

University of Groningen

Albime triangles and Guy's favourite elliptic curve

Bakker, Erika; Chahal, Jasbir S; Top, Jaap

Published in:
Expositiones Mathematicae

DOI:
[10.1016/j.exmath.2014.12.005](https://doi.org/10.1016/j.exmath.2014.12.005)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2016

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Bakker, E., Chahal, J. S., & Top, J. (2016). Albime triangles and Guy's favourite elliptic curve. *Expositiones Mathematicae*, 34(1), 82-94. <https://doi.org/10.1016/j.exmath.2014.12.005>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.



Albime triangles and Guy's favourite elliptic curve

Erika Bakker^a, Jasbir S. Chahal^b, Jaap Top^{a,*}

^a *Johann Bernoulli Institute, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands*

^b *Department of Mathematics, Brigham Young University, Provo, UT 84602, USA*

Received 22 October 2014; received in revised form 28 October 2014

Abstract

This text discusses triangles with the property that a bisector at one vertex, the median at another, and the altitude at the third vertex are concurrent. It turns out that since the 1930s, such triangles appeared in the problem sections of various journals. We recall their well known relation with points on a certain elliptic curve, and we present an elementary proof of the classical result providing the group structure on the real points of such an elliptic curve. A consequence of this answers a question posed by John P. Hoyt in 1991.

© 2015 Elsevier GmbH. All rights reserved.

MSC 2010: primary 11G05; secondary 33E05; 11K06; 97G40

Keywords: Elliptic curve; Elliptic integral; Equidistribution

1. Introduction

There are many geometric problems which are related to questions concerning elliptic curves. Well-known examples include the congruent number problem [22,33] and various

* Corresponding author.

E-mail addresses: h.b.bakker.1@student.rug.nl (E. Bakker), jasbir@mathematics.byu.edu (J.S. Chahal), j.top@rug.nl (J. Top).

problems dealing with Heron triangles (see, e.g., [37,12]). In this note another such application of arithmetic of elliptic curves to a problem involving triangles is discussed. Some of the rather amusing history of the problem will be presented in Section 3, where we also formulate the main result. The same problem is discussed in Richard Guy’s paper [12]; in fact, he calls the particular curve appearing in the problem his ‘favourite elliptic curve’. Our note can be regarded as a supplement and completion to [12]: we answer the motivating geometric problem which led to Guy’s paper (Theorem 3.2, which appears to be new), and we present a proof using only some basic calculus of some properties of elliptic integrals which play a crucial role in the solution of the problem (Lemma 5.1). Although the latter result is certainly known to experts and in fact may be readily deduced from standard theory as sketched, e.g., in Larry Washington’s [38, pp. 274–275], the elementary approach given below does not appear in standard textbooks on elliptic curves such as [30,5,29,21,16,38].

Recall that in a triangle ABC , a *bisector* (or, angle bisector) is a line through one of the vertices which divides the corresponding angle into equal parts. Similarly, a *median* is a line through one of the vertices which divides the opposite side into equal parts. And finally, an *altitude* is a line through one of the vertices which is perpendicular to the opposite side. In college geometry one learns some facts concerning these lines in a triangle. For instance, the three medians of a triangle are concurrent, the three altitudes are concurrent, and so are the three (interior) bisectors.

For a triangle ABC the bisector in A , the median in B and the altitude in C are in general *not* concurrent. For further reference, we propose a name for triangles which have this concurrency property.

Definition 1.1. A triangle is called *albime concurrent* (altitude–bisector–median concurrent) or simply *albime*, if after possibly permuting the vertices $\{A, B, C\}$, a bisector in A , the median in B and the altitude in C are concurrent.

2. From triangles to points on a curve

It is easy to construct an albime triangle starting from an arbitrary angle CAB and an arbitrary length $|AC|$: the altitude in C and the bisector in A are determined from this data, hence also two points of the remaining median, namely the midpoint of AC and the intersection point of the given altitude and bisector. So we know the median, and hence the point B .

Let \mathcal{S} be the set of equivalence classes of similar triangles. Note that scaling an albime triangle by any positive real number produces another one, so ‘albime’ is a property of a class in \mathcal{S} . Let $\mathcal{A} \subset \mathcal{S}$ be the set of equivalence classes of similar triangles which are albime. We shall identify a given equivalence class with any of its members. Let I be the open interval $(0, 2)$. We will prove the following fact.

