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Germs, Remco; Foreest, Nicky D. van

Publication date:
2014

university of groningen / faculty of economics and business

14001-OPERA

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Remco Germs
Nicky D. van Foreest

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Research Institute SOM
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Visiting address:
Nettelbosje 2
9747 AE Groningen
The Netherlands

Postal address:
P.O. Box 800
9700 AV Groningen
The Netherlands

T +31 50 363 7068/3815

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Remco Germs
Wageningen University

Nicky D. van Foreest
University of Groningen
n.d.van.foreest@rug.nl

Optimal Control of Production-Inventory Systems with Constant and Compound Poisson Demand

Remco Germs[1] and Nicky D. Van Foreest[2]

[1]Operations Research and Logistics Group, Wageningen University, Hollandseweg 1, 6706 KN Wageningen
[2]Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700 AV Groningen

December 18, 2013

Abstract

In this paper, we study a production-inventory systems with finite production capacity and fixed setup costs. The demand process is modeled as a mixture of a compound Poisson process and a constant demand rate. For the backlog model we establish conditions on the holding and backlogging costs such that the average-cost optimal policy is of (s, S)-type. The method of proof is based on the reduction of the production-inventory problem to an appropriate optimal stopping problem and the analysis of the associated free-boundary problem. We show that our approach can also be applied to lost-sales models and that inventory models with unconstrained order capacity can be obtained as a limiting case of our model. This allows us to analyze a large class of single-item inventory models, including many of the classical cases, and compute in a numerical efficient way optimal policies for these models, whether these optimal policies are of (s, S)-type or not.

1 Introduction

Production-inventory systems with a finite, constant production rate are used to produce a wide variety of products in the food-processing industry, such as sugar (Grunow et al., 2007), coffee liquids (Pool et al., 2011), sorbitol, and modified starches (Rajaram and Karmarkar, 2004). Other examples of (process) industries are glass manufacturing, the electronic computer industry, and the pharmaceutical industry (Shi et al., 2012). Due to the finite production capacity, inventory levels in these production facilities can only increase gradually over time. Hence, these facilities require a sufficient amount of on-hand inventory to achieve a certain service level. Clearly, in practice, the production rate of these facilities is higher than the average demand. As a consequence, the production facility has to be switched off once in a while to prevent inventory from growing without bound. Typically, there are significant costs associated with switching the facility on, so that, from a financial point of view, it is critical to establish a good production-inventory policy to decide when to setup the machine to produce, and when to shut it down.

In the special case of constant demand, the production-inventory system reduces to the classical Economic Production Quantity (EPQ) model of Taft (1918), in which holding and shortage costs are balanced against setup costs to minimize the long-run average cost. When holding and backlogging costs are linear, it is intuitive that a long-run average-cost optimal policy to control production is of a so-called (s, S)-type. Such policies decide to start production when the inventory level I_t is at or below a level s and switch off production when $I_t \geq S$, where $S > s$. Explicit formulas to compute the optimal policy parameters s and S can be derived algebraically, in fact, without using derivatives, see, e.g., Cárdenas-Barrón (2001). One important generalization of the constant demand process is the Brownian motion demand process, as considered by Vickson (1986). Under the assumption of linear holding and backlogging cost, Vickson (1986) shows that an (s, S)-policy is average-cost optimal. Recently, Wu and

Chao (2013) extend this result to a production-inventory model in which the cumulative production as well as the cumulative demand process are modeled by a two-dimensional Brownian motion process.

In contrast with these purely continuous (stochastic) demand processes, many inventory theory models assume that customer orders arrive in batches at random epochs. One batch-wise demand process that has been extensively studied is the compound Poisson process. In this scenario, demands arrive in accordance with a Poisson process, and the demand sizes form a set of independently and identically distributed (i.i.d.) random variables. This case has been studied in a very recent paper by Van Foreest and Wijngaard (2013).

The next step in the generalization of the EPQ model is to consider a demand process that is a mixture of continuous constant demand and compound Poisson demand. This demand model is of practical interest, as in many applications of production-inventory systems, demand is generated from a deterministic source as well as a stochastic source. Sobel and Zhang (2001) discuss an example of a manufacturer who sells products through two channels. Large customers with long-term supply contracts generate a more or less constant demand, while smaller customers generate stochastic (unscheduled) demand. Thus, for these situations, it is necessary to model the demand as a mixture of deterministic demand, for the first channel, and a stochastic (compound Poisson) demand, for the second channel. There has also been theoretical interest in this demand model. Hordijk and Van der Duyn Schouten (1986) and Presman and Sethi (2006) prove for inventory systems with *infinite* ordering capacity that (s, S)-policies are average-cost optimal for this demand model. The work on this model, which we henceforth call the *stochastic EOQ model*, clearly merges the EOQ formula and the classical results on optimal (s, S)-policies for infinite capacity stochastic inventory models, c.f. Beyer et al. (2010).

In this paper, we consider the *stochastic EPQ model*, which is the stochastic generalization of the EPQ model: a continuous review production-inventory model with finite, constant production rate and a demand process that is a mixture of continuous and constant demand and a compound Poisson process. Our paper makes the following contributions to the prior work on optimal control of infinite horizon continuous review single-item production-inventory models:

(i) We establish for the first time conditions on the inventory costs and the demand distribution such that average-cost optimal policies exist for production-inventory systems with backlogging. We also consider conditions that guarantee that the optimal policy is of (s, S)-type.

(ii) We demonstrate the generality of the approach by applying it to a production-inventory model with lost sales and by proving that the stochastic EOQ model can be obtained as a limiting case of the stochastic EPQ model.

(iii) We discuss how the approach can be used to actually compute optimal policies in a numerically efficient way.

Let us elaborate on each of these points.

