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Strong Targeted Controllability of Dynamical Networks

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***Abstract*— Network controllability is the ability to control the entire network, meaning that we can drive the network from any initial state to any desired final state in finite time by using appropriate inputs which are applied to a subset of nodes of the network. Despite obvious advantages, network controllability is not always feasible as it may ask for a considerable portion of the nodes to be controlled. Moreover, there are cases where controllability of the entire network is not of interest, but rather we are interested in controllability properties of certain parts of the network. This motivates us to investigate the so-called "targeted controllability" of the network, where controllability is only required for a subset of nodes in the network. Noting that targeted controllability can be treated as an output controllability problem, we investigate the (strong) structural output controllability properties of the network from a topological viewpoint. In addition, we examine the structural properties of the reachable subspace of the network. To this end, we use the notion of zero forcing sets, which has been recently exploited in the context of structural controllability.**

I. Introduction

The study of systems evolving on graphs and networks of dynamical agents have attracted a lot of attentions in the last two decades. In this context, it is customary to represent the infrastructure of a dynamical network by a graph where the agents are located at the nodes, and the physical coupling or the communication takes place over the edges of the graph. Hence, graph theory has become an indispensable tool for analysis and control of complex networks.

Clearly, we cannot solely rely on purely algebraic methods for analysis and design of dynamical networks, and we need to adopt a topological viewpoint to deal with numerical errors, uncertainties and changes in the network parameters. Motivated by this fact, a topological approach has been taken to study consensus [3], model reduction [14], [15], and controllability see e.g. [12], [7], [20], [19], [17], [18], [4], [5].

In the controllability framework, agents are labeled as *leaders* and *followers*. Leaders are agents through which external input signals are injected to the network, and the rest of the agents are called followers. Then, controllability analysis amounts to investigate the possibility of deriving the states of the agents to a desired point by appropriate input signals applied to the leaders. The mainstream of research in this direction has been devoted to controllability analysis of networks with symmetric unweighted Laplacian matrix, see e.g. [7], [20], [19], [17], [18], [22]. To broaden the scope of the analysis and to cope with the inherent uncertainties in complex networks, an emerging thread in the study of controllability of complex network is centered around structural controllability. Structural controllability deals with a family of systems rather than a particular instance and asks whether the family contains a controllable pair (weak structural controllability) [11] or all members of the family are controllable (strong structural controllability) [2], [16]. In particular, it has been shown that weak structural controllability of complex networks can be fully characterized in terms of maximum matching [11], and strong structural controllability has a one-to-one correspondence to zero forcing sets of the graph [16]. For a more general look at control properties of structured linear systems, see e.g. [6].

Note that network controllability is not always present in complex networks, or it may ask for considerable number of nodes to be directly controlled which is not always feasible. In addition, one can postulate the cases where steering the network to any arbitrary state is not necessary, and the domain of interest is restricted from the whole state space to a particular subspace. Likewise, we may ask for controllability properties in a subset of the nodes of the network, rather than the entire network. The latter, under the title of "targeted controllability", has been recently studied in [8], and exact topological condition for targeted controllability (in the weak structural sense) is reported in case the leader set is singleton.

In this paper, we study the "strong structurally reachable subspace" of the network, which is a strong structural extension of ordinary reachability subspace. In particular, a point in the strongly reachable subspace can always be reached for the whole family of systems defined on a graph. We show that the strong structurally reachable subspace is topologically equivalent to the so called "derived set" of the leader set. Then, noting that targeted controllability is essentially an output controllability problem, we conclude that each node in the derived set is controllable from the leader set, in a strong structural manner. This provides a sufficient condition for strong structural targeted controllability of the network in terms of the derived set. We also provide a shaper version of this sufficient condition by extending the derived set. An exact topological condition for targeted controllability is a subject of future research.

The structure of this paper is as follows. Preliminaries and problem motivation are provided In Section II. In Section III and Section V, we recap the notion of output controllability and zero forcing sets, respectively. The main results of the paper are reported in Section VI. The paper ends with conclusions in Section VII.