Theorem 2.1. *The map*

$$\Delta : I \longrightarrow \mathcal{A}$$

given by $\Delta(c) =$ the triangle with side lengths c , $2 - c$ and $\sqrt{c^3 - 4c + 4}$ is a bijection.

The proof of this requires two classical results from elementary geometry. The first one is usually attributed to Giovanni Ceva, who proved it in 1678. However, as is explained in [14, pp. 9–10], the result was in fact proven in the 11th century by Al-Mu'taman ibn Hūd in his *Kitab al-Istikmāl*.

Ceva's Theorem. *Given a triangle ABC with a point D on side BC, a point E on side AC and a point F on side AB, then the lines AD, BE and CF are concurrent, if and only if*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

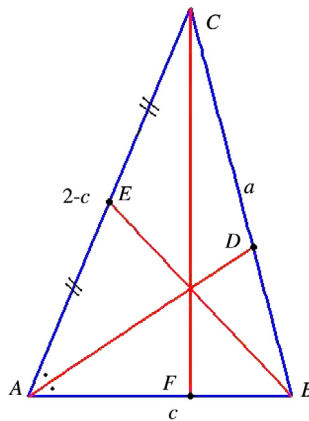
Several simple proofs of this result appear in the literature, as an example see [10, Theorems 1.21 and 1.22].

The second classical result we will use, is the so-called angle bisector theorem ([10, Theorem 1.33]).

Proposition 3 in Euclid, Elements, VI. *Suppose that in triangle ABC the point D is on the side BC. Then AD is a bisector, if and only if*

$$\frac{|AB|}{|AC|} = \frac{|BD|}{|CD|}.$$

Now we proceed in describing all possible albime triangles. Let $\triangle ABC$ be albime and put $a = |BC|$, $b = |AC|$, and $c = |AB|$. Without loss of generality, by an appropriate scaling, assume $b + c = 2$.



The two classical results mentioned above show

$$\frac{|AF|}{c - |AF|} = \frac{2 - c}{c},$$

which implies $|AF| = c(2 - c)/2$ and $|BF| = c^2/2$.

Using Pythagoras' theorem in the triangles AFC and BFC gives

$$a^2 = c^3 - 4c + 4.$$

Conversely, any pair of real numbers (a, c) satisfying $0 < c < 2$ and $a^2 = c^3 - 4c + 4$ gives a triangle as desired, namely with sides of length $|a|$, $b = 2 - c$ and c , respectively (the fact that these numbers satisfy the triangle inequality, is a nice exercise in standard calculus using the equality $a^2 = c^3 - 4c + 4$). Note that for different c 's, we obtain dissimilar triangles. This finishes the proof of [Theorem 2.1](#). \square

The equation

$$y^2 = x^3 - 4x + 4$$

together with a point O at infinity defines an elliptic curve. Many textbooks dealing with the theory of such curves were already mentioned in the Introduction. The points of any elliptic curve E form an abelian group with zero element O , in which points are added by the so called chord and tangent method: points P_1, P_2, P_3 add to O precisely when a line ℓ exists such that $\ell \cap E$ equals $\{P_1, P_2, P_3\}$ (points counted with the appropriate multiplicity in case not all P_i are distinct, and O is on a line only for all vertical lines, i.e., those with equation $x = c$ for any constant c).

For a field K over which the equation of E is defined, let $E(K)$ denote the subset of E consisting of O and all points on E with coordinates in K . Then $E(K)$ is a subgroup of E . In the special case of $y^2 = x^3 - 4x + 4$, the point $P = (2, 2)$ is in $E(\mathbb{Q})$. A calculation shows $-4P = (1, 1)$, which corresponds to the equilateral albime triangle.

3. Rational albime triangles: history and main result

The problem of classifying albime triangles gets more interesting when one adds the following number theoretic restriction:

Describe the rational albime triangles, i.e., such that the lengths of the three sides are rational numbers.

To our knowledge, the earliest appearance of albime triangles happened in 1937, when D.L. MacKay of Evander Childs High School in New York posed the following problem in the February issue of the American Mathematical Monthly [\[23\]](#).

In the triangle ABC , the bisector of angle A , the median from vertex B , and the altitude from vertex C , are concurrent. Show that the triangle may be constructed with ruler and compasses if the lengths of sides b and c are given.

In November of the same year 1937, the Monthly published the solution [\[20\]](#) by J.W. Kitchens, at the time a student at the University of Oklahoma. Moreover, it is remarked that 8 others including the proposer also solved the problem. The solution by Kitchens is based on Ceva's theorem and the angle bisector theorem, precisely the ideas also used in the proof of [Theorem 2.1](#).

