Consensus dynamics in distribution networks and nonlinear multi-agent systems

Wei, Jieqiang

Consensus Dynamics in Distribution Networks and Nonlinear Multi-Agent Systems

Jieqiang Wei

The research described in this dissertation has been carried out at the Faculty of Mathematics and Natural Sciences, University of Groningen, The Netherlands, within collaboration between the research Institute of Technology and Management (ITM) and the Johann Bernoulli Institute for Mathematics and Computer Science (JBI).

This dissertation has been completed in partial fulfillment of the requirements of the Dutch Institute of Systems and Control (DISC) for graduate study.

Printed by Ipskamp Drukkers

ISBN (book): 978-90-367-8701-7

Consensus Dynamics in Distribution Networks and Nonlinear Multi-Agent Systems

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. E. Sterken
and in accordance with the decision by the College of Deans.

This thesis will be defended in public on

Friday 18 March 2016 at 11.00 hours

by

Jieqiang Wei

born on 15 May 1987
in Shandong, China

Supervisor
Prof. A.J. van der Schaft

Assessment committee
Prof. C. De Persis
Prof. F. Blanchini
Prof. Y. Hong

To my family,

Liangzhu, Yinqin,
Weiyan and Liwen

Acknowledgments

First of all, I would like to thank my supervisor, Arjan van der Schaft for the careful guidance. In the last five years, your understanding, support and a great sense of humor make it much easier for me to go through all the difficulties in my research and personal life. I enjoyed and learned a lot from our discussions. What I admire you the most are the enthusiasm, diligence and deep insight for research, which have a great influence on my life. I feel truly honored to be a student of you.

I greatly appreciate the assessment committee, Prof. Yiguang Hong, Prof. Claudio De Persis and Prof. Franco Blanchini for their constructive comments and discussions on the feasible future research.

Many thanks to my dear paranymphs, Anneroos Everts and Pooya Monshizadeh. Since the CDC trip to Los Angeles in 2014, we started to call each other honey, and I hope we will keep using this title. Dear Anneroos, it was very lucky to have a officemate like you. Thank you for patiently helping me out with all kinds of daily problems. I cherish all the great memories with you: the trip to LA and Benelux meetings, the chats about culture differences (not in a shocking way), about life and hobbies and so on. I also enjoyed our fruitful discussions and collaborations very much. Special thank to you for the Dutch translations and English polishing. Dear Pooya, thank you for being such a nice brother to me all the time, and all the delightful discussions about research, life philosophy and fashion. Also I enjoyed very much the parties and the badminton games we had. I can see that you are very talented in both.

Studying in Groningen is a wonderful experience for me, and it is my pleasure to thank my colleagues and friends for making the university such a great place: Dear Claudio, thank you for your encouragement about my research and job interview in Stockholm. Dear Nima, thank you for being so cheerful and for setting such a good example for me. Dear Ming, I appreciate very much that you have been very supportive to both me and my wife. Dear Weiguo, thank you for the good suggestions during my PhD and pre-postdoc. After you, I will be the continuity of Groningen spirit in KTH, especially with the simple dutch style lunch. Dear Desti, Olexander Ivanov, Hector, thank you for our nice collaborations of DISC courses. Moreover, I would like to thank Jacquelien Scherpen, Bayu Jayawardhana, Anton Proskurnikov, James Riehl, Tjardo Scholten, Tjerk Stegink, Sebastian Trip, Hadi Tagvafard and Rully Tri Cahyono from DTPA and SMS group. Dear Ineke Schelhaas, I appreciate it very much for all your help in my work and life. Dear Harry, it has been a blessing to know someone like you who is all at once wise, humorous and intelligent. Dear Kanat, thank you for your kind words when my wife was ill. That really gave me the courage to face all the difficulties. I appreciate all the useful discussions about discontinuous dynamical systems. Don't really know what to report to the Chinese government without you being around. Dear Devrim, I miss the way you talk about math and football. I am also very grateful to Ewoud Vos, Hidde-Jan, Bao Nguyen, Xiaodong Cheng, Shuo Zhang, Matin Jafarian, Monika Jozsa, Noorma Megawati, Mauricio Munoz Arias, Pouria Ramazi, Rodolfo Reyes Baez, Fan Zhang, Jianlei Zhang, Tao Liu, Sourabh Kotnala for the invaluable friendship.

It has been pleasant for being in our badminton team which is like a big family for all the time. Dear Sun Zhe, my captain, you are such a modest gentleman. Thank you for forewarning me all the problems might happen when man reach his 30. And I do wish you can defeat one of the ladies in your current club. Dear Ding Ning, thank you for being such a great spiritual leader, I've learned so many valuable things from you. And thank you for introducing me into ACSSNL. Dear Xie Qibing, thank you for being such a funny and energetic teammate. I enjoyed very much all the game we played. Dear Zhang Lu, Zhang Shuhai, Loria, Li Yixian, Wang Kexin, Zhang Xuewen, Peng Suping, He Tao, Xiao Chengyong, Deng Chenglong, thank you all for being together with me almost every weekend during last four years, and for such a incredible tournament in 2014 which we won the second place.

I want to thank the officers in the Chinese embassy in Den Haag. Counselor Fang, Mr. Xia Lei, Mr. Wang Yiwei, thank you for being supportive to my job in ACSSNL and ACSSG, and giving me the opportunity to meet President Xi in the Spring of 2014. I would also like to thank my friends in ACSSNL, Bao Xinfeng, Xia Zhihan, Li Tianqi, Xieli, Xiao Teng, Qian Xiaochen. It was a great pleasure to work with you all.

Last, but not least, I would like to thank my family members. Dear Baba mama, there is no words to express my gratitude for your unconditional love for all these years and how guilty I feel for not being with you when you need me. Dear Weiyan, thank you for being compatible and supportive, and for bringing so many joys into my life. I will never forget our two magic experiences in UMCG. Magic indeed, aren't they? I don't like to admit, but I really start to believe more and more in the old saying that "this women is always right". Dear Liwen, my son, my heart, this thesis is partly dedicated to you, and I really wished to have something you would understand in the near future, but just the materials I have now don't fit with "Bob the builder" very well.

Confucian once said, "A son not ought to go abroad to a distance where he will not be able to pay the due services to his parents. If he does go abroad, he must have a fixed place to which he goes." This place, for me, is Groningen, which will be a special, warm place in my heart forever.

Jieqiang Wei
Stockholm
February 2016

Contents

List of Symbols

$\mathcal{F}[X]$	the Filippov set-valued map associated with a vector field $X : \mathbb{R}^d \to \mathbb{R}^d$	23
$\mathbb{1}_n$	the n dimensional vector with all components being equal to one.	11
$\mathrm{co}\{\mathcal{S}\}$	convex hull of a set $\mathcal{S} \subset \mathbb{R}^d$	23
$\overline{\mathrm{co}}\{\mathcal{S}\}$	convex closure of a set $\mathcal{S} \subset \mathbb{R}^d$	23
∂f	generalized gradient of the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$	23
supp	the set of indices of nonzero components	13
$\tilde{\mathcal{L}}_{\mathcal{F}} f$	set-valued Lie derivative of the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the set-valued map $\mathcal{F} : \mathbb{R}^d \to 2^{\mathbb{R}^d}$	23
$\xi_{\mathcal{C}}$	vectorized representation of a circuit $\mathcal{C}$	11
e_{ij}	the edge with tail vertex v_i and head vertex v_j	11
f^*	convex conjugate of a convex function f	13

Chapter 1

Introduction

1.1 Introduction

Two classes of complex systems are considered in this thesis: distribution networks and multi-agent systems.

Distribution networks can be seen everywhere in our daily life, such as public transportation systems, telephone networks and electrical power systems. Typically, a distribution network is depicted as a graph where resources can enter the network via supply vertices (e.g. power plants, factories) and leave the network via demand vertices (e.g. cities, consumption centers), together with edges (e.g. routes, cable lines, roads) that connect the supply, demand and additional internal vertices. Often, flow capacity constraints and (transportation) cost functions are assigned to the edges.

Generally speaking, distribution networks can be divided into two classes, depending on whether the vertices can store resources or not. If the vertices can only distribute resources but not store, we refer to this type of distribution networks as static ones. The study of static distribution networks is a broad research topic which has a long history and a large number of applications [7]. One celebrated result is the max-flow min-cut theorem [35, 36]. The static distribution problem is closely related to monotropic programming problems which enjoy a complete and symmetric duality theory [61].

Differently from static distribution networks, vertices can have storage of resources in dynamical distribution networks. This type of models has many applications in e.g. communication networks [32, 66], transportation networks [9, 50], and production distribution networks [12].

In this thesis, we consider dynamical distribution networks where we assign a set of nonlinear integrators to the vertices (with state variables corresponding to storage). All the integrators are controlled by the flows on the edges. Furthermore, unknown but constant in/outflows may enter or leave the network through some of the vertices. The control aim is to regulate the outputs of the vertices to consensus by controlling the flows on the edges. This is called the output agreement problem. The problem can be solved by a continuous distributed controller defined on the edges. In the case without flow capacity constraints, the solution to the output agreement problem is straightforward by formulating the closed-loop system as a port-Hamiltonian system [6]. In the case with flow capacity constraints, the constraint intervals and supplies/demands need to satisfy certain conditions to guarantee convergence to output agreement. The proof is done by the Lyapunov method and by the analysis of the graphical structure of the network. Similar to the monotropic programming problem in [61], we can relate the case with flow capacity constraints to a pair of dual optimization problems. This leads to an equivalent expression of the necessary and sufficient condition for output agreement with flow capacity constraints.

The other problem about dynamical distribution networks studied in this thesis is the case with state constraints on the vertices. More precisely, we consider the model used in the first problem with a Proportional-Integral (PI) controller assigned to all the edges. Due the oscillatory behavior introduced by the PI controller, the state constraints can be violated. The control objective in this problem is to achieve output agreement of the vertices while the state constraints remain satisfied for all time. This objective can be accomplished by designing state-based flow constraints. The result is proved for all Filippov trajectories of the discontinuous closed-loop system.

In the second part of this thesis, we study *multi-agent systems* which occur in several applications including formation flight of unmanned air vehicles (UAVs), clusters of satellites, automated highway systems and task assignment (see e.g. [25], [33], [60], [64]). In these applications, an agent represents a system which can interact autonomously with other systems according to the communication law to perform specific tasks. The communication structure is usually given by a (directed) graph called the communication graph, where the vertices are corresponding to the agents, while the edges represent the information exchanging or sensing channels among the agents. See Figure 1.1 for a graphical explanation.

One fundamental and benchmark problem in multi-agent systems is the *(state) consensus problem* (see e.g. [37], [51], [57], [59]). Roughly speaking, consensus and the like (synchronization, rendezvous) refer to the group behavior that all agents asymptotically reach a certain state of interest. Although consensus problems have a history in computer science [47], we focus on their applications in cooperative control of multi-agent systems.

The simplest and most well-studied consensus problem is the continuous-time linear time-invariant consensus protocol. The operation mechanism is that each agent moves towards the weighted average of the states of its neighbors. From the classical results (see e.g. [48], [51]), it is known that this consensus protocol will drive the states of the agent to consensus if and only if the underlying communication graph contains a directed spanning tree.

Apart from the linear consensus protocol, nonlinear versions have attracted attention of many researchers in the last decade. The nonlinear consensus protocols may arise due to the nature of the controller. For instance, the measurement

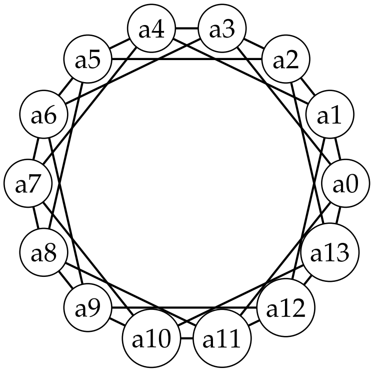


Figure 1.1: This figure depicts a multi-agent system with 14 agents placed on the nodes. The edge between agent i and j indicates that the information of i is available to j and vice versa. If the undirected edges are replaced by directed ones, it means that the information of the end vertex of the edge is available to the head vertex but not vice versa.

of the state of each agent can be nonlinear (see for example [20] about quantized consensus protocols), or the comparison of the states of two agents can be nonlinear, see e.g. [21], [22] about sign-based control protocols. The nonlinear consensus protocols may also describe the physical coupling existing in the network (see e.g. [26], [53]).

In this thesis we consider a general mathematical model of nonlinear consensus protocols which cover all cases of nonlinearity. The nonlinearities are assumed to be sign-preserving with possible discontinuities. For these nonlinear consensus protocols, we provide sufficient conditions on the nonlinear functions and the topology of the underlying graph such that consensus is asymptotically achieved. The analysis is performed within the framework of Filippov solutions. An important source of inspiration of this work is [26]. Specifically, the result in Section 5.2.2 modifies and extends the result in [26] from signum functions to general nonlinear functions under the weakest topological condition, namely a digraph containing a directed spanning tree.

1.2 Thesis outline

The outline of this thesis is as follows.

In Chapter 2, preliminaries are given on graph theory, convex analysis, network optimization theory, port-Hamiltonian systems, equilibrium-independent passivity and non-smooth analysis.

The main part of the thesis is divided into two parts.

The first part of this thesis, containing Chapter 3 and 4, deals with dynamical distribution networks.

In Chapter 3, we propose distributed PI controllers defined on the edges which regulate the flows of the network. It is proved that the output agreement of the vertices in the distribution network is asymptotically stable for the cases with and without flow constraints. This chapter is based on [70], [72] and [74].

In Chapter 4, we consider the case that the distribution network has state constraints. By modifying the controller proposed in Chapter 3 with state-based flow constraints, we prove that all the Filippov (as well as Krasovskii) trajectories meet the state constraints and converge to output agreement. Chapter 4 is an extended and modified version of [73].

The second part of this thesis is Chapter 5 and deals with several general mathematical models of nonlinear multi-agent systems. Sufficient conditions about the nonlinear function in the dynamics and topology of the network are provided to guarantee the states of the agent to achieve consensus. In this chapter, the nonlinear functions are assumed to be sign-preserving with possible discontinuities. All the results are derived for Filippov solutions. This chapter is partially based on [75] which is submitted for journal publication.

1.3 Publications

Journal articles

1. J. Wei, A.J. van der Schaft. " Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows ". *Systems & Control Letters*, 62(11):1001-1008, 2013.

2. J. Wei, A.R.F. Everts, M.K. Camlibel and A.J. van der Schaft. "Consensus problems with arbitrary sign-preserving nonlinearities". *Submitted to Automatica*.

3. J. Wei, A.J. van der Schaft. "A graph-theoretic condition for the stability of dynamical distribution networks with flow constraints". *In preparation*.

Conference papers

1. A.J. van der Schaft, J. Wei. "A Hamiltonian Perspective on the Control of Dynamical Distribution Networks". In *Proceedings of the 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, Bertinoro, Italy,

pp.24-29, 2012.

2. J. Wei, A.J. van der Schaft. “Stability of Dynamical Distribution Networks with Arbitrary Flow Constraints and Unknown In/outflows”. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, pp.55-60, 2013.

3. J. Wei, A.J. van der Schaft. “A graphic condition for the stability of dynamical distribution networks with flow constraints”. In *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*, Groningen, The Netherland, 2014.

4. J. Wei, A.J. van der Schaft. “Constrained proportional integral control of dynamical distribution networks with state constraints”. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, California, USA, pp.6056-6061, 2014.

Conference abstracts

1. J. Wei, A.J. van der Schaft. “Control of transportation networks modeled as port-Hamiltonian systems on graphs”. *Benelux Meeting on Systems and Control*, Heijden, The Netherlands, 2012

2. J. Wei, A.J. van der Schaft. “A Hamiltonian perspective on the control of dynamical distribution networks”. *Benelux Meeting on Systems and Control*, Houffalize, Belgium, 2013.

3. J. Wei, A.J. van der Schaft. “A graphic condition for the stability of dynamical distribution networks with flow constraints”. *Benelux Meeting on Systems and Control*, Heijden, The Netherlands, 2014.

4. J. Wei, A.J. van der Schaft. “Constrained proportional integral control of dynamical distribution networks with state constraints”. *Benelux Meeting on Systems and Control*, Lommel, Belgium, 2015.

Chapter 2

Preliminaries

In this chapter we present the tools from graph theory, convex analysis, network optimization theory, port-Hamiltonian systems, equilibrium-independent passivity and non-smooth stability analysis, which will be used in the subsequent chapters.

2.1 Graph theory

In this section, we first provide some essentials from the field of graph theory as can be found e.g. in [11, 14].

A *graph* $\mathcal{G}$ is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ such that $\mathcal{E}$ is a subset of $\mathcal{V} \times \mathcal{V}$ of unordered pairs of $\mathcal{V}$ where no self-loops are allowed. The finite sets $\mathcal{V}$ and $\mathcal{E}$ (also denoted as $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$) are the set of *vertices* (also called *nodes*) and *edges* (of $\mathcal{G}$) respectively. The map $\mathcal{W} : \mathcal{E} \to \mathbb{R}_+$ defines the *weights* on each edge. Such weights might represent, for example, costs, lengths or capacities, etc. In this thesis, let us denote $\mathcal{V} = \{v_1, \ldots, v_n\}$ and $|\mathcal{E}| = m$. If the edges are ordered pairs of vertices, the graph $\mathcal{G}$ is called a *directed graph*, or *digraph* for short. An edge of a digraph $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j)$ (with $v_i \neq v_j$) representing the tail vertex v_i and the head vertex v_j of this edge. Given a digraph $\mathcal{G}$, a corresponding graph, denoted as $\mathcal{G}^o$, can be obtained by neglecting the direction of the edges. Note that $\mathcal{G}^o$ may have multiple edges. We say that $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{W}')$ is a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ if $\mathcal{V}' \subset \mathcal{V}$, $\mathcal{E}' \subset \mathcal{E}$, and $\mathcal{W}' = \mathcal{W}|_{\mathcal{E}'}$.

For a graph $\mathcal{G}$, a *path* P from vertex v_i to vertex v_{i+l} is a finite sequence of edges which connect a sequence of vertices $\{v_i, v_{i+1}, \ldots, v_{i+l}\}$ which are all distinct from one another. This path is denoted by $P = v_i v_{i+1} \cdots v_{i+l}$. If a path $P = v_i v_{i+1} \cdots v_{i+l}$ is such that $v_i = v_{i+l}$ and the vertices $v_{i+j}, 0 \leqslant j < l$, are distinct from each other, then P is said to be a *circuit*. A graph is *acyclic* if it does not contain any circuit. A graph is *connected* if for every two vertices v_i and v_j there is a path from v_i to v_j. A subgraph $\mathcal{T} \subseteq \mathcal{G}$ is a tree if it is connected and acyclic, and a spanning tree if it is a tree and contains all the vertices of $\mathcal{G}$.

For a digraph $\mathcal{G}$, we define a *directed path* from vertex v_i to vertex v_j to be a path of $\mathcal{G}^o$ such that the first edge starts from v_i, the last edge ends at v_j and every edge starts where the previous edge ends. A digraph is called *strongly connected* if for every two vertices v_i and v_j there is a directed path from v_i to v_j. A digraph $\mathcal{G}$

is called *weakly connected* if the graph $\mathcal{G}^o$ is connected. A subgraph $\mathcal{T}$ of $\mathcal{G}$ is called a *directed spanning tree* for $\mathcal{G}$ if $\mathcal{V}(\mathcal{T}) = \mathcal{V}(\mathcal{G})$, and for every vertex $v_i \in \mathcal{V}$ there is exactly one v_j such that $e_{ji} \in \mathcal{E}(\mathcal{T})$, except for one vertex, which is called the root of the spanning tree. Furthermore, we call a vertex $v \in \mathcal{V}$ a *root* of $\mathcal{G}$ if there is a directed spanning tree for $\mathcal{G}$ with v as a root. In other words, v is a root of $\mathcal{G}$ if there is a directed path from v to every other vertex in the graph.

A (di)graph is called *simple* if it has no multiple edges or self-loops. For a simple (di)graph $\mathcal{G}$ the weighted *adjacency matrix* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined in the following way: $a_{ij} = \mathcal{W}(e_{ji})$ if and only if $e_{ji} \in \mathcal{E}$, where $\mathcal{W}$ is the weight function. Moreover, $a_{ii} = 0$ for all $i = 1, \dots, n$. Notice that for undirected graphs, $A = A^T$. The set of neighbors of vertex v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. For each vertex v_i, its in-degree and out-degree is defined as

$$\begin{aligned} \deg_{\text{in}}(v_i) = &\ \textstyle\sum_{j=1}^{n} a_{ij}, \\ \deg_{\text{out}}(v_i) = &\ \textstyle\sum_{j=1}^{n} a_{ji}. \end{aligned}$$

For undirected graphs, the in-degree is equal to the out-degree for all the vertices, and hence will be called the degree. For a graph with all the weights on the edges being one, the adjacency matrix only has 0-1 elements, and in this case we have $\deg_{\text{in}}(v_i) = |\mathcal{N}_i|$. The degree matrix of the digraph $\mathcal{G}$ is a diagonal matrix Δ where $\Delta_{ii} = \deg_{\text{in}}(v_i)$. The *graph Laplacian* is defined as

$$L = \Delta - A.$$

This implies $L\mathbb{1} = 0$, where $\mathbb{1}_n$ is the n dimensional vector containing only ones. We omit the subscript if the dimension of the vector is unambiguous from the context. We say that a vertex v_i is *balanced* if its in-degree and out-degree are equal. The graph $\mathcal{G}$ is called balanced if all of its vertices are balanced or, equivalently, if

$$\mathbb{1}^T L = 0.$$

A digraph, with possibly multiple edges, is completely specified by its *incidence matrix* B, which is an $n \times m$ matrix, with $(i, j)^{\text{th}}$ element equal to -1 if the j^{th} edge is towards vertex i, and equal to 1 if the j^{th} edge is originating from vertex i, and 0 otherwise. The graph Laplacian is also given as $L = BWB^T$, where W is the diagonal matrix of the edge weights. A digraph is weakly connected if and only if $\ker B^T = \operatorname{span} \mathbb{1}_n$. A digraph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the weakly connected components. The number of weakly connected components is equal to $\dim\ker B^T$.

Lemma 2.1. *For a digraph with weights one on all the edges, it is balanced if and only if* $\mathbb{1}_m \in \ker B$.

Definition 2.2. For a (di)graph $\mathcal{G}$, the *vertex space* $C_0(\mathcal{G})$ is the vector space of all functions from $\mathcal{V}$ to a field $\mathcal{R}$. The *edge space* $C_1(\mathcal{G})$ is the vector space of all functions from $\mathcal{E}$ to a field $\mathcal{R}$.

If $\mathcal{G}$ has n vertices and m edges and we take the field $\mathcal{R}$ as $\mathbb{R}$, then $C_0(\mathcal{G})$ and $C_1(\mathcal{G})$ can be identified with $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Indeed, the elements of $\mathbf{x} \in C_0(\mathcal{G})$ can be written in the form $\mathbf{x} = \sum_{i=1}^n x_i v^i$ where $v^i : \mathcal{V} \to \mathbb{R}$ is equal to 0 everywhere except 1 at the vertex v_i. Then $\{v^1, \ldots, v^n\}$ is a basis of $C_0(\mathcal{G})$. In this way, the incidence matrix B of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$.

By replacing the field $\mathcal{R}$ by some other fields, such as $\mathcal{R} = \mathbb{R}^3$, many of the results of this thesis can be generalized to higher-dimensional spaces.

Given a digraph $\mathcal{G}$, let $\mathcal{C}$ be a circuit in $\mathcal{G}^o$, then the two possible cyclic orderings of the vertices of $\mathcal{C}$ induce two possible *circuit orientations* of the edges of $\mathcal{C}$. Let us choose one of these circuit orientations, and define a function $\xi_\mathcal{C} \in C_1(\mathcal{G})$ as follows. We take $\xi_\mathcal{C}(e) = +1$ if e belongs to $\mathcal{C}$ and its circuit orientation coincides with its orientation in $\mathcal{G}$, $\xi_\mathcal{C}(e) = -1$ if e belongs to $\mathcal{C}$ and its circuit orientation is the reverse of its orientation in $\mathcal{G}$, and we take $\xi_\mathcal{C}(e) = 0$ if e is not in $\mathcal{C}$. Furthermore, $\mathcal{C}$ together with a chosen orientation is called a *positive circuit* if $\xi_\mathcal{C} \in \mathbb{R}^m_{\geqslant 0}$.

Theorem 2.3. *For a weakly connected digraph $\mathcal{G}$, the kernel of the incidence matrix B is a vector space whose dimension is $m - n + 1$. If $\mathcal{C}$ is a circuit in $\mathcal{G}^o$, then $\xi_\mathcal{C} \in \ker B$.*

2.2 Convex analysis

In this section, we review some notations and definitions from convex analysis as can be found in [28, 62]. Afterwards some connections between graph theory and convex analysis are given.

The *support* of a vector, $\operatorname{supp}(x)$, is the set of indices i such that $x_i \neq 0$. If x and y are different points in $\mathbb{R}^k$, the set of points of the form $(1-\lambda)x + \lambda y$ where $\lambda \in \mathbb{R}$ is called the line through x and y. A *face* of a convex set C is a convex subset C' of C such that every line segment in C with a relative interior point in C' has both endpoints in C'. The empty set and C itself are faces of C. The zero dimensional faces of C are *extreme points* of C. Thus a point $x \in C$ is an extreme point if and only if there is no way to express x as a convex combination $(1-\lambda)y + \lambda z$ such that $y \in C, z \in C$ and $0 < \lambda < 1$. A subset K of $\mathbb{R}^n$ is called a *cone* if it is closed under positive scalar multiplication, i.e., $\lambda x \in K$ when $x \in K$ and $\lambda \geqslant 0$. A *convex cone* is a cone which is a convex set. For a convex cone, the origin is the only extreme point. The one-dimensional set

$$\{\lambda\Gamma + x \mid \lambda \geqslant 0\} \subset \mathbb{R}^k$$

where Γ is a non-zero vector in $\mathbb{R}^k$ defines a half-line called a *ray* in direction $\Gamma \in \mathbb{R}^k$ emanating from $x \in \mathbb{R}^k$. An *extreme direction* Γ of a convex cone K is a vector corresponding to a half-line face that is a ray emanating from the origin. A nonzero direction Γ in K is extreme if and only if

$$\lambda_1\Gamma_1 + \lambda_2\Gamma_2 \neq \Gamma, \forall\lambda_1, \lambda_2 \geqslant 0, \forall\Gamma_1, \Gamma_2 \in K \setminus \{\lambda\Gamma \mid \lambda \geqslant 0\}.$$

When the number of extreme points and directions of a convex set is finite, the set of extreme elements is called a set of *minimal generators* for that convex set. A convex cone with minimal generators, which possibly has empty interior, is called a *polyhedral cone*.

Example 2.1. Consider the convex polyhedral cone given as in Figure 2.1. The coordinations of the apexes A,B and C are $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ respectively. Then the vectors $(1, 1, 2)^T$, $(1, 2, 1)^T$ and $(2, 1, 1)^T$ are extreme directions. The set of minimal generators for this convex cone is composed of the origin O, which is the trivial extreme point, and these three extreme directions.

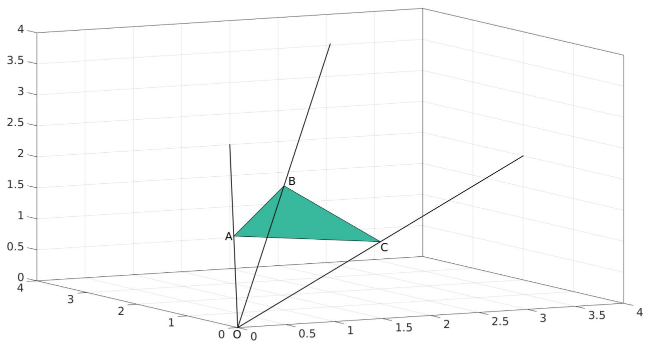


Figure 2.1: The convex polyhedral cone with three extreme directions.

Finally, let us recall some notations from linear algebra. A linearly dependent set is called minimal if every proper subset is linearly independent.

Proposition 2.4. *For a minimal linearly dependent set* $\{\xi^1, \ldots, \xi^k\}$ *there exists a unique (up to a multiplier) non-trivial linear combination* $c_1\xi^1 + \ldots + c_k\xi^k$ *which is equal to* 0 *while none of its coefficients* c_i *is* 0.

Lemma 2.5 (Lemma 3.2.9 in [38]). *Given a digraph* $\mathcal{G}$ *with incidence matrix* B*, then the set of minimal generators of the polyhedral cone* $\ker B \cap \mathbb{R}^m_{\geqslant 0}$ *is composed of positive circuits.*

Proof. First, from Theorem 2.3 we have that the kernel of B is spanned by the vectors which are defined as the circuits of $\mathcal{G}^o$. Hence all the positive circuits belong to $\ker B \cap \mathbb{R}^m_{\geqslant 0}$. Next, we need to show that $\xi_\mathcal{C}$ where $\mathcal{C}$ is a positive circuit is an extreme direction. Indeed, the columns of B corresponding to any strict subset of $\text{supp}(\xi_\mathcal{C})$ are linearly independent which implies that $\xi_\mathcal{C}$ is an extreme direction. In other words, any positive circuit corresponds to a minimal linearly dependent set of the columns of the incidence matrix. Hence a positive circuit is an extreme direction.

Next we will show that an extreme direction of $\ker B \cap \mathbb{R}^m_{\geqslant 0}$ is also a positive circuit (up to a multiplier). First we notice that an extreme direction must have a minimal support. In fact, suppose the vector $v \in \ker B \cap \mathbb{R}^m_{\geqslant 0}$ being an extreme direction does not have minimal support, i.e., there exists a circuit $\mathcal{C} \subsetneqq \{e_i \in \mathcal{E} \mid i \in \text{supp}(v)\}$. Then there exists a $\varepsilon > 0$ such that $v \pm \varepsilon\xi_\mathcal{C} \in \ker B \cap \mathbb{R}^m_{\geqslant 0}$. Therefore

$$v = \frac{1}{2}(v - \varepsilon\xi_\mathcal{C}) + \frac{1}{2}(v + \varepsilon\xi_\mathcal{C}), \tag{2.1}$$

which is a contradiction to the fact that v is an extreme direction. Hence for any extreme direction v, the columns of B corresponding to nonzero components of v are minimal linearly dependent, i.e., $\{e_i \in \mathcal{E} \mid i \in \text{supp}(v)\}$ forms a circuit in $\mathcal{G}^o$. By the fact that $v \in \mathbb{R}^m_{\geqslant 0}$, we have the set $\{e_i \in \mathcal{E} \mid i \in \text{supp}(v)\}$ forms a positive circuit in $\mathcal{G}$, i.e., v is equal to a vector defined by a positive circuit (up to a multiplier). □

Definition 2.6. A convex function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *proper* if the set

$$\{x \in \mathbb{R} \mid f(x) < +\infty\} \tag{2.2}$$

is nonempty and the function never attains $-\infty$.

Definition 2.7. The *convex conjugate* of a convex function f, denoted f^*, is defined as

$$f^*(x^*) = \sup_{x \in \mathbb{R}} \{\langle x, x^* \rangle - f(x)\}. \tag{2.3}$$

Definition 2.8. A relation $\mathcal{R} \subset \mathbb{R}^n \times \mathbb{R}^n$ is said to be *monotone* if

$$\langle x_1 - x_0, x_1^* - x_0^* \rangle > 0 \tag{2.4}$$

for every $(x_0, x_0^*), (x_1, x_1^*) \in \mathcal{R}$.

2.3 Network optimization theory

In this section, we review some notations and two classical optimal problems from network optimization theory as can be found in [61]. A *network* is given by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} = \{e_1, \ldots, e_m\}$. As in Section 2.1, the incidence matrix of $\mathcal{G}$ is denoted by $B \in \mathbb{R}^{n \times m}$.

By a *flow* of a network $\mathcal{G}$, we mean a vector $\mu = (\mu_1, \ldots, \mu_m)^T$ where the value μ_i, called the *flux* in the edge e_i, is interpreted as the quantity of material flowing through the edge e_j. Given a flow of a network, the total departing flow from v_i minus the total arriving flow at v_i is called the *divergence* of the flow at the vertex v_i and it will be denoted by u_i. We call the vector $u = (u_1, \ldots, u_n)^T$ the divergence vector associated with the flow μ. This definition is summarized by

$$u + B\mu = 0. \tag{2.5}$$

Duality in the study of flows is closely tied to the following notion. A *potential* in a network $\mathcal{G}$ is a vector $y = (y_1, \ldots, y_n)$ where the value y_i is called the potential at the vertex v_i. With an edge $e_k = (v_i, v_j)$, one associates the potential difference as $\zeta_k = y_i - y_j$. This defines the *tension* vector $\zeta = (\zeta_1, \ldots, \zeta_m)$ which is the differential of the potential y, i.e., ζ can be expressed as

$$\zeta = B^T y. \tag{2.6}$$

The relation between the pair of flows and tension and the pair of potentials and divergence is expressed by the *conversion formula*:

$$\mu^T \zeta = -y^T u. \tag{2.7}$$

The *optimal flow problem* attempts to optimize the flow and divergence in a network subject to the constraint (2.5). Each edge and node are assigned a convex *flux cost* $C_k^{flux}(\mu_k)$ and a convex *divergence cost* $C_i^{div}(u_i)$, i.e.,

$$\begin{aligned} &\min_{u,\mu} \sum_{i=1}^{n} C_i^{div}(u_i) + \sum_{k=1}^{m} C_k^{flux}(\mu_k) \\ &\text{s.t } \; u + B\mu = 0. \end{aligned} \tag{2.8}$$

The problem (2.8) admits a dual problem, called the *optimal potential problem*, which is given as

$$\begin{aligned}
\min_{y,\zeta} \ & \sum_{i=1}^{n} C_i^{pot}(y_i) + \sum_{k=1}^{m} C_k^{ten}(\zeta_k) \\
\text{s.t } \ & \zeta = B^T y,
\end{aligned} \tag{2.9}$$

where C_i^{pot} and C_k^{ten} are the conjugate functions of C_i^{div} and C_k^{flux} respectively, i.e., $C_i^{pot} = C_i^{div,*}$ and $C_k^{ten} = C_k^{flux,*}$. The duality of problem (2.8) and (2.9) is a well-known result, here for the sake of completeness, we attach the proof of it as follows.

Proof of the duality of (2.8) *and* (2.9). The optimal flow problem (2.8) is equivalent to the following problem

$$\begin{aligned}
\min_{u,\tilde{u},\mu,\tilde{\mu}} \ & \sum_{i=1}^{n} C_i^{div}(\tilde{u}_i) + \sum_{k=1}^{m} C_k^{flux}(\tilde{\mu}_k) \\
\text{s.t } \ & u + B\mu = 0, \\
& \tilde{u} = u, \\
& \tilde{\mu} = \mu.
\end{aligned} \tag{2.10}$$

By introducing the Lagrangian multipliers y and ζ for the constraints $\tilde{u} = u$ and $\tilde{\mu} = \mu$, the conjugate of (2.10) is

$$\begin{aligned}
\min_{u,\tilde{u},\mu,\tilde{\mu}} \ & \sum_{i=1}^{n} C_i^{div}(\tilde{u}_i) + \sum_{k=1}^{m} C_k^{flux}(\tilde{\mu}_k) - y^T(\tilde{u} - u) - \zeta^T(\tilde{\mu} - \mu) \\
\text{s.t } \ & u + B\mu = 0,
\end{aligned} \tag{2.11}$$

which is equal to

$$\min_{\tilde{u},\mu,\tilde{\mu}} \sum_{i=1}^{n} C_i^{div}(\tilde{u}_i) + \sum_{k=1}^{m} C_k^{flux}(\tilde{\mu}_k) - y^T(\tilde{u} + B\mu) - \zeta^T(\tilde{\mu} - \mu). \tag{2.12}$$

By the definition of convex conjugate function, the function (2.12) can be equivalently expressed as

$$\min_{\tilde{u},\mu,\tilde{\mu}} - \sum_{i=1}^{n} C_i^{div,*}(y_i) - \sum_{k=1}^{m} C_k^{flux,*}(\zeta_k) - \mu^T(B^T y - \zeta), \tag{2.13}$$

which is equal to

$$\begin{cases} -\sum_{i=1}^{n} C_i^{div,*}(y_i) - \sum_{k=1}^{m} C_k^{flux,*}(\zeta_k) & \text{if } \zeta = B^T y, \\ -\infty & \text{if } \zeta \neq B^T y. \end{cases} \tag{2.14}$$

Hence the Lagrange dual problem of (2.8) is given as

$$\begin{aligned} &\max_{y,\zeta} \; -\sum_{i=1}^{n} C_i^{div,*}(y_i) - \sum_{k=1}^{m} C_k^{flux,*}(\zeta_k) \\ &\text{s.t } \; \zeta = B^T y, \end{aligned} \tag{2.15}$$

which is equivalent to the optimal potential problem (2.9). □

2.4 Port-Hamiltonian systems

The port-Hamiltonian framework is aimed at providing a unified framework for the modeling of systems belonging to different physical domains (mechanical, electrical, hydraulic, thermal, etc.). This is achieved by recognizing *energy* as the 'lingua franca' between physical domains, and by identifying ideal system components capturing the main physical characteristics. This section is based on [69].