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II. Preliminaries and Motivation

For a given simple directed graph G, the vertex set of G is a nonempty set and is denoted by V. The arc set of G, denoted by E, is a subset of $V \times V$, and $(i,i) \notin E$ for all $i \in V$. The cardinality of a given set V is denoted by $|V|$. Also we sometimes use $|G|$ to denote in short the cardinality of V. We call vertex j an out-neighbor of vertex i if $(i,j) \in E$. The following family of matrices associated with G is called the *qualitative class* of G:

$$Q(G) = \{X \in \mathbb{R}^{|G|\times|G|} : \text{ for } i \neq j,\ X_{ij} \neq 0 \Leftrightarrow (j,i) \in E\} \quad (1)$$

For $V = \{1,2,\dots,n\}$ and $V' = \{v_1, v_2, \dots, v_r\} \subseteq V$, we define the $n \times r$ matrix $P(V;V') = [P_{ij}]$ by:

$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

We consider the following finite-dimensional linear input/state/output system defined on a graph G

$$\dot{x}(t) = Xx(t) + Uu(t) \quad (3a)$$

$$y = Hx(t) \quad (3b)$$

where $x \in \mathbb{R}^{|G|}$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in R^\ell$ is the output, $X \in Q(G)$, $U = P(V;V_L)$ for some given *leader set* $V_L \subseteq V$, and $H = P^T(V;V_T)$ for some given *target set* $V_T \subseteq V$.

Systems of the form (3) where $X \in Q(G)$ for a given graph G are encountered in various contexts. Examples include the cases where X is the adjacency matrix [9], the (in-degree or out-degree) Laplacian [13], normalized Laplacian [1], etc.

In this paper, we study structural controllability properties of the systems (3a), and we investigate the "structural output controllability problem" for systems of the form (3).

With a slight abuse of notation, we sometimes call $(X;V_L)$ controllable if the pair (X,U) is controllable. For a given graph G and a leader set V_L we say $(G;V_L)$ is controllable if the pair $(X;V_L)$ is controllable for all $X \in Q(G)$. In this case, we say that the network (3) is *strongly structurally controllable*.

For simplicity, we use calligraphic notation to denote the image of a matrix induced by a subset $V' \subseteq V$. More precisely, $\mathcal{V}'$ denotes, in short, the subspace $\operatorname{im} P(V;V')$.

As mentioned before, in this paper we are primarily interested in the case where strong structural controllability does not hold in the network. Then, clearly, driving the entire network from any initial state to any desired final state may not be possible. However, an interesting problem is to quantify the "partial controllability" which is still present in the network. In particular, we address the question which states of the network are reachable by applying appropriate input signals to the leaders? What is the subset of the nodes that can be driven to an arbitrary state, given a leader set? To formalize the aforementioned questions, we recap some notion from geometric control theory in the next section.

III. Review: Reachable subspace and Output controllability

In this section we will review the notion of output controllability for general linear input-state-output systems.

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (4)$$

with state space $\mathbb{R}^n$. For a given initial state x_0 and input function u, we denote the resulting state trajectory of the system by $x_u(t,x_0)$. The smallest A-invariant subspace containing the image $\operatorname{im} B$ of the input matrix B is denoted by $\langle A \mid \operatorname{im} B\rangle$. This subspace, called the *reachable subspace*, consists of all points in the state space that can be reached from the origin in finite time by choosing an appropriate input function, i.e., all points $x_1 \in \mathbb{R}^n$ for which there exists $T > 0$ and u such that $x_1 = x_u(T,0)$. It is well known that the system is controllable if and only if the reachable subspace $\langle A \mid \operatorname{im} B\rangle$ is equal to the entire state space $\mathbb{R}^n$. In turn, this is equivalent to the condition