The *rational* albime triangles (although the term was invented in the present paper) made their appearance in 1939, when the same D.L. MacKay asked in the Monthly [\[24\]](#):

What relationship exists between the sides of a triangle ABC if the bisector of angle A , the median from vertex B , and the altitude from vertex C are concurrent? Can the three sides be commensurable if the triangle is not equilateral?

Here ‘commensurable’ means precisely that the side lengths a, b, c should be such that a/c and b/c are rational numbers, in other words, up to a scaling factor we have a rational albime triangle. As before, the relationship between a, b, c is found using Ceva’s theorem and the bisector theorem. It reads

$$a^2 = b^2 + c^2 - \frac{2bc^2}{b+c} = b^2 - c^2 + \frac{2c^3}{b+c},$$

a formula which in the scaled version with $b+c=2$ we encountered in the proof of [Theorem 2.1](#). In 1940 the Monthly published a solution [34] proposed by Charles W. Trigg, who much later in 1967 authored the problem book *Mathematical Quickies* [35]. According to the Monthly, three others found partial solutions. Trigg derives the relationship between a, b, c and points out how this relates to the earlier Monthly problem [23]. To describe all *rational* albime triangles, he first scales so that a, b, c are coprime integers. The formula then shows that $2c^3/(b+c)$ is an integer, and Trigg argues that a, b, c being coprime implies that $b+c$ cannot be a divisor of c^3 . He concludes $b=c=1$, from which it easily follows that the triangle is equilateral. So the only rational albime triangle, up to scaling, is the equilateral one! Unfortunately, the argument contains a flaw, as the example $a=13, b=12$, and $c=15$ shows.

Instead of considering the usual interior bisector of angle A , one can also take the exterior bisector. An interesting (and correct) remark by Trigg is that this amounts to replacing c by $-c$ in the formula. To construct triangles with an exterior bisector, an altitude, and a median concurrent becomes a problem that appeared in National Mathematics Magazine in the early forties at least twice [7,19,8,32]. In 1949 Victor Thébault proposes in Mathematics Magazine (the new name of the National Math. Magazine) another, equivalent criterion [31] for a triangle ABC to be albime: taking as before a bisector of angle A and a median from vertex B , the criterion reads

$$\sin(C) = \pm \tan(A) \cdot \cos(B),$$

with the sign \pm depending on whether one considers the interior or the exterior bisector. No less than four proofs were published [11] in the same journal in 1950.

Apparently, the mistake in Trigg’s classification of rational albime triangles remained unnoticed. In 1971, Stanley Rabinowitz [25] asked in Mathematics Magazine:

- (1) Find all triangles ABC such that the median to side a , the bisector of angle B , and the altitude to side c are concurrent.
- (2) Find all such triangles with integral sides.

Two solutions [4] are presented in the subsequent volume of the journal, and six people who also solved the problem are listed. One of the published solutions is by Trigg, who repeats the argument he published in the Monthly 32 years earlier [34], including the error. The other solution is by Leon Bankoff (1908–1997), a talented problem solver especially known as a prominent Los Angeles dentist who had many Beverly Hills–Hollywood celebrities among his patients [2], [1, pp. 79–95]. Bankoff argues as follows: with a reference to a problem in the celebrated USSR Olympiad Problem Book [28], he states that if a triangle ABC has integral sides, then one of its angles is $60^\circ, 90^\circ$, or 120° . Then he shows that if a rational albime triangle has an angle of such size, it necessarily is

equilateral. Bankoff's solution is incorrect, due to the fact that he misquotes the result stated and proved in the Olympiad Book. Namely, Problem 238 in this book asks:

Prove that in a triangle whose sides have integral length it is not possible to find angles differing from 60° , 90° , and 120° , and commensurable with a right angle.

The condition “commensurable with a right angle” means that the angle should have (measured in degrees) a rational number as its size. With a little algebra, this problem is easy to solve. Indeed, suppose the angle in question has size $\pi m/n$ (in radians). Since the triangle has integral sides, by the cosine law $2 \cos(\pi m/n)$ is a rational number. It can be written as $e^{\pi im/n} + e^{-\pi im/n}$, a sum of two algebraic integers, so it is itself an algebraic integer. As a consequence, $2 \cos(\pi m/n) \in \mathbb{Z}$ which immediately solves the problem.

For Bankoff's argument to be correct, he would need that in a rational albime triangle at least one of the angles is commensurable with a right one. This is not true as the example with $a = 13$, $b = 12$, and $c = 15$ shows. His argument in fact shows the following weaker result.

