant. Obviously, $\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} d_i = 2\pi$. It follows that $c = 1$. Then any solution to system (8) converges to the equilibrium

$$L(d)=\begin{bmatrix} \frac{d_2}{d_2+d_1}+\frac{d_N}{d_1+d_N} & -\frac{d_1}{d_2+d_1} & 0 & \cdots & \cdots & -\frac{d_1}{d_1+d_N} \\ -\frac{d_2}{d_2+d_1} & \frac{d_3}{d_3+d_2}+\frac{d_1}{d_2+d_1} & -\frac{d_2}{d_3+d_2} & 0 & \cdots & 0 \\ 0 & & & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & & \\ -\frac{d_N}{d_1+d_N} & 0 & \cdots & 0 & -\frac{d_N}{d_N+d_{N-1}} & \frac{d_1}{d_1+d_N}+\frac{d_{N-1}}{d_N+d_{N-1}} \end{bmatrix}. \tag{9}$$

point $y^*$ satisfying $y^* = d$. Thus, the Circle Formation Problem is solved under the proposed control law (5).

Furthermore, the solution to system (8) is

$$y(t) = e^{-L(d)t}y(0), \qquad t \geq 0. \tag{15}$$

We have

$$e^{-L(d)} = e^{2I_N - L(d)}e^{-2I_N} = e^{Q(d)}e^{-2} = e^{-2}\sum_{k=0}^{\infty}\frac{1}{k!}Q^k(d). \tag{16}$$

Since $Q(d)$ is nonnegative and primitive, $Q^k(d)$ is nonnegative for all $k = 1, 2, \ldots$ because of Lemma 5. Furthermore, $\sum_{k=0}^{\infty}\frac{1}{k!}Q^k(d)$ is positive in view of Lemma 3. Thus, one can check that $e^{-L(d)}$ is positive. The initial condition (1) ensures $y(0) > 0$ because of the construction of $y_i$. Thus, under any initial conditions the solution to system (8) satisfies $y(t) > 0$ for all $t \geq 0$. This ends the proof. ■

*Remark 2:* The Uniform Circle Formation Problem is solved with order preservation under the following control law

$$u_i(t) = \frac{y_i(t) - y_{i^-}(t)}{2}, \qquad i = 1, 2, \ldots, N. \tag{17}$$

## IV. Sampled-data control

In the previous section, a distributed control law (5) for the continuous-time model (3) has been proposed and proved to solve the Circle Formation Problem with the property of order preservation. However, considering the applications in real-robot systems, the continuous-time model cannot be directly implemented because of the limitation of system hardware constraints, such as communication bandwidth, rise time, and computation load. A sampled-data model is usually required for such real-robot systems [22].

In this section, we investigate distributed solutions to the Circle Formation Problem by sampled-data control.

### A. Control law

A sampled-data control law is proposed by using period sampling technology and zero-order hold circuit [22]. Let $h > 0$ be the sampling period, the obtained sampled-data control law is given by

$$u_i(t) = \frac{d_{i^-}}{d_i + d_{i^-}}y_i(kh) - \frac{d_i}{d_i + d_{i^-}}y_{i^-}(kh), \tag{18}$$
$$t \in [kh, kh+h), k = 0, 1, 2, \ldots; i = 1, 2, \ldots, N.$$

Under the control law (18), the dynamics of the overall system with the new states $y_i$'s can be described by

$$y(kh+h) = P(d)y(kh),\ k = 0, 1, 2, \ldots, \tag{19}$$

where

$$P(d) = I_N - hL(d). \tag{20}$$

### B. Analysis

We first give a sufficient and necessary condition without considering order preservation. Let $\lambda_i$ denote the $i$th eigenvalue of $L(d)$ given by (9), and $Re(\cdot)$ and $Im(\cdot)$ represent, respectively, the real and imaginary parts of a number.

*Theorem 2:* The Circle Formation Problem is solved under the sampled-data control law (18) if and only if

$$h < \min_{1\leq i\leq N, \lambda_i \neq 0}\frac{2Re(\lambda_i)}{Re(\lambda_i)^2 + Im(\lambda_i)^2}. \tag{21}$$

*Proof*: Let $\mu_i$ denote the $i$th eigenvalue of $P(d)$ corresponding to $\lambda_i$ of $L(d)$. Then we have $\mu_i = 1 - h\lambda_i$, $i = 1, 2, \ldots, N$. It holds that $\mu^* = 1$ is a single eigenvalue of $P(d)$ since $\lambda^* = 0$ is a single eigenvalue of $L(d)$ from Theorem 1.