The term *port* in the name stands for power ports which provide an interface for the sub-models within the model to interact with each other. Each port is composed by pairs (f, e) of equally dimensioned vectors of *flow* and *effort* variables. The flow vector f belongs to the flow space $\mathcal{F}$ and the effort vector e belongs to the effort space $\mathcal{E}$ which is the dual space of $\mathcal{F}$, i.e., $\mathcal{E} := \mathcal{F}^*$. The duality product $< e \mid f >$, that is, the linear functional $e \in \mathcal{E} = \mathcal{F}^*$ acting on $f \in \mathcal{F}$, denotes the instantaneous power transmit through the link. The space of the *port variables* is defined as $\mathcal{F} \times \mathcal{E}$.

Central in the definition of a port-Hamiltonian system is the notion of a *Dirac structure*. Basic property of a Dirac structure is power conservation: the Dirac structure links the various port (flow and effort) variables f and e in such a way that the total power $< e \mid f >$ is equal to zero.

Definition 2.9 (Dirac structure)**.** Consider a finite-dimensional linear space $\mathcal{F}$ with $\mathcal{E} = \mathcal{F}^*$. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a *Dirac structure* if

1. $< e \mid f >= 0, \text{ for all } (f, e) \in \mathcal{D}$,

2. $\dim \mathcal{D} = \dim \mathcal{F}$.

The port-Hamiltonian system consists of energy-storing elements with ports (f_S, e_S), energy-dissipating (resistive) elements with ports (f_R, e_R), and external ports (f_P, e_P) which are interconnected by a Dirac structure. The formal definition is given as follows.

Definition 2.10. Consider a state space manifold $\mathcal{X}$ and a Hamiltonian

$$H : \mathcal{X} \to \mathbb{R}, \tag{2.16}$$

defining energy-storage. A port-Hamiltonian system on $\mathcal{X}$ is defined by a Dirac structure

$$\mathcal{D} \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P, \tag{2.17}$$

having energy-storing port $(f_S, e_S) \in T_x\mathcal{X} \times T_x^*\mathcal{X}$, where $T_x\mathcal{X}$ is the tangent space of $\mathcal{X}$ at $x \in \mathcal{X}$, and a resistive structure

$$\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R, \tag{2.18}$$

corresponding to an energy-dissipating port $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$. Its dynamics are specified by

$$\big(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_R(t), e_R(t), f_P(t), e_P(t) \big) \in \mathcal{D}(x(t)), \tag{2.19}$$

$$(f_R(t), e_R(t)) \in \mathcal{R}(x(t)), t \in \mathbb{R}. \tag{2.20}$$

In the remainder, we only consider finite-dimensional port Hamiltonian systems. Under certain assumptions on the Dirac structure and the resistive relation, the port-Hamiltonian system can be written as

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned} \tag{2.21}$$

with skew-symmetric interconnection matrix $J(x) = -J^T(x)$, positive semi-definite dissipation matrix $R(x) = R^T(x) \geqslant 0$, input matrix $g(x)$, and Hamiltonian $H(x)$. The system (2.21) is called an input-state-output port-Hamiltonian system.

If H is nonnegative, then it can be verified that system (2.21) is passive [2] with respect to the input-output (u, y) with the storage function $H(x)$, i.e.,

$$\begin{aligned}\dot{H}(x(t)) &= \frac{\partial^T H}{\partial x}(x)(J(x) - R(x))\frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x)g(x)u \\ &= -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x)g(x)u \\ &\leqslant y^T u.\end{aligned} \tag{2.22}$$

This inequality expresses the basic fact that the increase of the internally stored energy (the Hamiltonian) is always less than or equal to the externally supplied power.

2.5 Equilibrium-independent passivity

Port-Hamiltonian systems are closely related to several notions of passivity. In this thesis we focus on the Equilibrium-independent passivity. The name first appeared in [40]; however the essence of this idea has been proposed by several researchers, see e.g. [43] and the references therein. This section is based on [40].

Consider a general dynamical system Σ of the form

$$\begin{aligned}\dot{x} &= f(x, u), \\ y &= h(x, u)\end{aligned} \tag{2.23}$$

with $x \in \mathcal{X} \subset \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m, y \in \mathcal{Y} \subset \mathbb{R}^m$..

Assume that there exists a set $\mathcal{U}^* \subset \mathcal{U}$ such that for every $u^* \in \mathcal{U}^*$ there exists a unique $x^* \in \mathcal{X}$ such that $f(x^*, u^*) = 0$. Define

$$k_{\mathcal{X}} : \mathcal{U}^* \to \mathcal{X} \text{ such that } x^* = k_{\mathcal{X}}(u^*) \tag{2.24}$$

and assume it to be once differentiable. We call this function the *equilibrium input-state map*.

We also define the *equilibrium input-output map*

$$k_{\mathcal{Y}}(u) : \mathcal{U}^* \to \mathcal{Y} \tag{2.25}$$

by $y^* = k_{\mathcal{Y}}(u^*) = h(k_{\mathcal{X}}(u^*), u^*)$.

Definition 2.11 ([40]). Σ is *equilibrium-independent passive* on $\mathcal{U}^*$ if for every $u^* \in \mathcal{U}^*$ there exists a once differentiable and positive definite storage function $V_{u^*} : \mathcal{X} \to \mathbb{R}$ such that $V_{u^*}(x^*) = 0$ and

$$\frac{\partial^T V_{u^*}}{\partial x}(x)f(x, u) \leqslant (u - u^*)^T(y - y^*) \tag{2.26}$$

for all $u \in \mathcal{U}, x \in \mathcal{X}$, where $y = h(x, u)$ and $y^* = k_{\mathcal{Y}}(u^*)$.

Definition 2.12 ([40]). Σ is *output strictly equilibrium-independent passive* on $\mathcal{U}^*$ if

$$\frac{\partial^T V_{u^*}}{\partial x}(x) f(x, u) \leqslant (u - u^*)^T (y - y^*) - \rho(y - y^*) \tag{2.27}$$

for some positive definite function $\rho(\cdot)$.

Lemma 2.13 ([40]). *If Σ is equilibrium-independent passive, then the equilibrium input-output map $k_{\mathcal{Y}}(u)$ is monotonically increasing.*

In Section 3.4, we will present a more general definition of equilibrium-independent passivity where we replace the equilibrium input-output map with a monotone relation. This extension is motivated by [18].

The port-Hamiltonian system (2.21) is equilibrium-independent passive if the Hamiltonian $H \in \mathcal{C}^2$ is strictly convex and $J(x) - R(x)$ is an invertible constant matrix. In this case, the equilibrium input-output map is

$$k_{\mathcal{Y}}(u) = -G^T (J - R)^{-1} G u. \tag{2.28}$$

Since $J = -J^T$ and $R \leqslant 0$, we have

$$\begin{aligned} (J - R)^{-1} + (J - R)^{-T} &= (J - R)^{-T} \big((J - R) + (J - R)^T \big) (J - R)^{-1} \\ &\leqslant 0 \end{aligned} \tag{2.29}$$

which implies that $k_{\mathcal{Y}}$ is monotonically increasing. Furthermore, the storage function V_{u^*} can be taken as $H(x) - \frac{\partial^T H}{\partial x}(x^*)(x - x^*) - H(x^*)$, where x^* satisfies $(J - R)\frac{\partial H}{\partial x}(x^*) + G u^* = 0$.

2.6 Non-smooth analysis and stability

In the rest of this chapter we give some definitions and notations regarding Filippov solutions and non-smooth stability analysis which are used in this thesis (see [24] for further information). Let X be a map from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $2^{\mathbb{R}^d}$ be the collection of all subsets of $\mathbb{R}^d$. Consider the differential equation

$$\dot{x}(t) = X(x(t)), \tag{2.30}$$

with X possibly discontinuous. We interpret the solution of (2.30) in the Filippov sense. For each $x \in \mathbb{R}^d$, the Filippov set-valued map is defined as follows.

Definition 2.14. Define the *Filippov set-valued map* of X $\mathcal{F}[X] : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ as

$$\mathcal{F}[X](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mathfrak{m}(S)=0} \overline{\text{co}}\{X(\mathcal{B}(x,\delta)\backslash S)\}, \tag{2.31}$$

where $\mathcal{B}(x,\delta)$ is the open ball centered at x with radius $\delta > 0$, S is a subset of $\mathbb{R}^d$, $\mathfrak{m}$ denotes the Lebesgue measure and co denotes convex hull.

Notice that if X is continuous at x, then $\mathcal{F}[X](x) = \{X(x)\}$.

Example 2.2. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows

$$f(x_1, x_2) = \begin{bmatrix} \text{sign}(x_2 - x_1) \\ \text{sign}(x_1 - x_2) \end{bmatrix}. \tag{2.32}$$

Denote the set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$ which has zero Lebesgue measure in $\mathbb{R}^2$. Notice that at any point $x \in \mathbb{R}^2 \setminus S$, the function f is continuous. Hence the Filippov set-valued map of f is equal to the singleton $\{f(x)\}$, i.e., $\mathcal{F}[f](x) = \{f(x)\}$. However, the function f is discontinuous at any point $x \in S$. In this case, for any $\delta > 0$, there exist points $y, z \in \mathcal{B}(x,\delta)\backslash S$ such that $f(y) = (1,-1)^T$ and $f(z) = (-1,1)^T$. Hence the Filippov set-valued map $\mathcal{F}[f](x) = \overline{\text{co}}\{(1,-1)^T, (-1,1)^T\}$ for any $x \in S$.

Definition 2.15. A *Filippov solution* of the differential equation (2.30) on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^d$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[X](x(t)) \tag{2.33}$$

for almost all $t \in [0, t_1]$.

A Filippov solution $t \to x(t)$ is *maximal* if it cannot be extended forward in time, that is, if $t \to x(t)$ is not the result of the truncation of another solution with a larger interval of definition. Since the Filippov solutions of a discontinuous system (2.30) are not necessarily unique, we need to specify two types of invariant set. A set $S \subset \mathbb{R}^d$ is *weakly invariant* for (2.33) if, for each $x_0 \in S$, S contains at least one maximal solution of (2.33) with initial condition x_0. Similarly, $S \subset \mathbb{R}^d$ is *strongly invariant* for (2.33) if, for each $x_0 \in S$, S contains all the maximal solutions of (2.33) for initial condition x_0.

A time-invariant set-valued map $F : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is upper semi-continuous (respectively, lower semi-continuous) at $x \in \mathbb{R}^d$ if, for all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that $F(y) \subset F(x) + \mathcal{B}(0, \varepsilon)$ (respectively, $F(x) \subset F(y) + \mathcal{B}(0, \varepsilon)$) for all $y \in \mathcal{B}(x, \delta)$, where $\mathcal{B}(x, \delta)$ is the open ball centered at x with radius $\delta > 0$. It is proved in [34] that the Filippov set-valued map $\mathcal{F}[X]$ is upper semi-continuous.

Some properties about computing the Filippov set-valued maps are summarized below.

Proposition 2.16. *([24])* Product Rule*: If $X_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $X_2 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_3}$ are locally bounded at $x \in \mathbb{R}^{d_1}$, then*

$$\mathcal{F}[(X_1, X_2)^T](x) \subseteq \mathcal{F}[X_1](x) \times \mathcal{F}[X_2](x). \tag{2.34}$$

Moreover, if either X_1 or X_2 is continuous at x, then equality holds.

Chain Rule*: If $X_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is continuously differentiable at $x \in \mathbb{R}^{d_1}$ with Jacobian rank d_2, and $X_2 : \mathbb{R}^{d_2} \to \mathbb{R}^{d_3}$ is locally bounded at $X_1(x) \in \mathbb{R}^{d_2}$, then*

$$\mathcal{F}[X_2 \circ X_1](x) = \mathcal{F}[X_2](X_1(x)). \tag{2.35}$$

Matrix Transformation Rule*: If $X_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is locally bounded at $x \in \mathbb{R}^{d_1}$ and $Z : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1 \times d_2}$ is continuous at $x \in \mathbb{R}^{d_1}$, then*

$$\mathcal{F}[ZX_1](x) = Z(x)\mathcal{F}[X_1](x). \tag{2.36}$$

The stability analysis of the differential inclusion (2.33) is done by the Lyapunov method with a candidate function which is possibly non-smooth. This Lyapunov function is assumed to be *regular* and locally Lipschitz in the following sense.

Definition 2.17. Let f be a map from $\mathbb{R}^d$ to $\mathbb{R}$. The *right directional derivative* $f'(x; v)$ of f at x in the direction of $v \in \mathbb{R}^d$ is defined as

$$f'(x; v) = \lim_{h \to 0^+} \frac{f(x + hv) - f(x)}{h},$$

whenever this limits exists.

Definition 2.18. Let f be a map from $\mathbb{R}^d$ to $\mathbb{R}$. The *generalized derivative* $f^o(x; v)$ of f at x in the direction of $v \in \mathbb{R}^d$ is given by

$$f^o(x; v) = \limsup_{\substack{y \to x \\ h \to 0^+}} \frac{f(y + hv) - f(y)}{h} = \lim_{\substack{\delta \to 0^+ \\ \epsilon \to 0^+}} \sup_{\substack{y \in \mathcal{B}(x, \delta) \\ h \in [0, \epsilon)}} \frac{f(y + hv) - f(y)}{h}.$$

Definition 2.19. We call a function f *regular* at x if $f'(x; v) = f^o(x; v)$ for all $v \in \mathbb{R}^d$.

Notice that a convex function is regular [23]. In order to study the evolution of the Lyapunov function along the trajectories of (2.33), we also need to introduce the definitions of *generalized gradient* and *set-valued Lie derivative*.

Definition 2.20 ([23])**.** If $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, then its *generalized gradient* $\partial f : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is defined by

$$\partial f(x) := \mathrm{co}\{\lim_{i \to \infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \cup \Omega_f\}, \tag{2.37}$$

where $\Omega_f \subset \mathbb{R}^d$ denotes the set of points where f fails to be differentiable and $S \subset \mathbb{R}^d$ is a set of measure zero that can be arbitrarily chosen to simplify the computation.

Notice that from Rademacher's Theorem [23], we have that locally Lipschitz functions are differentiable almost everywhere, i.e., $\mathfrak{m}(\Omega_f) = 0$.

Definition 2.21. Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, the *set-valued Lie derivative* of f with respect to the Filippov set-valued map $\mathcal{F}[X]$ at x is defined as

$$\tilde{\mathcal{L}}_{\mathcal{F}[X]} f(x) = \{a \in \mathbb{R} \mid \exists \nu \in \mathcal{F}[X](x) \text{ such that } \zeta^T \nu = a \text{ for all } \zeta \in \partial f(x)\}. \quad (2.38)$$

The following result is a generalization of LaSalle's invariance principle for discontinuous differential equations (2.30) with non-smooth Lyapunov functions.

Theorem 2.22 (LaSalle Invariance Principle). *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for* (2.33), *and assume that $\max \tilde{\mathcal{L}}_{\mathcal{F}[X]} f(y) \leqslant 0$ for each $y \in S$. Then, all solutions $x : [0, \infty) \to \mathbb{R}^d$ of* (2.33) *starting at S converge to the largest weakly invariant set M contained in*

$$S \cap \overline{\{y \in \mathbb{R}^d \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[X]} f(y)\}}. \quad (2.39)$$

where we define $\max \varnothing = -\infty$. Moreover, if the set M consists of a finite number of points, then the limit of each solution starting in S exists and is an element of M.

Besides the Filippov solution, another commonly used notion is that of *Krasovskii solution*.

Definition 2.23. We define the *Krasovskii set-valued map* of X $\mathcal{K}[X] : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ as

$$\mathcal{K}[X](x) \triangleq \bigcap_{\delta > 0} \overline{\text{co}}\{X(\mathcal{B}(x, \delta))\}. \quad (2.40)$$

Definition 2.24. A *Krasovskii solution* of the differential equation (2.30) on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^d$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{K}[X](x(t)) \quad (2.41)$$

for almost all $t \in [0, t_1]$.

Obviously, if X is locally bounded, any Filippov solution is also a Krasovskii solution. The following proposition provides a condition such that any Krasovskii solution is also a Filippov solution.

Proposition 2.25 ([19]). *If there exists a disjoint decomposition $\mathbb{R}^n = \cup \Omega_i$, with $\Omega_i = \overline{\text{int}\, \Omega_i}$ and continuous functions $X_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $X = X_i$ on Ω_i, then each Krasovskii solution of* (2.30) *is a Filippov solution.*

2.7 Notation

Given a vector x, the notation x_i denotes its i-th entry. Given a matrix D, the notation $D_{i,k}$ denotes its ik-th entry. The ith row and jth column of matrix D are denoted as $D_{i\cdot}$ and $D_{\cdot j}$ respectively.

For $\mu^-, \mu^+ \in \mathbb{R}^m$ the notation $\mu^- \leqslant \mu^+$ will denote element-wise inequality $\mu_i^- \leqslant \mu_i^+$, $i = 1, \ldots, m$. For $\mu^- \leqslant \mu^+$ the multidimensional saturation function $\mathrm{sat}(\cdot\,; \mu^-, \mu^+) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\mathrm{sat}(\mu\,; \mu^-, \mu^+)_i = \begin{cases} \mu_i^- & \text{if } \mu_i < \mu_i^-, \\ \mu_i & \text{if } \mu_i^- \leqslant \mu_i \leqslant \mu_i^+, \quad i = 1, \ldots, m, \\ \mu_i^+ & \text{if } \mu_i > \mu_i^+, \end{cases} \tag{2.42}$$

where μ^- and μ^+ are the vectors of lower and upper bounds of the saturation respectively. If $M(t) = \big(m_{ij}(t)\big)_{m\times n}$ is integrable, i.e., $m_{ij}(t)$ is integrable for all i, j, then $\int_a^b M(t)dt = \big(\int_a^b m_{ij}(t)dt\big)_{m\times n}$.

With $\mathbb{R}_-$, $\mathbb{R}_+$ and $\mathbb{R}_{\geqslant 0}$ we denote the sets of negative, positive and nonnegative real numbers, respectively.

Definition 2.26. We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is *sign preserving* if

(i) $y\varphi(y) > 0$ for all $y \in \mathbb{R} \setminus \{0\}$ while $\varphi(0) = 0$, and

(ii) for $\forall y \neq 0$, $\exists \varepsilon, \delta > 0$, such that for $\forall y' \in B(y, \delta)$, $|\varphi(y')| > \varepsilon$.

For the empty set, we adopt the convention that $\max \varnothing = -\infty$. The vectors $e_1, e_2, \ldots, e_n$ denote the canonical basis of $\mathbb{R}^n$.

Chapter 3

Output agreement of dynamical distribution networks with flow constraints and unknown in/outflows

3.1 Introduction

In this chapter we study a basic model for the dynamics of a distribution network. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*, which is constrained to take value in a given closed interval. Furthermore, some of the vertices serve as terminals where an unknown but constant flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. The control problem to be studied is to derive necessary and sufficient conditions for a distributed control structure (the control input corresponding to a given edge only depending on the difference of the state variables of the adjacent vertices) which will ensure that the outputs associated to all vertices will converge to the same value equal to the average of the initial condition, irrespective of the values of the constant unknown inflows and outflows.

We will first show how in the absence of constraints on the flow input variables a distributed proportional-integral (PI) controller structure, associating with every edge of the graph a controller state, will solve the problem if and only if the graph is weakly connected. This will be shown by identifying the closed-loop system as a port-Hamiltonian system, with state variables associated both to the vertices and the edges of the graph, in line with the general definition of port-Hamiltonian systems on graphs [3, 4, 5, 6]; see also [16, 77]. The proof of asymptotic load balancing will be given by modifying, depending on the vector of constant inflows and outflows, the total Hamiltonian function into a Lyapunov function. In examples the obtained PI-controller often has a clear physical interpretation, emulating the physical action of adding energy storage and damping to the edges.

The main contribution of this chapter resides in Section 3.3, where the same problem is addressed for the case of *constraints* on the flow input variables. In Section 3.3 we derive criteria, only depending on the structure of the graph and the flow constraints, to decide for what kind of in/outflows the system will reach asymptotic output agreement using the distributed PI controller proposed in Section 3.2.

The proof in Section 3.3 is based on Lyapunov theory and LaSalle's invariance principle. The Lyapunov function has a clear energy-based explanation which follows from the notion of equilibrium-independent passivity. This is discussed in Section 3.4.

The distribution network is closely connected to the theory of optimization, see e.g. [61]. In Section 3.5, we establish the relation between the dynamical distribution network with flow constraints and a pair of dual network optimization problems. This is done by extending the static input-output gains of the plant systems at the vertices and the controller systems at the edges to maximal monotone relations in a specific manner. We derive an equivalent expression of the main result in Section 3.3 from an optimization perspective.

In Section 3.6 and 3.7 we present some extensions and applications of the main result. In Section 3.6, we propose a modified PI controller which can drive the outputs of the vertices to an arbitrary feasible vector for the cases with and without flow constraints. In Section 3.7, we consider the case that the in/outflows are zero and the flow constraints are not too much different from each other, i.e., the intersection of all the flow constraints contains an open interval. It will be shown that the outputs of the vertices converge to consensus if and only if the network is weakly connected and *balanced*.

Finally, Section 3.8 contains the conclusions.

Related work can be summarized as follows. In [54], a class of cooperative control algorithms is proposed for distribution networks with time-varying exogenous in/outflows. However, the constraint intervals for the control inputs are assumed to be symmetric with respect to the origin, which turns out to be a major simplifying assumption with regard to the general constraint intervals considered in the present thesis. Furthermore, the output functions at each vertex are assumed to be linear. In [12], the main problem is the joint presence of buffer/flow capacity and of the unknown in/outflows. A discontinuous control strategy is proposed to drive the state variables, corresponding to storage at the vertices, to consensus for all constant in/outflows, by using a constrained control input of proportional type. Similarly, in [10] a saturated proportional controller is employed to achieve practical stability and optimality at steady-states. In Section III of [58], the author considered vehicles with double-integrator dynamics. In order to let all the vehicles reach consensus in position and velocity, a smoothly saturated PI controller is employed. More precisely, the input constraints, which are assumed to be between -1 and 1, are imposed on the proportional and integral part separately. The results from our previous work [68, 71] are sufficient conditions for stability of output agreement of the model we considered in this chapter. Finally, in [73] we investigated the asymptotic output agreement problem of dynamical distribution network with state constraints.

3.2 Dynamical model of the distribution network

In this section we introduce the dynamical distribution network defined on a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$. On the vertices, we consider nonlinear integrators, given as

$$\begin{aligned} \dot{x} &= u, & x, u &\in \mathbb{R}^n, \\ y &= \frac{\partial H}{\partial x}(x), & y &\in \mathbb{R}^n, \end{aligned} \tag{3.1}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of H. Here the i^{th} element of x and u, i.e. x_i and u_i, are the state and input variable associated to the i^{th} vertex of the graph respectively. System (3.1) defines a port-Hamiltonian system [2, 67], satisfying the energy-balance

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = u^T y. \tag{3.2}$$

As a next step we will extend the dynamical system (3.1) with a vector d of *inflows and outflows*

$$\begin{aligned} \dot{x} &= u + E\bar{d}, & \bar{d} &\in \mathbb{R}^k, \\ y &= \frac{\partial H}{\partial x}(x), \end{aligned} \tag{3.3}$$

with E an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or -1 (outflow), while the rest of the elements is zero. Thus E specifies the k terminal vertices where flows can enter or leave the network.

In this chapter we will regard $\bar{d}$ as a vector of constant *disturbances*, and we want to investigate control schemes which ensure asymptotic output agreement of system (3.1) irrespective of the (unknown) disturbance $\bar{d}$. By asymptotic output agreement, we mean that there exists a constant $\alpha \in \mathbb{R}$ such that

$$\frac{\partial H}{\partial x}(x(t)) \to \alpha \mathbb{1}, \text{ as } t \to \infty. \tag{3.4}$$

Furthermore, we denote the set

$$\Omega = \{x \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R}\} \tag{3.5}$$

as the output agreement set. A necessary condition for output agreement is

Assumption 3.1. There exists $\bar{x}^{oa} \in \mathbb{R}^n$ such that $\frac{\partial H}{\partial x}(\bar{x}^{oa}) \in \operatorname{span}\{\mathbb{1}\}$.

We consider the following general controllers defined on the edges of the digraph $\mathcal{G}$

$$\begin{aligned} \dot{\eta}_k &= f_k(\eta_k, \zeta_k), \\ \mu_k &= g_k(\eta_k, \zeta_k), \qquad k = 1, 2, \ldots, m \end{aligned} \tag{3.6}$$

where η_k, ζ_k, μ_k are respectively the states, input and output of the controller on the kth edge of $\mathcal{G}$. Denote the stacked vectors of η_k, ζ_k, μ_k as η, ζ, μ respectively. With the controller (3.6), the state variables $x_i, i = 1, 2, \ldots, n$, are controlled by the controller output $\mu_k, k = 1, 2, \ldots, m$, in the following manner

$$u + B\mu = 0, \tag{3.7}$$

where $B \in \mathbb{R}^{n\times m}$ is the incidence matrix of the digraph $\mathcal{G}$. In addition, the controller is driven by the relative output of the systems (3.1) on vertices, i.e

$$\zeta = B^T y \tag{3.8}$$

The closed-loop system of (3.1), (3.6), (3.7), (3.8) is given in Figure 3.1.

Note that this defines a *distributed* control scheme if H is of the form $H(x) = H_1(x_1) + \ldots + H_n(x_n)$. For the case $\bar{d} = 0$ we have $\frac{d}{dt}H = -\mu^T\zeta$.

Example 3.1 (Hydraulic network)**.** Consider a hydraulic network, modeled as a digraph with vertices corresponding to reservoirs, and edges corresponding to pipes. Let x_i be the volume of fluid stored at vertex i, and μ_j the flow through edge j. Then the mass-balance of the network is summarized as (3.1) and (3.7), i.e.,

$$\dot{x} = -B\mu + E\bar{d}, \tag{3.9}$$

where B is the incidence matrix of the graph. Let furthermore $H(x)$ denote the stored energy in the reservoirs (e.g., gravitational energy). Then $P_i := \frac{\partial H}{\partial x_i}(x), i = 1, \ldots, n$, are the *pressures* at the vertices, and the vector $\zeta = B^T \frac{\partial H}{\partial x}(x)$ is the vector whose kth element is the pressure *difference* $P_i - P_j$ across the k^{th} edge linking vertex i to vertex j. Output agreement in this case indicates that the pressures at the vertices are the same.

Example 3.2 (Thermodynamical systems)**.** Consider the thermal network, modeled as an undirected graph with vertices corresponding to masses with different temperatures, and edges corresponding to heat exchanging channels. Let x_i be the entropy of the mass i, and μ_j be the entropy flow through edge j. Then the energy balance of the masses is summarized as (3.1) and (3.7). Let $H_i(x_i) = Q_i$ denote the energy in the mass i. For thermodynamically reversible processes, $\frac{\partial H_i}{\partial x_i} = T_i$

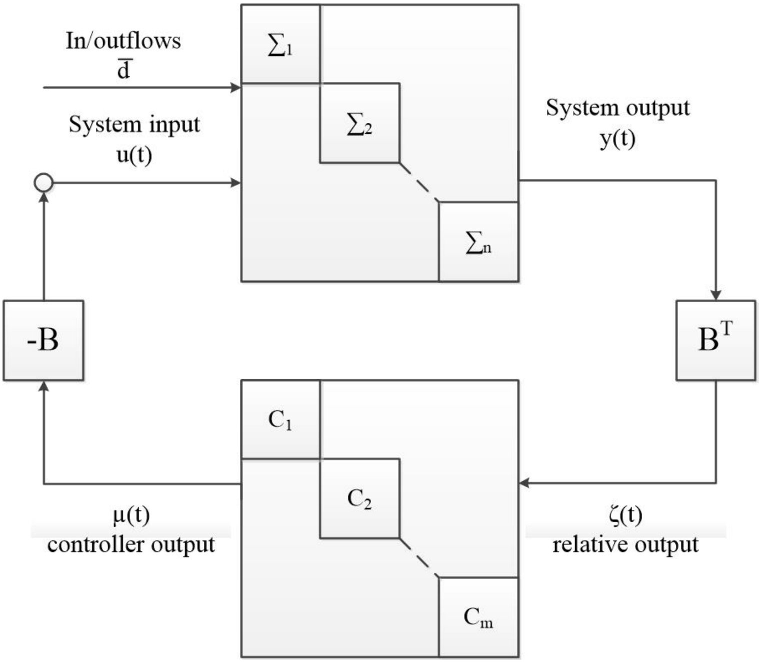


Figure 3.1: The closed-loop system of (3.1), (3.6), (3.7), (3.8) where Σ_i is the nonlinear integrator (3.1) on ith vertex, C_k is the controller (3.6) defined on kth edge, and B is the incidence matrix.

is the absolute temperature of mass i. Hence the j^{th} element of the input to the controller given as in (3.8) is the temperature *difference* $T_i - T_k$ across the j^{th} edge. If the controller Hamiltonian $H_c(\cdot) = \frac{1}{2}\|\cdot\|_2^2$, then the proportional and integral part of the controller (3.14) is the temperature difference and the accumulated temperature difference respectively.

The simplest control possibility is to apply a proportional output feedback

$$\mu = R\zeta = RB^T \frac{\partial H}{\partial x}(x), \tag{3.10}$$

where R is a diagonal matrix with strictly positive diagonal elements $r_1, \ldots, r_m$. Note that this control protocol is decentralized (see e.g.,[41]), namely the control action on each edge is only based on the vertices it affects. This control scheme leads to the closed-loop system

$$\dot{x} = -BRB^T \frac{\partial H}{\partial x}(x) + E\bar{d}. \tag{3.11}$$

In case of *zero* in/outflows $\bar{d} = 0$ this implies the energy-balance

$$\frac{d}{dt}H = -\frac{\partial^T H}{\partial x}(x) B R B^T \frac{\partial H}{\partial x}(x) \leqslant 0. \tag{3.12}$$

Hence if H is radially unbounded it follows that the trajectories of the closed-loop system (3.11) will converge to the set

$$\Omega := \{x \mid B^T \frac{\partial H}{\partial x}(x) = 0\}.$$

and thus to the output agreement set

$$\Omega = \{x \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R}\} \tag{3.13}$$

if and only if $\ker B^T = \operatorname{span}\{\mathbb{1}\}$, or equivalently [14], if and only if the graph is *weakly connected*.

In particular, for the standard Hamiltonian $H(x) = \frac{1}{2}\|x\|^2$ this means that the state variables $x_i(t), i = 1, \ldots, n$, converge to a common value α as $t \to \infty$. Since $\frac{d}{dt}\mathbb{1}^T x(t) = 0$ it follows that this common value is given as the average value $\alpha = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$.

For $\bar{d} \neq 0$, a proportional control $\mu = R\zeta$ will not be sufficient to reach load balancing, since the disturbance $\bar{d}$ can only be attenuated at the expense of increasing the gains in the matrix R. Hence we consider the *proportional-integral* (PI) control given by the dynamic output feedback [1]

$$\begin{aligned} \dot{\eta} &= \zeta, \\ \mu &= R\zeta + \tfrac{\partial H_c}{\partial \eta}(\eta), \end{aligned} \tag{3.14}$$

where $H_c(\eta)$ denotes the storage function (energy) of the controller. Indeed, we have

$$\begin{aligned} \frac{d}{dt}H_c(\eta) &= \frac{\partial^T H_c}{\partial \eta}(\eta)\dot{\eta}, \\ &\leqslant \mu^T \zeta \end{aligned} \tag{3.15}$$

which implies that the controller (3.14) is passive with respect to the input-output pair (ζ, μ). Note that this PI controller is of the same distributed nature as the static output feedback $\mu = R\zeta$.

The j^{th} element of the controller state η can be regarded as an additional state variable corresponding to the j^{th} edge of $\mathcal{G}$. Thus $\eta \in \mathbb{R}^m$, the edge space of the

[1] The same strategy and analysis for handling constant disturbances in port-Hamiltonian systems was already given in [63].

network. The closed-loop system resulting from the PI control (3.14) is given as

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -BRB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \eta}(\eta) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} \bar{d}, \tag{3.16}$$

This is again a port-Hamiltonian system[2], with total Hamiltonian $H_{\text{tot}}(x, \eta) := H(x) + H_c(\eta)$, and satisfying the energy-balance

$$\frac{d}{dt} H_{\text{tot}} = -\frac{\partial^T H}{\partial x}(x) BRB^T \frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x) E\bar{d} \tag{3.17}$$

Consider now a constant disturbance $\bar{d}$ satisfying the following assumption.

Assumption 3.2. There exists a controller state $\bar{\eta}$ such that

$$E\bar{d} = B\frac{\partial H_c}{\partial \eta}(\bar{\eta}). \tag{3.18}$$

By using Assumption 3.2, the closed-loop (3.16) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -BRB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial V_{\bar{d}}}{\partial x}(x) \\ \frac{\partial V_{\bar{d}}}{\partial \eta}(\eta) \end{bmatrix},$$

with

$$V_{\bar{d}}(x, \eta) := H(x) + H_c(\eta) - \frac{\partial^T H_c}{\partial \eta}(\bar{\eta})(\eta - \bar{\eta}) - H_c(\bar{\eta}).^3 \tag{3.19}$$

The function $V_{\bar{d}}$ will serve as a candidate Lyapunov function; leading to the following theorem.

Theorem 3.3. *Consider the system (3.3) on the graph $\mathcal{G}$ in closed loop with the PI-controller* (3.14)*. Let the constant disturbance $\bar{d}$ be such that there exists a $\bar{\eta}$ satisfying the matching equation (3.18). Assume that $V_{\bar{d}}(x, \eta)$ is radially unbounded. Then the trajectories of the closed-loop system (3.16) will converge to an element of the load balancing set*

$$\mathcal{E}_{\text{tot}} = \{(x, \eta) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1},\ \alpha \in \mathbb{R},\ B\frac{\partial H_c}{\partial \eta}(\eta) = E\bar{d}\}. \tag{3.20}$$

if and only if $\mathcal{G}$ is weakly connected.

[2]See [3, 4, 5, 6] for a general definition of port-Hamiltonian systems on graphs. The addition of a PI-controller can be also interpreted as 'control by interconnection', see e.g. [52]

[3]This function was introduced for passive systems with constant inputs in [43].

Proof. Suppose $\mathcal{G}$ is weakly connected. By (3.17) we obtain, making use of (3.18),

$$\begin{aligned}\frac{d}{dt}V_{\bar{d}} =& -\frac{\partial^T H}{\partial x}(x)BRB^T\frac{\partial H}{\partial x}(x) + \frac{\partial^T H}{\partial x}(x)E\bar{d} \\ &- \frac{\partial^T H_c}{\partial \eta}(\bar{\eta})B^T\frac{\partial H}{\partial x}(x) \\ =& -\frac{\partial^T H}{\partial x}(x)BRB^T\frac{\partial H}{\partial x}(x) \leqslant 0.\end{aligned} \tag{3.21}$$

Hence by LaSalle's invariance principle the system trajectories converge to the largest invariant set contained in

$$\{(x,\eta) \mid B^T\frac{\partial H}{\partial x}(x) = 0\}. \tag{3.22}$$

Substitution of $B^T\frac{\partial H}{\partial x}(x) = 0$ in the closed-loop system equations (3.16) yields η constant and $-B\frac{\partial H_c}{\partial \eta}(\eta) + E\bar{d} = 0$. The last equality holds because of the radial unboundedness of $V_{\bar{d}}$. Indeed, if $-B\frac{\partial H_c}{\partial \eta}(\eta) + E\bar{d} \neq 0$, we have $\|x(t)\| \to \infty$ as $t \to \infty$ within the invariant set. By the radial unboundedness of $V_{\bar{d}}$, this is a contradiction to the fact $V_{\bar{d}}$ is decreasing along the trajectories of the system (3.16). Since the graph is weakly connected $B^T\frac{\partial H}{\partial x}(x) = 0$ implies $\frac{\partial H}{\partial x}(x) = \alpha\mathbb{1}$. If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value α will be different for different components. □

Corollary 3.4. *If* $\ker B = 0$*, which is equivalent ([14]) to the graph having no* circuits, *then for every* $\bar{d}$ *there exists a unique* $\bar{\eta}$ *satisfying (3.18), and convergence is towards the set* $\mathcal{E}_{\text{tot}} = \{(x,\bar{\eta}) \mid \frac{\partial H}{\partial x}(x) = \alpha\mathbb{1}, \alpha \in \mathbb{R}, \eta = \bar{\eta}\}$.