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

If, in addition to the state equation, we specify an output equation

$$y(t) = Cx(t), \quad (5)$$

where the output $y(t)$ takes its values in the output space $\mathbb{R}^p$, we may introduce the notion of *output controllability*. Denote the output trajectory corresponding to the initial state x_0 and input function u by $y_u(t,x_0)$. The system (4), (5) is then called *output controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input function u and a $T > 0$ such that $y_u(T,x_0) = y_1$. We also say that the triple (A,B,C) is output controllable meaning that the system (4), (5) is output controllable. It is well known (see e.g. [21, Exc. 3.22]) that (A,B,C) is output controllable if and only if the rank condition

$$\operatorname{rank}\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} = p.$$

holds. In turn this is equivalent to the condition

$$C\langle A \mid \operatorname{im} B\rangle = \mathbb{R}^p,$$

i.e. the image under C of the reachable subspace is equal to the output space $\mathbb{R}^p$. Obviously, this condition is equivalent to $\ker C + \langle A \mid \operatorname{im} B\rangle = \mathbb{R}^n$. Finally, by taking orthogonal complements, the latter holds if and only if

$$\operatorname{im} C^\top \cap \langle A \mid \operatorname{im} B\rangle^\perp = \{0\}.$$

IV. Problem formulation

In this section, we formally define the problems mentioned in Section II. Note that for a given $X \in Q(G)$ and a given leader set V_L, the subspace $\langle X \mid \mathcal{V}_L\rangle$ contains all the states that can be reached from the origin by applying inputs to the nodes in the leader set V_L. Recall that we are interested in strong structural properties, i.e. properties which are valid for the whole qualitative class of a given graph. Hence, we define the *strong structurally reachable* as the

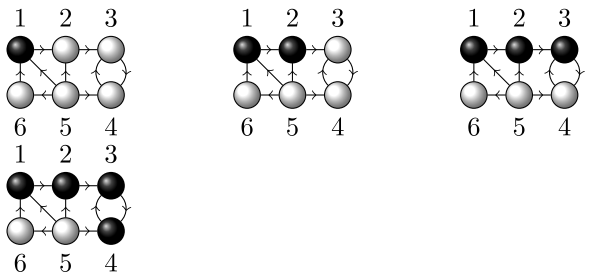

Fig. 1. An example for the coloring rule

subspace containing all the states which can be reached by applying appropriate input signals to the nodes in the leader set V_L, independent of the choice of $X \in Q(G)$. Clearly, this subspace is equal to $\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle$. As for the first problem, we investigate a topological characterization of the strong structurally reachable subspace. To this end, we need the notion of zero forcing sets.

V. Zero forcing sets

In this section, we review the notion of zero forcing sets together with the notations involved and terminology that will be used in the sequel. For more details see e.g. [10].

Let G be a graph, and suppose that each vertex is colored either white or black. Consider the following coloring rule:

◐ : If u is a black vertex and exactly one out-neighbor v of u is white, then change the color of v to black.

The following terminology will be used when we apply the color-change rule above to a graph G:

- If the color-change rule is applied to $u \in V$ to change the color of $v \in V$, we say u *forces* or *infects* v, and write $u \to v$.
- Given a coloring set $C \subseteq V$, i.e. C indexes the initially black vertices of G, the *derived set* set of C is denoted by $D(C)$, and is the set of black vertices obtained by applying the color-change rule until no more changes are possible. Noting that $C \subseteq D(C)$, we call $D(C) \setminus C$ the *strict derived set* of C.
- The set $Z \subseteq V$ is a *zero forcing set* (ZFS) for G if $D(Z) = V$.

Figure 1 illustrates the coloring rule, where vertex 1 is initially colored black. Then, by the color-change rule it is clear that $1 \to 2$. Consequently, $2 \to 3$, and $3 \to 4$. Therefore, the derived set of $\{1\}$ is equal to $\{1, 2, 3, 4\}$, and thus $\{1\}$ is not a zero forcing set. It is easy to see that $\{1, 5\}$ constitutes a zero forcing set for the depicted graph.

It is worth mentioning that there is a one to one correspondance between zero forcing sets and sets of leaders rendering the network strongly structurally controllable, more precisely $(G; V_L)$ is controllable if and only if V_L is a zero forcing set [16].