Theorem 3.1 (Bankoff). *One of the angles in a rational albime triangle has size a rational multiple of π if and only if the triangle is equilateral.* \square

A picture showing both Charles W. Trigg and Leon Bankoff is presented on p. 112 of [2], and page 108 of the same paper shows Bankoff with above mentioned Victor Thébault. As far as we know, neither the Monthly nor Mathematics Magazine published an erratum concerning the erroneous claims in the published solutions.

For almost 20 years apparently no new interest in (rational) albime triangles arose, until in 1991 John P. Hoyt poses Problem E 3434 in the Monthly [15]:

Prove (or disprove) that there are infinitely many triples of positive integers (a, b, c) with no common factor (greater than one) such that the triangle ABC with sides a, b, c has the following property: the median from A , the angle bisector from B , and the altitude from C are concurrent. Examples of such triples are $(12, 13, 15)$, $(35, 277, 308)$, and $(26598, 26447, 3193)$.

Hoyt's examples evidently show that the earlier “solutions” by Trigg and by Bankoff are incorrect. However, Hoyt presents no remarks or references to show that he is aware of this. The difficulty of the problem (to our knowledge, the Monthly nor any other journal to date ever published a complete answer) possibly triggered Richard Guy to look into it, resulting in the very nice paper [12]. Guy presents all essential tools needed to answer Hoyt's question. He states (quote from [12, p. 772]): “At the time of writing, no solution has been published, though I have seen an interesting one due to J.G. Mauldon, which makes no explicit use of an elliptic curve”. James Mauldon (1920–2002) was a distinguished British war veteran who taught mathematics at Oxford University and later became Walker Professor of Mathematics at Amherst College [27]. We are not aware of any details concerning his solution. We answer Hoyt's question in [Theorem 3.2](#).

Scaling a rational albime triangle by a positive rational one easily reduces Hoyt's problem to:

find all rational points (c, a) on the elliptic curve E with equation $y^2 = x^3 - 4x + 4$, which have the additional property $0 < c < 2$.

Guy remarks that this elliptic curve contains infinitely many rational points, and moreover he states without proof (p. 774 of [12]): “About 4/11 of the points give genuine triangles”. Although this clearly answers Hoyt’s question, Guy’s paper does not explain where the fraction 4/11 comes from. We do it in the following theorem, which is the main result of our paper. Parts (a) and (b) were also stated by Guy but (c) actually proves a precise version of his assertion about the fraction 4/11. Note that apart from mentioning Hoyt and Mauldon, Guy makes no reference to any of the earlier persons who looked into the problem.

Theorem 3.2. (a) *Up to scaling by a rational factor, any rational albime triangle ABC is obtained as follows.*

Take a rational point (c, a) on the elliptic curve E with equation $y^2 = x^3 - 4x + 4$, such that $0 < c < 2$. Then take the triangle with side lengths $|AB| = c$ and $|AC| = 2 - c$ and $|BC| = |a|$.

(b) *All rational points in $E(\mathbb{Q})$ are multiples of the point $P = (2, 2)$.*

(c) *The proportion of points $Q \in E(\mathbb{Q})$ with x -coordinate $x(Q)$ between 0 and 2, equals*

$$\lim_{k \rightarrow \infty} \frac{\#\{n : |n| \leq k \text{ \& } 0 < x(nP) < 2\}}{2k} = 0.361208\dots$$

In particular, there are infinitely many dissimilar rational albime triangles.

Before proving the theorem, we will try to connect some more dots in the history of the problem. This intends to remedy a disconnect between some recent activities on the problem in the Netherlands and the earlier work elsewhere.

Frederik van der Blij, mathematics professor at Utrecht University from 1954 until his retirement in 1988, discussed the rational albime triangles problem during a conference for Dutch high school mathematics teachers in Groningen in 2003 [36] and more extensively during a similar conference in Noordwijkerhout in 2005 [9, pp. 24–25]. Van der Blij writes (see [36, p. 4]) that a retired math teacher and former principal, Dr. Th.J. Kletter (Gorssel, the Netherlands) told him “this problem from his youth”. In fact, Kletter showed in 1957 that if the points A, B (where one takes a bisector and a median, respectively) are fixed, then the locus of points C such that ABC is albime, describes a conchoid of Nicomedes and the locus of points where the three concurrent lines meet, a cissoid of Diocles. Kletter knew no examples of rational albime triangles except the equilateral one. In 2005 van der Blij presents several examples, such as the one with side lengths (587783, 610584, 482143). He recommends the problem as a suitable one, with appropriate instructions, for advanced high school students. Note that the same example with a misprint (4832143 instead of 482143) also appears in Guy’s paper. Interestingly, van der Blij makes no reference to any earlier work on the problem. Indeed, the problem was picked up by high school student Tjitske Starkenburg. Her discussion of it won her the 2005 University of Groningen Jan Kommandeur prize for best high school science project and a third prize at the 2005 International Conference for Young Scientists in Katowice (Poland). One of us used the problem during a course on elliptic curves intended for high school teachers in the Fall of 2007 [18]. M. de Nijs, a teacher participating in the course, wrote a version of it which she successfully presented to a class of grade 9 students. The first author of the present paper in 2012 finished a research project [3] as part of an Education and Communication