A necessary condition for Assumption 3.2 being satisfied for all $\bar{d}$ is

$$\operatorname{im} E \subset \operatorname{im} B. \tag{3.23}$$

Furthermore, a necessary (and in case the graph is weakly connected necessary *and* sufficient) condition for the inclusion $\operatorname{im} E \subset \operatorname{im} B$ is that $\mathbb{1}^T E = 0$. In its turn $\mathbb{1}^T E = 0$ is equivalent to the fact that for every $\bar{d}$ the total inflow into the network equals to the total outflow. The condition $\mathbb{1}^T E = 0$ also implies

$$\mathbb{1}^T\dot{x} = -\mathbb{1}^T B\mu + \mathbb{1}^T E\bar{d} = 0, \tag{3.24}$$

yielding (as in the case $d = 0$) that $\mathbb{1}^T x$ is a *conserved quantity* for the closed-loop system (3.16). In particular it follows that the limit value $\lim_{t\to\infty} x(t) \in \text{span}\{\mathbb{1}\}$ is determined by the initial condition $x(0)$.

Corollary 3.5. *In case of the standard quadratic Hamiltonians* $H(x) = \frac{1}{2}\|x\|^2$, $H_c(\eta) = \frac{1}{2}\|\eta\|^2$ *there exists for every* $\bar{d}$ *a controller state* $\bar{\eta}$ *such that (3.18) holds if and only if*

$$\operatorname{im} E \subset \operatorname{im} B. \tag{3.25}$$

Furthermore, in this case $V_{\bar{d}}$ *equals the radially unbounded function* $\frac{1}{2}\|x\|^2 + \frac{1}{2}\|\eta - \bar{\eta}\|^2$, *while convergence will be towards the load balancing set* $\mathcal{E}_{\text{tot}} = \{(x, \eta) \mid x = \alpha\mathbb{1}, \alpha \in \mathbb{R},\ B\eta = E\bar{d}\}$.

Example 3.3 (Hydraulic network continued)**.** The proportional part $R\zeta$ of the controller corresponds to adding *damping* to the dynamics (proportional to the pressure differences along the edges). The integral part of the controller has the interpretation of adding *compressibility* to the hydraulic network dynamics. Using this emulated compressibility, the PI-controller is able to regulate the hydraulic network to a load balancing situation where all pressures P_i are equal, irrespective of the constant inflow and outflow $\bar{d}$ satisfying the matching condition (3.18). Note that for the Hamiltonian $H(x) = \frac{1}{2}\|x\|^2$ the pressures P_i are equal to each other if and only if the fluid levels x_i are equal.

In this example we show the simulation of the closed-loop system (3.16), where $H(x) = \frac{1}{2}\|x\|^2$, $H_c(\eta) = \frac{1}{2}\|\eta\|^2$ and $R = I$. The underlying network is given as in Figure 3.2 and the in/outflows $\bar{d}$ satisfy $\mathbb{1}^T E\bar{d} = 0$. The time evolutions of the state $x_1(t), \ldots, x_5(t)$ of the closed-loop are shown in Figure 3.3.

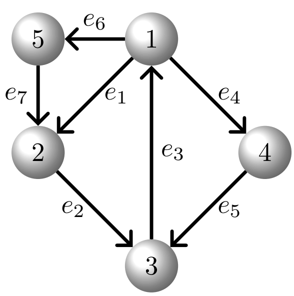


Figure 3.2: Network used in the examples of this chapter

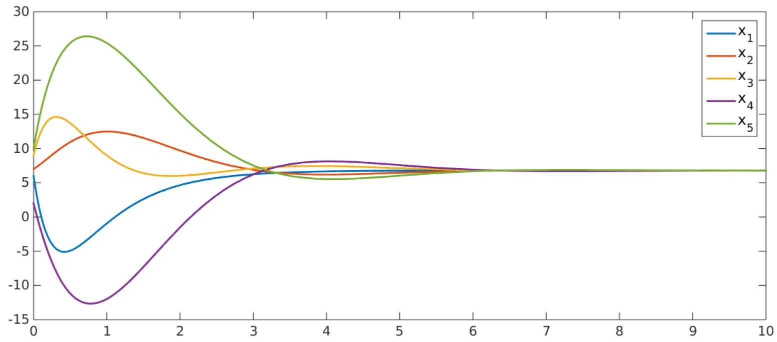


Figure 3.3: The trajectories of the state variables of dynamical system in Example 3.3.

3.3 Dynamical distribution network with constrained flows

3.3.1 Main results

In many practical cases, the elements of the vector of flows $\mu \in \mathbb{R}^m$ corresponding to the edges of the graph will be *constrained*, that is

$$\mu \in \mathcal{M} := \{\mu \in \mathbb{R}^m \mid \mu^- \leqslant \mu \leqslant \mu^+\} \tag{3.26}$$

for certain vectors μ^- and μ^+ satisfying $\mu_i^- \leqslant \mu_i^+, i = 1, \ldots, m$. A general constrained version of the PI controller is given as

$$\begin{aligned} \dot{\eta} &= \zeta, \\ \mu &= \operatorname{sat}\left(\frac{\partial H_c}{\partial \eta}(\eta) + R\zeta\,; \mu^-, \mu^+\right) \end{aligned} \tag{3.27}$$

Before we continue with constrained flows, we assume throughout this section that the controller Hamiltonian H_c satisfies the following assumption.

Assumption 3.6. The controller Hamiltonian H_c is convex and $\{\frac{\partial H_c}{\partial \eta}(\eta) \mid \eta \in \mathbb{R}^m\} = \mathbb{R}^m$.

Notice that the previous assumption is sufficient to guarantee the radial unboundedness of $H_c(\eta) - \frac{\partial^T H_c}{\partial \eta}(\bar{\eta})(\eta - \bar{\eta}) - H_c(\bar{\eta})$ for any $\bar{\eta} \in \mathbb{R}^m$. Notice that the controller (3.27) is different from the one in [58] where the saturation is imposed separately on the proportional and integral part of the controller. Moreover, the constrained controller (3.27) does not satisfy *incremental observability* as defined in [55]. Hence the methodology in [55] for output synchronization is not applicable here.

The closed-loop system resulting from the interconnection of (3.3), (3.7), (3.8) with the constrained PI controller (3.27) is given as

$$\begin{aligned} \dot{x} &= -B\,\mathrm{sat}\left(RB^T\tfrac{\partial H}{\partial x}(x) + \tfrac{\partial H_c}{\partial \eta}(\eta)\,;\mu^-,\mu^+\right) + E\bar{d}, \\ \dot{\eta} &= B^T\tfrac{\partial H}{\partial x}(x), \end{aligned} \tag{3.28}$$

The problem studied in the present section concerns the following question. Given arbitrary (but fixed) flow constraints, for what kind of in/outflows does the closed-loop system(3.28) achieve asymptotic output agreement.

Similar to Assumption 3.2 in the unconstrained case, if output agreement is achieved for (3.28) then the constrained controller needs to satisfy the following condition.

Assumption 3.7. There exists a vector $z^{\bar{d}} \in \mathbb{R}^m$ such that $z^{\bar{d}} \in \mathcal{M}$ and $E\bar{d} = Bz^{\bar{d}}$.

For any vector $z \in \mathcal{M}$, we define the subset $\bar{\mathcal{E}}(z;\mu^-,\mu^+)$ of $\mathcal{E}$ as

$$\bar{\mathcal{E}}(z;\mu^-,\mu^+) = \{e_i \in \mathcal{E} \mid z_i \in (\mu_i^-,\mu_i^+)\}. \tag{3.29}$$

Furthermore, if $z \in \mathcal{M} \cap \ker B$, we denote $\bar{\mathcal{E}}(z;\mu^-,\mu^+)$ as $\mathcal{E}_0(z;\mu^-,\mu^+)$ and $\mathcal{E} \setminus \mathcal{E}_0(z;\mu^-,\mu^+)$ as $\mathcal{E}_1(z;\mu^-,\mu^+)$. The subgraphs $\bar{\mathcal{G}}(z;\mu^-,\mu^+)$ and $\mathcal{G}_0(z;\mu^-,\mu^+)$ are defined as $\{\mathcal{V},\bar{\mathcal{E}}(z;\mu^-,\mu^+)\}$ and $\{\mathcal{V},\mathcal{E}_0(z;\mu^-,\mu^+)\}$ respectively. We omit the constraint intervals $[\mu^-,\mu^+]$ from the previous notations if they are unambiguous from the context.

The following condition on the in/outflows is stronger than the one in Assumption 3.7, and will turn out to be crucial for formulating the main results of this paper.

Definition 3.8 (Manageable in/outflows)**.** For a given network with multi-dimensional constraint interval $[\mu^-,\mu^+]$ for the edges, we say that the vector of in/outflows $E\bar{d}$ is *manageable* with respect to $[\mu^-,\mu^+]$ if there exists $z^{\bar{d}} \in \mathcal{M}$ such that $E\bar{d} = Bz^{\bar{d}}$ and the subgraph $\bar{\mathcal{G}}(z^{\bar{d}};\mu^-,\mu^+)$ is weakly connected.

The intuitive explanation of *manageable* in/outflows is explained in the following example.

Example 3.4. Consider the production-distribution network as formulated by system (3.1) and (3.7). In this example the Hamiltonians are given as $H(x) = \frac{1}{2}\|x\|^2$ and $H_c(\eta) = \frac{1}{2}\|\eta\|^2$. The vertices of the network, representing the system warehouses, are fed by the flows of the incident edges. Thus, $x(t)$ represents the amount of resources in the warehouses. These resources are raw materials, intermediate and finished products, as well as any other resource used in the production processes. $\mu(t)$ is the vector of controlled resource flows between warehouses, and $E\bar{d}$ represents the vector of *supplies* and *demands* depending on external factors where the positive components of $E\bar{d}$ correspond to supplies and the negative ones correspond to demands. As follows from Theorem 3.3, the controller (3.14) is able to let the storage of the resources among all the warehouses reach consensus with the supply/demand satisfying Assumption 3.2. In many practical cases, the flow μ is bounded. Hence the supply/demand can not be arbitrarily large. Instead, the notion of *manageability* is introduced. Intuitively, the network can provide the manageable supply/demand without violating the flow constraints while it has enough flow capacity to regulate the storage among the warehouses. More precisely, the network with *manageable* in/outflows is able to, first transfer the in/outflows through the network, and second have enough spare capacity (on a weakly connected subnetwork) to regulate the storage.

The following theorem is the main result of this chapter.

Theorem 3.9. *Consider the closed-loop system* (3.28) *defined on a weakly connected directed graph with flow constraints* $[\mu^-, \mu^+]$ *on the edges. Suppose the Hamiltonian* $H(x) \in \mathcal{C}^1$ *is radially unbounded and the controller Hamiltonian* H_c *satisfies Assumption 3.6. Then the trajectories converge to*

$$\mathcal{E}_{\text{tot}} = \{(\bar{x}, \bar{\eta}) \mid \frac{\partial H}{\partial x}(\bar{x}) = \alpha \mathbb{1}_n,\ B\,\text{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta})\,; \mu^-, \mu^+) = E\bar{d}\}, \tag{3.30}$$

if and only if the in/outflows $E\bar{d}$ *are manageable.*

Instead of proving the theorem directly, we shall prove an equivalent formulation of it.

Definition 3.10 (Interior Point Condition)**.** Given a directed graph with multi-dimensional constraint interval $[\mu^-, \mu^+]$, the network will be said to satisfy the Interior Point Condition if there exists a vector $z \in \mathcal{M} \cap \ker B$ such that the subgraph $\mathcal{G}_0(z)$ is weakly connected.

Denote the subset $\mathcal{E}' \subset \mathcal{E}$ as

$$\mathcal{E}' = \{e_i \mid \mu_i^- < \mu_i^+\}. \tag{3.31}$$

Notice that satisfaction of the Interior Point Condition implies that the subgraph $\{\mathcal{V}, \mathcal{E}'\}$ is weakly connected.

Lemma 3.11. *Given a network with multi-dimensional constraint interval* $[\mu^-, \mu^+]$ *and in/outflows* $E\bar{d}$*. Then* $E\bar{d}$ *is manageable if and only if for any* w *satisfying* $E\bar{d} = Bw$*, the network with multi-dimensional constraint interval* $[\mu^- - w, \mu^+ - w]$ *satisfies the Interior Point Condition.*

Proof. Sufficiency: Suppose the network with constraint interval $[\mu^- - w, \mu^+ - w]$ satisfies the interior point condition, i.e., there exists $z \in [\mu^- - w, \mu^+ - w] \cap \ker B$ such that the subgraph $\{\mathcal{V}, \mathcal{E}_0(z; \mu^- - w, \mu^+ - w)\}$ is weakly connected. This implies that the network with constraint interval $[\mu^- - w - z, \mu^+ - w - z]$ satisfies the interior point condition with respect to the zero vector. Hence $w + z \in [\mu^-, \mu^+]$, and the subgraph $\{\mathcal{V}, \bar{\mathcal{E}}(w + z, \mu^-, \mu^+)\}$ is weakly connected. Since $E\bar{d} = B(w + z) = Bw$, this implies that $E\bar{d}$ is manageable.

Necessity: Let $E\bar{d}$ be manageable with $z^{\bar{d}}$ such that $Bz^{\bar{d}} = E\bar{d}$ and $z^{\bar{d}} \in \mathcal{M}$. Then it is straightforward to see that the network with constraint interval $[\mu^- - z^{\bar{d}}, \mu^+ - z^{\bar{d}}]$ satisfies the interior point condition with zero vector z. Then for any w such that $E\bar{d} = Bw$, $w = z^{\bar{d}} + w_0$ for some $w_0 \in \ker B$. Hence the network with constraint interval $[\mu^- - w, \mu^+ - w]$ satisfies the interior point condition with $-w_0$. □

Next we will present an equivalent statement of Theorem 3.9. For any vector of in/outflows $E\bar{d}$ satisfying Assumption 3.7 for a vector $z^{\bar{d}}$, the closed-loop system (3.28) can be rewritten as

$$\begin{aligned}\dot{x} &= -B \operatorname{sat}\big(B^T \frac{\partial H}{\partial x}(x) + \frac{\partial \tilde{H}_c}{\partial \eta}(\eta)\,; \mu^- - \frac{\partial H_c}{\partial \eta}(\bar{\eta}^{\bar{d}}), \mu^+ - \frac{\partial H_c}{\partial \eta}(\bar{\eta}^{\bar{d}})\big),\\ \dot{\eta} &= B^T \frac{\partial H}{\partial x}(x),\end{aligned} \tag{3.32}$$

where $\frac{\partial H_c}{\partial \eta}(\bar{\eta}^{\bar{d}}) = z^{\bar{d}}$ and $\tilde{H}_c(\eta) = H_c(\eta) - \frac{\partial^T H_c}{\partial \eta}(\bar{\eta}^{\bar{d}})(\eta - \bar{\eta}^{\bar{d}})$. Furthermore, by Lemma 3.11, the vector of in/outflows $E\bar{d}$ being manageable is equivalent to the constraint interval $[\mu^- - \frac{\partial H_c}{\partial \eta}(\bar{\eta}^{\bar{d}}), \mu^+ - \frac{\partial H_c}{\partial \eta}(\bar{\eta}^{\bar{d}})]$ satisfying the interior point condition. Hence the system (3.28) with manageable in/outflows $E\bar{d}$ is equivalent to system (3.28) with $E\bar{d} = 0$ and constraint interval $[\mu^-, \mu^+]$ satisfying the interior point condition. As a consequence, Theorem 3.9 is equivalent to the following theorem.

Theorem 3.12. *Consider the dynamical system (3.28) defined on a weakly connected directed graph with flow constraint interval $[\mu^-, \mu^+]$ and in/outflows $E\bar{d} = 0$. Suppose the Hamiltonian $H(x) \in \mathcal{C}^1$ is radially unbounded, then the trajectories will converge to*

$$\mathcal{E}_{\text{tot}} = \{(\bar{x}, \bar{\eta}) \mid \frac{\partial H}{\partial x}(\bar{x}) = \alpha \mathbb{1}_n,\ B \operatorname{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta})\,; \mu^-, \mu^+) = 0\}, \tag{3.33}$$

if and only if the network satisfies the interior point condition.

The proof of Theorem 3.12 will be given in the next section.

3.3.2 Convergence analysis

In this section we shall prove Theorem 3.12 for (3.28) with in/outflows $E\bar{d} = 0$, i.e.,

$$\begin{aligned} \dot{x} &= -B\,\mathrm{sat}\left(RB^T \tfrac{\partial H}{\partial x}(x) + \tfrac{\partial H_c}{\partial \eta}(\eta)\,;\mu^-,\mu^+\right), \\ \dot{\eta} &= B^T \tfrac{\partial H}{\partial x}(x). \end{aligned} \tag{3.34}$$

In order to simplify the structure of the proof, we will assume throughout the rest of this section that

$$\mu_i^+ \geqslant \mu_i^- \geqslant 0, \quad i = 1, 2, \ldots, m, \tag{3.35}$$

where the two equality signs do not hold at the same time. We will say that the orientation of the graph is *compatible* with the flow constraints if (3.35) holds. The condition (3.35) can be assumed without loss of generality by considering two types of modifications of the network.

(i) Replacing any *bi-directional* edge whose constraint interval satisfies $\mu_i^- < 0 < \mu_i^+$ by two *uni-directional* edges with constraint intervals $[\mu_i^-, 0]$, $[0, \mu_i^+]$ and the same orientation. This follows from the equality

$$\mathrm{sat}(\nu\,;\mu_i^-, \mu_i^+) = \mathrm{sat}(\nu\,;\mu_i^-, 0) + \mathrm{sat}(\nu\,;0, \mu_i^+) \tag{3.36}$$

for any $\mu_i^- < 0 < \mu_i^+$. Replacing the bi-directional edges by the uni-directional ones does not change the dynamics of x in system (3.34). Hence, the output agreement property of (3.34) is not affected. This will be illustrated by the following example.

Example 3.5. (Compatibility between orientation and flow constraints) Consider the system (3.34) defined on the graph in Fig. 3.4 (left) with constraint interval $[-1, 2]$ and initial condition $(x_1(0), x_2(0), \eta(0))^T \in \mathbb{R}^3$. The dynamic of $x_1(t)$ and $x_2(t)$ are the same as the ones of the system defined on the graph in Fig.3.4 (middle) with constraint intervals $[-1, 0]$ and $[0, 2]$ on the edge e_1^1 and e_1^2 respectively, and initial condition $(x_1(0), x_2(0), \eta(0), \eta(0))^T \in \mathbb{R}^4$.

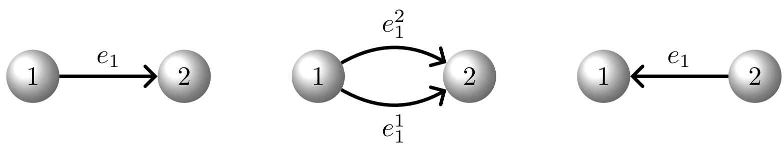


Figure 3.4: Compatibility between orientation and flow constraints

(ii) Changing the orientation of the graph. This follows from the equality

$$\mathrm{sat}(-\nu\,; \mu_i^-, \mu_i^+) = -\,\mathrm{sat}(\nu\,; -\mu_i^+, -\mu_i^-). \tag{3.37}$$

Changing the orientation of the graph does not change the dynamics of the state variables x in system (3.34) as illustrated by the following example.

Example 3.6. (Compatibility between orientation and flow constraints continued) Consider the dynamical system (3.34) defined on a graph with constraint interval $[\mu^-, \mu^+]$ and initial condition $(x_1(0), x_2(0), \eta(0))$. Then the dynamics is the same as the dynamics of the following system

$$\begin{aligned} \dot{x} &= -\tilde{B}\,\mathrm{sat}\left(R\tilde{B}^T \tfrac{\partial H}{\partial x}(x) + \tfrac{\partial \tilde{H}_c}{\partial \tilde{\eta}}(\tilde{\eta})\,; -\mu^+, -\mu^-\right), \\ \dot{\eta} &= \tilde{B}^T \tfrac{\partial H}{\partial x}(x). \end{aligned} \tag{3.38}$$

where $\tilde{H}_c(\tilde{\eta}) = H_c(-\tilde{\eta})$ and $\tilde{B} = -B$ with initial condition $(x_1(0), x_2(0), -\eta(0))$. In other words, by modifying the controller Hamiltonian H_c and initial condition, we can reverse the orientations of the edges such that the constraints intervals are $[-\mu^+, -\mu^-]$. Notice that $\tilde{H}_c$ satisfies Assumption 3.6 if and only if H_c does.

After splitting bi-directional edges into uni-directional edges, and changing orientations, we can therefore assume, without loss of generality, that the orientation of the graph is compatible with the flow constraints.

Next, we note that if $\mathcal{G}$ is strongly connected, then it must contain positive circuits. Indeed, by the definition of strong connectedness, for any two vertices v_i and v_j, there exists a directed path from v_i to v_j. Similarly, there exists a directed path from v_j to v_i. Then these two directed paths must contain a positive circuit. Hence any circuit can be written as a linear combination of positive circuits, with possibly non-positive coefficients. More precisely, consider a circuit $\mathcal{C}$ with $\xi_\mathcal{C} \notin \mathbb{R}^m_{\geqslant 0}$ nor $\mathbb{R}^m_{\leqslant 0}$. Without loss of generality, suppose $\xi_{\mathcal{C}_k} < 0$ where $e_k = (v_i, v_j)$ while the other components of $\xi_\mathcal{C}$ are nonnegative. By the strong connectedness of $\mathcal{G}$ there exists a directed path from v_j to v_i denoted as $\mathcal{P}$. Then $\mathcal{P}$ with the edge e_k and with the rest of edges from $\mathcal{C}$ can form two positive circuits, denoted as $\mathcal{C}'$ and $\mathcal{C}''$, satisfying $\xi_{\mathcal{C}'}, \xi_{\mathcal{C}''} \in \mathbb{R}^m_{\geqslant 0}$ respectively. Then $\xi_\mathcal{C} = \xi_{\mathcal{C}''} - \xi_{\mathcal{C}'}$. The previous analysis leads to the following lemma.

Lemma 3.13 ([14]). *For a strongly connected digraph $\mathcal{G}$, the positive circuits compose a basis of the space* $\ker B$.

Let us denote all the positive circuits of $\mathcal{G}$ as $\mathcal{PC} = \{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ with $\xi_{\mathcal{C}_i} \in \mathbb{R}^m_{\geqslant 0}$.

Lemma 3.14. *The vector z in the interior point condition can be chosen such that $\mathcal{E}_0(z) = \mathcal{E}'$.*

Proof. Recall the notations

$$\mathcal{E}' = \{e_i \in \mathcal{E} \mid \mu_i^- < \mu_i^+\}$$

and

$$\mathcal{E}_0(z;\mu^-,\mu^+) = \{e_i \in \mathcal{E} \mid z_i \in (\mu_i^-,\mu_i^+)\}.$$

where $z \in \mathcal{M} \cap \ker B$. Furthermore $\mathcal{E}_1(z;\mu^-,\mu^+) := \mathcal{E} \setminus \mathcal{E}_0(z;\mu^-,\mu^+)$.

Let $z \in \mathcal{M} \cap \ker B$ be such that $\mathcal{G}_0(z)$ is weakly connected, and $\mathcal{E}_1(z) \neq \varnothing$. If the edge $e_k \in \mathcal{E}_1(z;\mu^-,\mu^+)$ and $z_k = \mu_k^-$, then, since the subgraph $\mathcal{G}_0(z)$ is weakly connected, the edge e_k together with some of the edges belonging to $\mathcal{E}_0(z)$ compose a circuit, denoted as $\mathcal{C}$. Furthermore, take the vector $\xi_{\mathcal{C}}$ as the representation of $\mathcal{C}$ such that $\xi_{\mathcal{C}k} = 1$. Then there exists a sufficient small $\varepsilon > 0$ such that $\mathcal{E}_0(z + \varepsilon\xi_{\mathcal{C}}) = \mathcal{E}_0(z) \cup \{e_k\}$. Hence after a finite number of steps, there exists a vector $z' \in \mathcal{M} \cap \ker B$ such that $\mathcal{E}_0(z') = \mathcal{E}'$. □

Remark 3.15. By using the previous lemma and Lemma 3.11, the notion of manageable in/outflows coincides with Assumption 1 in [12] which is given as follows

$$E\bar{d} \in \operatorname{int} B\mathcal{M}. \tag{3.39}$$

In view of the previous lemma, we assume in the rest of this chapter that the vector z in the interior point condition satisfies $\mathcal{E}_0(z) = \mathcal{E}'$. Furthermore, by Lemma 2.5

$$z = \sum_{i=1}^{r} \alpha_i \xi_{\mathcal{C}_i}, \tag{3.40}$$

with $\alpha_i \geqslant 0$. This implies

Lemma 3.16. *Let $\mathcal{G}$ be a weakly connected directed graph with compatible constraint intervals $[\mu^-,\mu^+]$. Then $\mathcal{G}$ is strongly connected if it satisfies the interior point condition.*

The main tool in the proof of Theorem 3.12 is LaSalle's invariance principle which asks for a compact invariant set. One option for providing such a compact invariant set is proposed in the following lemma.

Lemma 3.17. *Suppose H is radially unbounded and H_c satisfies Assumption 3.6. Consider the function*

$$V(x,\eta) = \mathbb{1}^T S(\eta) + H(x), \tag{3.41}$$

where

$$S_i(\eta_i) = \int_0^{\eta_i} \operatorname{sat}(\frac{\partial H_c}{\partial \eta_i}(\tau)\,;\mu_i^-,\mu_i^+)d\tau. \tag{3.42}$$

If the network satisfies the interior point condition, then the set $\mathcal{S} = \{(x,\eta) \mid -D_i \leqslant \xi_{\mathcal{C}_i}^T\eta \leqslant D_i, i = 1,\dots,r\} \cap \{(x,\eta) \mid V(x,\eta) \leqslant D_{r+1}\}$, where $D_i, i = 1,\dots,r+1$, are *constants such that $\mathcal{S} \neq \varnothing$, is compact and forward invariant for* (3.34).

Proof. Denote the projections of $\mathcal{S}$ on $\mathbb{R}^n$ and $\mathbb{R}^m$ as

$$\mathcal{S}_\eta = \{\eta \mid \exists x \text{ such that } (x,\eta) \in \mathcal{S}\}, \tag{3.43}$$

$$\mathcal{S}_x = \{x \mid \exists \eta \text{ such that } (x,\eta) \in \mathcal{S}\}. \tag{3.44}$$

First we notice that for any vector $z \in \mathcal{M} \cap \ker B$ satisfying Lemma 3.14, the function $z^T\eta : \mathbb{R}^m \to \mathbb{R}$ is bounded on $\mathcal{S}_\eta$. This can be seen from (3.40).

Since $\mathcal{S}$ is closed by definition, the compactness of $\mathcal{S}$ is equivalent to boundedness. We will prove this by showing that the function $V(x,\eta)$ is radially unbounded on $\mathcal{S}$ as follows.

Firstly, observe that for any $(x,\eta) \in \mathcal{S}$ when $\|\eta\| \to \infty$, there must exist at least one edge $e_i \in \mathcal{E}'$ such that $|\eta_i| \to \infty$. Otherwise, suppose $|\eta_j| \to \infty$ only on the edge $e_j \in \mathcal{E} \setminus \mathcal{E}'$. By definition of $\mathcal{S}$, along any positive circuit $\mathcal{C}_k$ containing the edge e_j, there exists at least one edge belonging to $(\mathcal{E} \setminus \mathcal{E}') \setminus \{e_j\}$. This implies that the set $\mathcal{E} \setminus \mathcal{E}'$ contains a cut of the graph. Indeed, if we denote $e_j \sim (v_p, v_q)$, then there will be no path from v_q to v_p only using the edges from $\mathcal{E}'$. This is a contradiction with the fact that $\{\mathcal{V}, \mathcal{E}'\}$ is weakly connected.

Secondly, note that if $|\eta_i| \to \infty$, then by Assumption 3.6

$$S_i(\eta_i) = \begin{cases} \mu_i^- \eta_i & \text{if } \eta_i < 0, \\ \mu_i^+ \eta_i & \text{if } \eta_i > 0, \end{cases} \tag{3.45}$$

up to a constant. If $e_i \in \mathcal{E}'$, then $(\mu_i^- - z_i)\eta_i \to \infty$ whenever $\eta_i \to -\infty$ and $(\mu_i^+ - z_i)\eta_i \to \infty$ whenever $\eta_i \to \infty$. Hence $\mathbb{1}^T S(\eta) - z^T\eta \to \infty$ as $\|\eta\| \to \infty$. Furthermore, by the fact that $H(x)$ is bounded from below on $\mathbb{R}^n$ and the boundedness of the function $z^T\eta$, the function $V(x,\eta)$ tends to infinity as $\|\eta\| \to \infty$. Hence the set $\mathcal{S}_\eta$ is bounded.

Finally, the boundedness of $\mathcal{S}_\eta$ implies that the function $\mathbb{1}^T S(\eta)$ is bounded from below on $\mathcal{S}$. Hence the boundedness of the set $\mathcal{S}_x$ can be derived by the radial unboundedness of $H(x)$. In conclusion, the set $\mathcal{S}$ is bounded, hence compact.

To prove the forward invariance of the set $\mathcal{S}$, we notice that for (3.34) we have $\xi_{\mathcal{C}_i}^T \dot{\eta} = 0, i = 1, \dots, r$. Hence $\xi_{\mathcal{C}_i}^T \eta$ is constant along the trajectories of (3.34). Furthermore, $\dot{V} = \frac{\partial^T H}{\partial x} B\big(\operatorname{sat}(\frac{\partial H_c}{\partial \eta}; \mu^-, \mu^+) - \operatorname{sat}(RB^T \frac{\partial H}{\partial x} + \frac{\partial H_c}{\partial \eta}; \mu^-, \mu^+)\big) \leqslant 0$. Hence $\mathcal{S}$ is forward invariant.

□

We are ready to prove the main theorem.

Proof of Theorem 3.12. Sufficiency:

Suppose the network satisfies the interior point condition with vector $z \in \mathcal{M} \cap \ker B$. Consider (3.41) as a Lyapunov function. By Lemma 3.17 and LaSalle's invariance principle, it follows that $(x(t), \eta(t))$ converges to the largest invariant set $\mathcal{I}$ contained in

$$\{(x,\eta) \mid \dot{V} = 0\} = \{(x,\eta) \mid \operatorname{sat}(\frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+) = \operatorname{sat}(RB^T \frac{\partial H}{\partial x} + \frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+)\}.$$

We claim that in $\mathcal{I}$ the output of the controller, i.e., $\operatorname{sat}(RB^T \frac{\partial H}{\partial x} + \frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+) = \operatorname{sat}(\frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+)$ is constant. Indeed, if not, then suppose there exists $t_1 < t_2$ such that $\operatorname{sat}(\frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+) = \operatorname{sat}(RB^T \frac{\partial H}{\partial x} + \frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+)$ is time-varying on the time interval $[t_1, t_2]$. Then by the continuity of $\frac{\partial H}{\partial x}$ and $\frac{\partial H_c}{\partial \eta}$, it must hold that $\frac{\partial H_c}{\partial \eta}(\eta(t)) = RB^T \frac{\partial H}{\partial x}(x(t)) + \frac{\partial H_c}{\partial \eta}(\eta(t)) \in \mathcal{M}$ for $t \in [t_1, t_2]$. Hence $B^T \frac{\partial H}{\partial x}(x(t)) = 0$ which implies that $\eta(t)$ is constant on $[t_1, t_2]$. Then the vector $\frac{\partial H_c}{\partial \eta}(\eta(t))$ is constant as well on $[t_1, t_2]$, which yields a contradiction.

For the trajectories contained in $\mathcal{I}$, let us denote the output of the controller as $\bar{\mu} = \operatorname{sat}(RB^T \frac{\partial H}{\partial x} + \frac{\partial H_c}{\partial \eta}\,; \mu^-, \mu^+)$ which is constant. Then $x(t)$ is also a constant vector, denoted as $\bar{x}$. Indeed, since $\dot{x}(t) = -B\bar{\mu}$ with $\bar{\mu}$ constant, if $\dot{x}(t) \neq 0$, we would have $\|x\| \to \infty$, which is a contradiction to the boundedness of the trajectories. So $x(t) = \bar{x}$ and $B\bar{\mu} = 0$. Then $\dot{\eta} = B^T \frac{\partial H}{\partial x}(\bar{x})$ is constant. This implies that $B^T \frac{\partial H}{\partial x}(\bar{x})$ has to be zero by Lemma 3.17. In conclusion, for the weakly connected network the largest invariant set is given as

$$\mathcal{I} = \{(\bar{x}, \bar{\eta}) \mid \frac{\partial H}{\partial x}(\bar{x}) = \alpha \mathbb{1}_n,\ B \operatorname{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta})\,; \mu^-, \mu^+) = 0\}.$$

Necessity:

First of all, if $\mathcal{M} \cap \ker B = \varnothing$, then the dynamical system (3.34) does not have any equilibrium. Suppose now the network does not satisfy the interior point condition, i.e., for any vector $z \in \mathcal{M} \cap \ker B$ the subgraph $\mathcal{G}_0(z)$ is not weakly connected. In this case, we will show that the outputs of the dynamical system (3.34) will *cluster*, by selecting suitable initial conditions $(x(0), \eta(0))$ such that $B^T \frac{\partial H}{\partial x}(x(0)) \neq 0$ and $\dot{x}(t) = 0, t \geqslant 0$. 'Clustering' means that the outputs break up in groups converging to different values.

Consider the map $|\mathcal{E}_0(\cdot)| : \mathcal{M} \cap \ker B \to \{1, 2, \ldots, m\}$. Suppose $|\mathcal{E}_0(\cdot)|$ attains its maximum at a vector $\bar{z}$. The first observation about $\bar{z}$ that is for any positive circuit $\mathcal{C}$ satisfying $\mathcal{C} \cap \mathcal{E}_1(\bar{z}) \neq \varnothing$, there exist $e_i, e_j \in \mathcal{C}$ such that $\bar{z}_i = \mu_i^+$ and $\bar{z}_j = \mu_j^-$ where i and j are possibly equal. Indeed, if there exists a positive circuit $\mathcal{C}$ satisfying $\mathcal{C} \cap \mathcal{E}_1(\bar{z}) \neq \varnothing$ and $\mu_i^- < \bar{z}_i \leqslant \mu_i^+$ for any $e_i \in \mathcal{C}$, then there exists a sufficient small $\varepsilon > 0$ such that $\mathcal{E}_0(\bar{z} - \varepsilon\xi_{\mathcal{C}}) = \mathcal{E}_0(\bar{z}) \cup \mathcal{C}$ where $\xi_{\mathcal{C}} \in \mathbb{R}^m_{\geqslant 0}$. Hence $\mathcal{E}_1(\bar{z} - \varepsilon\xi_{\mathcal{C}}) \cap \mathcal{C} = \varnothing$ and $|\mathcal{E}_0(\bar{z} - \varepsilon\xi_{\mathcal{C}})| > |\mathcal{E}_0(\bar{z})|$. Similarly, if there exists a positive circuit $\mathcal{C}$ satisfying $\mathcal{C} \cap \mathcal{E}_1(\bar{z}) \neq \varnothing$ and $\mu_i^- \leqslant \bar{z}_i < \mu_i^+$ for any $e_i \in \mathcal{C}$, we have $|\mathcal{E}_0(\bar{z} + \varepsilon\xi_{\mathcal{C}})| > |\mathcal{E}_0(\bar{z})|$ for small enough $\varepsilon > 0$ and $\xi_{\mathcal{C}} \in \mathbb{R}^m_{\geqslant 0}$.