VI. Main results

Let $\mathcal{V}_L$ denote the subspace $\operatorname{im} P(V; V_L)$, and let $\mathcal{D}(V_L)$ denote the subspace $\operatorname{im} P(V; D(V_L))$, where P is defined in (2). Then, $\mathcal{D}(V_L)$ is contained in the reachability subspace as stated in the following lemma.

Lemma VI.1 *For a given $X \in Q(G)$ and a leader set $V_L \subseteq V$, we have $\mathcal{D}(V_L) \subseteq \langle X \mid \mathcal{V}_L \rangle$*

Proof. First note that in case $D(V_L) = V_L$, and thus $\mathcal{D}(V_L) = \mathcal{V}_L$, the statement of the lemma trivially holds. Now, suppose that $D(V_L) \neq V_L$, and vertex $v \in V_L$ forces vertex $w \notin V_L$. Then, we *claim* that

$$\operatorname{im} P(V; V_L \cup \{w\}) \subseteq \langle X \mid \mathcal{V}_L \rangle. \tag{6}$$

where P is given by (2). Clearly, the subspace inclusion (6) holds if and only if

$$\langle X \mid \mathcal{V}_L \rangle^{\perp} \subseteq P(V; V_L \cup \{w\})^{\perp} \tag{7}$$

Without loss of generality, assume that $V_L = \{1, 2, \ldots, m\}$, $v = m$, and $w = m + 1$. Then, the matrix X can be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \tag{8}$$

where the diagonal blocks/elements X_{11}, X_{22}, X_{33}, and X_{44} correspond to the vertices in $V_L \setminus \{v\}$, the vertex v, the vertex w, and the rest of the vertices, respectively.

Let $\xi \in \mathbb{R}^n$ be a vector in $\langle X \mid \mathcal{V}_L \rangle^{\perp}$. Clearly, we have $\xi^T X^{k-1} P(V; V_L) = 0$ for each $k \in \mathbb{N}$. We write $\xi = \operatorname{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ compatible with the partitioning of X. Note that $P(V; V_L \cup \{w\})$ now reads as

$$P(V; V_L \cup \{w\}) = \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the equality $\xi^T P(V_L; V) = 0$, we obtain that $\xi_1 = 0$ and $\xi_2 = 0$. Then, the equality $\xi^T X P(V_L; V) = 0$ yields

$$\begin{bmatrix} \xi_3^T & \xi_4^T \end{bmatrix} \begin{bmatrix} X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{bmatrix} = 0 \tag{9}$$

Observe that, since $v \to w$, the vertex v has exactly one out-neighbor in $V \setminus V_L$, and thus we have $X_{32} \neq 0$ and $X_{42} = 0$. Therefore, by (9), we obtain that the scalar ξ_3 is equal to zero. Clearly, $\xi = \operatorname{col}(0, 0, 0, \xi_4)$ is orthogonal to the subspace $P(V; V_L \cup \{w\})$. Hence, the subspace inclusion (7), and thus (6) holds. By repeating the argument above, we conclude that $\mathcal{D}(V_L) \subseteq \langle X \mid \mathcal{V}_L \rangle$. ■

Next, we show that the reachable subspace is invariant under the coloring rule.

Lemma VI.2 *For any given $X \in Q(G)$ and leader set $V_L \subseteq V$, we have $\langle X \mid \mathcal{V}_L \rangle = \langle X \mid \mathcal{D}(V_L) \rangle$.*

Proof. As $\mathcal{V}_L \subseteq \mathcal{D}(V_L)$, we obtain that

$$\langle X \mid \mathcal{V}_L \rangle \subseteq \langle X \mid \mathcal{D}(V_L) \rangle$$

In addition, by Lemma VI.1, we have $\mathcal{D}(V_L) \subseteq \langle X \mid \mathcal{V}_L \rangle$. Hence, $\langle X \mid \mathcal{D}(V_L) \rangle \subseteq \langle X \mid \mathcal{V}_L \rangle$, and thus the equality $\langle X \mid \mathcal{V}_L \rangle = \langle X \mid \mathcal{D}(V_L) \rangle$ holds. ∎

Now, the following theorem provides an exact topological characterization of the strong structurally reachable subspace.