Denote the corresponding eigenvector of $\mu^*$ by $\zeta^*$. Any solution to system (19) converges to $\zeta^*$ if and only if $\mu^*$ is a single eigenvalue and the other $N-1$ eigenvalues of $P(d)$ are located inside the unit circle, i.e., $|\mu_i| < 1$ for $\mu_i \neq \mu^*$. So $[1 - hRe(\lambda_i)]^2 + [hIm(\lambda_i)]^2 < 1$ for $\lambda_i \neq 0$, which implies that $h < \frac{2Re(\lambda_i)}{Re(\lambda_i)^2 + Im(\lambda_i)^2}$ for $\lambda_i \neq 0$. This sufficient and necessary condition can be summarized into (21).

Furthermore, one can check that $\zeta^* = cd$, where $c$ is a constant. Obviously, $\sum_{i=1}^N y_i = \sum_{i=1}^N d_i = 2\pi$. It follows that $c = 1$. Finally, any solution to system (19) converges to the equilibrium point $y^*$ satisfying $y^* = d$ if and only if the condition (21) is satisfied. This ends the proof. ■

With the requirement of order preservation, a more strict sufficient and necessary condition is given as follows.

*Theorem 3:* The Circle Formation Problem is solved with order preservation under the sampled-data control law (18) if and only if $0 < h \leq \frac{1}{2}$.

*Proof*: (sufficiency) When $0 < h \leq \frac{1}{2}$, we have $P(d) \geq 0$. The column sums of $P(d)$ are the constant 1. Then from Lemma 2, it holds that $\rho(P) = 1$. Furthermore, one can check that 1 is one of the eigenvalues of $P(d)$. Since $\Gamma(P)$ is strongly connected, $P(d)$ is irreducible from Lemma 1. Furthermore, all the main diagonal entries of $P(d)$ are positive. Then in view of Lemma 4 and Lemma 3 $P(d)$ is primitive, which means $P(d)$ has only one eigenvalue of the maximum modulus. More precisely, the largest eigenvalue of $P(d)$ is $\mu^* = 1$ and the other $N-1$ eigenvalues of $P(d)$ are located inside the unit circle.

Denote the corresponding eigenvector of $\mu^*$ by$\zeta^*$. One can check that $\zeta^* = cd$, where $c$ is a constant. Obviously,

$\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} d_i = 2\pi$. It follows that $c = 1$. Finally, when $0 < h \leq \frac{1}{2}$, any solution to system (19) converges to the equilibrium point $y^*$ satisfying $y^* = d$.

Furthermore, when $0 < h \leq \frac{1}{2}$, all the entries of the matrix $P(d)$ are nonnegative because of the restrictions of $d_i$. Moreover, no row only has zero entries. The initial condition (1) ensures $y(0) > 0$ because of the construction of $y_i$. Thus, under the initial condition (1), when $0 < h \leq \frac{1}{2}$, any solution to system (19) satisfies $y(kh) > 0$ for all $k = 0, 1, 2, \ldots$

(necessity) Consider the case when $h > \frac{1}{2}$. For any $h > \frac{1}{2}$, consider the first element in the vector $y$ as follows:

$$\begin{aligned} y_1(kh+h) &= [1 - h(\frac{\bar{d}_2}{\bar{d}_2 + \bar{d}_1} + \frac{\bar{d}_N}{\bar{d}_1 + \bar{d}_N})]y_1(kh) \\ &+ \frac{h\bar{d}_1}{\bar{d}_2 + \bar{d}_1} y_2(kh) + \frac{h\bar{d}_1}{\bar{d}_1 + \bar{d}_N} y_N(kh), \\ & k = 0, 1, 2, \ldots \end{aligned} \qquad (22)$$

One can always construct a circle formation $\bar{d}$ such that $\bar{d}_1 \to 0$. Thus we have $y_1(h) \to (1 - 2h)y_1(0)$. So there must exist a vector $y(0)$ satisfying $y(0) > 0$ and $\sum_{i=1}^{N} y_i(0) = 2\pi$ such that $y_1(h) \leq 0$. This ends the proof. ■