master's degree at Groningen University, also discussing this problem and its possible use in education. None of the Dutch authors mentioned above makes any reference to earlier texts dealing with (rational) albime triangles, not even to Guy's paper [12]. It seems likely that none of them was even aware of the existence of such texts. Our present paper hopefully remedies this omission.

4. The proof: parts (a) and (b)

Part (a) of [Theorem 3.2](#) follows easily using the ideas explained in [Section 2](#), see also [12, p. 773]. We now give some comments concerning Part (b).

For an elliptic curve defined over the rational numbers \mathbb{Q} , Mordell proved in 1922 that the group of points with coordinates in \mathbb{Q} has a finite number of generators. The textbooks [30,6] contain a proof for this, originally due to Tate, in the special case that the group contains a point of order 2. This does not apply for the curve discussed here. The proof presented in [29] applies in general. In fact, using the magma calculator [17] which is freely available on the internet, the following MAGMA-code

```
E:=EllipticCurve([-4,4]);
MordellWeilGroup(E);
Generators(E);
```

produces the output

```
Abelian Group isomorphic to Z
Defined on 1 generator (free)
[ (2 : 2 : 1) ]
```

So the group $E(\mathbb{Q})$ of rational points on E is infinite cyclic, with $P = (2, 2)$ as generator:

$$E(\mathbb{Q}) = \{nP : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

The verification of this result is in the present case a quite standard one. The fact that the ring $\mathbb{Z}[x]/(x^3 - 4x + 4)$ is a principal ideal domain greatly facilitates the calculation.

A few lines of MAGMA-code such as

```
Z:=Integers(); E:=EllipticCurve([-4,4]);
P:=E![2,2]; Albs:={@ @}; upb:=30;
for n in [1..upb]
  do Q:=n*P; c:=Q[1];
    if c gt 0 and c lt 2
      then a:=Abs(Q[2]); d:=Denominator(a);
          a:=Z!(d*a); b:=Z!(d*(2-c)); c:=Z!(d*c);
          g:=Gcd(Gcd(a,b),c);
          Albs:=Albs join {[a/g,b/g,c/g]};
    end if;
end for; Albs;
```

will produce a list of examples; changing the value of upb makes this list longer or shorter. Moreover, any rational albime triangle will appear in the list (at least in theory), by making the value of upb sufficiently large.

5. From elliptic curve to circle

In order to prove the last assertion of [Theorem 3.2](#), we state and prove a result ([Lemma 5.1](#)) which requires some preliminary remarks. Let

$$f(t) := t^3 + at^2 + bt + c$$

be a cubic polynomial with real coefficients a, b, c and precisely one real zero α of multiplicity one. Let

$$E/\mathbb{R} : y^2 = f(x)$$

denote the elliptic curve over \mathbb{R} defined by the polynomial $f(t)$.

We denote by

$$\mathbb{T} := \{z \in \mathbb{C}^*; |z| = 1\}$$

the unit circle group under multiplication. Finally, with

$$\Omega := 2 \int_{\alpha}^{\infty} \frac{dt}{\sqrt{f(t)}},$$

for any point $P = (x, y) \neq O$ in $E(\mathbb{R})$, we put

$$\varphi(P) := \begin{cases} \frac{1}{\Omega} \int_x^{\infty} \frac{dt}{\sqrt{f(t)}} & \text{for } y \geq 0; \\ 1 - \frac{1}{\Omega} \int_x^{\infty} \frac{dt}{\sqrt{f(t)}} & \text{for } y \leq 0, \end{cases}$$

whereas $\varphi(O) = 0$. We then define a map $\Psi : E(\mathbb{R}) \longrightarrow \mathbb{T}$ by

$$\Psi(P) := e^{2\pi i \varphi(P)}.$$

Because $f(t)$ has only one real zero, it is obvious that $E(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ is, topologically, connected and compact and thus homeomorphic to the unit circle \mathbb{T} . Moreover $E(\mathbb{R})$ and \mathbb{T} are real topological groups. The following lemma compares these groups. A similar result over the complex numbers is well-known and may be found, e.g., in [\[29, Chapter VI Theorem 5.2\]](#). It is possible to deduce our result from this complex case. Note however that the argument below only uses some real calculus, hence it is much more elementary.