Theorem VI.3 *For any given leader set $V_L \subseteq V$, we have*

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle = \mathcal{D}(V_L)$$

Proof. First, note that by Lemma VI.1 it follows that

$$\mathcal{D}(V_L) \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle \tag{10}$$

Now, we claim that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle \subseteq \mathcal{D}(V_L) \tag{11}$$

We define the set S as

$$S = \{s \in \mathbb{R}^n : s_i = 0 \Leftrightarrow i \in D(V_L)\} \tag{12}$$

Let s be a vector in S. Without loss of generality, let $D(V_L) = \{1, 2, \ldots, d\}$. Then, s can be written as $\operatorname{col}(0_d, s_2)$ where each element of $s_2 \in \mathbb{R}^{n-d}$ is nonzero. Let the matrix X be partitioned accordingly as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

Clearly, we have

$$s^T X = s_2^T \begin{bmatrix} X_{21} & X_{22} \end{bmatrix}$$

Observe that X_{21} corresponds to the arcs from a vertex $v \in D(V_L)$ to a vertex $w \notin D(V_L)$. Hence, by the coloring rule, each column of X_{21} is either identically zero or contains at least two nonzero elements. We choose these nonzero elements, if any, such that $s_2^T X_{21} = 0$. Noting that the diagonal elements of X_{22} are free parameters, we conclude that, for any vector $s \in S$, there exists a matrix $X \in Q(G)$ such that $s^T X = 0$. Therefore, we obtain that

$$s \in \langle X \mid \mathcal{D}(V_L) \rangle^{\perp}$$

for some matrix $X \in Q(G)$. Now, let $\xi \in \mathbb{R}^n$ be a vector in $\bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle$. Hence, by definition,

$$\xi \in \langle X \mid \mathcal{D}(V_L) \rangle$$

for all $X \in Q(G)$. Therefore, we have $s^T \xi = 0$ which yields $s_2^T \xi_2 = 0$, by writing $\xi = \operatorname{col}(\xi_1, \xi_2)$. As this conclusion holds for an arbitrary choice of $s \in S$, we obtain that $\xi_2 = 0$, and thus $\xi \in \mathcal{D}(V_L)$ which proves (11). Now, by Lemma VI.2, the subspace inclusion (11) is equivalent to

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle \subseteq \mathcal{D}(V_L)$$

This together with (10) completes the proof. ∎

Next, we consider the "strong targeted controllability" problem for systems of the form (3). For a given leader set V_L and a target set V_T, we call the system (3) *strongly targeted controllable* if the triple (X, U, H) is output controllable for all $X \in Q(G)$. In this case, we also say that $(G; V_L; V_T)$ is targeted controllable.

Observe that (strong) targeted controllability is basically a (strong) structural output controllability property. Indeed, in case $(G; V_L; V_T)$ is targeted controllable, then the output of the network can be steered to any desired state in $\mathbb{R}^{|V_T|}$, irrespective of the choice of $X \in Q(G)$.

Let $\mathcal{V}_T = \operatorname{im} P(V; V_T)$. Then, by Section III, geometric conditions for strong targeted controllability can be given as follows.

Proposition VI.4 *The following statements are equivalent:*

(i) $(G; V_L; V_T)$ *is targeted controllable*

(ii) $\operatorname{rank} \begin{bmatrix} HU & HXU & HX^2U & \cdots & HX^{n-1}U \end{bmatrix} = \ell$ *for all* $X \in Q(G)$

(iii) $H \langle X \mid \mathcal{V}_L \rangle = \mathbb{R}^{\ell}$ *for all* $X \in Q(G)$

(iv) $\mathcal{V}_T \cap \langle X \mid \mathcal{V}_L \rangle^{\perp} = \{0\}$ *for all* $X \in Q(G)$

Then, by using the results established previously in this section, we have the following theorem.