We further consider the Uniform Circle Formation Problem. Given the uniform circle formation $\hat{d} = \frac{2\pi}{N}\mathbf{1}_N$, we simplify the system (19) to be

$$y(kh+h) = \hat{P}y(kh), \ k = 0, 1, 2, \ldots, \qquad (23)$$

where

$$\hat{P} = \mathrm{Circ}_n(\frac{h}{2}, 1-h, \frac{h}{2}). \qquad (24)$$

*Theorem 4:* The Uniform Circle Formation Problem is solved with order preservation under the sampled-data control law (18) if and only if $0 < h < 1$ for even $N$ and $0 < h \leq 1$ for odd $N$.

*Proof*: From Lemma $A$.1 of [23], since $N \geq 2$ and $h > 0$, all the eigenvalues of $\hat{P}$ are

$$\hat{\mu}_i = 1 - h + h\cos(\frac{i}{N}2\pi), \ \ i = 1, 2, \ldots, N. \qquad (25)$$

(sufficiency) When $0 < h \leq 1$, one can check that $-1 \leq 1 - 2h \leq \hat{\mu}_i \leq 1$ for all $i = 1, 2, \ldots, N$ and $\hat{\mu}_N = 1$. We discuss $\hat{\mu}_i$, $i = 1, 2, \ldots, N-1$ as follows. One can easily check that $\hat{\mu}_i \neq 1$ for all $i = 1, 2, \ldots, N-1$ when $h > 0$. Suppose that there exists a $k$, $k \in \{1, 2, \ldots, N-1\}$, such that $\hat{\mu}_k = -1$. When $0 < h < 1$, such a $k$ doesn't exist. When $h = 1$, it holds that $k = \frac{N}{2}$ for even $N$ and $k$ doesn't exist for odd $N$. To sum up, when $0 < h < 1$ for even $N$ and $0 < h \leq 1$ for odd $N$, $\hat{\mu}_N = 1$ is the only eigenvalue of the maximum modulus. More precisely, the largest eigenvalue of $\hat{P}$ is $\hat{\mu}_N = 1$ and the other $N-1$ eigenvalues of $\hat{P}$ satisfy $-1 < \hat{\mu}_i < 1$.

Denote the corresponding eigenvector of $\hat{\mu}_N$ by $\hat{\zeta}_N$. One can check that $\hat{\zeta}_N = \hat{c}\mathbf{1}_N$, where $\hat{c}$ is a constant. From the definition of $y_i$, $\sum_{i=1}^{N} y_i = 2\pi$. Finally, when $0 < h < 1$ for even $N$ and $0 < h \leq 1$ for odd $N$, any solution to system (19) converges to the equilibrium $y^*$ satisfying $y^* = \hat{d}$.

Furthermore, when $0 < h \leq 1$, all the entries of matrix $\hat{P}$ are nonnegative. Moreover, no row only has zero entries. The initial condition (1) ensures $y(0) > 0$ because of the construction of $y_i$. Thus, under the initial condition (1), when $0 < h < 1$ for even $N$ and $0 < h \leq 1$ for odd $N$, any solution to system (19) satisfies $y(kh) > 0$ for all $k = 0, 1, 2, \ldots$

(necessity) Consider the case when $h > 1$. For any $h > 1$, consider the first element in the vector $y$

$$\begin{aligned} y_1(kh+h) &= (1-h)y_1(kh) + \frac{h}{2}y_2(kh) + \frac{h}{2}y_N(kh), \\ & k = 0, 1, 2, \ldots \end{aligned} \qquad (26)$$

One can easily check that there exists a vector $y(0)$ satisfying $y(0) > 0$ and $\sum_{i=1}^{N} y_i(0) = 2\pi$ such that $y_1(h) \leq 0$. This ends the proof. ■

## V. Simulations

To verify the effectiveness of our proposed control laws in the previous two sections, we carry out some numerical simulations. In this section, we show the simulation results of the way-point control law (5) and the sampled-data control law (18) in Fig. 2 and Fig. 3, respectively.

The initial angular positions of the $N$ robots in the simulations are generated randomly following the initial condition (1). The desired circle formation in the simulation of Circle Formation Problem is given randomly. The angular position of each robot and the angular distance between each pair of neighbors are shown in each case.