Lemma 5.1. *The map*

$$\Psi : E(\mathbb{R}) \longrightarrow \mathbb{T}$$

defines an isomorphism of groups.

Proof. From the definitions, it is clear that if $\alpha \leq x_1 < x_2$, then for $y_i := \sqrt{f(x_i)}$ we have

$$\frac{1}{2} \geq \varphi(x_1, y_1) > \varphi(x_2, y_2) > 0$$

and

$$\frac{1}{2} \leq \varphi(x_1, -y_1) < \varphi(x_2, -y_2) < 1.$$

It follows that Ψ is injective. Moreover, since φ is continuous (away from O) and

$$\lim_{x \rightarrow \infty} \varphi(x, \sqrt{f(x)}) = 0,$$

we conclude that Ψ is also surjective, hence Ψ is invertible.

It remains to show that Ψ is a homomorphism of groups. To prove this, we only have to show that if $P_1 + P_2 + P_3 = O$, in other words $\{P_1, P_2, P_3\}$ is the set of intersection points of E with a line ℓ (counted with the appropriate multiplicity if some of these points coincide), then $\varphi(P_1) + \varphi(P_2) + \varphi(P_3) \in \mathbb{Z}$.

This follows from the well-known invariance of the differential $\frac{dt}{\sqrt{f(t)}}$ on E : if Q is any point on E and $(t, \sqrt{f(t)}) + Q = (t', s')$ using the group law on E , then

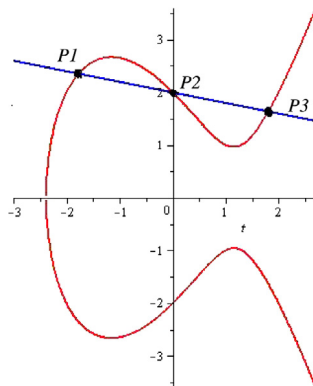
$$\frac{dt}{\sqrt{f(t)}} = \frac{dt'}{s'}.$$

This follows easily using a straightforward calculation; see also [29, III, Section 5]. Note that since (t', s') satisfies the equation of our elliptic curve, one has $s' = \pm\sqrt{f(t')}$. The choice of the sign here depends on the point Q and on the interval in which one chooses $t \geq \alpha$.

The case in which one of the points P_i equals the point at infinity O , is easy and will be omitted. So we assume $P_i = (a_i, b_i)$ are the three intersection points of E with a line. Order the points so that $a_3 \geq a_2 \geq a_1 \geq \alpha$. Since $\varphi(-P) = 1 - \varphi(P) \equiv -\varphi(P) \pmod{\mathbb{Z}}$, we may and will assume that $b_1 \geq 0$. Distinguish the following cases.

(1) If all points P_i are on or above the x -axis, then consider the point $(t_2, s_2) = (\alpha, 0) + P_1$. Split the integral

$$\int_{a_2}^{\infty} \frac{dt}{\sqrt{f(t)}} = \int_{a_2}^{t_2} \frac{dt}{\sqrt{f(t)}} + \int_{t_2}^{\infty} \frac{dt}{\sqrt{f(t)}}.$$



Now change variables in this integration using

$$(t, \sqrt{f(t)}) + P_1 = (t', s')$$

One observes that if $x_1 < t < t_2$, then $s' = -\sqrt{f(t')}$ and if $t > t_2$, then $s' = \sqrt{f(t')}$. Moreover, by the choice of the points P_i , one verifies that $t = a_2$ corresponds to $t' = a_3$ while $t = \infty$ corresponds to $t' = a_1$. Similarly, $t = t_2$ corresponds to $t' = \alpha$.