Theorem VI.5 *Given a leader set V_L and a target set V_T, we have that $(G; V_L; V_T)$ is targeted controllable if $V_T \subseteq D(V_L)$.*

Proof. Assume that $V_T \subseteq D(V_L)$, and thus $\mathcal{V}_T \subseteq \mathcal{D}(V_L)$. By Theorem VI.3, this is equivalent to

$$\mathcal{V}_T \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle \tag{13}$$

Therefore, it is easy to observe that

$$\mathcal{V}_T \cap \langle X \mid \mathcal{V}_L \rangle^{\perp} = \{0\} \tag{14}$$

for all $X \in Q(G)$, which results in targeted controllability of $(G; V_L; V_T)$ by the fourth statement of Proposition VI.4. ∎

Theorem VI.5 provides a sufficient condition for targeted controllability. In particular, targeted controllability of $(G; V_L; V_T)$ is guaranteed provided that the target nodes belong to the derived set of V_L. The absence of necessary and sufficient condition in Theorem VI.5 is associated with the gap between the conditions (13) and (14).

Consider the graph depicted in Figure 2, and let $V_L = \{1, 2\}$. It is easy to observe that by the color-change rule the derived set of V_L is obtained as $D(V_L) = \{1, 2, 3, 4\}$. By Theorem VI.5, we have that $(G; V_L; V_T)$ is targeted controllable for any

$$V_T \subseteq \{1, 2, 3, 4\}. \tag{15}$$

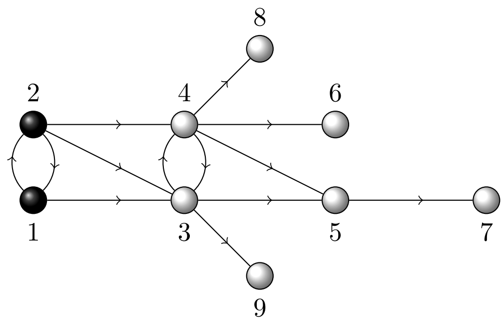

Fig. 2. The graph $G = (V, E)$

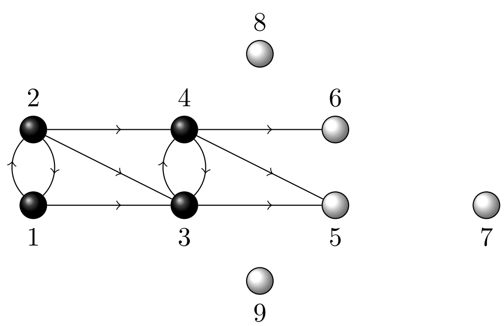

Fig. 3. The subgraph $G' = (V, E')$

However, this is not necessary as one can show that $(G; V_L; V_T)$ is targeted controllable with

$$V_T = \{1, 2, 3, 4, 5, 6, 7\}. \tag{16}$$

Next, we show that the sufficient condition provided by Theorem VI.5 can be sharpened by extending the derived set of V_L. In particular, our goal is to conclude that $(G; V_L; V_T)$ is targeted controllable if

$$V_T \subseteq D(V_L) \cup V_E \tag{17}$$

for some appropriate, to be chosen, subset $V_E \subseteq V \setminus D(V_L)$. To this end, we define the subgraph $G' = (V, E')$ with

$$E' = \{(i, j) : \ \ i \in D(V_L) \text{ and } j \in V_T\} \tag{18}$$

We choose V_E as the *strict* derived set of $D(V_L)$ in the subgraph G'. This means that the vertices in $D(V_L)$ are initially colored black, and we apply the color-change rule based on the arc set E'.

Let $D(V_L) \cup V_E$ be denoted by $D'(V_L)$, and note that $D'(V_L)$ is equal to the derived set of $D(V_L)$ in G'. Then, by construction, we have $V_E \subseteq V_T$.