The second observation about the vector $\bar{z}$ is that, along any circuit $\mathcal{C}$ satisfying

$\mathcal{C} \cap \mathcal{E}_1(\bar{z}) \neq \varnothing$ and $\xi_{\mathcal{C}_i} = \xi_{\mathcal{C}_j}$ for any $e_i, e_j \in \mathcal{C} \cap \mathcal{E}_1(\bar{z})$, at least one component of $\bar{z}$ on the edge in $\mathcal{C} \cap \mathcal{E}_1(\bar{z})$ is equal to the upper saturation boundary and at least one component of $\bar{z}$ on the edge in $\mathcal{C} \cap \mathcal{E}_1(\bar{z})$ is equal to the lower saturation boundary. Indeed, if $\bar{z}_i = \mu_i^+$ and $\xi_{\mathcal{C}_i} = 1$ for all $e_i \in \mathcal{C} \cap \mathcal{E}_1(\bar{z})$, then there exists $\varepsilon > 0$ small enough such that $\mathcal{E}_0(\bar{z} - \varepsilon\xi_{\mathcal{C}}) = \mathcal{E}_0(\bar{z}) \cup \mathcal{C}$. This follow from the fact that $\bar{z}_j \in (\mu_j^-, \mu_j^+)$ for all $e_j \in \mathcal{C} \cap \mathcal{E}_0(\bar{z})$. Hence we have a contradiction to $|\mathcal{E}_0(\bar{z})|$ being the maximum. The same conclusion holds if $\bar{z}_i = \mu_i^-$ for all $e_i \in \mathcal{C} \cap \mathcal{E}_1(\bar{z})$. The graphical explanation of the previous two observations is given in Figure 3.5.

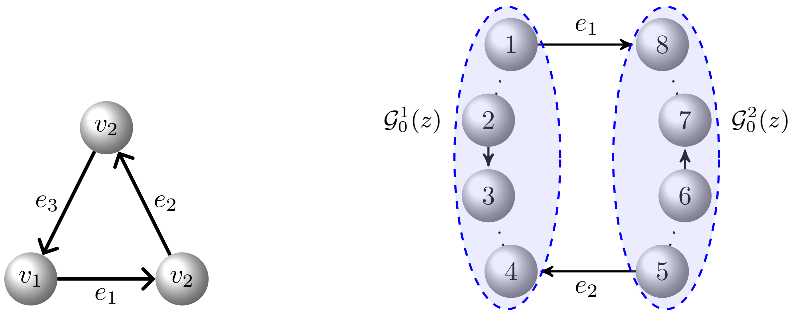
Denote the weakly connected components of $\mathcal{G}_0(\bar{z})$ by $\mathcal{G}_0^i(\bar{z}), i = 1, \ldots, \ell$. By the second observation, we have that on all the edges from $\mathcal{G}_0^i(\bar{z})$ to $\mathcal{G}_0^j(\bar{z})$, i.e., $e_k \sim (v_p, v_q)$ satisfying $v_p \in \mathcal{G}_0^i(\bar{z})$ and $v_q \in \mathcal{G}_0^j(\bar{z})$, the corresponding components $\bar{z}_k$ of $\bar{z}$ reach the same saturation boundaries (upper or lower) at the same time.

Notice that the partitioning $\mathcal{G}_0^i(\bar{z}), i = 1, \ldots, \ell$, is the same for any choice of $\bar{z}$ which maximize the map $|\mathcal{E}_0(\cdot)|$. In fact, suppose there exists a vector $z' \in \mathcal{M} \cap \ker B$ such that $|\mathcal{E}_0(z')| = |\mathcal{E}_0(\bar{z})|$ being the maximum, but there exist two vertices v_i, v_j which belong to the same component of $\mathcal{G}_0(\bar{z})$ but two different components of $\mathcal{G}_0(z')$. In other words, there exists $e_k \in \mathcal{E}_1(z') \cap \mathcal{E}_0(\bar{z})$. Then for the vector $\frac{\bar{z}+z'}{2}$ which belongs to $\mathcal{M} \cap \ker B$, we have $e_k \in \mathcal{E}_0(\frac{\bar{z}+z'}{2})$ and $|\mathcal{E}_0(\frac{\bar{z}+z'}{2})| - |\mathcal{E}_0(z')| \geqslant 1$. This is a contradiction to the fact that the map $|\mathcal{E}_0(\cdot)|$ reaches its maximum at z'. Hence the partitioning is unique.

Finally we can select a suitable initial condition of the system (3.34) on $\mathcal{G}$ to be such that $\dot{x} = 0, B^T \frac{\partial H}{\partial x}(x) \neq 0, \forall t > 0$. Based on the vector $\bar{z}$, we can set $\frac{\partial H_c}{\partial \eta}(\eta(0)) = \bar{z}$. Furthermore, we can assign to $\frac{\partial H}{\partial x}(0)$ the same value within each weakly connected component of $\mathcal{G}_0(\bar{z})$. More precisely, if there exists an edge $e_k \in \mathcal{E}_1(\bar{z})$ from $\mathcal{G}_0^i(\bar{z})$ to $\mathcal{G}_0^j(\bar{z})$ such that $\bar{z}_k = \mu_k^+$, then we set $\frac{\partial H}{\partial x_p}(0) > \frac{\partial H}{\partial x_q}(0)$ for any $v_p \in \mathcal{G}_0^i(\bar{z})$ and $v_q \in \mathcal{G}_0^j(\bar{z})$; if there exists an edge $e_k \in \mathcal{E}_1(\bar{z})$ from $\mathcal{G}_0^i(\bar{z})$ to $\mathcal{G}_0^j(\bar{z})$ such that $\bar{z}_k = \mu_k^-$, then we set $\frac{\partial H}{\partial x_p}(0) < \frac{\partial H}{\partial x_q}(0)$ for any $v_p \in \mathcal{G}_0^i(\bar{z})$ and $v_q \in \mathcal{G}_0^j(\bar{z})$. As a result, we have $\dot{\eta}_k > 0$ for $e_k \in \mathcal{E}_1(\bar{z})$ and $\bar{z}_k = \mu_k^+$, $\dot{\eta}_k < 0$ for $e_k \in \mathcal{E}_1(\bar{z})$ and $\bar{z}_k = \mu_k^-$, and $\operatorname{sat}\left(RB^T \frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial \eta}(\eta)\,;\mu^-, \mu^+\right) = \bar{z}$. Furthermore, since $\bar{z} \in \mathcal{M} \cap \ker B$, we have that $\dot{x} = 0$ for all t.

□

Example 3.7. Consider the dynamical system (3.34) defined on the network given in Figure 3.2 with $H(x)$ given as Fig.3.6 (above). The constraint intervals are $\mu^- = [0, 1, 2, 0, 0, 0, 0]^T$ and $\mu^+ = [1, 3, 3, 2, 2, 1, 2]^T$. In order to check the interior point condition, we can take $z = [1, 2, 3, 1, 1, 1, 1]^T$ in which case $\mathcal{E}_0(z) = \{e_2, e_4, e_5, e_7\}$ and $\mathcal{G}_0$ is weakly connected. As can be seen from Fig.3.6 (below), the outputs converge to consensus. Here we emphasize that radial unboundedness of the Hamiltonian H is sufficient for output agreement, i.e., H is not necessarily convex.



(a) A positive circuit for the first observation

(b) A circuit for the second observation

Figure 3.5: The graphical explanation for the two observations in the proof of Theorem 3.12. In (a), suppose the saturation boundaries are $[\mu_i^-, \mu_i^+] = [0, 1], i = 1, 2, 3$, then the components of the vector $z = \mathbb{1}_3$ reach the upper saturation bounds on the corresponding edges. Hence $|\mathcal{E}_0(z)| = 0$ is not the maximum of the map $|\mathcal{E}_0(\cdot)|$. In this case, the maximum is 3. In (b), $\mathcal{G}_0^1(z)$ and $\mathcal{G}_0^2(z)$ are two weakly connected components of $\mathcal{G}_0(z)$. Along the circuit, denoted as $\mathcal{C}$, we have $\xi_{\mathcal{C}_1} = \xi_{\mathcal{C}_2}$. Suppose $z_1 = \mu_1^+$ and $z_2 = \mu_2^+$, then by the fact that $z_k \in (\mu_k^-, \mu_k^+)$ for any $e_k \in \mathcal{C} \setminus \{e_1, e_2\}$, there exists small enough ε such that $(z - \varepsilon\xi_{\mathcal{C}})_i \in (\mu_i^-, \mu_i^+)$ for any $e_i \in \mathcal{C}$. Hence, $|\mathcal{E}_0(z)|$ is not the maximum over all the vectors in $\mathcal{M} \cap \ker B$.

3.4 Connection to Equilibrium Independent Passivity

As we will see in this section, the stability of the closed-loop system (3.16) can be alternatively derived from the theory of *passivity*.

We start by introducing the definition of Equilibrium Independent Passivity on the set Ω which is modified from the original definition in [18, 40] (see Section 2.5).

Definition 3.18. The system

$$\begin{aligned} \dot{x} &= f(x, u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \\ y &= h(x, u), \qquad y \in \mathbb{R}^m \end{aligned} \tag{3.46}$$

is called *equilibrium independent passive* (EIP) on the set $\Omega \subset \mathbb{R}^n$ if there exists a monotone relation

$$\mathcal{R} \subset \mathbb{R}^m \times \mathbb{R}^m,$$

called the steady state input-output gain, such that for every pair of $(\bar{u}, \bar{y}) \in \mathcal{R}$ for which there exists an $\bar{x} \in \Omega$ satisfying

$$0 = f(\bar{x}, \bar{u}), \bar{y} = h(\bar{x}, \bar{u}), \tag{3.47}$$

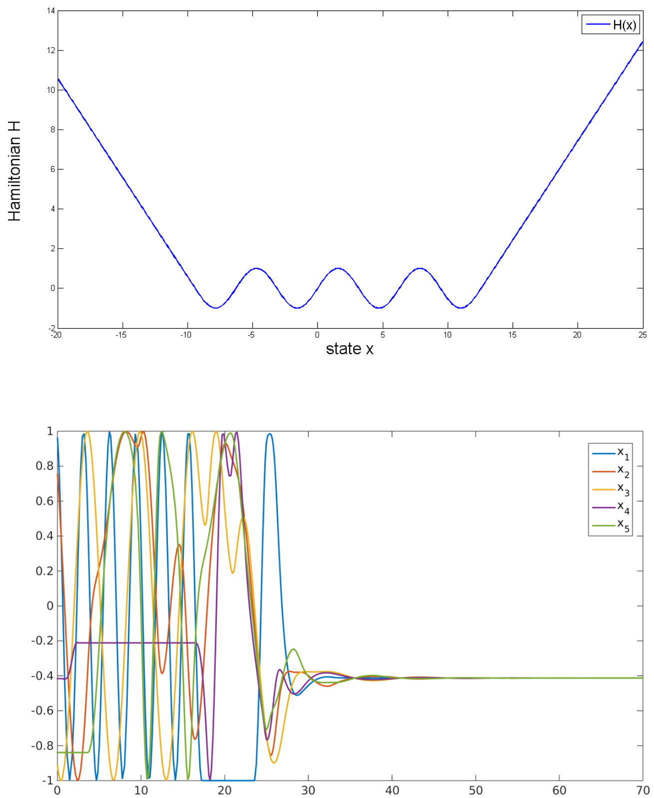


Figure 3.6: In Example 3.7. Above, the Hamiltonian function H. Below, the simulation result showing the trajectories of the outputs of the plants.

there exists a function $V_{\bar{x}}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ that is bounded from below, called the *storage function* corresponding to $\bar{x}$, such that along the trajectories of (3.46)

$$\frac{d}{dt} V_{\bar{x}}(x(t)) \leqslant (u - \bar{u})^T (y - \bar{y}). \tag{3.48}$$

Furthermore, the system (3.46) is output, respectively input, strictly EIP on Ω if

$$\frac{d}{dt} V_{\bar{x}}(x(t)) \leqslant (u - \bar{u})^T (y - \bar{y}) - \rho(y - \bar{y}) \tag{3.49}$$

$$\frac{d}{dt} V_{\bar{x}}(x(t)) \leqslant (u - \bar{u})^T (y - \bar{y}) - \rho(u - \bar{u}) \tag{3.50}$$

respectively, for some positive definite function $\rho(\cdot)$.

In the absence of constraints we obtain the following proposition about the EIP property of the system composed of (3.1) together with (3.7) and (3.8) which is given as in the dashed box in Figure 3.7 (right), and of the controller system (3.14).

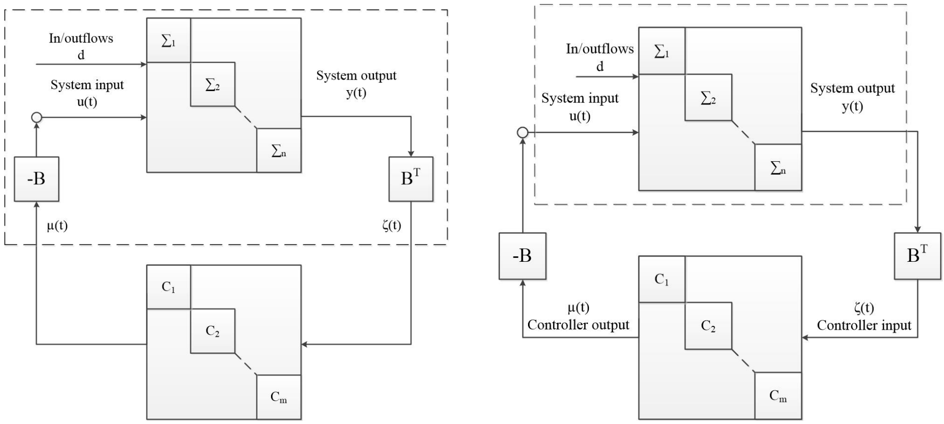


Figure 3.7: Right: within the dashed box, we have the system composed of (3.1) together with (3.7) and (3.8). In this case, the input is μ and the output is ζ. Left: within the dashed box, there is the system (3.1) with input u and output y.

Proposition 3.19. *Suppose the function* (3.19) *is radially unbounded. The system composed of* (3.1) *together with* (3.7) *and* (3.8)*, i.e.,*

$$\begin{aligned} \dot{x} &= -B\mu + E\bar{d} \\ \zeta &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{3.51}$$

with input μ and output ζ, is EIP on $\{x \mid \frac{\partial H}{\partial x}(x) \in \operatorname{span} \mathbb{1}_n\}$ while the controller system (3.14) is input strictly EIP on $\{\eta \mid B\frac{\partial H_c}{\partial \eta}(\eta) = E\bar{d}\}$.

Proof. Consider any $(-\bar{\mu}, \bar{\zeta})$ for which there exists a steady state $\bar{x}$ satisfying

$$\begin{aligned} 0 &= -B\bar{\mu} + E\bar{d}, \\ \bar{\zeta} &= B^T \frac{\partial H}{\partial x}(\bar{x}), \end{aligned} \tag{3.52}$$

and $\frac{\partial H}{\partial x}(\bar{x}) \in \operatorname{span} \mathbb{1}_n$. Then $V_{\bar{x}}(x) := H(x)$ satisfies

$$\frac{d}{dt} V_{\bar{x}}(x) = (-\mu + \bar{\mu})^T (\zeta - \bar{\zeta}). \tag{3.53}$$

With regard to the controller system (3.14) take any pair of $(\bar{\zeta}, \bar{\mu})$ such that $\exists \bar{\eta}$ for which

$$0 = \bar{\zeta}, \bar{\mu} = \frac{\partial H_c}{\partial \eta}(\bar{\eta}), \tag{3.54}$$

and $B\frac{\partial H_c}{\partial \eta}(\bar{\eta}) = E\bar{d}$. Then define the storage function as $V_{\bar{\eta}}(\eta) := H_c(\eta) - \frac{\partial^T H_c}{\partial \eta}(\bar{\eta})(\eta - \bar{\eta})$. It follows that

$$\frac{d}{dt}V_{\bar{\eta}}(\eta) = (\mu - \bar{\mu})^T(\zeta - \bar{\zeta}) - (\zeta - \bar{\zeta})^T R(\zeta - \bar{\zeta}) \tag{3.55}$$

Finally, from the radial unboundedness of (3.19), it follows that $V_{\bar{x}}(x)$ and $V_{\bar{\eta}}(\eta)$ are bounded from below. Hence, the system (3.51) and (3.14) are EIP and input strictly EIP respectively. □

Notice that the function $V_{\bar{x}}(x) + V_{\bar{\eta}}(\eta)$ is the same as the modified Hamiltonian function in (3.19). Hence the same conclusion in Theorem 3.3 can be derived from the point of view of equilibrium independent passivity.

Remark 3.20. The equilibrium independent passivity of (3.51) is only based on the radial unboundedness of (3.19). However, in order to prove the equilibrium independent passivity of (3.1), which is given as in Figure 3.7 (left) on $\{x \mid \frac{\partial H}{\partial x}(x) \in \operatorname{span} \mathbb{1}_n\}$, the radial unboundedness of H is not sufficient. Hence a stronger condition is needed, for example strict convexity of H. In this case a possible storage function is $V(x) = H(x) - \frac{\partial^T H}{\partial x}(\bar{x})(x - \bar{x}) - H(\bar{x})$, where $\bar{x}$ satisfies $\frac{\partial^T H}{\partial x}(\bar{x}) \in \operatorname{span} \mathbb{1}_n$. Note that the radial unboundedness of H is not sufficient to guarantee that V is bounded from below.

We obtain the following extension to the case with constraint intervals.

Proposition 3.21. *The system* (3.27) *with* $\mu^+ \geqslant \mu^-$ *where Assumption 3.6 satisfied is EIP on* $\mathbb{R}^m$*.*

Proof. For any steady-state pair $(\bar{\zeta}, \bar{\mu})$ of the constrained controller (3.27) for which there exists $\bar{\eta}$ such that

$$0 = \bar{\zeta}, \;\; \bar{\mu} = \operatorname{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta}); \mu^-, \mu^+), \tag{3.56}$$

we define the function

$$\hat{V}_{\bar{\eta}}(\eta) := \mathbb{1}^T S\big(\eta\,; \mu^-, \mu^+\big) - \operatorname{sat}^T\big(\frac{\partial H_c}{\partial \eta}(\bar{\eta}); \mu^-, \mu^+\big)(\eta - \bar{\eta}), \tag{3.57}$$

where $S\big(\eta\,; \mu^-, \mu^+\big)$ is given as in (3.42). Then

$$\frac{d}{dt}\hat{V}_{\bar{\eta}}(\eta) \leqslant (\mu - \bar{\mu})^T(\zeta - \bar{\zeta}) \tag{3.58}$$

which follows from $\text{sat}^T(\frac{\partial H_c}{\partial \eta}(\eta))\zeta \leqslant \text{sat}^T(\frac{\partial H_c}{\partial \eta}(\eta) + R\zeta)\zeta$. By Assumption 3.6 each component of the function $S\big(\eta\,;\mu^-,\mu^+\big)$ is convex. Hence the function $\hat{V}_{\bar{\eta}}(\eta)$ is bounded from below. In conclusion, the controller (3.27) is EIP. □

Remark 3.22. The controller (3.27) is EIP but *not* input strictly EIP, which makes the analysis of the convergence of the closed-loop system (3.28) to output agreement less straightforward than in the unconstrained case.

Remark 3.23. In [58] a constrained controller is considered where the saturation is present on the proportional and integral part separately, i.e.,

$$\begin{aligned} \dot{\eta} &= \zeta, \\ \mu &= \text{sat}\left(\tfrac{\partial H_c}{\partial \eta}(\eta)\right) + \text{sat}(R\zeta) \end{aligned} \tag{3.59}$$

with saturation intervals $\mu^- = -\mu^+ = -\mathbb{1}_n$. In this case, for any steady-state input-output pair $(\bar{\zeta}, \bar{\mu})$, the function $\hat{V}^d_{\bar{\eta}}(\eta)$ satisfies

$$\frac{d}{dt} S_{\bar{\eta}}(\eta) \leqslant (\mu - \bar{\mu})^T(\zeta - \bar{\zeta}) - (\text{sat}(R\zeta) - \text{sat}(R\bar{\zeta}))^T(\zeta - \bar{\zeta}), \tag{3.60}$$

yielding input strictly EIP.

3.5 An optimization perspective on the Interior Point Condition

In Section 3.3, it is shown that for system (3.34), the interior point condition is essential for its stability. In this section we will establish the relation between the system (3.34) and a static optimization problem. This optimization problem can be also used as a test for the interior point condition. Specifically, in this section we consider the dynamics on each vertex given as

$$\begin{aligned} \dot{x}_i &= u_i, \\ y_i &= \frac{\partial H}{\partial x_i}(x). \end{aligned} \tag{3.61}$$

Our approach is based on the following classical result [61] on maximal monotonicity.

Definition 3.24 ([61]). A relation $\Lambda \subset \mathbb{R}^2$ is said to be *maximally monotone* in $\mathbb{R}^2$ if and only if

(i) For any $(\rho, \sigma) \in \Lambda$ and $(\rho', \sigma') \in \Lambda$, one has either $(\rho, \sigma) \leqslant (\rho', \sigma')$ or $(\rho', \sigma') \leqslant (\rho, \sigma)$, and

(ii) For an arbitrary $(\rho, \sigma) \notin \Lambda$, there exists $(\rho', \sigma') \in \Lambda$ such that neither $(\rho, \sigma) \leqslant (\rho', \sigma')$ nor $(\rho', \sigma') \leqslant (\rho, \sigma)$. That is, Λ is maximal with respect to property (i).

Proposition 3.25. *Any maximal monotone relation $\mathfrak{M} \subset \mathbb{R} \times \mathbb{R}$ is the graph of the sub-gradient of a proper, convex, lower semi-continuous function $K : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$.*

In [17], the authors already established the relation between EIP systems with the set of steady-state input-output values being a maximal monotone set. However, in our case the set of steady-state input-output values of the plant (3.61) and the constrained controller (3.27) is monotone but in general not maximal.

In the following part of this section, we first show that for a general single-input single-output (SISO) system with a monotone set of steady-state input-output values, this set can be extended to a maximal monotone relation. Consider a general SISO system given as

$$\begin{aligned} \dot{x}_i = &\quad f_i(x_i, u_i), \quad x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}, \\ y_i = &\quad g_i(x_i, u_i), \quad y_i \in \mathbb{R}, \end{aligned} \tag{3.62}$$

with set of steady-state input-output values defined as

$$\begin{aligned} \kappa_i :&= \{(\bar{u}_i, \bar{y}_i) \mid \exists \bar{x}_i \text{ s.t. } 0 = f_i(\bar{x}_i, \bar{u}_i) \text{ and } \bar{y}_i = g_i(\bar{x}_i, \bar{u}_i)\} \\ &\subset \mathbb{R} \times \mathbb{R} \end{aligned} \tag{3.63}$$

being a monotone relation. Denote

$$\begin{aligned} (u_i^+, y_i^+) &= \max_{(\bar{u}_i, \bar{y}_i) \in \kappa_i} (\bar{u}_i, \bar{y}_i) \\ (u_i^-, y_i^-) &= \min_{(\bar{u}_i, \bar{y}_i) \in \kappa_i} (\bar{u}_i, \bar{y}_i). \end{aligned} \tag{3.64}$$

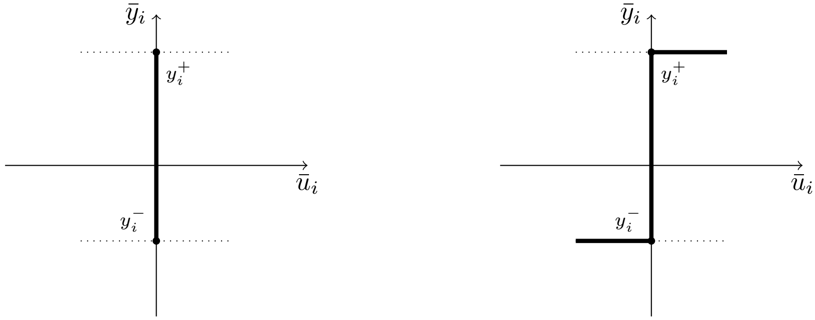
If u_i^+, y_i^+, u_i^- and y_i^- are finite, i.e., $|u_i^+| + |y_i^+| + |u_i^-| + |y_i^-| < \infty$, we define the following sets

$$\begin{aligned} \kappa_i^r &= \{(\bar{u}_i, \bar{y}_i) \mid \bar{u}_i \geqslant u_i^+, \bar{y}_i = y_i^+\} \\ \kappa_i^\ell &= \{(\bar{u}_i, \bar{y}_i) \mid \bar{u}_i \leqslant u_i^-, \bar{y}_i = y_i^-\} \end{aligned} \tag{3.65}$$

Then the extended set of steady-state input-output values, denoted as κ_i^e, is define as

$$\kappa_i^e = \begin{cases} \kappa_i \cup \kappa_i^r \cup \kappa_i^\ell & \text{if } |u_i^+| + |y_i^+| + |u_i^-| + |y_i^-| < \infty, \\ \kappa_i \cup \kappa_i^r & \text{if } |u_i^-| + |y_i^-| = \infty \text{ and } |u_i^+| + |y_i^+| < \infty, \\ \kappa_i \cup \kappa_i^\ell & \text{if } |u_i^+| + |y_i^+| = \infty \text{ and } |u_i^-| + |y_i^-| < \infty, \end{cases} \tag{3.66}$$

The specific procedure of the extension of the input-output gain of (3.61) when $\{(0, \frac{\partial H}{\partial x_i}(\bar{x})) \mid x \in \mathbb{R}^n\} \subset \{0\} \times (\mathbb{R} \setminus \{\pm\infty\})$ can be explained by Figure 3.8. With



(a) the original static input-output gain

(b) the extended input-output gain being a maximal monotone

Figure 3.8: The extension of the set of steady-state input-output values given in (a), which is monotone, to a maximal monotone relation given as in (b).

a slight abuse of the notation, we denote the extended steady state input-output gain of (3.61) by κ_i^e. By Proposition 3.25, for each $\kappa_i^e \subset \mathbb{R} \times \mathbb{R}$ there exists a convex function K_i^e such that $\nabla K_i^e = \kappa_i^e$. For example, when y_i^+ and y_i^- are finite

$$K_i^e(\bar{u}_i) = \begin{cases} y_i^+ \bar{u}_i & \bar{u}_i \geqslant 0 \\ y_i^- \bar{u}_i & \bar{u}_i < 0, \end{cases} \tag{3.67}$$

When $y_i^+ = +\infty$ and y_i^- is finite, $K_i^e(\bar{u}_i) = y_i^- \bar{u}_i + I_{(-\infty,0)}(\bar{u}_i)$ where I_A is the indicator function of the set A. When $y_i^- = -\infty$ and y_i^+ is finite, $K_i^e(\bar{u}_i) = y_i^+ \bar{u}_i + I_{(0,\infty)}(\bar{u}_i)$. When y_i^+ and y_i^- are both equal to infinity, $K_i^e(\bar{u}_i) = I_{\{0\}}(\bar{u}_i)$.

For controllers (3.27), the extension of the definition of static input-output gain can be done in the same manner. We denote the static input-output gain and the extended one as $\gamma_k(\bar{\zeta}_k)$ and $\gamma_k^e(\bar{\zeta}_k)$ respectively. Furthermore $\Gamma_k^e(\cdot) : \mathbb{R} \to \mathbb{R}$

$$\Gamma_k^e(\bar{\zeta}_k) = \begin{cases} \mu_k^+ \bar{\zeta}_k & \bar{\zeta}_k \geqslant 0 \\ \mu_k^- \bar{\zeta}_k & \bar{\zeta}_k < 0, \end{cases} \tag{3.68}$$

is the convex function satisfying $\nabla \Gamma_k^e = \gamma_k^e$. Notice that the convex conjugate of Γ_k^e is $\Gamma_k^{e*}(\bar{\mu}_k) = I_{[\mu_k^-, \mu_k^+]}(\bar{\mu}_k)$.

Employing the terminologies of Section 2.3, we consider the functions Γ_k^{e*} and K_i^e as the costs for the flux $\bar{\mu}_k$ and the divergence $\bar{u}_i$ respectively. Hence the optimal flow problem (2.8) is given as

$$\begin{aligned} &\min_{\bar{u},\bar{\mu}} \sum_{i=1}^{n} K_i^e(\bar{u}_i) + \sum_{k=1}^{m} \Gamma_k^{e*}(\bar{\mu}_k) \\ &\text{s.t } \bar{u} + B\bar{\mu} = 0 \end{aligned} \tag{3.69}$$

Furthermore, the optimal potential problem, dual to (3.69), is given as

$$\begin{aligned} &\min_{\bar{y},\bar{\zeta}} \sum_{i=1}^{n} K_i^{e*}(\bar{y}_i) + \sum_{k=1}^{m} \Gamma_k^{e}(\bar{\zeta}_k) \\ &\text{s.t } \bar{\zeta} = B^T\bar{y} \end{aligned} \tag{3.70}$$

where $K_i^{e*}(\bar{y}_i) = I_{[y_i^-, y_i^+]}(\bar{y}_i)$. The optimal solutions of (3.69) and (3.70) are denoted as (u, μ) and (y, ζ) respectively.

By using the optimization problem (3.70), we have an equivalent condition to the interior point condition.

Theorem 3.26. *Consider a network, defined as a digraph, with dynamics* (3.61) *on the vertices and flow constraints* $[\mu^-, \mu^+]$ *on the edges. Suppose* $y_i^- < y_j^+$ *for all* $i, j = 1, 2, \ldots, n$. *Then the network satisfies the interior point condition if and only if any* (y, ζ) *which is an optimal solution of* (3.70) *satisfies* $\zeta = 0$.

Remark 3.27. The condition $\zeta = 0$ for any (y, ζ) being an optimal solution of (3.70) implies that $\mathrm{y} \in \mathrm{span}\{\mathbb{1}\}$, i.e., the steady-state output of (3.61) achieves consensus. In Theorem 3.26, we will show that this condition is equivalent to the interior point condition.

Proof of Theorem 3.26. Notice that the condition $y_i^- < y_j^+$ for all $i, j = 1, 2, \ldots, n$, implies that

$$\exists \bar{y} \in \mathrm{span}\{\mathbb{1}\} \cap (y^-, y^+). \tag{3.71}$$

Necessity: Suppose the network satisfies the interior point condition, and by Lemma 3.14, we take a vector $z \in \mathcal{M} \cap \ker B$ such that $\mathcal{E}' = \mathcal{E}_0(z)$. Then for any $\bar{\zeta} = B^T\bar{y}$, we have $z^T\bar{\zeta} = 0$ which implies that (3.70) is equivalent to

$$\begin{aligned} &\min_{\bar{\zeta},\bar{y}} \sum_{i=1}^{n} I_{[y_i^-, y_i^+]}(\bar{y}_i) + \sum_{k=1}^{m} \Gamma_k^{e}(\bar{\zeta}_k) - z_k^T\bar{\zeta}_k \\ &s.t.\ \bar{\zeta} = B^T\bar{y}. \end{aligned} \tag{3.72}$$

Furthermore, since on the edge $e_k \in \mathcal{E} \setminus \mathcal{E}'$, $\Gamma_k(\zeta_k) = z_k\zeta_k$, the optimization problem (3.72) can be rewritten as

$$\begin{aligned}&\min_{\bar{\zeta},\bar{y}} \sum_{i=1}^{n} I_{[y_i^-,y_i^+]}(\bar{y}_i) + \sum_{e_k\in\mathcal{E}'} \Gamma_k^e(\bar{\zeta}_k) - z_k^T\bar{\zeta}_k \\ &s.t.\ \bar{\zeta} = B^T\bar{y}.\end{aligned} \tag{3.73}$$

Notice that on the edge $e_k \in \mathcal{E}'$, we have $z_k \in (\mu_k^-, \mu_k^+)$. Thus $\Gamma_k(\zeta_k) - z_k\zeta_k$ is convex and has a unique minimum at $\zeta_k = 0$. Furthermore, by (3.71), the minimum of (3.73) which is zero is attained. Moreover, since $\{\mathcal{V}, \mathcal{E}'\}$ is weakly connected, ζ in the optimal solution of (3.70) can only be zero.

Sufficiency: In (3.70), the objective functions $\Gamma_k^e, k = 1, 2, \ldots, m$, are convex and the constraint $\bar{\zeta} = B^T\bar{y}$ is affine, so the Karush–Kuhn–Tucker (KKT) condition is sufficient and necessary, i.e. $(\zeta, \mathrm{y}, \lambda)$ is an optimal solution of (3.70) if and only if

$$0 \in \partial\Gamma_k^e(\zeta_k) - \lambda_k, \quad k = 1, 2, \ldots, m \tag{3.74}$$

$$0 \in B\lambda + \partial I_{[y_i^-,y_i^+]}(\mathrm{y}_i) \tag{3.75}$$

$$\zeta = B^T\mathrm{y} \tag{3.76}$$

where $\lambda \in \mathbb{R}^m$ is the corresponding Lagrangian multiplier. Since $(\mathrm{y}, 0)$ where $\mathrm{y} \in \mathrm{span}\{\mathbb{1}\} \cap (y^-, y^+)$ is one optimal solution of (3.70) and $\partial\Gamma_k^e(\bar{\zeta}_k)|_{\bar{\zeta}_k=0} = [\mu_k^-, \mu_k^+]$ and $\partial I_{[y_i^-,y_i^+]}(\bar{y}_i)|_{\bar{y}_i=\mathrm{y}_i} = 0$, the conditions (3.74) and (3.75) imply that $\ker B \cap \mathcal{M} \neq \varnothing$.

Now we will prove the conclusion by contradiction. Recall that for a vector $z \in [\mu^-, \mu^+] \cap \ker B$, the subgraph $\mathcal{G}_0(z; \mu^-, \mu^+)$ is defined as $\{\mathcal{V}, \mathcal{E}_0(z; \mu^-, \mu^+)\}$ with $\mathcal{E}_0(z; \mu^-, \mu^+) = \{e_i \in \mathcal{E} \mid z_i \in (\mu_i^-, \mu_i^+)\}$. We omit μ^- and μ^+ from the previous notations in the remaining part of this proof. Suppose the network does not satisfy the interior point condition, i.e., for any vector $z \in \mathcal{M} \cap \ker B$, the graph $\mathcal{G}_0(z)$ is not weakly connected. In this case we show that there exists a optimal solution of (3.70) with corresponding Lagrangian multiplier, denoted as $(\bar{\mathrm{y}}, \bar{\zeta}, \bar{\lambda})$, satisfying (3.74),(3.75),(3.76) and $\bar{\zeta} \neq 0$. Indeed, similarly as in the necessity part of the proof of Theorem 3.12, we assume that the maximal value of the map $|\mathcal{E}_0(\cdot)| : \mathcal{M} \cap \ker B \mapsto \{1, 2, \ldots, m\}$ is attained at $\bar{\lambda}$. Hence (3.75) is satisfied. Denote the weakly connected components of $\mathcal{G}_0(\bar{\lambda})$ as $\mathcal{G}_0^1(\bar{\lambda}), \ldots, \mathcal{G}_0^\ell(\bar{\lambda})$. We construct a vector $\bar{\mathrm{y}}$ such that

(i) $\bar{\mathrm{y}}_i = \bar{\mathrm{y}}_j$ for any v_i, v_j belonging to the same weakly connected component of $\mathcal{G}_0(\bar{\lambda})$;

(ii) $\bar{\mathrm{y}}_i > \bar{\mathrm{y}}_j$ if there exists an edge $e_k \sim (v_i, v_j)$ and $\bar{\lambda}_k = \mu_k^+$;

(iii) while $\bar{\mathrm{y}}_i < \bar{\mathrm{y}}_j$ if there exists an edge $e_k \sim (v_i, v_j)$ and $\bar{\lambda}_k = \mu_k^-$.

It can be verified that $\bar{\mathrm{y}}$ satisfies the conditions (3.74) and (3.76). This is a contradiction. In conclusion, the network satisfies the interior point condition. □

Remark 3.28. The radial unboundedness of the Hamiltonian H implies that $0 \in (y_i^-, y_i^+)$. For the dynamical system (3.34), the radial unboundedness of H and the interior point condition imply the convergence of the trajectories of (3.34) to output agreement. At the same time, for the optimization (3.70), $y_i^- < y_j^+$ for all $i, j = 1, 2, \ldots, n$, and the interior point condition imply that any optimal solution (y, ζ) of (3.70) satisfies $\zeta = 0$ and $\mathrm{y} \in \mathrm{span}\{\mathbb{1}\}$.