This means that the substitution given here, changes the above integral into

$$\int_{a_3}^{\alpha} \frac{dt}{-\sqrt{f(t)}} + \int_{\alpha}^{a_1} \frac{dt}{\sqrt{f(t)}}$$

Substituting this in the formula for $\varphi(P_2)$ it follows that

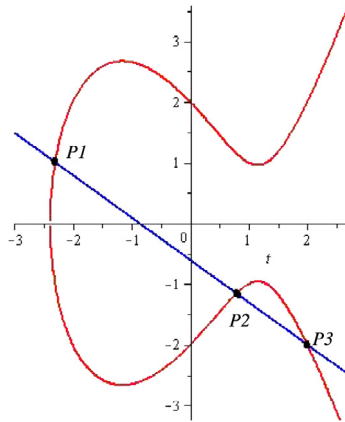
$$\varphi(P_1) + \varphi(P_2) + \varphi(P_3) = 1.$$

(2) In the remaining case, one of P_2, P_3 (and hence both) are located beneath the x -axis. Here one uses the substitution

$$(t, \sqrt{f(t)}) - P_1 = (t', -\sqrt{f(t')})$$

which is valid for all $t > a_1$. Hence, without the necessity to split the integral for $\varphi(P_2)$, one obtains

$$\int_{a_2}^{\infty} \frac{dt}{\sqrt{f(t)}} = \int_{a_1}^{a_3} \frac{dt}{\sqrt{f(t)}}$$



Using this, it follows that

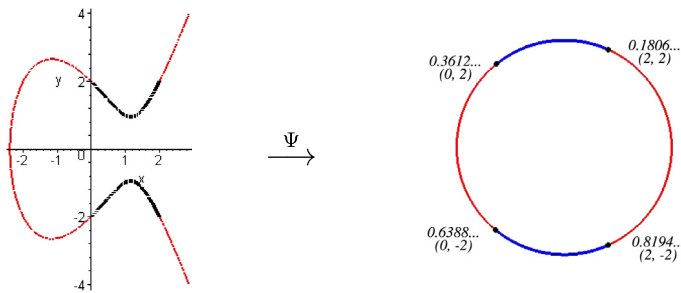
$$\varphi(P_1) + \varphi(P_2) + \varphi(P_3) = 2.$$

So in all cases one finds $\varphi(P_1) + \varphi(P_2) + \varphi(P_3) \in \mathbb{Z}$, which proves the lemma. \square

The cubic $f(t) = t^3 - 4t + 4$ satisfies the hypothesis of [Lemma 5.1](#). In fact, its unique real zero is

$$\alpha = -\frac{(54 + 6\sqrt{33})^{\frac{1}{3}}}{3} - \frac{4}{(54 + 6\sqrt{33})^{\frac{1}{3}}} \approx -2,38297576790623749412\dots$$

Denoting $P = (2, 2)$ which by [Theorem 3.2\(b\)](#) generates the infinite cyclic subgroup $E(\mathbb{Q})$ of $E(\mathbb{R})$, a consequence of [Lemma 5.1](#) is that $\Psi(P)$ generates an infinite cyclic subgroup of \mathbb{T} . A classical result of Kronecker (1884) then implies that this subgroup is dense in \mathbb{T} . It follows that $\mathbb{Z} \cdot P$ is dense in $C(\mathbb{R})$. In particular, for infinitely many integers n we have that $0 < x(nP) < 2$. So there exist infinitely many pairwise nonsimilar rational albime triangles, which answers Hoyt’s question.



Around 1909–1910 Kronecker’s density result was improved by several people, including Hermann Weyl. The resulting “equidistribution theorem” (see, e.g., [[13](#), Section 23.10]) states that the proportion of the set of all n such that $\Psi(P)^n$ is in a given arc of the unit circle, equals the ratio of the length of that arc over 2π . To see this, apply the (short and complete!) description of the equidistribution result as presented in [[26](#)] to the function that takes the value 1 on all points of the given interval, and 0 everywhere else. This shows the remaining assertion in [Theorem 3.2](#). \square