Clearly (17) can be written as

$$\mathcal{V}_T \subseteq \mathcal{D}'(V_L) = \mathcal{D}(V_L) \oplus \mathcal{V}_E \tag{19}$$

where $\mathcal{D}'(V_L) = \operatorname{im} P(V; D'(V_L))$ and $\mathcal{V}_E = \operatorname{im} P(V; V_E)$. Without loss of generality, assume that

$$D(V_L) = \{1, 2, \ldots, d\}$$

and

$$V_E = \{d + 1, d + 2, \cdots, d + e\}.$$

Also let

$$V_T = \{d - t, d - t + 1, \ldots, d, d + 1, \ldots, d + e\}$$

for some $t < d$. Consider the fourth statement in Proposition VI.4. Let $X \in Q(G)$ and ξ be a vector in the subspace $\mathcal{V}_T \cap \langle X \mid \mathcal{V}_L\rangle^\perp$. Hence, $\xi \in \mathcal{V}_T \cap \langle X \mid \mathcal{D}(V_L)\rangle^\perp$ by Lemma VI.2. We write $\xi \in \mathbb{R}^n$ as $\xi = \operatorname{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ by partitioning the vertices into the subsets $D(V_L) \setminus V_T$, $D(V_L) \cap V_T$, V_E, and $V \setminus D'(V_L)$, respectively. Now, compatible with ξ, let the matrix X be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \tag{20}$$

and assume that (17), and thus (19), holds. Then, we have $\xi \in \mathcal{D}'(V_L)$, and clearly we obtain that $\xi_4 = 0$. Moreover, we have

$$\xi^T X^{k-1} P(V; D(V_L) = 0 \tag{21}$$

for each $k \in \mathbb{N}$. The equality $\xi^T P(V; D(V_L)) = 0$ yields $\xi_1 = \xi_2 = 0$. Then, from $\xi^T X P(V; D(V_L)) = 0$, we obtain that

$$\xi_3^T \begin{bmatrix} X_{31} & X_{32} \end{bmatrix} = 0. \tag{22}$$

Note that X_{21}, X_{22}, X_{31}, and X_{32} correspond to the arcs from the vertices in the derived set to those in the target set V_T. Observe that the matrix

$$X' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ X_{31} & X_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

belongs to the qualitative class $Q(G')$, where the partitioning is compatible to (20). Noting that $D'(V_L)$ is equal to the derived set of $D(V_L)$ in G', by Lemma VI.1, we have

$$\mathcal{D}'(V_L) \subseteq \langle X' \mid \mathcal{D}(V_L)\rangle \tag{23}$$

It is straightforward to investigate that the subspace in the right hand side of (23) is computed as

$$\langle X' \mid \mathcal{D}(V_L)\rangle = \operatorname{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, (23) yileds

$$\mathcal{D}'(V_L) = \operatorname{im} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X_{31} & X_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the partitioning is again compatible to (20). This obviously implies that $\begin{bmatrix} X_{31} & X_{32} \end{bmatrix}$ is full row rank. Consequently, (22) results in $\xi_3 = 0$ which in turn implies targeted controllability of $(G; V_L; V_T)$ by the fourth statement of

Proposition VI.4. Therefore, we conclude that the sufficient condition provided by Theorem VI.5 can be sharpened as (17) with V_E being the strict derived set of $D(V_L)$ in the subgraph G'. This, using the same notation as above, is summarized in the following theorem.

Theorem VI.6 *Given a leader set V_L and a target set V_T, we have that $(G; V_L; V_T)$ is targeted controllable if $V_T \subseteq D'(V_L)$.*

To clarify, note that the set $D'(V_L)$ is constructed as a result of the following steps:

1) Compute the set $D(V_L)$, that is the derived set of V_L in the graph $G = (V, E)$
2) Construct the subgraph $G' = (V, E')$ from G, with E' given by (18)
3) Compute the derived set of $D(V_L)$ in G', and denote it by $D'(V_L)$. This means that the vertices in $D(V_L)$ are initially colored black, and we apply the color-change rule based on the arc set E'.