Remark 3.29. When the network satisfies the interior point condition, the set of optimal solutions of (3.69) and (3.70) is

$$\mathcal{O} := \{\mathrm{u}, \mu, \mathrm{y}, \zeta \mid \mathrm{u} = B\mu = 0, B^T\mathrm{y} = \zeta = 0, \mu \in [\mu^-, \mu^+], \mathrm{y} \in [y^-, y^+]\}, \quad (3.77)$$

which represents the set of equilibria for the dynamical network (3.7),(3.8),(3.27) and (3.61). Theorem 3.12 proves the convergence of $u(t), \mu(t), y(t), \zeta(t)$ to $\mathcal{O}$.

Example 3.8. Consider an optimization problem defined on a digraph given as in Fig.3.4 (left). Suppose the flow constraint on e_1 is $[0, 1]$. Then the network does not satisfy the interior point condition. Hence by Theorem 3.26, there exists an optimal solution (y, ζ) of (3.70) such that $\zeta \neq 0$. Indeed, it can be verified that any (y, ζ), satisfying $\zeta = \mathrm{y}_1 - \mathrm{y}_2$ where $\mathrm{y}_1 < \mathrm{y}_2$ and $\mathrm{y}_i \in (y_i^-, y_i^+), i = 1, 2$, is an optimal solution of (3.70).

3.6 Convergence to arbitrary output vector

In Section 3.2 and 3.3 we showed that in the unconstrained case the controller (3.14) and in the constrained case the controller (3.27) will drive the plant to output agreement. In the present section we will show how both controllers can be modified in such a way that the vector of outputs will converge to any desirable vector in an *admissible set* $\mathcal{A}$, which, for a connected graph, is defined as

$$\mathcal{A} = \{\frac{\partial H}{\partial x}(x) \mid \mathbb{1}^T x = \mathbb{1}^T x(0)\}, \quad (3.78)$$

where $x(0)$ is the initial condition of the system.

First we start with the unconstrained case. Define the modified controller

$$\begin{aligned} \dot{\eta} &= (\zeta - \zeta^*) \\ u &= R(\zeta - \zeta^*) + \frac{\partial H_c}{\partial \eta} \end{aligned} \quad (3.79)$$

where $\zeta^* = B^T \frac{\partial H}{\partial x}(x^*)$ is the desirable value with $\frac{\partial H}{\partial x}(x^*) \in \mathcal{A}$. Then the closed-loop resulting from (3.1), (3.7), (3.8) and (3.79) is given as

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -BRB^T & -B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(x^*) \\ \frac{\partial H_c}{\partial \eta}(\eta) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d, \tag{3.80}$$

Theorem 3.30. *Suppose $H \in \mathcal{C}^2$ has a diagonal positive definite Hessian matrix, the controller Hamiltonian H_c satisfies Assumption 3.6, and the in/outflows satisfy the matching condition* (3.18). *Consider the system (3.80) defined on a weakly connected digraph $\mathcal{G}$. Then for any desirable output vector $\frac{\partial H}{\partial x}(x^*)$ in the admissible set $\mathcal{A}$, the trajectories will converge to*

$$\mathcal{E}_{tot} = \{(x^*, \bar{\eta}) \mid B\frac{\partial H_c}{\partial \eta}(\bar{\eta}) = E\bar{d}\}. \tag{3.81}$$

Hence $\lim_{t\to\infty} \frac{\partial H}{\partial x}(x(t)) = \frac{\partial H}{\partial x}(x^*)$.

Proof. Consider the modified Hamiltonian

$$\begin{aligned} H^*(x,\eta) = H(x) - \frac{\partial^T H}{\partial x}(x^*)(x - x^*) - H(x^*) \\ + H_c(\eta) - H_c(\bar{\eta}) - \frac{\partial^T H_c}{\partial \eta}(\bar{\eta})(\eta - \bar{\eta}) \end{aligned} \tag{3.82}$$

as Lyapunov function. By the fact that $H \in \mathcal{C}^2$ has a diagonal positive definite Hessian matrix, which implies that H is strictly convex, and by Assumption 3.6, we have that $H^*(x,\eta)$ is radially unbounded. Then by LaSalle's Invariance principle it follows that the trajectories will converge to

$$\{(\bar{x}, \bar{\eta}) \mid B\frac{\partial H_c}{\partial \eta}(\bar{\eta}) = E\bar{d}, \frac{\partial H}{\partial x}(\bar{x}) - \frac{\partial H}{\partial x}(x^*) \in \operatorname{span}\{\mathbb{1}\}\}. \tag{3.83}$$

By the fundamental theorem of calculus applied to $H \in \mathcal{C}^2$

$$\frac{\partial H}{\partial x}(\bar{x}) - \frac{\partial H}{\partial x}(x^*) = (\int_0^1 \frac{\partial^2 H}{\partial x^2}(x^* + th)dt) \cdot h \tag{3.84}$$

where $h = \bar{x} - x^*$ is orthogonal to $\operatorname{span}\{\mathbb{1}\}$. Since $\frac{\partial H}{\partial x}(\bar{x}) - \frac{\partial H}{\partial x}(x^*) \in \operatorname{span}\{\mathbb{1}\}$, we have that

$$0 = h^T(\frac{\partial H}{\partial x}(\bar{x}) - \frac{\partial H}{\partial x}(x^*)) = h^T \cdot (\int_0^1 \frac{\partial^2 H}{\partial x^2}(x^* + th)dt) \cdot h \tag{3.85}$$

Furthermore, since $\frac{\partial^2 H}{\partial x^2}$ is positive definite and therefore also $\int_0^1 \frac{\partial^2 H}{\partial x^2}(x^* + th)dt$, this implies $h = 0$. In conclusion we proved that the state x will converge to x^*, and thus $\lim_{t\to\infty} \frac{\partial H}{\partial x}(x(t)) = \frac{\partial H}{\partial x}(x^*)$. □

Remark 3.31. In [12], the authors considered the problem of driving the state of system (3.1) and (3.7), i.e., $\dot{x} = -B\mu + Ed$, to an arbitrary desirable state $\bar{x}$ satisfying $\mathbb{1}^T\bar{x} = \mathbb{1}^T x(0)$ by designing feedback $\mu = \mu(x, \bar{x})$. The resulting controller is discontinuous (bang-bang type strategy).

Remark 3.32. If the underlying network is not weakly connected, then the admissible set can be defined for each connected component. In fact, suppose the graph has k components, and $x_{n_i} \in \mathbb{R}^{n_i}, i = 1, 2, \dots, k$, represent the corresponding state variables in each component. The admissible set is now defined as

$$\mathcal{A}' = \{\frac{\partial H}{\partial x}(x) \mid \mathbb{1}_{n_i}^T(x_{n_i} - x_{n_i}(0)) = 0, i = 1, 2, \dots, k\}, \tag{3.86}$$

and for any value within the admissible set $\mathcal{A}'$ Theorem 3.30 remains to hold.

For the constrained case the definition of the controller system is further modified to

$$\begin{aligned} \dot{\eta} &= (\zeta - \zeta^*) \\ \mu &= \text{sat}(R(\zeta - \zeta^*) + \frac{\partial H_c}{\partial \eta}(\eta), \mu^-, \mu^+) \end{aligned} \tag{3.87}$$

and we obtain the following result. The closed-loop system composed of (3.1), (3.7), (3.8) and (3.87) is given as

$$\begin{aligned} \dot{x} &= -B\,\text{sat}\,(RB^T(\tfrac{\partial H}{\partial x}(x) - \tfrac{\partial H}{\partial x}(x^*)) + \tfrac{\partial H_c}{\partial \eta}(\eta)\,; \mu^-, \mu^+) + E\bar{d}, \\ \dot{\eta} &= B^T(\tfrac{\partial H}{\partial x}(x) - \tfrac{\partial H}{\partial x}(x^*)), \end{aligned} \tag{3.88}$$

Corollary 3.33. *Consider the same assumptions on H and H_c as in Theorem 3.30. Then for any output vector $\frac{\partial H}{\partial x}(x^*)$ belonging to the admissible set $\mathcal{A}$, the trajectories of the closed-loop system* (3.88) *converge to*

$$\mathcal{E}_{\text{tot}} = \{(x^*, \bar{\eta}) \mid B\,\text{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta})\,; \mu^-, \mu^+) = E\bar{d}\}. \tag{3.89}$$

if and only if the in/outflows $E\bar{d}$ are manageable.

Proof. By Lemma 3.11, it is equivalent to prove that the closed-loop system (3.1),(3.7),(3.8),(3.87) converges to (3.89) if and only if the network satisfies the interior point condition with $E\bar{d} = 0$. By using the function

$$V(x, \eta) = \mathbb{1}^T S(\eta) + H(x) - \frac{\partial^T H}{\partial x}(x^*)(x - x^*) - H(x^*), \tag{3.90}$$

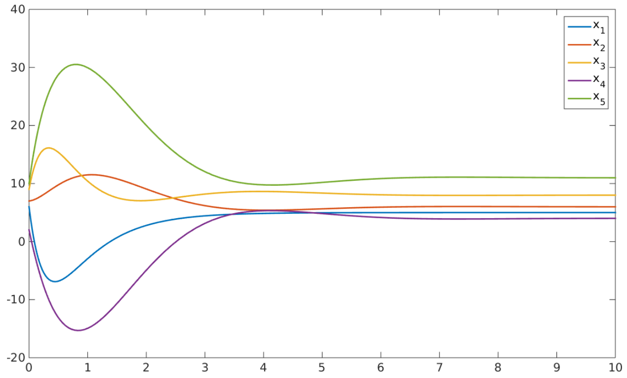


Figure 3.9: The trajectories of the state variables of dynamical system in Example 3.9.

as Lyapunov candidate function, where $S(\eta)$ is given as in (3.41), the conclusion follows from Theorem 3.12 and the fundamental theorem of calculus. $\square$

Example 3.9 (Hydraulic network of Example 3.3 continued)**.** Instead of driving the levels of the fluid in the reservoirs to consensus, the controller (3.79) is able to drive the fluid in the reservoirs to arbitrary levels belonging to the admissible set $\mathcal{A}$. Assume the initial levels are $6, 7, 9, 2$, and 10, while the desirable levels are $5, 6, 8, 4$, and 11. It can be seen from Figure 3.9 that the fluid levels in the reservoirs converge to the desirable values.

If the flows are constrained, the dynamical system is given as in (3.88). It is shown in Figure 3.10 that, the fluid in the reservoirs converge to the desirable levels $5, 6, 8, 4$, and 11 still.

3.7 Two corollaries of the interior point condition

In this section, we consider two corollaries of the interior point condition.

The first corollary concerns a special type of networks with bidirectional flow constraints on all the edges, i.e., $\mu_i^- < 0 < \mu_i^+, i = 1, 2, \ldots, m$. For this case, the result in Theorem 3.12 can be formulated as follows.

Corollary 3.34. *Consider the closed-loop system* (3.34) *and constraint intervals* $[\mu_i^-, \mu_i^+]$ *satisfying* $\mu_i^- < 0 < \mu_i^+, i = 1, \ldots, m$. *Then the trajectories will converge to the set* (3.33) *if and only if the network is* weakly connected.

Proof. If the constraint intervals satisfy $\mu_i^- < 0 < \mu_i^+, i = 1, \ldots, m$, then the network satisfies the interior point condition, by taking $z = 0$ in Definition 3.10, if

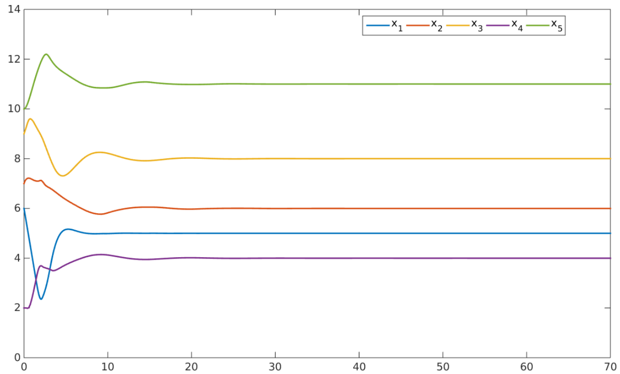


Figure 3.10: The trajectories of the state variables of dynamical system in Example 3.9.

and only if it is weakly connected. Hence the conclusion follows from Theorem 3.12. □

For the rest of this section we consider the closed-loop system (3.34) with more specific constraint boundaries μ^-, μ^+, i.e., we assume the following.

Assumption 3.35. The constraint intervals satisfy $0 \leqslant \mu_i^- < \mu_i^+, i = 1, \ldots, m$, and $\cap_{i=1}^m (\mu_i^-, \mu_i^+) \neq \varnothing$.

The next result can be seen as an extension of Theorem 3.12.

Corollary 3.36. *Consider the system* (3.34) *defined on a digraph $\mathcal{G}$. Then for any flow constraints μ^- and μ^+ satisfying Assumption 3.35, the trajectories of (3.34) converge to*

$$\Omega = \{(\bar{x}, \bar{\eta}) \mid \frac{\partial H}{\partial x}(\bar{x}) = \alpha \mathbb{1},\ \alpha \in \mathbb{R},\ B\,\mathrm{sat}(\frac{\partial H_c}{\partial \eta}(\bar{\eta})\,; \mu^-, \mu^+) = 0\,\} \tag{3.91}$$

if and only if the digraph $\mathcal{G}$ is weakly connected *and* balanced.

In order to prove Theorem 3.36 we need the following lemma. Recall that a digraph is *balanced* if every vertex has in-degree (number of incoming edges) equal to its out-degree (number of outgoing edges). Furthermore, we will say that two *circuits* of a graph are *non-overlapping* if they do not have any edges in common.

Lemma 3.37. *A weakly connected digraph $\mathcal{G}$ is balanced if and only if it can be covered by non-overlapping positive circuits, i.e., there exist positive circuits $\mathcal{C}_1, \ldots, \mathcal{C}_k$, satisfying $\xi_{\mathcal{C}_i} \in \mathbb{R}^m_{\geqslant 0}$ and $\mathbb{1}_m = \xi_{\mathcal{C}_1} + \ldots + \xi_{\mathcal{C}_k}$, while $\mathrm{supp}(\xi_{\mathcal{C}_i}) \cap \mathrm{supp}(\xi_{\mathcal{C}_j}) = \varnothing$ for all $i, j = 1, \ldots, k$ with $i \neq j$.*

Proof. Sufficiency: If a graph can be covered by non-overlapping positive circuits, then $\mathbb{1}_m \in \ker B$, i.e., every vertex has the same in-degree and out-degree, hence by Lemma 2.1 this graph is balanced.

Necessity: Since any weakly connected and balanced digraph is also strongly connected, the digraph $\mathcal{G}$ contains positive circuits. Consider a circuit $\mathcal{C}_1 \subset \mathcal{G}$, and suppose the set of the columns of the incidence matrix B is minimal linearly dependent. I.e., if $\mathcal{G}$ itself is a cycle, then $\mathbb{1}_m = \xi_{\mathcal{C}_1}$. Otherwise we continue with the vector $\mathbb{1}_m - \xi_{\mathcal{C}_1} \in \ker B \cap \mathbb{R}^m_{\geqslant 0}$. If the set $\{B_i \mid i \in \text{supp}(\mathbb{1}_m - \xi_{\mathcal{C}_1})\}$ is minimal linearly dependent, then $\mathbb{1}_m - \xi_{\mathcal{C}_1} = \xi_{\mathcal{C}_2}$ where $\mathcal{C}_2$ is a positive circuit. Otherwise we conduct the previous analysis with $\mathbb{1}_m - \xi_{\mathcal{C}_1} - \xi_{\mathcal{C}_2}$. Since the digraph $\mathcal{G}$ is finite, this analysis will stop after a finite number of steps. Hence we have $\mathbb{1}_m = \xi_{\mathcal{C}_1} + \ldots + \xi_{\mathcal{C}_k}$. □

Proof of Corollary 3.36. Notice that by Theorem 3.12, it is equivalent to prove that for any constraint interval satisfying Assumption 3.35, the network satisfies the interior point condition if and only if the digraph $\mathcal{G}$ is weakly connected and balanced.
Sufficiency:

Since the graph $\mathcal{G}$ is balanced, it follows that $\mathbb{1} \in \ker B$. By Assumption 3.35, there exists $\alpha \in \mathbb{R}$ such that $\alpha\mathbb{1} \in (\mu^-, \mu^+)$. Hence the network satisfies the interior point condition for $\alpha\mathbb{1}$, i.e., $\mathcal{E}_0(\alpha\mathbb{1}; \mu^-, \mu^+)$ is weakly connected. Then the conclusion follows from Theorem 3.12.
Necessity: It is equivalent to prove that if for any constraint interval satisfying Assumption 3.35, the network satisfies the interior point condition, then the digraph is weakly connected and balanced. The weak connectedness is straightforward. In fact it is proved in Lemma 3.16 that the digraph $\mathcal{G}$ is actually strongly connected.

Since the digraph is strongly connected, we have that the set $\ker B \cap \mathbb{R}^m_{\geqslant 1}$ is nonempty. Let us consider the convex programming problem

$$\min_{\omega \in \ker B \cap \mathbb{R}^m_{\geqslant 1}} \|\omega\|_\infty \tag{3.92}$$

where the minimal value, denoted as T, is finite and is attained (see [62], Theorem 27.2). If $T > 1$, clearly $[\mathbb{1}, T\mathbb{1}]$ satisfies Assumption 3.35. Furthermore, it is proved in Lemma 3.14 that there exists a vector $z \in \ker B \cap (\mathbb{1}, T\mathbb{1})$ if the network satisfies the interior point condition. Then we have that

$$\max_i z_i < T.$$

This is a contradiction to the fact that T is the minimal value of (3.92). Hence $T = 1$ which implies $\mathbb{1} \in \ker B$, i.e., the digraph $\mathcal{G}$ is balanced.

□

The above proof is illustrated by the following example.

Example 3.10. Consider the dynamical system (3.34) defined on a digraph given as in Figure 3.2 with $H(x) = \frac{1}{2}\|x\|_2^2$, $H_c(\eta) = \frac{1}{2}\|\eta\|_2^2$ and $[\mu^-, \mu^+] = [\mathbb{1}_7, 3\mathbb{1}_7]$, that is

$$\begin{aligned} \dot{x} &= -B\,\text{sat}(RB^T x + \eta\,; \mathbb{1}_7, 3\mathbb{1}_7), \\ \dot{\eta} &= B^T x. \end{aligned} \tag{3.93}$$

The purpose of this example is to show that there exists an initial condition $x(0)$ and $\eta(0)$ such that $x(t)$ does not converge to consensus as $t \to \infty$. By taking $x(0) = (3, 7, 5, 1, 4)^T$ and $\eta(0) = (1, -1, -1, 1, 1, 1, 1)^T$, the state x in system (3.93) will converge to the vector ν with $\nu_2 = \nu_3 > \nu_5 > \nu_4 > \nu_1$ as is illustrated by the numerical simulation in Figure 3.11. This is also deduced by the following

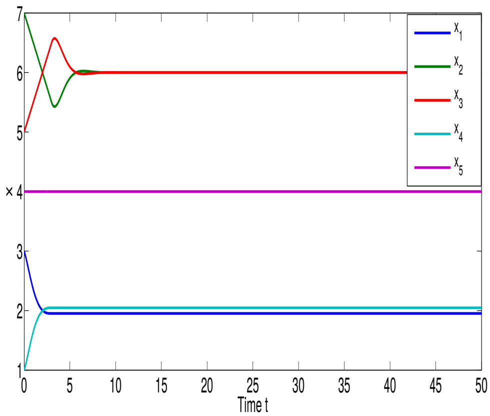


Figure 3.11: The time-evolutions $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$ of the system (3.93).

analysis. Indeed, a minimizing vector for $\min_{\omega \in \ker B \cap \mathbb{R}^m_{\geqslant 1}} \|\omega\|_\infty$ is

$$\omega = (1, 2, 3, 1, 1, 1, 1)^T$$

Hence, if we take the lower and upper bound as 1 and 3 respectively, then $\mathcal{E}_1 = \{e_3\}, \mathcal{E}_2 = \{e_1, e_4, e_5, e_6, e_7\}$. By setting $\nu_2 = \nu_3 > \nu_5 > \nu_4 > \nu_1$, the flow in e_3 reaches its upper bound, while the flows in e_1, e_4, e_5, e_6, e_7 reach their lower bounds, i.e.,

$$\text{sat}(-B^T\nu - B^T\nu t - \tilde{\eta}(0)\,; \mathbb{1}, 3\mathbb{1}) = T,\ \forall t > 0. \tag{3.94}$$

Thus there exists an equilibrium ν satisfying $B^T\nu \neq 0$.

3.8 Conclusions

We have discussed a basic model of dynamical distribution networks where the flows through the edges are controlled by distributed PI controllers, and satisfy flow constraints. The main result of this chapter is that the *manageability* of the in/outflows is a sufficient and necessary condition for asymptotic output agreement. For the case without in/outflows, manageability is equivalent to an interior point condition which depends on the graphical structure of the network and on the constraint intervals. The analysis is based on the construction of Lyapunov functions and LaSalle's Invariance principle. By a modification of the PI controller, under the additional assumption of convexity of the Hamiltonian function, it is shown how the outputs of the system can be driven to any desirable vector in an admissible set.

We also established a relation between a distribution network with flow constraints and a static optimization problem, where the latter can be used as a test for the interior point condition. Furthermore, by employing the notion of equilibrium-independent passivity, it is shown how the Lyapunov function can be obtained as the total storage function of the plant and controller system.

An obvious open problem is how to handle the distribution network with time-varying flow constraints. Another problem concerns the extension of the obtained results to the case where the in/outflows are not assumed to be constant, but are e.g. periodic functions of time; see already [54].

Chapter 4

Output agreement of dynamical distribution networks with state constraints

4.1 Introduction

In this chapter we continue our study of the dynamics of distribution networks. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to *storage*, and with every edge a control input variable corresponding to *flow*, possibly subject to constraints. In the previous chapter it has been shown under which conditions a constrained proportional-integral (PI) controller will regulate the system towards output agreement in the presence of unknown constant external disturbances (corresponding to constant in/outflows of the network).

In many cases of practical interest it is natural to require that the state variables of the distribution network will remain larger than a given minimal value, e.g. zero. A hydraulic network with state variables being the storage of fluid is a clear example of such a situation. On the other hand, the previously developed PI-controller can give rise to damped oscillatory behavior which may violate such state constraints. The aim of the current chapter is to modify the PI-controller in such a way that the lower bounds for the state variables will be satisfied for all time, while the system will still converge to output agreement. This is done by adapting the constraints of the PI controller.

The main related work for this chapter can be summarized as follows. In [54] the same problem is studied. However, the approach taken in [54] does not respect mass conservation. In [13], the authors considered a different model by distinguishing two types of flows. By using a saturated network-decentralized control protocol, the distribution network can be asymptotically stable to consensus.

The structure of this chapter is as follows. In Section 4.2, we formulate the problem as the adaptation of the constraints of the PI-controller such that the system will reach output agreement while the state variables satisfy the state constraints, and introduce the method we will use in this chapter. In Section 4.3 an optimal control protocol for the adaptation of the flow (control) constraints is developed. For the distribution network with the resulting flow constraints, we prove that all Filippov solutions satisfy the state constraints and converge to output agreement. The conclusions are contained in Section 4.4.

4.2 Problem formulation

As summarized in Sections 3.2 and 3.3, both the PI controller with flow constraints and without flow constraints are successful in obtaining output agreement for the plant (3.1). However, the PI controller introduces oscillatory behavior which may cause the state variables x to become smaller than some given lower bounds. For certain applications this may be undesirable or infeasible, as is illustrated by the following example.

Example 4.1 (Hydraulic network continued)**.** Consider the hydraulic network described in Examples 3.1 and 3.3. Here we consider the specific case that the reservoirs at each vertex of the network are cylindric. Then $x_i = S_i h_i, H_i = \frac{1}{2}\rho S_i g h_i^2 = \frac{\rho g}{2S_i} x_i^2$ where S_i is the bottom area, h_i is the height of liquid of ith reservoir respectively, and g is the gravity coefficient. Hence the *pressures* at the vertices are $P_i := \frac{\partial H}{\partial x_i}(x) = \rho g h_i = \frac{\rho g x_i}{S_i}, i = 1, \dots, n$. Although the PI controller (3.14) can regulate the pressure differences among the reservoirs, it also introduces an oscillatory behavior which can result in a negative value of the amount of fluid x (see e.g. Figure 3.3).

Motivated by the previous example, in order to keep the state variable of (3.16) satisfying certain constraints, the flow provided by the controller (3.14) needs to be regulated. In this section we focus our attention on system (3.1) with zero in/outflows $\bar{d}$ and controller (3.14), where $H_c(\eta) \in \mathbb{C}^1$. Furthermore, the Hamiltonian H is assumed to be of the form $H(x) = \sum_{i=1}^{n} H_i(x_i) \in \mathbb{C}^2$, where $H_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly convex. Moreover, we only consider the case that the graph $\mathcal{G}^o$ is a tree and weakly connected. Hence the number of the edges is $n-1$. Denote the set

$$\mathcal{Q} = \{(x, \eta) \in \mathbb{R}^{2n-1} \mid x_i \geqslant \gamma_i, i = 1, 2, \dots, n\}. \tag{4.1}$$

where $\gamma_i := \arg\min H_i(x_i)$. The control aim is to regulate the flow provided by the PI controller (3.14) such that output agreement of (3.1) is achieved asymptotically, while the evolution of the state variables $x_i(t)$ is such that $x_i(t) \in \mathcal{Q}$ for $t \geqslant 0$. More specifically, the flow we will design in this section is of the following form

$$\mathrm{sat}(\mu(x, \eta); -|\phi^*(x, \eta)|, |\phi^*(x, \eta)|), \tag{4.2}$$

where μ is the output of (3.14), i.e.,

$$\mu(x, \eta) = RB^T \frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial \eta}(\eta),$$

and $\phi^*(x, \eta)$ is the state-dependent saturation boundary to be designed.

Remark 4.1. Note that when $H(x) = H_1(x_1) + \ldots + H_n(x_n)$ and H_i are convex, the control aim can be equivalently formulated as keeping the vector $\frac{\partial H}{\partial x}$ in the positive orthant $\mathbb{R}^n_{\geqslant 0}$. Note that the *proportional* controller (3.10) automatically fulfills this control aim. In this case the closed-loop system (3.11) with $\bar{d} = 0$, which amounts to

$$\dot{x} = -BRB^T \frac{\partial H}{\partial x}(x),$$

remains in the set $\mathcal{Q}$ whenever $x_i(0) \geqslant \gamma_i$. This directly follows from the properties of the weighted Laplacian matrix BRB^T: whenever at a certain moment $\frac{\partial H_i}{\partial x_i}(x_i(t)) = 0$, then $\dot{x}_i(t) = -\sum_j r_k(\frac{\partial H_i}{\partial x_i}(x_i(t)) - \frac{\partial H_j}{\partial x_j}(x_j(t))) \geqslant 0$, where r_k is the k^{th} diagonal element of R and $e_k \sim (v_i, v_j)$. However for the second-order system (3.16), in order to achieve the control aim the flows on the edges need to be regulated.

4.3 The design of the flow constraints

This section is devoted to the design of the flow constraints $\phi^*(x, \eta)$ shown in (4.2). The main result of this section is stated as Theorem 4.5.

We start the analysis by dividing the vertices of the network into the following two subsets, referred to as *white* and *gray* vertices

$$\begin{aligned} \mathcal{V}^W(x) &= \{v_i \in \mathcal{V} \mid x_i > \gamma_i\} \\ \mathcal{V}^G(x) &= \{v_i \in \mathcal{V} \mid x_i \leqslant \gamma_i\}. \end{aligned} \tag{4.3}$$

Notice that this division is time-dependent.

The basic idea is to saturate the flows in the edges which are adjacent to the gray vertices, i.e., $v_i \in \mathcal{V}^G$, such that the state variable x_i is not decreasing. In the other edges the flows are unconstrained, i.e., are equal to μ which is the output of the PI-controller. This idea is formulated in Algorithm 4.1, which solves a finite number of optimization problems. Furthermore, the final solution of Algorithm 4.1, denoted as $\phi^*(x, \eta)$, gives us the flow constraints in (4.2).

Algorithm 4.1 Flow constraints algorithm

Initialization: $\phi^0 = \mu$,

while $\mathcal{V}^B(x, \phi^\ell) = \{v_i \mid x_i \leqslant \gamma_i \text{ and } B_{i\cdot}\phi^\ell < 0\} \neq \varnothing$ **do**

$$\phi^{\ell+1} = \arg\min_{\omega} \sum_{e_j \in \mathcal{E}^B_{out}(t)} \frac{1}{2|\phi^\ell_j|}\left(\left(\omega_j - \phi^\ell_j\right)^2 + \omega_j^2\right) \tag{4.4}$$

$$\text{s.t. } B_{i\cdot}\omega = 0, \forall v_i \in \mathcal{V}^B(x, \phi^\ell), \tag{4.5}$$

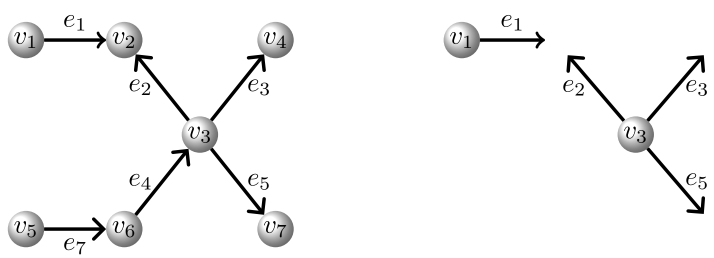
$$\omega_k = \phi^\ell_k, \forall e_k \in \mathcal{E} \setminus \mathcal{E}^B_{out}(x, \phi^\ell), \tag{4.6}$$

where

$$\begin{aligned} \mathcal{E}^B_{out}(x, \phi^\ell) &= \cup_{v_i \in \mathcal{V}^B(x)} f^{out}_{v_i}, \\ f^{out}_{v_i}(\phi^\ell) &= \{e_j \in \mathcal{E} \mid B_{ij}\phi^\ell_j > 0\}. \end{aligned} \tag{4.7}$$

Remark 4.2. We first show that in each step of Algorithm 4.1, the optimal solution exists. In fact, since each weakly connected component of the subgraph $\mathcal{G}^B = \{\mathcal{V}^B(x, \eta), \mathcal{E}^B_{out}(x, \eta)\}$ has at least one edge with one end node belonging to $\mathcal{V}^B(x, \eta)$, while the other one belongs to $\mathcal{V} \setminus \mathcal{V}^B(x, \eta)$. This implies that the coefficient matrix in the system of linear equations (4.5) has full row rank. Hence the vector $\phi^{\ell+1}$ exists. Furthermore, it can be verified that $0 < |\phi^{\ell+1}_k| < |\phi^\ell_k|$, for $e_k \in \mathcal{E}^B_{out}(x, \phi^\ell)$.

Example 4.2. Consider the digraph given as in Figure 4.1(a), and suppose the positive flows μ are assigned on each edge. If the black nodes are v_1, v_3, then the subgraph $\mathcal{G}^B = \{\mathcal{V}^B(x, \eta), \mathcal{E}^B_{out}(x, \eta)\}$ is given as in Figure 4.1(b). Notice that each component of $\mathcal{G}^B$ has at least one edge, from $\mathcal{V}^B(x, \eta)$ to $\mathcal{V} \setminus \mathcal{V}^B(x, \eta)$ or conversely, on which the flows are saturated.



(a) The whole graph

(b) The subgraph $\mathcal{G}^B = \{\mathcal{V}^B(x, \eta), \mathcal{E}^B_{out}(x, \eta)\}$

Figure 4.1: Network structure of Example 4.2.

Remark 4.3. Since the digraph $\mathcal{G}$ is acyclic, Algorithm 4.1 stops after a finite number of steps (at most $n-1$).

We denote the final result of Algorithm 4.1 as $\phi^*(x,\eta)$.

We can write Algorithm 4.1 in a more precise manner. Notice that the vectors $B_{i\cdot}$ for $v_i \in \mathcal{V}^B$ are linearly independent. Furthermore, for each node in $\mathcal{V}^B$, the $\phi_k^{\ell+1}$s on the outgoing edges e_ks only depend on the ϕ^ℓ on the incoming edges. Hence the optimization problem (4.4),(4.5),(4.6) can be solved in a distributed fashion, i.e., it can be solved independently for each vertex in $\mathcal{V}^B$. Indeed, for each $v_i \in \mathcal{V}^B(x,\phi^\ell)$ we denote $f_{v_i}^{in}(\phi^\ell) = \{e_j \in \mathcal{E} \mid B_{ij}\phi_j^\ell \leqslant 0\}$. Then the optimal solution $\phi_j^{\ell+1}$ with $e_j \in f_{v_i}^{out}(\phi^\ell)$ satisfies

$$\phi^{\ell+1}|_{f_{v_i}^{out}(\phi^\ell)} = \arg\min_{\omega} \sum_{e_j \in f_{v_i}^{out}(\phi^\ell)} \frac{1}{2|\phi_j^\ell|}\Big((\omega_j - \phi_j^\ell)^2 + \omega_j^2\Big) \tag{4.8}$$

$$\text{s.t.} \sum_{e_j \in f_{v_i}^{out}(\phi^\ell)} B_{ij}\omega_j + \sum_{e_k \in f_{v_i}^{in}(\phi^\ell)} B_{ik}\phi_k^\ell = 0. \tag{4.9}$$

By the standard Lagrange multiplier method, the optimal solution $\phi_j^{\ell+1}$ is given as

$$\phi_j^{\ell+1} = \frac{\sum_{e_k \in f_{v_i}^{in}(\phi^\ell)} |\phi_k^\ell|}{\sum_{e_j \in f_{v_i}^{out}(\phi^\ell)} |\phi_j^\ell|}\phi_j^\ell,\ e_j \in f_{v_i}^{out}(\phi^\ell). \tag{4.10}$$

Hence Algorithm 4.1 can be written as follows.

Algorithm 4.2 Flow constraints algorithm

1: Initialization: $\phi^0 = \mu$,
2: **while** $\mathcal{V}^B(x,\phi^\ell) = \{v_i \mid x_i \leqslant \gamma_i \text{ and } B_{i\cdot}\phi^\ell < 0\} \neq \varnothing$ **do**
3:

$$\phi_j^{\ell+1} = \frac{\sum_{e_k \in f_{v_i}^{in}(\phi^\ell)} |\phi_k^\ell|}{\sum_{e_j \in f_{v_i}^{out}(\phi^\ell)} |\phi_j^\ell|}\phi_j^\ell, \qquad e_j \in f_{v_i}^{out}(\phi^\ell), \tag{4.11}$$

$$\phi_k^{\ell+1} = \phi_k^\ell, \qquad e_k \in \mathcal{E} \setminus \mathcal{E}_{out}^B(x,\phi^\ell). \tag{4.12}$$

An intuitive example of the previous analysis is given as follows.

Example 4.3. Let us consider a part of the network given as given in Fig.4.2. This example shows how the flow is regulated at each step of Algorithm 4.2. Suppose that at time t the state variable at v_2, i.e., $x_2(t)$, is equal to γ_2, and the flows on e_1, e_2, e_3 take the positive values $\phi_1^\ell, \phi_2^\ell, \phi_3^\ell$. Furthermore, assume $\phi_1^\ell - \phi_2^\ell - \phi_3^\ell < 0$. Then at the $\ell+1$th step of Algorithm 4.2, the flows on e_2 and e_3 are saturated to

$\frac{|\phi_1^\ell|}{|\phi_2^\ell|+|\phi_3^\ell|}\phi_2^\ell$ and $\frac{|\phi_1^\ell|}{|\phi_2^\ell|+|\phi_3^\ell|}\phi_3^\ell$ respectively.

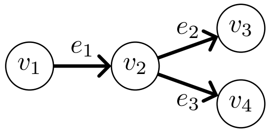


Figure 4.2: Explanation about how the outgoing flows of *black* vertices are regulated.

Lemma 4.4. *For the acyclic digraph $\mathcal{G}$, the flow constraint function $\phi^*(x,\eta)$ given by Algorithm 4.1 is continuous at almost every $(x,\eta) \in \mathbb{R}^{2n-1}$. More precisely, the function $\phi^*(x,\eta)$ is continuous on $\mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$ where*

$$\mathcal{Q}_x = \mathcal{Q}_{x_1} \cup \cdots \cup \mathcal{Q}_{x_n} \tag{4.13}$$

and

$$\mathcal{Q}_{x_i} = \{(x,\eta) \mid x_i = \gamma_i\}. \tag{4.14}$$

Proof. Notice that $\mathcal{Q}_x$ is composed of a finite number of hyperplanes. Hence it has zero measure in $\mathbb{R}^{2n-1}$. We will prove that ϕ^* is continuous at any $(x,\eta) \in \mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$.