References

- [1] Donald J. Albers, Gerald L. Alexanderson, *Fascinating Mathematical People: Interviews and Memoirs*, Princeton University Press, 2011.
- [2] G.L. Alexanderson, A conversation with Leon Bankoff, *College Math. J.* 23 (1992) 98–117.
- [3] Erika Bakker, *Congruente getallen en concurrente lijnen* (Master’s thesis), in *Education and Communication University of Groningen*, 2012.
- [4] Leon Bankoff, Charles W. Trigg, A special triangle. Two solutions to problem 790, *Math. Mag.* (1972) 49–51.
- [5] J.W.S. Cassels, *Lectures on Elliptic Curves*, in: *London Mathematical Society Student Texts*, vol. 24, Cambridge University Press, Cambridge, 1991.
- [6] J.S. Chahal, *Topics in Number Theory*, Plenum Press, New York, London, 1988.
- [7] Walter B. Clarke, Problem 354, *Nat. Math. Mag.* 14 (1940) 422.
- [8] Walter B. Clarke, Problem 504, *Nat. Math. Mag.* 17 (1943) 233.
- [9] Conference guide of the Nationale Wiskunde Dagen 2005. Available at <http://www.fisme.science.uu.nl/nwd/>.
- [10] H.S.M. Coxeter, S.L. Greitzer, *Geometry revisited*, *New Math. Library* 19 (1967) MAA.
- [11] B.K. Gold, L.M. Kelly, D.L. MacKay, P. Na Nagara, Four solutions to problem 43, *Math. Mag.* 23 (1950) 275–276.
- [12] Richard K. Guy, My favorite elliptic curve: a tale of two types of triangles, *Amer. Math. Monthly* 102 (1995) 771–781.
- [13] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1938, (6th edition 2008).
- [14] Jan P. Hogendijk, Al-Mu’taman ibn Hūd, 11th century king of Saragossa and brilliant mathematician, *Historia Math.* 22 (1995) 1–18.

- [15] John P. Hoyt, Problem E 3434, *Amer. Math. Monthly* 98 (1991) 365.
- [16] Dale Husemöller, *Elliptic Curves*, second ed., in: *Graduate Texts in Mathematics*, vol. 111, Springer, New York, 2004.
- [17] <http://magma.maths.usyd.edu.au/calc/>.
- [18] <http://www.math.rug.nl/~top/NCC.html>.
- [19] E.C. Kennedy, Solution to problem 354, *Nat. Math. Mag.* 15 (1940) 149.
- [20] J.W. Kitchens, Solution to problem E 263, *Amer. Math. Monthly* 44 (1937) 599–600.
- [21] Anthony W. Knap, *Elliptic Curves*, in: *Mathematical Notes*, vol. 40, Princeton University Press, Princeton, 1993.
- [22] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*. Vol. 97, second ed., in: *Graduate Texts in Mathematics*, vol. 106, Springer, New York, 1993.
- [23] D.L. MacKay, Problem E 263, *Amer. Math. Monthly* 44 (1937) 104.
- [24] D.L. MacKay, Problem E 374, *Amer. Math. Monthly* 46 (1939) 168.
- [25] Stanley Rabinowitz, Problem 790, *Math. Mag.* 44 (1971) 106.
- [26] Jeffrey Rauch, <http://www.math.lsa.umich.edu/~rauch/558/Kronecker.pdf>.
- [27] Remembering Prof. James Mauldon. *Amherst Magazine*, Summer 2002.
- [28] D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom, *The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics*, Dover Publications, Inc., New York, 1993, (first Published by W.H. Freeman and Co., San Francisco, 1962).
- [29] J.H. Silverman, *The Arithmetic of Elliptic Curves*, in: *Graduate Texts in Mathematics*, vol. 106, Springer, New York, 1986.
- [30] J.H. Silverman, J. Tate, *Rational Points on Elliptic Curves*, in: *Undergraduate Texts in Mathematics*, Springer, New York, 1992.
- [31] Victor Thébault, Problem 43, *Math. Mag.* 23 (1949) 103.
- [32] Paul D. Thomas, Solution to problem 504, *Nat. Math. Mag.* 18 (1943) 91.
- [33] Jaap Top, Noriko Yui, Congruent number problems and their variants, in: J.P. Buhler, P. Stevenhagen (Eds.), *Algorithmic Number Theory*, in: *MSRI Publications*, vol. 44, Cambridge University Press, 2008, pp. 613–640.
- [34] Charles W. Trigg, Solution to problem E 374, *Amer. Math. Monthly* 47 (1940) 176.
- [35] Charles W. Trigg, *Mathematical Quickies: 270 Stimulating Problems with Solutions*, McGraw-Hill Book Company, New York, 1967, (Republished by Dover Publ., Mineola, NY, 1985).
- [36] F. van der Blij, Wiskunde moet uitdagender en spannender (Mathematics should be more challenging and exciting), *Nieuwe Wiskrant* 23 (2004) 4–6.
- [37] Ronald van Luijk, An elliptic K3 surface associated to Heron triangles, *J. Number Theory* 123 (2007) 92–119.
- [38] Lawrence C. Washington, *Elliptic Curves: Number Theory and Cryptography*, Chapman & Hall/CRC, Boca Raton, 2003.