The simulation results show that the groups of robots can eventually converge to the desired circle formation. In particular, the figures show clearly that the robots preserve ordering under our proposed control laws.

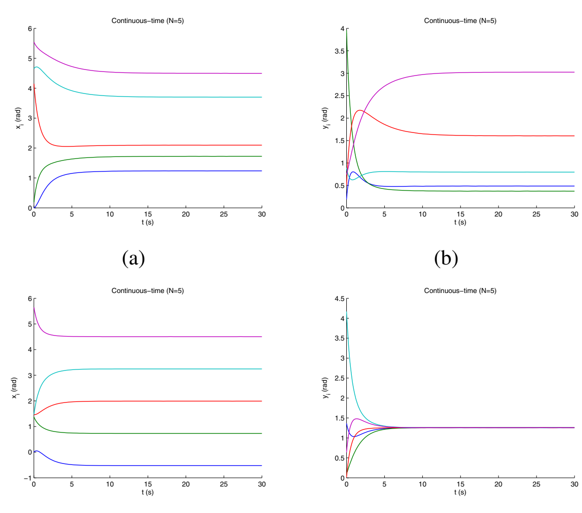


Fig. 2. Simulation results of the proposed way-point control law (5) for the continuous-time case when $N = 5$. (a)(b) the Circle Formation Problem; (c)(d) the Uniform Circle Formation Problem. (a)(c) angular position of each robot; (b)(d) angular distance between each pair of neighbors.

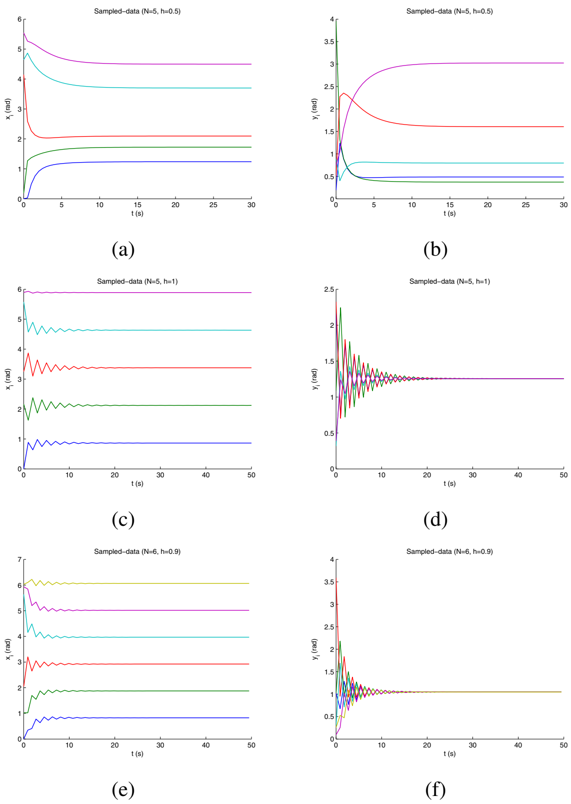


Fig. 3. Simulation results of the sampled-data control law (18) for the sampled-data case. (a)(b) the Circle Formation Problem when $N = 5, h = 0.5$s; (c)(d) the Uniform Circle Formation Problem when $N = 5, h = 1$s; (e)(f) the Uniform Circle Formation Problem when $N = 6, h = 0.9$s. (a)(c)(e) angular position of each robot; (b)(d)(f) angular distance between each pair of neighbors.

## VI. Conclusion

In this paper, we have proposed a distributed control law for a group of robots to realize any given circle formation on a circle of a prescribed radius. A sampled-data control law, which is easy to be used in real-robot systems, has also been proposed. In particular, we have given the theoretical analysis for both cases as well as the numerical simulations to show the effectiveness of our proposed control laws and the property of order preservation.

From a theoretical point of view, the results of this paper have provided a simple but effective method that can be proved to solve the circle formation problem on a circle. From a practical point of view, our control laws allow the robots to preserve the ordering such that it is easy to be used in real-robot systems to avoid collision.

However, the circle formation problem we considered in this paper is under the constraint that the robots are already located on the circle in the beginning. This naturally leads to the task of designing a control law to guide the group of robots to form to a circle from any configuration in the plane. We are also interested in using robotic testbed to implement the designed control strategies.

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