We start the analysis with the first step of Algorithm 4.1 where $\phi^0 = \mu$.

For the flow ϕ^0, we define the following two sets

$$\begin{aligned} \mathcal{Q}_{B\phi^0} &= \mathcal{Q}_{B_1.\phi^0} \cup \cdots \cup \mathcal{Q}_{B_n.\phi^0}, \\ \mathcal{Q}_{\phi^0} &= \mathcal{Q}_{\phi_1^0} \cup \cdots \cup \mathcal{Q}_{\phi_m^0}, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \mathcal{Q}_{B_i.\phi^0} &= \{(x,\eta) \mid B_i.\phi^0 = 0\}, \\ \mathcal{Q}_{\phi_i^0} &= \{(x,\eta) \mid \phi_i^0 = 0\}. \end{aligned} \tag{4.16}$$

We will show that the vector-valued function ϕ^1 is continuous at $(x,\eta) \in \mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$ by analyzing three cases:

1. For any $(x,\eta) \in \mathbb{R}^{2n-1} \setminus (\mathcal{Q}_x \cup \mathcal{Q}_{B\phi^0} \cup \mathcal{Q}_{\phi^0})$, there exists a δ such that for any $(x',\eta') \in \mathcal{B}\big((x,\eta),\delta\big)$, the sets $\mathcal{V}_B$ and $\mathcal{E}_{out}^B$ are constant, i.e.,

$$\begin{aligned} \mathcal{V}_B(x',\phi^0(x',\eta')) &= \mathcal{V}_B(x,\phi^0(x,\eta)) \\ \mathcal{E}_{out}^B(x',\phi^0(x',\eta')) &= \mathcal{E}_{out}^B(x,\phi^0(x,\eta)). \end{aligned} \tag{4.17}$$

For fixed sets $\mathcal{V}_B(x,\phi^0)$ and $\mathcal{E}_{out}^B(x,\phi^0)$, the optimal solution ϕ^1 is a continuous function of ϕ^0, hence a continuous function at $(x,\eta) \in \mathbb{R}^{2n-1} \setminus (\mathcal{Q}_x \cup$

$\mathcal{Q}_{B\phi^0} \cup \mathcal{Q}_{\phi^0}$). Furthermore, $\phi_i^1 \neq 0$ for all $i = 1, \ldots, m$.

2. For any $(x, \eta) \in \mathcal{Q}_{\phi^0} \cap (\mathbb{R}^{2n-1} \setminus \mathcal{Q}_x)$, by the fact that $|\phi_k^1| \leqslant |\phi_k^0|$, we have that
$$\lim_{(x',\eta')\to(x,\eta)} \phi^1(x', \eta') = \phi^1(x, \eta).$$

3. For any $(x, \eta) \in \mathcal{Q}_{B\phi^0} \cap (\mathbb{R}^{2n-1} \setminus \mathcal{Q}_x)$, i.e., $B_{i\cdot}\phi^0(x, \eta) = 0$ for some vertices $v_i \in \mathcal{V}$, then for any $e_k = f_{v_i}^{out}$ we have $\phi_k^1 = \phi_k^0$. For any $(x', \eta') \in \mathcal{B}((x, \eta), \delta)$, if $B_{i\cdot}\phi^0(x', \eta') > 0$, then $\phi_k^1(x', \eta') = \phi_k^0(x', \eta')$ for any $e_k \in f_{v_i}^{out}$; if $B_{i\cdot}\phi^0(x', \eta') < 0$, then $\phi_k^1(x', \eta') = \frac{\sum_{e_j \in f_{v_i}^{in}} |\phi_j^0(x',\eta')|}{\sum_{e_j \in f_{v_i}^{out}} |\phi_j^0(x',\eta')|} \phi_k^0(x', \eta')$ for $e_k \in f_{v_i}^{out}$. They all converge to $\phi_k^1(x, \eta) = \phi_k^0(x, \eta)$ as $(x', \eta') \to (x, \eta)$.

Hence ϕ^1 is continuous at $(x, \eta) \in \mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$. Repeating the previous analysis, we have that $\phi^{\ell+1}$ is continuous at any $(x, \eta) \in \mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$. Since the Algorithm 4.1 stops after a finite number of steps (at most $n - 1$), we have that ϕ^* is continuous at any $(x, \eta) \in \mathbb{R}^{2n-1} \setminus \mathcal{Q}_x$. □

Now the modification of the closed-loop system (3.16) with the time-varying flow constraints can be written as

$$\begin{aligned} \dot{x} &= -B\,\mathrm{sat}_{\phi^*(x,\eta)}\big(RB^T \frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial \eta}(\eta)\big) \\ \dot{\eta} &= B^T \frac{\partial H}{\partial x}(x), \end{aligned} \tag{4.18}$$

where $\mathrm{sat}_{\phi^*(x,\eta)}(\mu) = \mathrm{sat}(\mu; -|\phi^*(x, \eta)|, |\phi^*(x, \eta)|)$.

Since the right-hand side of (4.18) is discontinuous, we consider its solution in the Filippov sense, i.e., an absolutely continuous function $(x(t), \eta(t))$ for $t \geqslant 0$ which satisfies the following differential inclusion almost everywhere.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{bmatrix} \in \begin{bmatrix} \mathcal{F}[-B\,\mathrm{sat}_{\phi^*(x,\eta)}(RB^T \frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial \eta}(\eta))](x,\eta) \\ \mathcal{F}[B^T \frac{\partial H}{\partial x}](x) \end{bmatrix} \tag{4.19}$$

By the matrix transformation rule (2.36) and the continuity of $\frac{\partial H}{\partial x}(x)$, the previous differential inclusion can be equivalently written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{bmatrix} \in \begin{bmatrix} -B\mathcal{F}[\mathrm{sat}_{\phi^*(x,\eta)}(RB^T \frac{\partial H}{\partial x}(x) + \frac{\partial H_c}{\partial \eta}(\eta))](x,\eta) \\ B^T \frac{\partial H}{\partial x}(x). \end{bmatrix}$$

The existence of a complete solution of (4.19) follows from the fact that $\mathrm{sat}_{\phi^*(t)}(\mu(t)) \leqslant \mu(t)$ and the system (3.16) has complete solutions.

We can delete the points in $\mathcal{Q}_x$ to simplify the calculation of the Filippov set-valued maps in (4.19).

Before continuing, let us make the following observation. By Lemma 4.4, the saturation bound functions $\phi^*(x,\eta)$ are piece-wise continuous in $\mathbb{R}^{2n-1}$. Thanks to Proposition 2.25, any Krasovskii solution of (4.19) is a Filippov solution. Hence the Carathéodory solution of (4.18) [1] being a Krasovskii solution is also a Filippov solution.

Theorem 4.5. *Consider the system (4.19) on a digraph $\mathcal{G}$ with saturation bounds $\phi^*(x,\eta)$ given by Algorithm 4.1. Assume that $H(x) = \sum_{i=1}^n H_i(x_i) \in \mathbb{C}^2$ and $H_c \in \mathbb{C}^1$ are radially unbounded. Furthermore, assume H_i is strictly convex with* $\arg\min H_i(x_i) = \gamma_i, i = 1, \dots, n$. *Then*

(i) for any Filippov solution $(x(t),\eta(t))$ satisfying $x(0) \in \operatorname{int}\mathcal{Q}$, we have $x(t) \in \mathcal{Q}$ for all $t > 0$;

(ii) all the trajectories will converge to an element of the output agreement set

$$\mathcal{E}_{\text{tot}} = \{(x,\eta) \mid \frac{\partial H}{\partial x}(x) = \alpha\mathbb{1},\ \alpha \in \mathbb{R}^+,\ B\frac{\partial H_c}{\partial \eta}(\eta) = 0\,\}.$$

if and only if $\mathcal{G}$ is weakly connected.

Proof. In order to simplify the exposition, we denote the right-hand side of (4.18) as $X(x,\eta)$, and denote

$$\mathcal{F}_x[X](x,\eta) := -B\mathcal{F}[\text{sat}_{\phi^*(x,\eta)}\left(\mu\right)](x,\eta). \tag{4.20}$$

(i) Recall the definitions (4.14) of $\mathcal{Q}_{x_i}$ and (4.1) of $\mathcal{Q}$. Denote the open regions of $\mathbb{R}^{2n-1}$ divided by the hyperplanes $\mathcal{Q}_{x_1},\dots,\mathcal{Q}_{x_n}$ as $\mathcal{R}_1,\dots,\mathcal{R}_N$ with $\mathcal{R}_1 = \operatorname{int}\mathcal{Q}$.

For any $(x,\eta) \in \mathcal{R}_i \setminus \mathcal{Q}_\mu, i = 1,\dots,N$, by the continuity of $\mathcal{E}^B_{out}$ and ϕ^* we have that the right-hand side of (4.18) is continuous. More specifically, for any $(x,\eta) \in \mathcal{R}_i \setminus \mathcal{Q}_\mu, i = 2,\dots,N$, without loss of generality, suppose $x_{i_j} < \gamma_{i_j}, j = 1,\dots,k$ and $x_{i_j} > \gamma_{i_j}, j = k+1,\dots,n$, we have

$$\mathcal{F}_x[X](x,\eta)_{i_j} = \{0\}, j = 1,\dots,k. \tag{4.21}$$

Hence for any $(x,\eta) \in \mathcal{Q}_x$, there exists a vector $\nu \in \mathcal{F}[X](x,\eta)$ which is tangent to $\mathcal{Q}_x$. Therefore, by the result in [24, p. 52] (in the section "Piecewise Continuous Vector Fields and Sliding Motions"), the trajectory $(x(t),\eta(t))$ will not cross $\mathcal{Q}_x$ but slide on it or return to $\mathcal{Q}$.

[1] A Carathéodory solution of (4.18) is an absolutely continuous function that satisfies (4.18) for almost all $t > 0$.

(ii) *Sufficiency.*

Consider the Hamiltonian function

$$V(x,\eta) := H(x) + H_c(\eta), \tag{4.22}$$

as candidate Lyapunov function. Notice that V is differentiable. Hence the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}[X]}V : \mathbb{R}^{2n-1} \to 2^{\mathbb{R}}$ of V with respect to $\mathcal{F}[X]$ at $(x(t),\eta(t))$ is given as

$$\begin{aligned}
\tilde{\mathcal{L}}_F V &= \{(\nabla V)^T \nu \mid \nu \in \mathcal{F}[X](x(t),\eta(t))\} \\
&= \frac{\partial^T H}{\partial x}(x(t))\mathcal{F}_x[X](x(t),\eta(t)) + \frac{\partial^T H}{\partial x}(x)B\frac{\partial H_c}{\partial \eta}(\eta(t)) \\
&= \sum_{i=1}^{n} -\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\mathcal{F}[\mathrm{sat}_{\phi^*(x,\eta)}(\mu_i)](x(t),\eta(t)) \\
&+ \frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\frac{\partial H_c}{\partial \eta_i}(\eta(t))
\end{aligned} \tag{4.23}$$

We calculate the Filippov set-valued map $\mathcal{F}[\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)](x(t),\eta(t))$ by considering two cases, i.e., $|\phi_i^*(x,\eta)| = |\mu_i(x,\eta)|$ and $|\phi_i^*(x,\eta)| < |\mu_i(x,\eta)|$.

For the first case, consider two subcases. If $\mu_i(x,\eta) = 0$, by the fact that $|\phi_i^*(x,\eta)| \leqslant |\mu_i(x,\eta)|$ we have that $\mathcal{F}[\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)](x(t),\eta(t)) = \{\mu_i\} = \{0\}$. If $|\mu_i(x,\eta)| > 0$ and $e_i \in f_{v_j}^{out}$ with $x_j > \gamma_j$, then $\mathcal{F}[\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)](x(t),\eta(t)) = \{\mu_i\}$. If $|\mu_i(x,\eta)| > 0$ and $e_i \in f_{v_j}^{out}$ with $x_j \leqslant \gamma_j$, then by Algorithm 4.1 we have $B_{j\cdot}\mu \geqslant 0$. Hence $\mathcal{F}[\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)](x(t),\eta(t)) = \{\mu_i\}$, which follows from Lemma 4.4. In conclusion, the Filippov set-valued map in this case is the singleton $\{\mu_i\}$, and

$$\begin{aligned}
&-\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\mathcal{F}\big[\,\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)\big](x(t),\eta(t)) + \frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\frac{\partial H_c}{\partial \eta_i}(\eta(t)) \\
&= -\frac{\partial^T H}{\partial x}(x)B_{\cdot i}B_{\cdot i}^T\frac{\partial H}{\partial x}(x).
\end{aligned} \tag{4.24}$$

For the second case, we have that any element ν_i in the set $\mathcal{F}\big[\,\mathrm{sat}_{\phi_i^*(x,\eta)}(\mu_i)\big](x(t),\eta(t))$ can be written as

$$\nu_i = (1-\lambda_i)0 + \lambda_i(\mu_i(t)), \text{ for some } \lambda_i \in [0,1]. \tag{4.25}$$

Then

$$\begin{aligned}
&-\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\mathcal{F}\big[\operatorname{sat}_{\phi_i^*(x,\eta)}(\mu_i)\big](x(t),\eta(t))\\
&+\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\frac{\partial H_c}{\partial \eta_i}(\eta(t))\\
=&\{-\lambda_i\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}B_{\cdot i}^T\frac{\partial H}{\partial x}(x(t))+(1-\lambda_i)\frac{\partial^T H}{\partial x}(x(t))B_{\cdot i}\frac{\partial H_c}{\partial \eta_i}(\eta(t))\\
&\mid \lambda_i\in[0,1]\}
\end{aligned} \tag{4.26}$$

By Algorithm 4.1, e_i is an outgoing edge of a *grey* vertex. Then we have either

(a) $B_{\cdot i}^T\frac{\partial H}{\partial x}(x)\geqslant 0$ and $\mu_i<0$, or

(b) $B_{\cdot i}^T\frac{\partial H}{\partial x}(x)\leqslant 0$ and $\mu_i>0$.

Both (a) and (b) imply that $\frac{\partial^T H}{\partial x}(x)B_{\cdot i}\frac{\partial H_c}{\partial \eta_i}(\eta)\leqslant -\frac{\partial^T H}{\partial x}(x)B_{\cdot i}B_{\cdot i}^T\frac{\partial H}{\partial x}(x)$. Hence any element in the set (4.26) is smaller than or equal to $-\frac{\partial^T H}{\partial x}(x)B_{\cdot i}B_{\cdot i}^T\frac{\partial H}{\partial x}(x)$.

Hence we conclude that $\max\tilde{\mathcal{L}}_F V_{\bar{d}}(x,\eta)\leqslant -\frac{\partial^T H}{\partial x}(x)BB^T\frac{\partial H}{\partial x}(x)$.

By LaSalle's Invariance Principle any Filippov solution of (4.19) will converge to the largest invariant set, denoted as Ω, within the set $\{(x,\eta)\mid \dot{V}=0\}$, i.e., $\{(x,\eta)\mid B^T\frac{\partial H}{\partial x}(x)=0\}$. Within Ω we have

$$-B^T\frac{\partial^2 H}{\partial x^2}B\operatorname{sat}_{\phi^*(x(t),\eta(t)}(\frac{\partial H_c}{\partial \eta}(\eta(t)))=0, \tag{4.27}$$

which implies that x remains at a constant value in Ω, denoted by $\bar{x}$, for which $\frac{\partial H}{\partial x}(\bar{x})=\alpha\mathbb{1}$. Furthermore, in view of $\mathbb{1}^T\bar{x}>\mathbb{1}^T\gamma$ and the convexity of H, it follows that $\alpha>0$. By the optimal control protocol given in Algorithm 4.1, we have that all the vertices will be *white* for large enough t, which implies that at steady state $B\frac{\partial H_c}{\partial \eta}(\eta)=0$.

Necessity. If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value α will be different for different components.

□

Example 4.4. Consider the dynamical system (4.19) defined on the graph given in Figure 4.1(a), with flow constraints given as in Algorithm 4.1. Take $H_i(x_i)=\frac{1}{2}x_i^2$ and $R=I$. The x-part of a Filippov solution of one of the trajectories of (4.19) for initial condition $[x(0),\eta(0)]=[0,0.5,1,2,0,5,9,3,0,-1,-2,-4]$ is given in Figure 3.6. It can be seen that the volume of each reservoir is kept nonnegative for all times. Furthermore the pressures of the reservoirs converge to a common value.

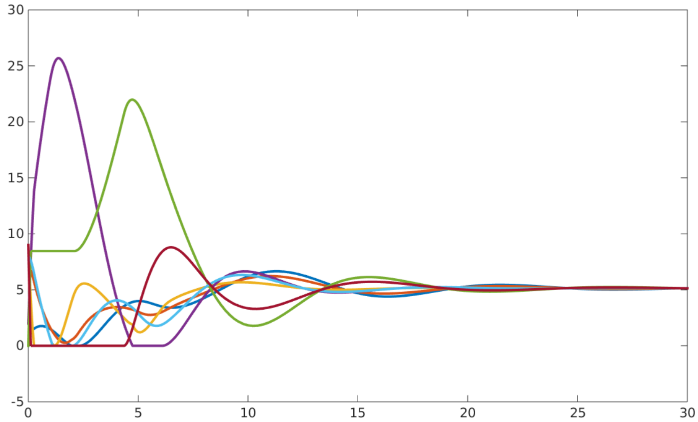


Figure 4.3: One solution of system (4.19) defined on the graph as in Figure 4.1(a) using the flow constraints given by Algorithm 4.1.

4.4 Conclusions

We have considered a basic model of dynamical distribution networks with state inequality constraints. We have formulated a distributed PI controller structure with time-varying flow constraints which achieves consensus and maintains the state constraints. The flow constraints have been expressed in terms of solutions of an optimization problem. We have discussed the existence of solutions for the system in the sense of Filippov, and carried out the stability analysis of the network by taking the Hamiltonian of the system as the Lyapunov function.

The results of this chapter can be extended in a straightforward way to the case where the flows on the edges obey a priori constraints. It is of interest to investigate the extension of the results of this chapter to general graphs containing circuits. One problem in such an extension is the fact that if the graph contains circuits, then Algorithm 4.1 requires an infinite number of steps. In order to guarantee the continuity of ϕ^*, we need the uniform convergence of ϕ^ℓ which is not clear yet.

Chapter 5

Consensus dynamics with arbitrary sign-preserving nonlinearities

5.1 Introduction

The consensus problem for multi-agent systems is one of the most active fields of systems and control theory since at least one decade. The basic continuous-time linear consensus protocol can be summarized as

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{J}_i} \alpha_{ij}(x_i(t) - x_j(t)) \tag{5.1}$$

where $\mathcal{J}_i$ represents the set of agents whose information is available to agent i, and α_{ij} denotes a positive weight. The protocol (5.1) can be written in matrix form as $\dot{x} = -Lx$ where L is the Laplacian matrix of the underlying digraph. It is well-known that consensus is asymptotically achieved if the digraph contains a directed spanning tree. This result has been extended to several other scenarios, including switching topologies and communication delays [51, 57, 59].

Apart from linear consensus protocols, also nonlinear consensus protocols have recently attracted attention of many researchers. The nonlinear consensus protocols may arise due to the nature of the controller, see e.g. [42, 46, 65] or may describe the physical coupling existing in the network, see e.g. [17, 49]. In general, starting from the model (5.1), the nonlinearity can appear in three places. Firstly, the measurement of the state x_i and x_j in the right-hand side of (5.1) can be nonlinear, i.e., $f_i(x_i)$ and $f_j(x_j)$ instead of x_i and x_j, for some nonlinear functions f_i, f_j. Secondly, we can replace the difference $x_i(t) - x_j(t)$ by a nonlinear function $f(x_i(t) - x_j(t))$. Thirdly, the dynamics of each agent can be nonlinear, i.e., $\dot{x}_i = f_i(-L_{i\cdot}x)$ instead of $\dot{x}_i = -L_{i\cdot}x$.

In this chapter, we investigate a general nonlinear consensus protocol which contains the previous three cases. The topology among the agents is assumed to be a directed graph containing a directed spanning tree, which, as mentioned above, for the linear consensus protocol is known to be a sufficient and necessary condition for reaching consensus.

There are many related works on nonlinear consensus protocols. In [45, 53], continuous nonlinear functions are considered, which is a special case of our set-up where the nonlinear functions are allowed to be discontinuous. Nonlinearities in the form of sign functions were considered in [26]. In [29], the authors considered a similar control protocol as in [26] in a hybrid dynamical systems framework with a self-triggered communication policy. In addition, in [29] practical consensus is considered, that is, the states of the agents are not exactly equal to but close enough to each other. The results in [26, 29] are restricted to undirected graphs.

As a special type of nonlinear consensus protocols, quantized consensus protocols have been studied from different viewpoints. In [20], the authors considered the case when the measurements of the states of the agents are quantized. In [31, 39], the authors investigate the case when the measurement of relative states of the agents are quantized. In this thesis, we will only focus on the first case.

The contributions of this chapter are twofold. Firstly, a general nonlinear consensus protocol is considered with the weakest topology assumption, i.e., a directed graph containing a directed spanning tree. Secondly, the nonlinear functions considered can be discontinuous. In order to study the behavior of the nonlinear consensus protocols for discontinuous nonlinearities, we employ the notion of Filippov solutions and prove the convergence to consensus.

The structure of this chapter is as follows.

The main results are presented in Section 5.2. The general problem is introduced in Section 5.2.1. Then in Sections 5.2.2 and 5.2.3, two important subcases are considered. Finally in Section 5.2.4, the results of Sections 5.2.2 and 5.2.3 are combined. In Section 5.3, we consider a special case when the underlying topology is strongly connected and the nonlinear functions are continuous. In this case we obtain a port-Hamiltonian formulation and prove stability within this framework. In Section 5.4, we consider the consensus protocol where we do not have precise measurement of the state of each agent, i.e., a quantized consensus protocol. In this section we extend the result in [20] to the directed graph case. The main results of the chapter are stated as Theorem 5.4, 5.9 and 5.15.

5.2 The model with precise measurement of the states

5.2.1 Problem formulation

In this chapter we consider a network of n agents, where the communication topology is given by a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. In this network, agent i receives information from agent j if and only if there is an edge from node v_j to node v_i in the graph $\mathcal{G}$. We denote the state of agent i at time t as $x_i(t) \in \mathbb{R}$, and consider the following dynamics for agent i

$$\dot{x}_i = f_i(\sum_{j=1}^{n} a_{ij} g_{ij}(x_j - x_i)) =: h_i(x), \tag{5.2}$$

where f_i and g_{ij} are functions from $\mathbb{R}$ to $\mathbb{R}$ and a_{ij} are the elements of the adjacency matrix A.

The functions f_i describe how agent i handles incoming information, while the functions g_{ij} are concerned with the flow of information along the edges. For the rest of the chapter, we assume the following.

Assumption 5.1. The functions f_i and g_{ij} are *sign-preserving* (see Definition 2.26) and *piecewise continuous*.

Notice that the previous assumption includes functions like the signum function sign and the saturation function sat.

To handle possible discontinuities in the right-hand side of (5.2), we consider Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)). \tag{5.3}$$

Here we assume the existence and completeness of Filippov solutions of (5.3) for any initial condition. Notice that when the functions h_i are globally bounded, e.g. if f_i and g_{ij} are chosen as signum or saturation functions, then the completeness of Filippov solution of (5.3) is guaranteed by Theorem 1 in Chapter 2 § 7 of [34].

The agents of the network are said to achieve *consensus* if they all converge to the same value, that is for any initial state x_0

$$\lim_{t\to\infty} x(t) = \eta \mathbb{1}$$

for some $\eta \in \mathbb{R}$, where $x(t) = [x_1(t), \ldots, x_n(t)]^T$ is any solution of (5.2) for $x(0) = x_0$. It is well known that if all functions f_i and g_{ij} are the identity function, then the agents will achieve consensus if and only if the graph $\mathcal{G}$ contains a directed spanning tree [1, 76]. In this chapter we investigate the consensus problem for general functions f_i and g_{ij} satisfying Assumption 5.1. First, in Section 5.2.2, we consider the special case that the functions g_{ij} are equal to the identity function, that is $\dot{x} = f_i(\sum_{j=1}^{n} a_{ij}(x_j - x_i))$. Thereafter, in Section 5.2.3, we consider the case where the functions f_i are the identity function, that is $\dot{x}_i = \sum_{j=1}^{n} a_{ij} g_{ij}(x_j - x_i)$. Finally, in section 5.2.4, we will combine these results.

The following examples motivate why the sign-preserving condition is needed for all functions f_i and g_{ij}.

(a) Digraph with spanning tree (b) Strongly connected digraph

Figure 5.1: Two digraphs with two nodes for Examples 5.1, 5.5 and 5.6.

Example 5.1. Consider the following system defined on the graph given in Fig. 5.1(a)

$$\begin{aligned} \dot{x}_1 &= f_1(0) \\ \dot{x}_2 &= f_2(x_1 - x_2), \end{aligned} \tag{5.4}$$

with $f_i = \mathrm{sat}(;0,1), i = 1,2$. Notice that the function $\mathrm{sat}(\cdot\,;0,1)$ is not sign preserving. In this case the existence of a directed spanning tree is not a sufficient condition for convergence to consensus. Indeed, if the initial condition satisfies $x_2(0) > x_1(0)$, then $x_1(t) = x_1(0)$ and $x_2(t) = x_2(0)$ for all $t \geqslant 0$. Hence, the agents do not reach consensus.

Example 5.2. The condition *(ii)* in Definition 2.26 can be motivated by the following counter example. Consider the system (5.4) defined on the digraph in Fig. 5.1(a) with f_i are defined as

$$f_i(y) = \begin{cases} y+1 & \text{if } y < -1, \\ y & \text{if } -1 \leqslant y \leqslant 1,\ i = 1,2. \\ y-1 & \text{if } y > 1, \end{cases} \tag{5.5}$$

Then the function f_i satisfies *(i)*, but not *(ii)* in Definition 2.26. Consider the point $x^* = [0,1]^T$, we have

$$\mathcal{F}[f](x^*) = \overline{\mathrm{co}}\{[0,-1]^T, [0,0]^T\}.$$

Hence x^* is a steady state of the differential inclusion $\dot{x}(t) \in \mathcal{F}[h](x(t))$. In fact, it can be verified that

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 1 + e^{-t} \end{aligned}$$

is a solution of (5.4) which is converging to x^*. However, if f_i is replaced by any functions which satisfies *(ii)* in Definition 2.26, $[0,0]^T \notin \mathcal{F}[h](x^*)$ which implies that the vector x^* is not a steady state of the differential inclusion $\dot{x}(t) \in \mathcal{F}[h](x(t))$.

5.2.2 Node nonlinearity

We first consider the system (5.2) where the functions g_{ij} are all the identity function, and focus our attention on the functions f_i, which describe how agent i handles the incoming information flow. In this case, the total dynamics of the agents can be written as

$$\dot{x} = f(-Lx), \tag{5.6}$$

where L is the graph Laplacian induced by the information flow digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$f(y) = [f_1(y_1), f_2(y_2), \ldots, f_n(y_n)]^T.$$

This leads to the consideration of the Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)), \tag{5.7}$$

where $h(x) := f(-Lx)$. Since L is a singular matrix, we have $\mathcal{F}[f](-Lx(t)) \subsetneqq \mathcal{F}[h](x(t))$ in general.

The aim of this section is to investigate under which conditions the Filippov solutions of the system (5.7) achieve consensus. Because of possible discontinuity of the right-hand side of (5.6), it turns out that there can be Filippov solutions that are unbounded. The following example illustrates this unwanted behavior.

Example 5.3. Consider a dynamical system (5.6) defined on an undirected graph with three nodes, as given in Fig. 5.3(a), where the functions f_i are all given by the signum function:

$$\begin{aligned} \dot{x}_1 &= \operatorname{sign}(x_2 + x_3 - 2x_1) \\ \dot{x}_2 &= \operatorname{sign}(x_1 + x_3 - 2x_2) \\ \dot{x}_3 &= \operatorname{sign}(x_1 + x_2 - 2x_3). \end{aligned}$$

Suppose that at time t_0 we have $x(t_0) \in \operatorname{span}\{\mathbb{1}\}$, then

$$\mathcal{F}[h](x(t_0)) = \overline{\text{co}}\left\{\nu_1, \nu_2, \nu_3, -\nu_1, -\nu_2, -\nu_3\right\}, \tag{5.8}$$

where $\nu_1 = [1, 1, -1]^T, \nu_2 = [1, -1, 1]^T, \nu_3 = [-1, 1, 1]^T$. Since $\frac{1}{3}\sum_{i=1}^{3} \nu_i = \frac{1}{3}\mathbb{1}$, we have that $\{\eta\mathbb{1} \mid \eta \in [-\frac{1}{3}, \frac{1}{3}]\} \subset \mathcal{F}[h](x(t_0))$. Indeed, the Filippov set-valued map $\mathcal{F}[h](\mathbb{1})$ is given as the polyhedron in Figure 5.2. Hence, any time-function $x(t) = \eta(t)\mathbb{1}$ with $\eta(t)$ differentiable almost everywhere and satisfying $\dot{\eta}(t) \in [-\frac{1}{3}, \frac{1}{3}]$ is a Filippov solution for this system. A complete analysis of this example is given in the Appendix.

Thus the values of x_1, x_2, x_3 converge to each other, but <u>not</u> to a <u>fixed</u> consensus

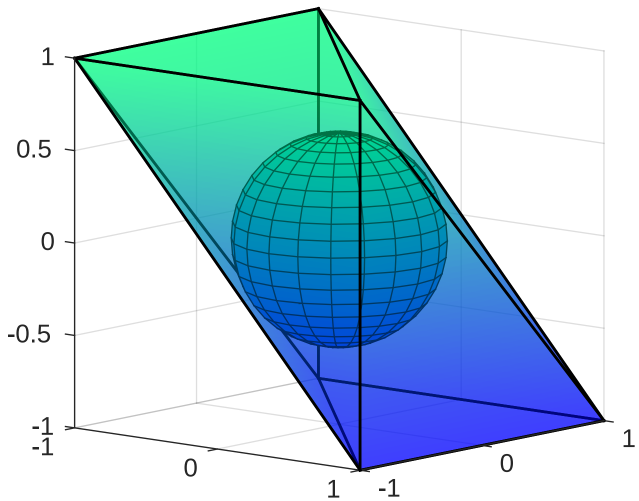


Figure 5.2: The polyhedron is the Filippov set-valued map of sign at $x \in \mathrm{span}\{\mathbb{1}\}$. The sphere centers at the origin with radius $\frac{\sqrt{3}}{3}$. The sphere is contained in the polyhedron and is tangent to two surfaces of it. Hence for any $\alpha \in [-\frac{1}{3}, \frac{1}{3}]$, the vector $\alpha\mathbb{1}$ belongs to the polyhedron.

(a) Undirected graph (b) Directed graph

Figure 5.3: Two graphs with three nodes, one undirected and one directed graph, used in Examples 5.3, 5.4, 5.7 and 5.9.

value. The undesirable behavior $x(t) = \eta(t)\mathbb{1}$ in the previous example will be called *sliding consensus*. Note that this example shows that for the validity of Theorem 11 in [26] we need extra conditions. In fact, it will turn out that the occurrence of sliding consensus can be excluded by replacing the signum function for at least one node by a function that is continuous at the origin. This motivates to introduce the following subsets of the node set of a digraph $\mathcal{G}$ with index set

$\mathcal{I} = \{1, 2, \ldots, n\}$:

$$\mathcal{I}_r = \{i \in \mathcal{I} \mid v_i \text{ is a root of } \mathcal{G}\} \tag{5.9}$$

$$\mathcal{I}_c = \{i \in \mathcal{I} \mid f_i \text{ is continuous at the origin}\}. \tag{5.10}$$

First we state two preparatory lemmas.

Lemma 5.2. *Consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. The subgraph corresponding to the roots of $\mathcal{G}$ is strongly connected.*

Proof. If there is just one root, the statement is trivial. If $\mathcal{G}$ has more than one root, let v_i and v_j be two distinct roots, and let $\mathcal{G}' = (\mathcal{I}_r, \mathcal{E}', A')$ denote the subgraph corresponding to the root set $\mathcal{I}_r$. Since v_i is a root, there is a directed path from v_i to v_j in the graph $\mathcal{G}$. Vice versa, there is a directed path from v_j to v_i. Combining these two paths, we get a directed cycle containing both v_i and v_j. Note that every node in this cycle is a root as well. Therefore, all the edges in this cycle are in $\mathcal{E}'$, and hence there is a directed path in $\mathcal{G}'$ from every root v_i to any other root v_j. □

Note that if a node v_k is a root, then $e_{kj} \in \mathcal{E}$ implies that v_j is a root as well. Hence, $(-Lx)_k = \sum_{j \in \mathcal{I}_r} a_{kj}(x_j - x_k)$.

Lemma 5.3. *The following functions are regular and Lipschitz continuous,*

$$V(x) := \max_{i \in \mathcal{I}} x_i, \qquad W(x) := -\min_{i \in \mathcal{I}} x_i. \tag{5.11}$$

Proof. Since any convex function is regular ([23], Prop. 2.3.6), it is enough to prove that $V(x)$ and $W(x)$ are convex. The projection function $p_j(x) = x_j$ is linear, hence convex and concave. Since the maximum of a finite number of convex functions is again convex, it follows that $V(x)$ is convex. Since $W(x)$ can be rewritten as $\max_{i \in \mathcal{I}}(-x_i)$, also W is convex. The function $V(x)$ is Lipschitz continuous since $|V(x) - V(y)| = |\max_i x_i - \max_i y_i| \leqslant \|x - y\|_\infty$ for all $x, y \in \mathbb{R}^n$. Similarly, $W(x)$ is Lipschitz continuous as well. □

The following theorem is the main result of this section.

Theorem 5.4. *Consider system* (5.7) *defined on a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. If one of the following three conditions holds, i.e.,*

(i) $\mathcal{I}_c \cap \mathcal{I}_r$ *is not empty,*

(ii) $|\mathcal{I}_r| = 1$,

(iii) $|\mathcal{I}_r| = 2$ *and* $f_i(0^-) = -f_i(0^+)$ *for* $i \in \mathcal{I}_r$,

where $f_i(0^-)$ *and* $f_i(0^+)$ *are the left and right-hand limits of* f_i *at the origin respectively, then all the trajectories of system* (5.7) *achieve consensus, for any initial condition. Furthermore, they will remain in the set* $[\min_i x_i(0), \max_i x_i(0)]^n$ *for all* $t \geqslant 0$.

Proof. Notice that in all cases $\mathcal{I}_r$ is nonempty, which implies that the graph $\mathcal{G}$ contains a directed spanning tree. Condition *(i)* implies that the digraph $\mathcal{G}$ has a root v_i for which f_i is continuous at the origin.

Consider the candidate Lyapunov functions V and W as given in (5.11). Eventually, we will apply LaSalle's Invariance principle to both V and W. By Lemma 5.3, V and W are regular and Lipschitz continuous. Let $x(t)$ be a trajectory of (5.7). Define $\alpha(t) := \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$.

Let the set Ψ be defined as

$$\Psi = \{t \geqslant 0 \mid \text{both } \dot{x}(t) \text{ and } \frac{d}{dt}V(x(t)) \text{ exist}\}. \tag{5.12}$$

Since x is absolutely continuous and V is locally Lipschitz, Ψ equals $[0, \infty)$ minus a set $\bar{\Psi}$ of measure zero. By Lemma 1 in [8], we have

$$\frac{d}{dt}V(x(t)) \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \tag{5.13}$$

for all $t \in \Psi$ and hence the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is nonempty for all $t \in \Psi$. For $t \in \bar{\Psi}$, we have $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \varnothing$. Hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = -\infty < 0$ by definition. Next we want to show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \leqslant 0$ for all $t \in \Psi$ by considering the two possible cases: $\mathcal{I}_r \nsubseteq \alpha(t)$ or $\mathcal{I}_r \subseteq \alpha(t)$.