Now, consider again the graph depicted in Figure 2 with $V_L = \{1, 2\}$. Recall that the derived set of V_L is given by $D(V_L) = \{1, 2, 3, 4\}$ in this case. Let V_T be given by

$$V_T = \{1, 2, 3, 4, 5, 6\} \tag{24}$$

Then, Figure 3 shows the subgraph $G' = (V, E')$ with E' given by (18). It is easy to observe that the derived set of $D(V_L)$ in G' is obtained as $D'(V_L) = \{1, 2, 3, 4, 5, 6\}$. Therefore, noting that $V_T = D'(V_L)$, we conclude that $(G; V_L; V_T)$ is targeted controllable by Theorem VI.6. Observe that by extending the derived set of V_L to the set $D'(V_L)$, the condition (15) has been replaced by a less conservative condition (24). However, the sufficient condition provided by Theorem VI.6 is still not exact, as evident by (16). In fact, the set $D(V_L)$ does not infect vertex 7 in G'.

VII. Conclusions

In this paper, we have studied the case where the network is not strongly structurally controllable, and we are interested in partial controllability or controllability properties in some parts of the network. We have exploited the notion of zero forcing sets equipped with a coloring rule. As observed, the reachability subspace is invariant under this coloring. We have also investigated strong structurally reachable subspace of the network and showed that this subspace is topoligically equivalent to the derived set of the leader set. Then, we have studied targeted controllability of the network from a strong structural perspective. We have established topological sufficient conditions guaranteeing the (strong) targeted controllability of the network. Investigating an exact topological characterization of targeted controllability is a subject of future research.

References

[1] A. Banerjee and J. Jost. On the spectrum of the normalized graph. *arXiv:0705.3772*, 2007.

[2] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Transactions on Automatic Control*, 58(9):2349–2354, 2013.

[3] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47(4):575–600, 2008.

[4] A. Chapman and M. Mesbahi. Strong structural controllability of networked dynamics. In *Proc. of American Control Conference*, pages 6141–6146, 2013.

[5] A. Chapman and M. Mesbahi. On symmetry and controllability of multi-agent systems. In *53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014*, pages 625–630, 2014.

[6] J.M. Dion, C. Commault, and J. van der Woude. Generic properties and control of linear structured systems: a survey. *Automatica*, 39(7):1125–1144, 2003.

[7] M. Egerstedt, S. Martini, M. Cao, M. K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus network? *Control Systems Magazine*, 32(4):66–73, 2012.

[8] J. Gao, Y.Y. Liu, R.M. D'Souza, and A.L. Barabási. Target control of complex networks. *Nature communications*, 5, 2014.

[9] C. D. Godsil. Control by quantum dynamics on graphs. *Physical Review A*, 81(5):052316–1:5, 2010.

[10] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications*, 432:1961–1974, 2010.

[11] Y. Y. Liu, J. J. Slotine, and A. L. Barabasi. Controllability of complex networks. *Nature*, 473:167–173, 2011.

[12] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[13] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton and Oxford, 2010.

[14] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Projection-based model reduction of multi-agent systems using graph partitions. *IEEE Transactions on Control of Network Systems*, 1(2):145–154, 2014.

[15] N. Monshizadeh and A.J. van der Schaft. Structure-preserving model reduction of physical network systems by clustering. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 4434–4440, Dec 2014.

[16] N. Monshizadeh, S. Zhang, and M.K. Camlibel. Zero forcing sets and controllability of dynamical systems defined on graphs. *IEEE Transactions on Automatic Control*, 59(9):2562–2567, 2014.

[17] M. Nabi-Abdolyousefi and M. Mesbahi. On the controllability properties of circulant networks. *IEEE Transactions on Automatic Control*, 58(12):3179–3184, 2013.

[18] G. Notarstefano and G. Parlangeli. Controllability and observability of grid graphs via reduction and symmetries. *IEEE Transactions on Automatic Control*, 58(7):1719–1731, 2013.

[19] G. Parlangeli and G. Notarstefano. On the reachability and observability of path and cycle graphs. *IEEE Transactions on Automatic Control*, 57(3):743–748, 2012.

[20] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[21] H.L. Trentelman, A.A. Stoorvogel, and M.L. J. Hautus. *Control Theory for Linear Systems*. Springer, London, 2001.

[22] S. Zhang, M. Cao, and M.K. Camlibel. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. *IEEE Transactions on Automatic Control*, 59(3):745–750, 2014.