If $\mathcal{I}_r \nsubseteq \alpha(t)$, then there exists an $i \in \mathcal{I}_r$ such that $x_i(t) < V(x(t))$. Furthermore, there exists $j \in \alpha(t)$ such that $-(Lx)_j < 0$. Indeed, the index $j \in \alpha(t)$ can be chosen such that the path from v_i to v_j has the least number of edges. By the definition of the Filippov set-valued map and the fact that the function f_j is sign-preserving, we have that if $\nu \in \mathcal{F}[h](x(t))$, then $\nu_j < 0$. The generalized gradient of V is given as [23, Example 2.2.8]

$$\partial V(x(t)) = \mathrm{co}\{e_j \in \mathbb{R}^n \mid j \in \alpha(t)\}. \tag{5.14}$$

Let $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$. By definition, there exists a $\nu^a \in \mathcal{F}[h](x(t))$ such that $a = \nu^a \cdot \zeta$ for all $\zeta \in \partial V(x(t))$. Consequently, this ν^a satisfies

$$\nu_i^a = \nu_j^a \qquad \forall i, j \in \alpha(t).$$

Since for any $\nu \in \mathcal{F}[h](x(t))$ there exist $j \in \alpha(t)$ such that $\nu_j < 0$, then any $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ satisfies $a < 0$. By the fact that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is a closed set, we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) < 0$.

If $\mathcal{I}_r \subseteq \alpha(t)$, we will consider the conditions *(i)*, *(ii)* and *(iii)* separately.

(i) In this case $\mathcal{I}_c \cap \mathcal{I}_r \subseteq \alpha(t)$. For any $i \in \mathcal{I}_c \cap \mathcal{I}_r$, we have that f_i is continuous at 0 and satisfies $f_i(0) = 0$. Hence for any $\nu \in \mathcal{F}[h](x(t))$ it satisfies $\nu_i = 0$. Using the same argument as in the first case, we can conclude that the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ equals the singleton $\{0\}$.

(ii) Let $\mathcal{I}_r = \{i\}$. Then we have that $L_i = 0$ where L_i is the ith row of L. Hence, for any $\nu \in \mathcal{F}[h](x(t))$, $\nu_i = 0$, which again implies that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$.

(iii) Let $\mathcal{I}_r = \{i, j\}$. By Lemma 5.2 the dynamics of x_i and x_j are given as $\dot{x}_i = f_i(a_{ij}(x_j - x_i))$ and $\dot{x}_j = f_j(a_{ji}(x_i - x_j))$, respectively. Since $\{i, j\} = \mathcal{I}_r \subseteq \alpha(t)$ we have $x_i(t) = x_j(t)$. By the fact that $f_k(0^-) = -f_k(0^+)$ for $k \in \mathcal{I}_r$, we see that for any $\nu \in \mathcal{F}[h](x(t))$ we have

$$\begin{bmatrix} \nu_i \\ \nu_j \end{bmatrix} \subseteq \overline{\mathrm{co}}\{\begin{bmatrix} f_i(0^-) \\ f_j(0^+) \end{bmatrix}, -\begin{bmatrix} f_i(0^-) \\ f_j(0^+) \end{bmatrix}\}. \tag{5.15}$$

Hence, any $\nu \in \mathcal{F}[h](x(t))$ with $\nu_i = \nu_j$ must satisfy $\nu_i = \nu_j = 0$. This implies that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$.

For all three conditions, we have that $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) = \{0\}$, and hence

$$\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \leqslant 0.$$

Define $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$. By using similar arguments, we find that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t)) < 0$ if $\mathcal{I}_r \nsubseteq \beta(t)$, and $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t)) \leqslant 0$ if $\mathcal{I}_r \subseteq \beta(t)$.

We conclude that $V(x(t))$ and $W(x(t))$ are non-increasing along the trajectories $x(t)$ of the system (5.7). Hence, the trajectories are bounded and remain in the set $[\min_i x_i(0), \max_i x_i(0)]^n$ for all $t \geqslant 0$. Therefore, for any $N \in \mathbb{R}_+$, the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant for (5.7). By Theorem 2.22, we have that all solutions of (5.7) starting from S_N converge to the largest weakly invariant set M contained in

$$S_N \cap \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x)\}} \cap \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x)\}}. \tag{5.16}$$

From the argument above we see that $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is possible only if $\mathcal{I}_r \subseteq \alpha(t)$, and $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t))$ can only happen if $\mathcal{I}_r \subseteq \beta(t)$. This implies that for every root v_i, the state x_i converges simultaneously to $\max_{i \in \mathcal{I}} x_i$ and to $\min_{i \in \mathcal{I}} x_i$, i.e., the trajectories $x(t)$ of the system achieve consensus for any initial condition. □

The conditions *(i)*, *(ii)* and *(iii)* in Theorem 5.4 all exclude the possibility of sliding consensus. Each condition will be illustrated by an example.

Example 5.4. Consider system (5.6) defined on the undirected graph in Fig. 5.3(a), defined as

$$\begin{aligned}\dot{x}_1 &= f_1(x_2 + x_3 - 2x_1)\\ \dot{x}_2 &= f_2(x_1 + x_3 - 2x_2)\\ \dot{x}_3 &= f_3(x_1 + x_2 - 2x_3).\end{aligned}$$

Suppose condition *(i)* in Theorem 5.4 is satisfied. For example, assume that f_1 is continuous at the origin. Then the sliding consensus is not a Fillippov solution. Indeed, if at time t_0 we have $x(t_0) \in \mathrm{span}\{\mathbb{1}\}$, then the first component of the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is equal to $\{0\}$. This implies that $x_1(t) = x_1(t_0)$ and therefore $x(t) = x(t_0)$, for all $t \geqslant t_0$.

Example 5.5. Consider system (5.6) defined on the digraph in Fig. 5.1(a) given by

$$\dot{x}_1 = f_1(0) \tag{5.17}$$

$$\dot{x}_2 = f_2(x_1 - x_2). \tag{5.18}$$

It satisfies condition *(ii)* of Theorem 5.4. Since $f_1(0) = 0$, the state of the root v_1 is constant. Consensus is achieved by the fact that f_2 is sign-preserving.

Example 5.6. Consider the system (5.6) defined on the digraph given in Fig. 5.1(b). The dynamics is defined as

$$\begin{aligned}\dot{x}_1 &= f_1(x_2 - x_1)\\ \dot{x}_2 &= f_2(x_1 - x_2).\end{aligned}$$

First, let f_1 and f_2 be signum functions, in which case condition *(iii)* of Theorem 5.4 is satisfied. If the trajectory achieves consensus at time t, then the image of the Filippov set-valued map $\mathcal{F}[h](x(t))$ is $\overline{\mathrm{co}}\{[1,-1]^T, [-1,1]^T\}$, which intersects $\mathrm{span}\{\mathbb{1}\}$ only at $[0,0]^T$. Hence $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x) = \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x) = 0$, which implies that the trajectory remains constant, i.e., there is no sliding consensus. See Figure 5.4 for a graphical explanation.

If the condition *(iii)* is not satisfied, i.e., $f_i(0^-) \neq -f_i(0^+)$ for $i = 1,2$, then sliding consensus can be a Filippov solution. For instance, take

$$f_i(x) = \begin{cases} 2 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \quad i = 1,2, \\ -1 & \text{if } x < 0, \end{cases}$$

Suppose that at t_0 the state x achieves consensus. Then the Filippov set-valued map at $x(t_0)$ is $\overline{\mathrm{co}}\{[-1,2]^T, [2,-1]^T\}$ which intersects $\mathrm{span}\{\mathbb{1}\}$ at $[\frac{1}{2}, \frac{1}{2}]^T$. Then $x(t) = \frac{1}{2}\mathbb{1}t + x(t_0)$ is a Filippov solution for $t \geqslant t_0$.

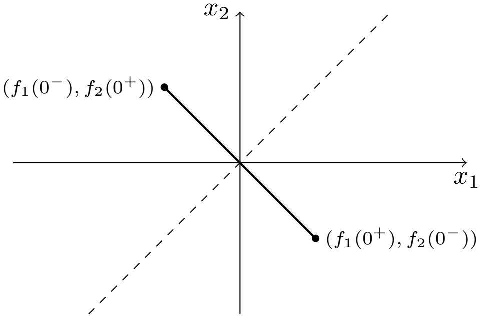


Figure 5.4: For a digraph with two nodes, the Filippov set-valued map at any point in $\mathrm{span}\{\mathbb{1}\}$ is the solid line segment which intersect $\mathrm{span}\{\mathbb{1}\}$ at the origin. Hence the sliding consensus can not be a Filippov solution.

5.2.3 Edge nonlinearity

In this section we consider the case where the functions f_i are all the identity function, that is,

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij} g_{ij}(x_j - x_i) =: h_i(x). \qquad i \in \mathcal{I} \tag{5.19}$$

We consider two cases, corresponding to the underlying graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ being undirected or directed.

We start with the undirected case.

Assumption 5.5. For all $(v_j, v_i) \in \mathcal{E}$, $g_{ij}(0^-) = -g_{ji}(0^+)$.

Theorem 5.6. *Consider the dynamics* (5.19) *defined on a connected undirected graph. Suppose the functions g_{ij} satisfy Assumption 5.5 (on top of Assumption 5.1). Then the trajectories of the system* (5.19) *achieve consensus asymptotically.*

Proof. Consider the Lyapunov candidate functions V and W as defined in (5.11). We use the same notations as in the proof of Theorem 5.4. Similarly, as in the proof of Theorem 5.4, we only prove that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) < 0$ for all $t \in \Psi$ where $\mathbb{R}_{\geqslant 0} \setminus \Psi$ is a set of measure zero and the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is nonempty for all $t \in \Psi$. Denote $S = \{x \in \mathbb{R}^n \mid \exists i, j \in \mathcal{I} \text{ such that } x_i = x_j\}$, which is a measure zero set in $\mathbb{R}^n$. We consider two possible cases for a given time t: $x(t) \notin \mathrm{span}\{\mathbb{1}\}$, and $x(t) \in \mathrm{span}\{\mathbb{1}\}$.

First, $x(t) \notin \mathrm{span}\{\mathbb{1}\}$. For small enough δ, all $y \in B(x(t), \delta) \setminus S$ satisfy $y_j - y_i < 0$ whenever $i \in \alpha(t)$ and $j \notin \alpha(t)$. Notice that $\alpha(t) = \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$. By Assumption 5.5, for $i, j \in \alpha(t)$ we have $g_{ij}(y_j - y_i) + g_{ji}(y_i - y_j) \to 0$ for all $y \in B(x(t), \delta) \setminus S$ as $\delta \to 0$. As we are considering undirected graphs, these two statements, together with the sign-preserving property of the functions g_{ij}, imply that for small enough δ there exists an $\epsilon > 0$ such that for all $y \in B(x(t), \delta) \setminus S$ we have

$$\sum_{i \in \alpha(t)} \sum_{j \in \mathcal{N}_i} a_{ij} g_{ij}(y_j - y_i) < -\epsilon.$$

Thus, for any $\nu \in \mathcal{F}[h](x(t))$ we have $\sum_{i \in \alpha(t)} \nu_i < 0$. It follows that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is contained in $\mathbb{R}_-$. By the fact that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is closed we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) < 0$.

Secondly, $x(t) \in \operatorname{span}\{\mathbb{1}\}$. Then by Assumption 5.5, for any $y \in B(x(t), \delta) \setminus S$, we have

$$\langle \mathbb{1}, h(y) \rangle = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}_i} a_{ij} g_{ij}(y_j(t) - y_i(t))$$

which approaches zero for $\delta \to 0$. On the other hand, by definition of the Filippov set-valued map, for any vector $\nu \in \mathcal{F}[h](x(t))$ it can be formulated as

$$\nu = \lim_{\substack{y \in B(x(t), \delta) \setminus S \\ \delta \to 0}} h(y). \tag{5.20}$$

Hence we have that any vector $\nu \in \mathcal{F}[h](x)$ is orthogonal to $\mathbb{1}$. Since $\mathbb{1} \in \partial V(x(t))$, we have $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$.

So far we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $x(t) \notin \operatorname{span}\{\mathbb{1}\}$ and $\tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) = \{0\}$ if $x(t) \in \operatorname{span}\{\mathbb{1}\}$.

The above analysis implies that the trajectories are bounded. Indeed for any $N \in \mathbb{R}_+$ the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant. By Theorem 2.22, the conclusion follows. □

Remark 5.7. The stronger assumption $g_{ij}(x) = -g_{ji}(-x)$ for all $\forall e_{ij} \in \mathcal{E}$ implies that $\mathbb{1}^T \dot{x} = 0$. In this case, the trajectories converge to a consensus value defined by the average of the initial conditions.

Example 5.7. If $g_{ij}(0^-) \neq -g_{ji}(0^+)$, then sliding consensus may occur. For instance, consider the system (5.19) defined on the undirected graph in Fig. 5.3(a) given by

$$\begin{aligned} \dot{x}_1(t) &= g_{12}(x_2(t) - x_1(t)) + g_{13}(x_3(t) - x_1(t)) \\ \dot{x}_2(t) &= g_{21}(x_1(t) - x_2(t)) + g_{23}(x_3(t) - x_2(t)) \\ \dot{x}_3(t) &= g_{31}(x_1(t) - x_3(t)) + g_{32}(x_2(t) - x_3(t)) \end{aligned} \tag{5.21}$$

where

$$g_{ij}(x) = \begin{cases} 1.5 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -0.5 & \text{if } x < 0, \end{cases} \quad \forall (v_j, v_i) \in \mathcal{E},$$

Suppose that at time t_0 the state satisfies $x(t_0) \in \operatorname{span}\{\mathbb{1}\}$. Then $\mathcal{F}[h](x(t_0))$ is the closed convex hull of

$$\{[-1,1,3]^T, [-1,3,1]^T, [1,-1,3]^T, [3,-1,1]^T, [1,3,-1]^T, [3,1,-1]^T\}. \tag{5.22}$$

Hence, $\mathbb{1}$ is an element in $\mathcal{F}[h](x(t_0))$ and thus $x(t) = t\mathbb{1} + x(t_0)$ is a Filippov solution for $t > t_0$. The graphical explanation is given in Figure 5.5.

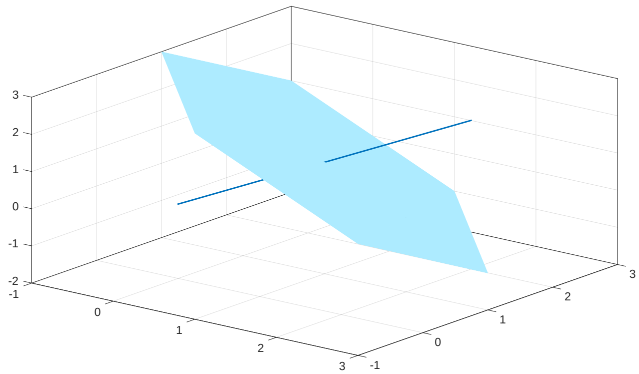


Figure 5.5: The surface is the convex hull of the vectors in (5.22) and the line spanned by $\mathbb{1}$. They intersect at the point $[1,1,1]^T$.

Example 5.8 (Example 5.7 continued)**.** Consider the dynamical system (5.21) with $g_{ij}(x) = \operatorname{sign}(x)$. In this case, the Filippov set-valued map at $x(t_0) \in \operatorname{span}\{\mathbb{1}\}$ is the closed convex hull of

$$\{[-2,0,2], [2,0,-2], [-2,2,0], [2,-2,0], [0,-2,2], [0,2,-2]\} \tag{5.23}$$

which intersects $\operatorname{span}\{\mathbb{1}\}$ at the origin. The graphical explanation is given in Figure 5.6.

Finally, let us briefly consider the case of a directed graph. In this case, Assumption 5.5 is not sufficient to guarantee convergence to consensus as shown by the following example.

Example 5.9. Consider system (5.19) on the directed graph as in Fig. 5.3(b), where the functions g_{ij} are the signum function. Hence the dynamics can be written as

$$\begin{aligned}
\dot{x}_1 &= \operatorname{sign}(x_3 - x_1)\\
\dot{x}_2 &= \operatorname{sign}(x_1 - x_2)\\
\dot{x}_3 &= \operatorname{sign}(x_2 - x_3).
\end{aligned}$$

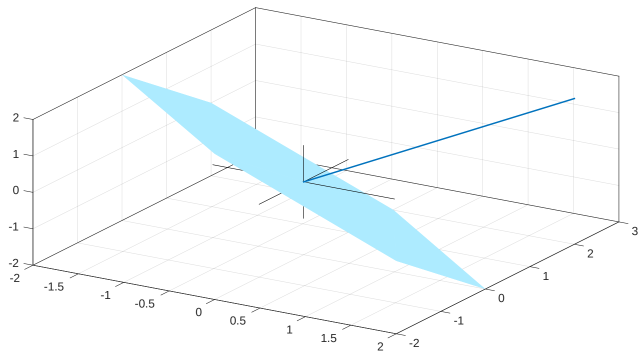


Figure 5.6: The surface is the convex hull of the vectors in (5.23) and the blue line is spanned by $\mathbb{1}$. They intersect at the origin.

Suppose that at time t_0, the state satisfies $x(t_0) \in \text{span}\{\mathbb{1}\}$. Then the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is the same as in (5.8). Hence by the same argument as in Example 5.3, sliding consensus is a Filippov solution, and thus there is no convergence to consensus.

For digraphs, we quote the following result from [53].

Theorem 5.8. *Consider the system* (5.19) *with continuous functions g_{ij}. If the underlying graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ contains a directed spanning tree, then the trajectories of* (5.19) *achieve consensus asymptotically.*

Extension of Theorem 5.8 to the case of discontinuous functions g_{ij} is a topic for further research.

5.2.4 Combining results

The multi-agent system given in (5.2) can be seen as a combination of system (5.6) and system (5.19). We have the following result.

Theorem 5.9. *Consider system* (5.2) *defined on a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with continuous functions g_{ij}. If one of the following three conditions holds, i.e.,*

(i) $\mathcal{I}_r \cap \mathcal{I}_c$ *is not empty,*

(ii) $|\mathcal{I}_r| = 1$,

(iii) $|\mathcal{I}_r| = 2$ *and* $f_i(0^-) = -f_i(0^+)$ *for* $i \in \mathcal{I}_r$,

then all Filippov solutions of system (5.2) *achieve consensus, for all initial conditions.*

Proof. Since the proof is similar to the proof of Theorem 5.4, we only provide a sketch of the proof. Recall that $\alpha(t) = \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$ and $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$.

Similarly we use V and W as Lyapunov functions. We will show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V \leqslant 0$ by considering two cases: $\mathcal{I}_r \nsubseteq \alpha(t)$ and $\mathcal{I}_r \subseteq \alpha(t)$.

When $\mathcal{I}_r \nsubseteq \alpha(t)$, there exists a $k \in \alpha(t)$ satisfying $\sum_{j=1}^{n} a_{kj} g_{kj}(x_j - x_k) < 0$. This implies that the kth component of $\mathcal{F}[h](x(t))$ is contained in $\mathbb{R}_-$. Hence, $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V < 0$.

When $\mathcal{I}_r \subseteq \alpha(t)$, we can use similar arguments as in the proof of Theorem 5.4 to see that the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is either $\{0\}$ or $\varnothing$ if one of the conditions *(i)*, *(ii)* and *(iii)* holds. Hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \leqslant 0$.

Similarly, we have that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $\mathcal{I}_r \nsubseteq \beta(t)$, and

$$\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) \leqslant 0$$

if $\mathcal{I}_r \subseteq \beta(t)$. Based on Theorem 2.22, the conclusion follows. □

5.3 Port-Hamiltonian formulation

In Section 5.2.2, we considered the system (5.6) defined on a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ containing a directed spanning tree. In this section we consider the special case when the functions f_i are continuous and the underlying digraph $\mathcal{G}$ is strongly connected. In this case, (5.6) admits classical solutions. We provide another point of view about the stability of system (5.6) by using a port-Hamiltonian formulation, which is of interest in itself. Some basic information on port-Hamiltonian systems can be found in [6, 67, 69].

First, we state a preparatory lemma. Consider the singular transformation $z = -Lx$. In [44], the following lemma was proved for the case $f_i(\cdot) = \operatorname{sat}(\cdot; -1, 1)$; here we consider the general case where we do not assume that the functions f_i are sign-preserving.

Lemma 5.10. *Assume the digraph $\mathcal{G}$ contains a directed spanning tree and the functions f_i are continuous. Then the following two statements are equivalent:*

1. *The trajectories of the system (5.6) asymptotically converge to* $\operatorname{span}\{\mathbb{1}\}$ *for any initial condition.*
2. *The trajectories of the system*

$$\dot{z} = -Lf(z) \tag{5.24}$$

with initial condition $z(0)$ satisfying

$$z(0) \in \operatorname{im} L \tag{5.25}$$

asymptotically converge to the origin.

Proof. (2 ⇒ 1). Let $x(t)$ be a solution of (5.6). Then $z(t) = -Lx(t)$ is a solution of $\dot{z}(t) = -L\dot{x}(t) = -Lf(-Lx(t)) = -Lf(z(t))$ with $z(0) = Lx(0) \in \operatorname{im} L$. Hence $z(t) = -Lx(t) \to 0$, and thus $x(t) \to \ker L$. Since the digraph G contains a directed spanning tree, we have that $\ker L = \operatorname{span}\{\mathbb{1}\}$ [76], that is, $x(t)$ converges to $\operatorname{span}\{\mathbb{1}\}$.

(1 ⇒ 2). Let $z(t)$ be a solution of (5.24) satisfying (5.25). Then $z(t) \in \operatorname{im} L$ for $t \geqslant 0$, and hence $z(t) = -L\tilde{x}(t)$ for some function $\tilde{x}$ satisfying

$$L\dot{\tilde{x}} = Lf(-L\tilde{x}(t)). \tag{5.26}$$

Define $\phi(t) = \dot{\tilde{x}} - f(-L\tilde{x}(t)) \in \ker L$ and $x(t) = \tilde{x}(t) - \int_0^t \phi(\tau)d\tau$. Then $z(t) = -Lx(t)$, while

$$\dot{x}(t) = \dot{\tilde{x}}(t) - \phi(t) = f(-L\tilde{x}(t)) = f(-Lx(t)). \tag{5.27}$$

Therefore, $x(t)$ is a solution of system (5.6), and hence $x(t)$ converge into $\operatorname{span}\{\mathbb{1}\} = \ker L$. Hence, $z(t) = -Lx(t) \to 0$ for $t \to \infty$.

□

Property 5.11 ([15], Theorem 1.37). *The graph Laplacian matrix L of a balanced and strongly connected graph $\mathcal{G}$ satisfies*

$$L + L^T = \frac{1}{2}L_0, \tag{5.28}$$

where L_0 represents the graph Laplacian matrix of the undirected graph $\mathcal{G}^o$ obtained by neglecting the orientation of the edges.

Theorem 5.12. *Suppose the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected and the functions f_i are continuous. Let $z = -Lx$. Then system (5.24) can be written in the port-Hamiltonian form*

$$\dot{w} = (J - R)\frac{\partial H}{\partial w}(w)$$

where $w = \Sigma z$ for some invertible $n \times n$ matrix Σ, $J \in \mathbb{R}^{n\times n}$ is skew-symmetric, $R \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite, and $H(w)$ is a positive definite and radially unbounded function. Furthermore, the trajectories of (5.24) *asymptotically converge to the origin, and hence solutions of* (5.6) *achieve consensus.*

Proof. Since $\mathcal{G}$ is strongly connected, there exists a vector $\sigma \in \mathbb{R}^n_+$ such that $\sigma^T L = 0$ [14, Theorem 14 on p.58]. Define the diagonal matrix $\Sigma := \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, then the matrix ΣL is the Laplacian matrix of a balanced graph.

Since the functions f_i are sign-preserving, for each f_i there exists a positive definite function $F_i : \mathbb{R} \to \mathbb{R}$ such that

$$\nabla F_i = f_i. \tag{5.29}$$

Denote $F = [F_1, F_2, \dots, F_n]^T$. By introducing new coordinates $w = \Sigma z$, we have

$$\dot{w} = -\Sigma L f(\Sigma^{-1} w) \tag{5.30}$$

$$= -\Sigma L \frac{\partial H}{\partial w}(w) \tag{5.31}$$

where $H(w) = \sigma^T F(\Sigma^{-1} w)$ is positive definite and radially unbounded. Hence system (5.30) can be written as

$$\dot{w} = (J - R)\frac{\partial H}{\partial w}(w), \tag{5.32}$$

where $J = \frac{1}{2}(L^T \Sigma^T - \Sigma L)$ is skew-symmetric and $R = \frac{1}{2}(L^T \Sigma^T + \Sigma L)$ is symmetric. Furthermore, by Property 5.11, R is positive semi-definite, representing a Laplacian matrix of an undirected graph. Finally,

$$\frac{d}{dt} H(w(t)) = -\frac{\partial^T H}{\partial w} R \frac{\partial H}{\partial w} \leqslant 0. \tag{5.33}$$

Denote $\Omega = \{w \mid \dot{H}(w(t)) = 0\}$. By $\frac{\partial H}{\partial w}(w) = f(z)$, we have $\Omega = \{z \mid f(z) \in \operatorname{span}\{\mathbb{1}\}\}$. Since $z \in \operatorname{im} L$ for all t, we see that $\sigma^T z = 0$. Because the functions f_i are sign-preserving and $\sigma \in \mathbb{R}^n_+$, we have that $\Omega = \{0\}$. By using LaSalle's Invariance principle, the trajectories of the system (5.24) will converge to the largest forward invariant set in Ω, i.e., $z \to 0$. Hence by Lemma 5.10, the solutions of the system (5.6) achieve consensus. □

Remark 5.13. If the underlying digraph $\mathcal{G}$ is balanced and weakly connected [1], then the vector σ in the proof of Theorem 5.12 can be chosen as $\mathbb{1}$. In this case the system (5.24) can be written in the port-Hamiltonian form

$$\dot{z} = (J_b - R_b)\frac{\partial H_b}{\partial z}(z) \tag{5.34}$$

where $J_b = \frac{1}{2}(L^T - L)$, $R_b = \frac{1}{2}(L^T + L)$, $H_b(z) = \sum_{i=1}^n F_i(z_i)$ and the functions F_i satisfy (5.29).

In Example 5.1 we showed that if the nonlinear functions f_i in (5.6) are equivalent to saturation functions with the lower or upper bounds being zero, then the

[1] Note that a balanced digraph G is strongly connected if and only if it is weakly connected.

condition that the underlying digraph $\mathcal{G}$ contains a spanning tree is not sufficient to guarantee the convergence of the states to consensus. A sufficient condition for consensus for the system

$$\dot{x} = \operatorname{sat}(-Lx(t)\,;u^-,u^+) \tag{5.35}$$

with either u^- or u^+ being equal to zero can be formulated as follows.

Proposition 5.14. *Consider system* (5.35) *and suppose the underlying digraph $\mathcal{G}$ is strongly connected. Let the saturation bounds satisfy $u^- = 0$ and $u^+ \in \mathbb{R}^n_+$ (respectively, $u^- \in \mathbb{R}^n_-$ and $u^+ = 0$), then the trajectories of the system* (5.35) *converge to the maximal (minimal) value of the initial condition, i.e., to* $\max_{i\in\mathcal{I}} x_i(0)$ $(\min_{i\in\mathcal{I}} x_i(0))$.

Proof. We only consider the case $u^- = 0$ and $u^+ \in \mathbb{R}^n_+$. For the case $u^- \in \mathbb{R}^n_-$ and $u^+ = 0$ the conclusion follows similarly.

By Lemma 5.10, it is equivalent to prove that the state of the system

$$\dot{z} = -L\operatorname{sat}(z\,;0,u^+) \tag{5.36}$$

with initial condition $z(0) \in \operatorname{im} L$ converges to zero asymptotically. By taking σ, Σ and $w = \Sigma z$ as in the proof of Theorem 5.12, we can write the dynamics of w in the port-Hamiltonian formulation as in (5.32), with Hamiltonian function $H(w) = \sigma^T \operatorname{Sat}(\Sigma^{-1}w)$ where $\operatorname{Sat} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\operatorname{Sat}(z)_i = \int_0^{z_i} \operatorname{sat}(\tau\,;0,u_i^+)d\tau. \tag{5.37}$$

Notice that each component of $\operatorname{Sat}(z)$ is not radially unbounded since u_i^- is zero. However, because $z \in \operatorname{im} L$ for all t and $\sigma^T L = 0$ we have that $\sigma^T z = 0$, which implies that $\sigma^T \operatorname{Sat}(z)$ is radially unbounded, and hence $H(w)$ is too. Indeed, suppose $\|z\| \to \infty$, then by $\sigma^T z = 0$ there must exist a component z_i with $z_i \to \infty$.

Let us denote $\Omega = \{w \mid \dot{H}(w(t)) = 0\} = \{z \mid \operatorname{sat}(z\,;0,u^+) \in \operatorname{span}\{\mathbb{1}\}\}$. Since $\sigma^T z = 0$, we have $\operatorname{sat}(z\,;0,u^+) = 0$. Hence $\Omega = \{z \mid z = 0_n\}$. By using LaSalle's Invariance principle, the trajectories of system (5.24) will converge to the largest invariant set in Ω, i.e., $z \to 0$.

Hence, the states of system (5.35) with $u^- = 0$ and $u^+ \in \mathbb{R}^n_+$ achieve consensus. From equation (5.35) we see that for every component x_k we have $\dot{x}_k(t) \geqslant 0$. Furthermore, if x_k satisfies $x_k(t) = \max_{i\in\mathcal{I}} x_i(t)$, then $\dot{x}_k(t) = 0$. Hence, by continuity of $x(t)$, all states converge to $\max_{i\in\mathcal{I}} x_i(0)$.

□

5.4 The model with imprecise measurement of the states: quantized consensus

The linear consensus protocol given as

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i(t) - x_j(t))$$

describes an idealized case in the sense that each agent has exact information about itself and its neighbors. A natural question is that what would happen if the information is imprecise for each agent. Specifically, in this section we consider the case that the measurements are quantized.

5.4.1 Problem formulation

In this section we consider a multi-agent system, which is different from system (5.2), in the sense that the nonlinearities are added on the measurement of the state of each agent. More precisely, we consider the following dynamics for agent i

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\big(q_j(x_j) - q_i(x_i)\big) \tag{5.38}$$

where $q_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n$ are sign-preserving and monotone. For general functions q_i, the stability is beyond the scope of this thesis. Here we consider the special case that all the functions q_i are all identical to the quantizer $\mathsf{q} : \mathbb{R} \to \Delta\mathbb{Z}$ which is defined as

$$\mathsf{q}(z) = \Big\lfloor \frac{z}{\Delta} + \frac{1}{2} \Big\rfloor \Delta. \tag{5.39}$$

Note that the quantizer satisfies $|\mathsf{q}(z) - z| \leqslant \frac{\Delta}{2}$. If $x \in \mathbb{R}^n$, we denote with some abuse of notation $\mathsf{q}(x) = (\mathsf{q}(x_1), \ldots, \mathsf{q}(x_n))^T$. Hence the dynamics (5.38) can be written in the vector form as

$$\dot{x} = -L\mathsf{q}(x). \tag{5.40}$$

Since the right-hand side of (5.40) is discontinuous, we interpret the solution of it in the Filippov sense, namely as any solution of the differential inclusion

$$\begin{aligned} \dot{x}(t) &\in \mathcal{F}[-L\mathsf{q}](x(t)) \\ &= -L\mathcal{F}[\mathsf{q}](x(t)), \end{aligned} \tag{5.41}$$

where the second equality is implied by the matrix transformation rule in Proposition 2.16. Moreover, by Proposition 1 in [19], the Krasovskii and Filippov solutions of (5.40) are equal.

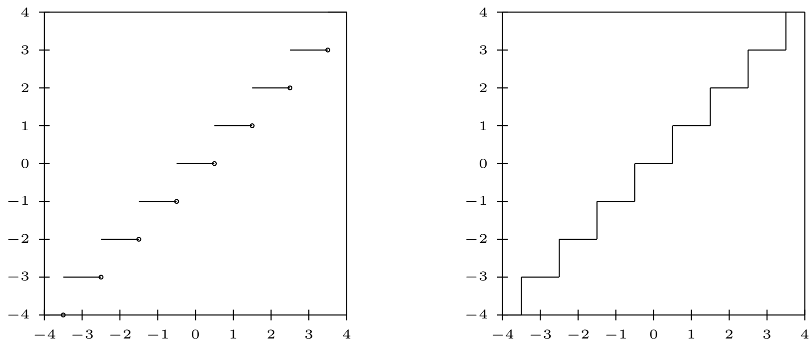


Figure 5.7: Diagram of the quantizer $\mathsf{q}(x)$ and the Filippov set-valued map $\mathcal{F}[\mathsf{q}](x)$ with $\Delta = 1$.

The contribution of this section is to extend the result in Section 3 of [20], which is about the stability of system (5.41) defined on a undirected graph, to the digraph case.

Let us denote $\mathcal{D}$ as

$$\mathcal{D} = \{x \in \mathbb{R}^n \mid \exists k \in \mathbb{Z} \text{ such that } \mathsf{q}(x_i) = \Delta k, i = 1, \dots, n\}. \tag{5.42}$$

By the definition of $\mathcal{F}[\mathsf{q}](x)$, the closure of $\mathcal{D}$ can be written as

$$\bar{\mathcal{D}} = \{x \in \mathbb{R}^N \mid \exists k \in \mathbb{Z} \text{ such that } \exists \nu \in \mathcal{F}[\mathsf{q}](x) \text{ satisfying } \nu = k\Delta \mathbb{1}\}. \tag{5.43}$$

It is known that without the precise measurement of the states, exact consensus can not be achieved. Instead, the notation of *practical consensus* will be employed. We say that the state variables of the agents converge to *practical consensus*, if $x(t) \to \bar{\mathcal{D}}$ as $t \to \infty$.

5.4.2 Stability analysis

In order to study the stability of system (5.41), we will use the functions V and W as given in (5.11) as candidate Lyapunov functions.

Consider the digraph $\mathcal{G}$ with at least one root. Let us denote the subgraph spanned by the roots as $\mathcal{G}_r$, which is by Lemma 5.2 strongly connected. Furthermore, let $n_r = |\mathcal{I}_r|$ where $\mathcal{I}_r$ is given as in (5.9), and denote the Laplacian of $\mathcal{G}_r$ as L_r. Hence there exists a vector $\sigma \in \mathbb{R}_+^{n_r}$ such that $\sigma^T L_r = 0$.

Theorem 5.15. *If the digraph $\mathcal{G}$ satisfies $\mathcal{I}_r \neq \varnothing$, then the trajectories of* (5.41) *converge to $\bar{\mathcal{D}}$ asymptotically.*

Proof. Consider the candidate Lyapunov functions V and W as given in (5.11). By Lemma 5.3, V and W are regular and Lipschitz continuous. Let $x(t)$ be a trajectory of (5.41). Similarly, as in the proof of Theorem 5.4, we only prove that $\max \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) < 0$ for all $t \in \Psi$ where $\mathbb{R}_{\geqslant 0} \setminus \Psi$ is a set of measure zero and the set $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t))$ is nonempty for all $t \in \Psi$.

Define $\alpha(t) = \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$. From [23, Example 2.2.8], we have

$$\partial V(x) = \text{co}\{e_j \in \mathbb{R}^n \mid j \in \alpha(t)\}. \tag{5.44}$$

For any $a \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t))$ for which there exists a $\nu \in -L\mathcal{F}[\mathsf{q}](x(t))$ such that $a = \nu \cdot \zeta$ for all $\zeta \in \partial V(x)$, we have that the vector ν satisfies $\nu_i = \nu_j$ for all $i, j \in \alpha(t)$.

Next we show that $\max \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) \leqslant 0$ by considering two possible cases $\mathcal{I}_r \nsubseteq \alpha(t)$ or $\mathcal{I}_r \subseteq \alpha(t)$.

If $\mathcal{I}_r \nsubseteq \alpha(t)$, then there exists $i \in \alpha(t)$ such that $\exists j \in \mathcal{N}_i$ satisfying $x_j < x_i$. Hence $\mathsf{q}(x_j) \leqslant \mathsf{q}(x_i)$ and $\nu_j \leqslant \nu_i$ for any $\nu_j \in \mathcal{F}[\mathsf{q}](x_j), \nu_i \in \mathcal{F}[\mathsf{q}](x_i)$. This implies that the ith component of the Filippov set-valued map, i.e., $-L_i\mathcal{F}[\mathsf{q}](x)$,is contained in $\mathbb{R}_{\leqslant 0}$, where L_i is the ith row of L. Hence $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) \subseteq \mathbb{R}_{\leqslant 0}$ which implies that $\max \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) \leqslant 0$.

If $\mathcal{I}_r \subseteq \alpha(t)$, then for any $a \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t))$ there exists $\omega \in -L\mathcal{F}[\mathsf{q}](x)$ such that $a = \omega^T\zeta, \forall \zeta \in \partial V$. Furthermore, $\omega = -L\nu$ for some $\nu \in \mathcal{F}[\mathsf{q}]$. If $|\mathcal{I}_r| = 1$, i.e., there is only one root denoted as i, then $\omega_i = 0$. Hence $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) = \{0\}$. If $|\mathcal{I}_r| > 1$, we have that $\nu_i = \nu_j, i, j \in \mathcal{I}_r$, which implies $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) = \{0\}$ again. Indeed, if not, $\{\omega_i : i \in \mathcal{I}_r\}$ has positive and negative components simultaneously. This implies $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) = \varnothing$, which is a contradiction to the fact that $t \in \Psi$. Hence $\tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x(t)) = \{0\}$.

By using a similar computation, we have that $\max \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} W(x(t)) \leqslant 0$.

So V and W are not increasing along the trajectories of the system (5.41). Hence the trajectories are bounded. Therefore for any $N \in \mathbb{R}_+$, $S_N := \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant N\}$ is strongly invariant for (5.41).

By Theorem 2.22, we have that all solutions of (5.41) starting at S_N converge to the largest weakly invariant set Ω contained in

$$S_N \cap \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x)\}} \cap \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} W(x)\}}. \tag{5.45}$$

At last, we analyze the set $\overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x)\}}$ in detail. We will show that for any $x \in \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]} V(x)\}}$, there exists $m \in \mathbb{Z}$ such that for any $i \in \mathcal{I}_r \cup \alpha(t)$ there exists $\nu_i \in \mathcal{F}[\mathsf{q}](x_i)$ satisfying $\nu_i = m\Delta$. Indeed, if $\mathcal{I}_r \subset \alpha(t)$, this is straightforward. If $\mathcal{I}_r \nsubseteq \alpha(t)$, there exists $m \in \mathbb{Z}$ such that for any $i \in \alpha(t)$ there exists $\nu_i \in \mathcal{F}[\mathsf{q}](x_i)$ satisfying $\nu_i = m\Delta$. Suppose there exists $j \in \mathcal{I}_r$ such that $\nu_j \leqslant (m-1)\Delta$ for any $\nu_j \in \mathcal{F}[\mathsf{q}](x_j)$, then there exists a vertex k on the directed path from the vertex j to $i \in \alpha(t)$ such that $\exists \nu_k \in \mathcal{F}[\mathsf{q}](x_k)$ satisfying $\nu_k = m\Delta$ and $-L_k\mathcal{F}[\mathsf{q}](x(t)) \leqslant -\Delta$. Hence x_k is strictly decreasing, and after finite time period, we have $\nu_k \leqslant (m-1)\Delta$ for any $\nu_k \in \mathcal{F}[\mathsf{q}](x_k)$. Then we repeat this analysis for the directed path from the vertex k to i. Eventually, there exists $i \in \alpha(t)$ such that $\nu_i \leqslant (m-1)\Delta$ for any $\nu_i \in \mathcal{F}[\mathsf{q}](x_i)$. This is a contradiction to the fact that $i \in \alpha(t)$. Hence, for any $x \in \overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]}V(x)\}}$, there exists $m \in \mathbb{Z}$ such that for any $i \in \mathcal{I}_r \cup \alpha(t)$ there exists $\nu_i \in \mathcal{F}[\mathsf{q}](x_i)$ satisfying $\nu_i = m\Delta$.

A similar conclusion holds for the set $\overline{\{x \in \mathbb{R}^n : 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[\mathsf{q}]}W(x)\}}$. Hence it is straightforward to see that $\Omega = \bar{\mathcal{D}}$. □

5.5 Conclusions

This chapter studies consensus problems for multi-agent systems defined on directed graphs where the consensus dynamics involves nonlinear and discontinuous functions. Since the right-hand sides of the differential equations are discontinuous, we interpret the solutions in the Filippov sense. We considered two types of nonlinear consensus protocols, namely with and without precise measurement of the states. For the first case, sufficient conditions, involving the nonlinear functions and the topology of the underlying graph, for the agents to converge to static consensus are provided. The result in Section 5.2.2 can be seen as a modification and extension of the result in [26, 27]. For a special case, namely the multi-agent system defined on a strongly connected graph with continuous functions, we showed the convergence by using a port-Hamiltonian formulation. For the second case, we considered specifically the quantized consensus and extended the result in [20] about undirected graphs to directed graphs.

Chapter 6

Conclusions

6.1 Conclusions

In Chapter 3, a basic model of dynamical distribution networks with flow constraints has been discussed. In order to address the output agreement problem of the vertices we propose a distributed PI controller which controls the flows through the edges. The main result is that the *manageability* of the in/outflows is a sufficient and necessary condition for asymptotic output agreement. This result is alternatively expressed into an interior point condition for an equivalent system without in/outflows. In this case the output agreement only depends on the flow constraints and the graphical structure of the network captured by the interior point condition. The stability proof of the closed-loop system is based on the construction of Lyapunov functions and LaSalle's Invariance principle. By employing the notion of equilibrium independent passivity, the Lyapunov function can be interpreted as the storage function of the network. By a modification of the PI controller, under the additional assumption of convexity of the Hamiltonian functions, it is shown how the outputs of the system can be driven to any desirable vector in an admissible set instead of to consensus. By establishing a relation between distribution networks with flow constraints and the monotropic optimization problem [61], an equivalent expression of the interior point condition is given from the viewpoint of optimization theory.

In Chapter 4, dynamical distribution networks with PI controllers on the edges and state inequality constraints have been investigated. The control aim is to design the flow constraints which regulate the flows through the edges such that the asymptotic output agreement is achieved, while state constraints are satisfied for all time. The design of flow constraints is given by an algorithm. Since the right-hand side of the resulting closed-loop system is discontinuous, the proof of stability is based on the use of Filippov solutions.

In Chapter 5, several general nonlinear consensus protocols have been discussed. These models are divided into two parts, namely with and without exact measurement of the states. For the first type, sufficient conditions, involving the nonlinear functions and the topology of the underlying graph, for the agents to converge to consensus are provided. The result in Section 5.2.2 can be seen as a modification and extension of the result in [26, 27]. For a special case, i.e., the consensus protocol defined on a strongly connected graph with continuous nonlinearities, we show the convergence by using a port-Hamiltonian formulation. For the second type, we consider a special case, namely quantized consensus, and extend the result in [20] on undirected graphs to directed ones. Notice that all the nonlinear functions are allowed to be discontinuous, and hence we interpret the solutions in the Filippov sense.

6.2 Future research and recommendations

For distribution networks:

1. It is of interest to extend the distribution network model considered in this thesis to the case with time-varying in/outflows. In this case, an open problem is how to design the distributed controller on the edges such that asymptotic output agreement of the vertices is achieved. A centralized continuous controller for this problem based on the internal model principle was given in [54].

2. It is of importance to investigate the distribution network under time-varying flow constraints, such as switching saturation levels, which includes many practical situations.

3. The results of Chapter 4 can be extended in a straightforward way to the case where the flows on the edges obey a priori constraints.

4. The extension of the results in Chapter 4 to general graphs containing circuits is an interesting topic for future investigations. One obstacle is that when the graph contains circuits, the execution of Algorithm 4.1 may involve an infinite number of steps. In order to guarantee the continuity of ϕ^*, we need the uniform convergence of ϕ^ℓ which is not straightforward.

For consensus protocols:

1. The problem formulation of Chapter 5 can be extended to allowing sliding consensus in the problem of achieving consensus. For this extended formulation, providing sufficient conditions on the nonlinearities is an open problem. Note that all the counterexamples in Chapter 5 correspond to sliding consensus. Clearly, sign-preservation is not enough, which already can be seen from Example 5.3. Another issue is that in this case LaSalle's Invariance principle may not be useful, since it asks for a compact invariant

set. However once there is sliding, boundedness of the trajectory is not guaranteed.

2. The extension of the consensus protocol with quantization to arbitrary sign-preserving nonlinear functions is under investigation.
3. For the node nonlinearity proposed in Section 5.2.2, the problem of finite-time convergence to consensus is open.
4. Higher-order consensus protocols are also commonly used in practice, e.g., acceleration control in formation control. It is of importance, both for theoretical and practical reasons, to provide a uniform framework for the stability of higher-order nonlinear consensus protocols.

Appendix

Appendix: Complete analysis of Example 5.3

In this section, we provide a complete analysis of the system

$$\begin{aligned} \dot{x} &= \text{sign}(-Lx), x \in \mathbb{R}^3 \\ &=: f(x) \end{aligned} \tag{A.1}$$

where

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{A.2}$$

as already given as Example 5.3 in Chapter 5.

Notice that the right-hand side of the dynamical system (A.1) is piece-wise continuous. Indeed, it is discontinuous at every x belonging to one of the three planes $\mathcal{P}_1 = \{x \mid x_2 + x_3 = 2x_1\}, \mathcal{P}_2 = \{x \mid x_1 + x_3 = 2x_2\}$ and $\mathcal{P}_3 = \{x \mid x_1 + x_2 = 2x_3\}$. These planes divide $\mathbb{R}^3$ into six parts, denote by $\mathcal{D}_1, \dots, \mathcal{D}_6$ (Figure A.1).

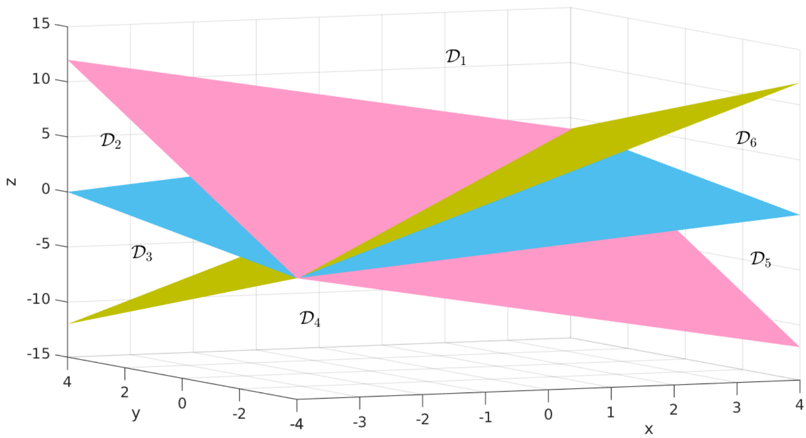
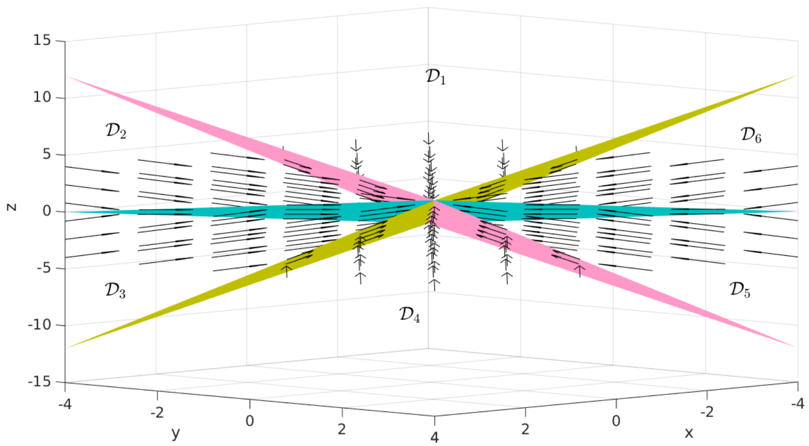
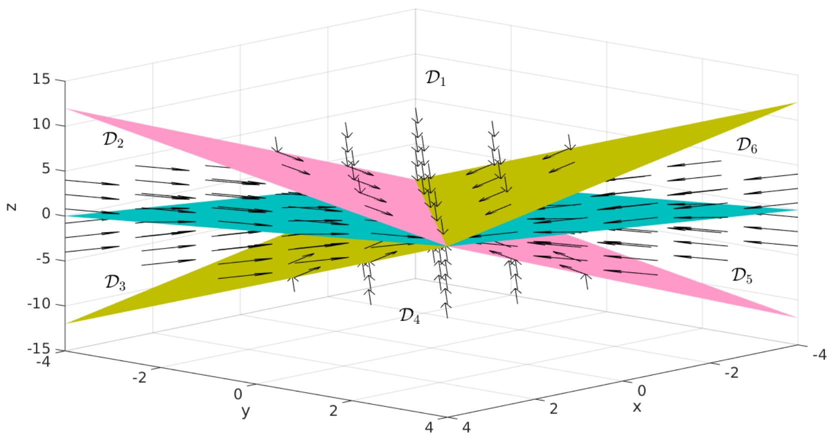


Figure A.1: The three planes on which the right-hand side of (A.1) is discontinuous.

Denote the restriction of f to $\mathcal{D}_i$ as $f_{\mathcal{D}_i}$ and the continuous extension of $f_{\mathcal{D}_i}$ to the closure $\overline{\mathcal{D}_i}$ as $f_{|\overline{\mathcal{D}_i}}$. Suppose the regions $\mathcal{D}_i$ and $\mathcal{D}_j$ are adjacent, then at any



(a)



(b)

Figure A.2: The vector field defined by $\mathrm{sign}(-Lx)$ from two different angles.

$x \in (\overline{\mathcal{D}_i} \cap \overline{\mathcal{D}_j}) \setminus \{\mathbb{1}\}$, the vector $f_{|\overline{\mathcal{D}_i}}(x)$ points into $\mathcal{D}_j$ while $f_{|\overline{\mathcal{D}_j}}(x)$ points into $\mathcal{D}_i$. This can be seen from description given in Figure A.2. Hence the trajectories of (A.1) will follow the sliding consensus once it reach one of $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$.

The evolution of the trajectories of (A.1) with initial condition $x_0 = [0, 6, 9]^T$ is given in Figure A.3.

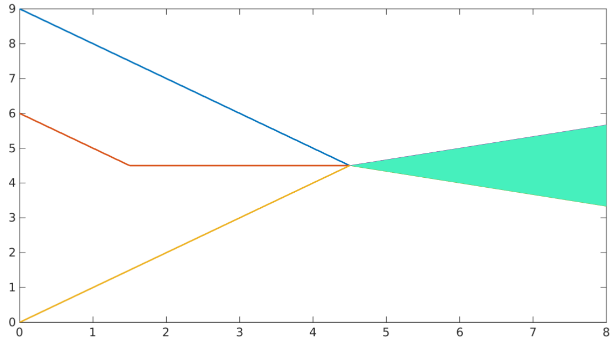


Figure A.3: For initial condition $x_0 = [0, 6, 9]^T$, the Filippov solution is unique before it reaches $\mathrm{span}\{\mathbb{1}\}$. The boundaries of the green area have slope $\frac{\sqrt{3}}{3}$ and $-\frac{\sqrt{3}}{3}$ respectively. In this example, the trajectory reaches $\mathrm{span}\{\mathbb{1}\}$ at $t = 4.5$. For any time $t > 4.5$, the trajectories can be anywhere in the green area with derivatives (if existing) belonging to $[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$. This is in contrast to the simulation given in [26].

Notice that the green area in Figure A.3 does not correspond to simulation of differential equations like in a standard ode package in Matlab. In fact, standard ode packages are not able to handle discontinuous dynamical systems with non-unique Filippov solutions. A special package like FilippovSim.zip [1] can be useful. The simulation of Filippov solutions is itself a challenging topic, see e.g.[30, 56].

[1]available at `http://www.staff.science.uu.nl/~kouzn101/MiniFilippov/FilippovP1.html`

Bibliography

[1] Rafig Agaev and Pavel Chebotarev. On the spectra of nonsymmetric Laplacian matrices. *Linear Algebra and its Applications*, 399(0):157 – 168, 2005.

[2] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*, volume 218, 2nd edition, Springer, London, 2000 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1996.

[3] A.J. van der Schaft and B.M. Maschke. Conservation laws on higher-dimensional networks. *Proc. 47th IEEE Conf. on Decision and Control*, pages 799–804, 2008.

[4] A.J. van der Schaft and B.M. Maschke. *Model-Based Control: Bridging Rigorous Theory and Advanced Technology, P.M.J. Van den Hof, C. Scherer, P.S.C. Heuberger, eds., chapter Conservation laws and lumped system dynamics*. Springer, Berlin-Heidelberg, 2009.

[5] A.J. van der Schaft and B.M. Maschke. Port-Hamiltonian dynamics on graphs: Consensus and coordination control algorithms. *Proc. 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems, Annecy, France*, pages 175–178, 2010.

[6] A.J. van der Schaft and B.M. Maschke. Port-Hamiltonian systems on graphs. *SIAM J. Control and Optimization*, 51(2):906–937, 2013.

[7] J. Aronson. A survey of dynamic network flows. *Annals of Operations Research*, 20(1):1–66, 1989.

[8] Bacciotti, Andrea and Ceragioli, Francesca. Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM. Control, Optim. Calculus Variations*, 4:361–376, 1999.

[9] J.C. Moreno Banos and M. Papageorgiou. A linear programming approach to large-scale linear optimal control problems. *IEEE Transactions on Automatic Control*, 40(5):971–977, 1995.

[10] Dario Bauso, Franco Blanchini, Laura Giarré, and Raffaele Pesenti. The linear saturated decentralized strategy for constrained flow control is asymptotically optimal. *Automatica*, 49(7):2206 – 2212, 2013.

[11] N. Biggs. *Algebraic Graph Theory*. Cambridge Mathematical Library. Cambridge University Press, 1993.

[12] F. Blanchini, S.Mian, and W.Ukovich. Control of production-distribution systems with unknown inputs and system failures. *IEEE Transactions on Automatic Control*, 45(6):1072–1081, 2000.

[13] Franco Blanchini, Elisa Franco, Giulia Giordano, Vahid Mardanlou, and Pier Luca Montessoro. Compartmental flow control: Decentralization, robustness and optimality. *Automatica*, 64:18 – 28, 2016.

[14] B. Bollobas. *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics*. Springer, New York, 1998.

[15] F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009.

[16] M. Bürger, D. Zelazo, and F. Allgöwer. Network clustering: A dynamical systems and saddle-point perspective. In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011*, pages 7825–7830, 2011.

[17] Mathias Bürger, Daniel Zelazo, and Frank Allgöwer. Duality and network theory in passivity-based cooperative control. *Automatica*, 50(8):2051 – 2061, 2014.

[18] M.K. Camlibel and A.J. van der Schaft. Incrementally port-Hamiltonian systems. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 2538–2543, Dec 2013.

[19] Francesca Ceragioli. *Discontinuous ordinary differential equations and stabilization*. PhD thesis, Università di Firenze, 2000.

[20] Francesca Ceragioli, Claudio De Persis, and Paolo Frasca. Discontinuities and hysteresis in quantized average consensus. *Automatica*, 47(9):1916 – 1928, 2011.

[21] Fei Chen, Yongcan Cao, and Wei Ren. Distributed average tracking of multiple time-varying reference signals with bounded derivatives. *IEEE Transactions on Automatic Control*, 57(12):3169–3174, Dec 2012.

[22] Fei Chen, Gang Feng, Lu Liu, and Wei Ren. Distributed average tracking of networked euler-lagrange systems. *IEEE Transactions on Automatic Control*, 60 (2):547–552, Feb 2015.

[23] Frank H. Clarke. *Optimization and Nonsmooth Analysis*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1990.

[24] J. Cortes. Discontinuous dynamical systems. *IEEE Control Systems magazine*, 28(3):36–73, 2008.

[25] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243–255, April 2004.

[26] Jorge Cortés. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42(11):1993 – 2000, 2006.

[27] Jorge Cortés. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44(3):726 – 737, 2008.

[28] Mεβoo Dattorro and Jon Dattorro. Convex optimization & euclidean distance geometry, 2005.

[29] Claudio De Persis and Paolo Frasca. Robust self-triggered coordination with ternary controllers. *IEEE Transactions on Automatic Control*, 58(12):3024–3038, 2013.

[30] Fabio Dercole and Yuri A. Kuznetsov. Slidecont: An auto97 driver for bifurcation analysis of Filippov systems. *ACM Trans. Math. Softw.*, 31(1):95–119, March 2005.

[31] Dimos V. Dimarogonas and Karl H. Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, 46(4):695 – 700, 2010.

[32] A. Ephremides and S. Verdú. Control and optimization methods in communication network problems. *IEEE Transactions on Automatic Control*, 34(9): 930–942, 1989.

[33] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[34] A.F. Filippov and F.M. Arscott. *Differential Equations with Discontinuous Right-hand Sides: Control Systems*. Mathematics and its Applications. Springer, 1988.

[35] L. R. Ford and D. R. Fulkerson. Maximal Flow through a Network. *Canadian Journal of Mathematics*, 8:399–404.

[36] L. R. Ford and D. R. Fulkerson. Constructing maximal dynamic flows from static flows. *Operations Research*, 6(3):419–433, 1958.

[37] R.A. Freeman, Peng Yang, and K.M. Lynch. Stability and convergence properties of dynamic average consensus estimators. In *Decision and Control, 2006 45th IEEE Conference on*, pages 338–343, Dec 2006.

[38] Karin Gatermann. Chemical reactions stoichiometric network analysis. *Vorlesung im Sommersemester 2002 an der Freien Universität Berlin*, 2002.

[39] Meng Guo and Dimos V. Dimarogonas. Consensus with quantized relative state measurements. *Automatica*, 49(8):2531 – 2537, 2013.

[40] G.H. Hines, M. Arcak, and A.K. Packard. Equilibrium-independent passivity: a new definition and implications. *American Control Conference (ACC), 2010*, pages 5584–5589, June 2010.

[41] Altug Iftar and E.J. Davison. Decentralized robust control for dynamic routing of large scale networks. In *American Control Conference, 1990*, pages 441–446, May 1990.

[42] Matin Jafarian and Claudio De Persis. Formation control using binary information. *Automatica*, 53(0):125 – 135, 2015.

[43] B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Casta nos. Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits. *Systems & Control Letters*, 56:618–622, 2007.

[44] Y Li, J Xiang, and W Wei. Consensus problems for linear time-invariant multi-agent systems with saturation constraints. *IET Control Theory & Applications*, 5(6):823–829, 2011.

[45] Zhiyun Lin, Bruce Francis, and Manfredi Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, 46(1):288–307, 2007.

[46] Hui Liu, Ming Cao, and Claudio De Persis. Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics. *Systems & Control Letters*, 61(12):1157 – 1167, 2012.

[47] Nancy A Lynch. *Distributed algorithms*. Morgan Kaufmann, 1996.

[48] Mehran Mesbahi and Magnus Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton series in applied mathematics. Princeton University Press, Princeton (N.J.), 2010.

[49] Nima Monshizadeh and Claudio De Persis. Output agreement in networks with unmatched disturbances and algebraic constraints. 2015.

[50] F.H. Moss and A. Segall. An optimal control approach to dynamic routing in networks. *IEEE Transactions on Automatic Control*, 27(2):329–339, 1982.

[51] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, Sept 2004.

[52] R. Ortega, A.J. van der Schaft, F. Castanos, and A. Astolfi. Control by interconnection and standard passivity-based control of port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, 53(11):2527–2542, 2008.

[53] A. Papachristodoulou, A. Jadbabaie, and U. Münz. Effects of delay in multi-agent consensus and oscillator synchronization. *IEEE Transactions on Automatic Control*, 55(6):1471–1477, June 2010.

[54] C. D. Persis. Balancing time-varying demand-supply in distribution networks: an internal model approach. *European Control Conference (ECC),Zürich, Switzerland.*, 2013.

[55] C. De Persis and B. Jayawardhana. On the internal model principle in the coordination of nonlinear systems. *IEEE Transactions on Control of Network Systems*, 1(3):272–282, Sept 2014.

[56] Petri T. Piiroinen and Yuri A. Kuznetsov. An event-driven method to simulate filippov systems with accurate computing of sliding motions. *ACM Trans. Math. Softw.*, 34(3):13:1–13:24, May 2008.

[57] R. Olfati-Saber and J.A. Fax and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215 –233, 2007.

[58] Wei Ren. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 53(6):1503–1509, 2008.

[59] Wei Ren, R.W. Beard, and E.M. Atkins. A survey of consensus problems in multi-agent coordination. In *Proceedings of American Control Conference, 2005*, pages 1859–1864 vol. 3, June 2005.

[60] Craig W. Reynolds. Flocks, herds and schools: A distributed behavioral model. *Computer Graphics*, 21(4):25–34, 1987.

[61] R.T. Rockafellar. *Network flows and monotropic optimization*. Pure and applied mathematics. Wiley, 1984.

[62] R.T. Rockafellar. *Convex Analysis*. Convex Analysis. Princeton University Press, 1997.

[63] J.G. Romero, A. Donaire, and R. Ortega. Robustifying energy shaping control of mechanical systems. *51st IEEE Conference on Decision and Control*, pages 4424–4429, 2012.

[64] J. Russell Carpenter. Decentralized control of satellite formations. *International Journal of Robust and Nonlinear Control*, 12(2-3):141–161, 2002.

[65] R.O. Saber and R.M. Murray. Consensus protocols for networks of dynamic agents. *Proceedings of American Control Conference, 2003*, 2:951–956, 2003.

[66] A. Segall. The modeling of adaptive routing in data-communication networks. *IEEE Transactions on Communications*, 25(1):85–95, 1977.

[67] A.J van der Schaft and B.M. Maschke. The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik und Übertragungstechnik*, 49:362–371, 1995.

[68] A.J. van der Schaft and J. Wei. A Hamiltonian perspective on the control of dynamical distribution networks. *In Proceedings of the 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, 2012.

[69] Arjan van der Schaft and Dimitri Jeltsema. *Port-Hamiltonian Systems Theory: An Introductory Overview*. Now Publishers Incorporated,Boston/Dordrecht, 2014.

[70] J. Wei and A.J. van der Schaft. Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows. *Systems & Control Letters*, 62(11):1001 – 1008, 2013.

[71] J. Wei and A.J. van der Schaft. Stability of dynamical distribution networks with arbitrary flow constraints and unknown in/outflows. *In Proceedings of the 52nd IEEE Conference on Decision and Control*, 2013.

[72] J. Wei and A.J. van der Schaft. A graphic condition for the stability of dynamical distribution networks with flow constraints. *In Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*, 2014.

[73] J. Wei and A.J. van der Schaft. Constrained proportional integral control of dynamical distribution networks with state constraints. *In Proceedings of the 53nd IEEE Conference on Decision and Control*, Dec 2014.

[74] J. Wei and A.J. van der Schaft. A graph-theoretic condition for the stability of dynamical distribution networks with flow constraints. *In preparation*, 2015.

[75] J. Wei, A.R.F. Everts, M.K. Camlibel, and A.J. van der Schaft. Consensus problems with arbitrary sign-preserving nonlinearities. *Submitted to Automatica*, 2015.

[76] Ren Wei, Randal W. Beard, and Timothy W. McLain. Coordination variables and consensus building in multiple vehicle systems. In Vijay Kumar, Naomi Leonard, and A.Stephen Morse, editors, *Cooperative Control*, volume 309 of *Lecture Notes in Control and Information Science*, pages 171–188. Springer Berlin Heidelberg, 2005.

[77] D. Zelazo and M. Mesbahi. Edge agreement: Graph-theoretic performance bounds and passivity analysis. *IEEE Transactions on Automatic Control*, 56(3): 544–555, 2011.

Summary

Dynamical distribution networks and nonlinear multi-agent systems are two main topics of this thesis. We use tools from port-Hamiltonian systems, discontinuous dynamical systems, passivity, optimization, and graph theory. The problems studied in this thesis are listed as follows.

For dynamical distribution networks, we consider the output agreement problem of the vertices with flow and state constraints.

Firstly, we study a basic model for the dynamics of a distribution network defined on a directed graph without any constraints. On each vertex, we assign a nonlinear integrator of which the state variable is controlled by the flows through the edges. Besides, some of the vertices serve as terminals where unknown but constant in/outflows may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. With any such type of in/outflows, we prove that a nonlinear PI controller defined on the edges, which controls the flows, can achieve output agreement among the vertices. This can be proved by making use of the port-Hamiltonian formulation of the closed-loop system.

Secondly, the model of the dynamical distribution network is extended to the more practically meaningful case that the flows on the edges are constrained. Based on the constraint intervals and topology of the network, we can define the notion of manageable in/outflows which coincides with Assumption 1 in [12]. We prove that with such manageable in/outflows and flow constraints on the edges, the PI controller providing the flows through the edges can still achieve output agreement of the vertices.

As a variation of the flow constraint problem, we study the case that the in/outflows are zero, in which instance the output agreement property of the vertices depends on a property of the constraint intervals. We formulate this property as an interior point condition and prove that asymptotic output agreement of the vertices is achieved if and only if the constraint intervals satisfy the interior point condition. The proof is based on a Lyapunov function and LaSalle's

Invariance principle. Several extensions and applications of the case with zero in/outflows considered in this thesis are given as follows. First, the stability of closed-loop system with constant flow constraints can be interpreted from the equilibrium-independent passivity point of view. Hence the Lyapunov function also can be interpreted as a storage function. Secondly, by using monotropic optimization theory we formulate an equivalent expression of the interior point condition. Thirdly, under certain conditions, the PI controller can be modified such that the output of the vertices can converge to an arbitrary feasible vector instead of to consensus. Finally, for any constraint intervals of which the intersection contains an open interval, the asymptotic output agreement is achieved if and only if the underlying directed graph is weakly connected and balanced.

The last problem about the dynamical distribution network is the case with state constraints. We provide a protocol which gives a state-based constraint for the flow. With such state-based constraints, we prove that the PI controller can achieve output agreement while the states do not violate the state constraints. The analysis is conducted on an acyclic graph, and this conclusion holds for all the Filippov trajectories.

In the last part of this thesis, we deal with nonlinear multi-agent systems where we focus on general nonlinear first-order consensus protocols. All the nonlinear functions are assumed to be sign-preserving and piecewise continuous, which includes many typical nonlinear functions used in practice, e.g., saturation, sign and quantization. The solutions of the models in this part are understood in the Filippov sense.

Firstly, the state of each agent is driven towards the direction which is given by the value of a nonlinear function of the weighted average of the states of its neighbors and itself. This model is called node nonlinearity. We provide a sufficient condition with respect to the nonlinear function to guarantee that all the solutions converge to constant consensus. The underlying directed graph is assumed to contain a directed spanning tree.

Secondly, in the linear first-order consensus protocol, the state of each agent is driven toward the direction which is given by the weighted summation of difference between the states of its neighbors and itself. In the analogous nonlinear model, we consider the case that the state difference is measured by an arbitrary sign-preserving nonlinear function. This model is called edge nonlinearity. In this case, we only provide sufficient conditions with respect to the nonlinear functions such that asymptotic consensus is achieved when the underlying graph is undirected. For the combined case, i.e., node and edge nonlinearity, sufficient conditions for constant consensus are also provided.

The last model of nonlinear multi-agent systems we consider in this thesis is the quantized consensus protocol. We extend the result in [29] on undirected graph case to directed graphs containing a directed spanning tree.

Samenvatting

Dynamische distributienetwerken en niet-lineaire multi-agentsystemen zijn twee hoofdonderwerpen van dit proefschrift. We gebruiken instrumenten van poort-Hamiltonse systemen, discontinue dynamische systemen, passiviteit, optimalisatie en grafentheorie. De volgende problemen worden in dit proefschrift behandeld.

Voor dynamische distributienetwerken beschouwen we het probleem van output consensus met beperkingen op de stroming en op de toestand.

We beginnen met het bestuderen van een basismodel voor de dynamica van een distributienetwerk op een gerichte graaf, zonder enige beperkingen. Op elke knoop van de graaf wijzen we een lineaire integrator toe, waarvan de toestandsvariabele wordt geregeld door de stroming door de takken. Daarnaast dienen sommige punten als aansluitpunten, waar een onbekende, maar constante, stroom het netwerk kan verlaten of kan instromen, op een zodanige manier dat de som van de in- en uitgaande stroom gelijk is aan nul. We bewijzen dat een niet-lineaire PI-regelaar, die gedefinieerd is op de takken van de graaf en de stroming regelt, output-consensus kan bewerkstelligen. Dit kan bewezen worden door gebruik te maken van de poort-Hamiltonse formulering van het gesloten-lus systeem.

Daarna breiden we het model van het dynamische distributienetwerk uit naar een praktischer geval, waarin de stroming door de takken van de graaf beperkt is. Gebaseerd op de berperkingsintervallen en de topologie van het netwerk, kunnen we een handelbare in- en uitgaande stroom definiëren, wat samenvalt met Aanname 1 in [12]. We bewijzen dat met deze handelbare in- en uitgaande stroom en stromingsbeperkingen op de takken, de PI-regelaar op de takken nog steeds output-consensus kan bereiken.

Als variatie op de stromingsbeperkingen, bestuderen we het geval waarbij de in- en uitgaande stroom nul is. In dit geval hangt de de output-consensus af van een eigenschap van de beperkingsintervallen: het hebben van een gemeenschappelijk inwendig punt. We bewijzen dat asymptotische output-consensus bereikt wordt dan en slechts dan als de beperkingsintervallen aan deze voorwaarde voldoen. Het bewijs is gebaseerd op een Lyapunov functie en het invariantieprincipe van LaSalle. Verscheidende uitbreidingen en toepassingen van het geval zonder in- en uitstroom worden in dit proefschrift behandeld. Ten eerste kan de stabiliteit van het gesloten-lus systeem met constante stomingsbeperkingen bekeken worden vanuit het perspectief van evenwichtsonafhankelijke passiviteit; de Lyapunov functie kan als opslag worden geïnterpreteerd. Ten tweede formuleren we een equivalente uitdrukking van de voorwaarde van het hebben van een gemeenschappelijk inwendig punt, met behulp van monotropische optimalisatietheorie. Ten derde kan, onder bepaalde voorwaarden, de PI-regelaar zo worden aangepast dat de output van de knopen convergeert naar een willekeurige vector, in plaats van naar consensus. Tenslotte, indien de doorsnijding van de beperkingsintervallen een open interval bevat, wordt output-consensus bereikt dan en slechts dan als de onderliggende gerichte graaf zwak samenhangend en gebalanceerd is.

Het laatste probleem dat we behandelen met betrekking tot dynamische distributienetwerken is het geval van beperkingen op de toestand. We presenteren een protocol dat een toestandsgebaseerde beperking voor de stroming geeft. We bewijzen dat de PI-regelaar dan output-consensus kan bereiken, terwijl de toestanden voldoen aan de beperkingen. De analyse is uitgevoerd op een acyclische graaf, en de conclusie geldt voor alle Filippov oplossingen.

In de rest van dit proefschrift bestuderen we niet-lineaire multi-agentsystemen, waarbij we ons richten op de eerste-orde consensus protocollen. We nemen aan dat alle niet-lineaire functies tekenbehoudend en stuksgewijs lineair zijn. Dit behelst onder andere veel typische niet-lineare functies die in de praktijk gebruikt worden, zoals verzadiging, de signum-functie en kwantisering. In dit deel beschouwen we Filippov oplossingen.

Eerst bestuderen we het geval waarin de toestand van iedere agent afhangt van een niet-lineare functie van een gewogen gemiddelde van de toestanden van zijn buren en zichzelf. Dit model noemen we niet-lineair in de knopen. We geven een voldoende voorwaarde met betrekking tot de niet-lineaire functie om te garanderen dat alle oplossingen naar constante consensus convergeren. We nemen aan dat de onderliggende gerichte graaf een gerichte opspannende boom bevat.

Als tweede beschouwen we het geval waarin de toestand van elke agent afhangt van een gewogen som van toestandsverschillen tussen zichzelf en zijn buren, waarbij het toestandsverschil wordt gemeten door een willekeurig tekenbehoudende, niet-lineare functie. Dit model noemen we niet-lineair in de takken. We geven voldoende voorwaarden met betrekking tot de niet-lineaire functies voor de asysmptotische stabiliteit van constante consensus voor het geval dat de onderliggende graaf ongericht is. Voor het geval dat het model niet-lineair is in zowel de knopen als in de takken geven we eveneens voldoende voorwaarden voor constante consensus.

Het laatste model van niet-lineaire multi-agentsystemen dat we in dit proefschrift behandelen is het gekwantificeerde consensusprotocol. We breiden het resultaat voor ongerichte grafen in [29] uit naar gerichte grafen die een gerichte opspannende boom bevatten.