Pertaining to (i), it is of importance to remark that the incorporation of the constant term in the demand process might appear to be innocuous, but, in fact, it complicates the analysis considerably. The point is that, when demand contains a constant demand term, the production-inventory process becomes a genuine continuous-time continuous state space process. As a result, the process can no longer be modeled as semi-Markov decision process. Hence, all the methods developed previously, such as the method of Van Foreest and Wijngaard (2013), are inadequate. More specifically, the most difficult part in the proof of Van Foreest and Wijngaard (2013) deals with an optimal stopping problem associated with the problem when to switch on production. To establish the existence of a solution to this optimal stopping problem, Van Foreest and Wijngaard (2013) use functional analytic methods. Adapting this line of proof to the present model requires, however, considerably more advanced concepts such as Sobolev spaces. Instead of continuing this functional analytic approach, we prefer to use free-boundary theory, as developed by Peskir and Shiryaev (2006), to establish the existence of a solution to the associated optimal stopping problem. Not only has this approach a more probabilistic flavor, we conjecture that it offers a possibility to incorporate also Brownian motion in the demand process, thereby potentially merging all the demand models discussed in the literature up to now.

It might be tempting to think that the (s, S)-structure of the optimal policy for our model can be obtained from the results for the stochastic EOQ model by incorporating constraints on the order size in the models of Hordijk and Van der Duyn Schouten (1986) and Presman and Sethi (2006). Recall, that the main difference with the stochastic EPQ model is that the stochastic EOQ model assumes unlimited ordering capacity rather than finite production capacity. This, however, is not true as follows from a counterexample by Wijngaard (1972).

Regarding (ii), the free-boundary formulation turns out to be very versatile as it allows us to analyze many other inventory models. To substantiate this, we address two cases in this paper. The first case considers a production-inventory model with lost sales. The second case shows that, similar to the fact that the EOQ model can be obtained as a limiting case of the EPQ model by taking the production rate $r \to \infty$, the stochastic EOQ can be obtained from the stochastic EPQ model. In fact, the quasi-variational inequalities (QVIs) derived by Presman and Sethi (2006) turn out to be particularly useful as they can be obtained as a limiting case of the QVIs of our model. Hence, our approach can be applied to find the optimal policy in a wide class of single-item inventory problems. For instance, the classical result of Scarf (1959) becomes a corollary of our work.

Related to (iii), and as an immediately consequence of point (ii), we can compute optimal policies for models with infinite ordering capacity such as the stochastic EOQ model. As these computational aspects were not addressed by Hordijk and Van der Duyn Schouten (1986) and Presman and Sethi (2006), we complement their work. Moreover, since the stochastic EOQ model is a clear extension of many of the classical inventory models with infinite ordering capacity, our numerical method is more generic than many of the presently known methods, for instance such as those discussed by Zheng and Federgruen (1991); Federgruen and Zheng (1993); Feng and Xiao (2000), and Chen and Feng (2006). Finally, the present work can also be applied, see Germs and Van Foreest (2013b), to compute optimal policies for production-clearing models, c.f., Perry et al. (2005) and Berman et al. (2005),

The paper has the following structure. In Section 2 we present the details of the model and state the optimal control problem under consideration. Section 3 proves the existence of an optimal stationary control policy. In Section 4 we derive structural properties of the optimal policy and provide conditions such that the optimal policy has an (s, S)-structure. In Section 5 we show how to apply our approach to the inventory problem of (1) lost sales, and (2), the stochastic EOQ model. In Section 6 we discuss the numerical issues involved in finding optimal policies for more complicated cases. With these numerical examples we address the issue how changing stochastic demand into more regular demand affects the long-run average cost. Finally, Section 7 concludes the paper and discusses a set of related inventory problems that may be tackled also by the techniques developed here.

2 Model and problem

We consider a one-product, single-machine production-inventory model in which the cumulative demand process $(X_t)_{t\geq 0}$ is a mixture of a constant demand rate and a compound Poisson process,

$$X_t = qt + \sum_{i\leq N_t} Y_i,$$

where $q \geq 0$ is the rate of constant demand, N_t counts the number of arrivals up to time t, and Y_i is the demand size of the ith customer. By assumption, N_t is Poisson distributed with parameter λt, and the random variables Y_i are i.i.d. as the generic random variable Y with distribution $F(y)$ and mean $\mu = \mathrm{E}[Y] > 0$. For later use we define the survivor function $G(\cdot) = 1 - F(\cdot)$.

The manufacturer can control the production facility by switching it on and off. Switching occurs instantaneously. When production is off, the inventory level decreases due to the demand; when production is on, inventory is replenished at a fixed rate r. To ensure that the demand can always be covered by producing for a sufficiently large amount of time, we require that

$$r > q + \lambda\mu. \tag{1}$$

Let $(P_t, I_t)_{t\geq 0}$ be the joint production-inventory process. When at time t, $P_t = 1$, production is on, and when $P_t = 0$, production is off. The inventory level at time t is given by I_t. We visualize the state space as two lines, an on-line at which $P_t = 1$ and an off-line at which $P_t = 0$, see Figure 1. We assume that all unfilled demand is backlogged. Thus, the inventory level I_t can take on any real value on both lines, where a negative value indicates the level of backlog.

The cost structure consists of two parts. The switching cost is such that each time the production switches on (i.e. the state of the system switches from $P_{t-} = 0$ to $P_t = 1$), a fixed cost $K > 0$ is incurred; any switch-off cost is absorbed in K. Inventory (backlogging) costs accrue at rate $h(x)$ when the inventory level is $x \geq 0$ ($x < 0$). The assumptions on $h(\cdot)$ are the following:

(i) $h(0) = 0$;

(ii) $h(x) = O(|x^n|)$ for some $n > 0$ as $x \to -\infty$;

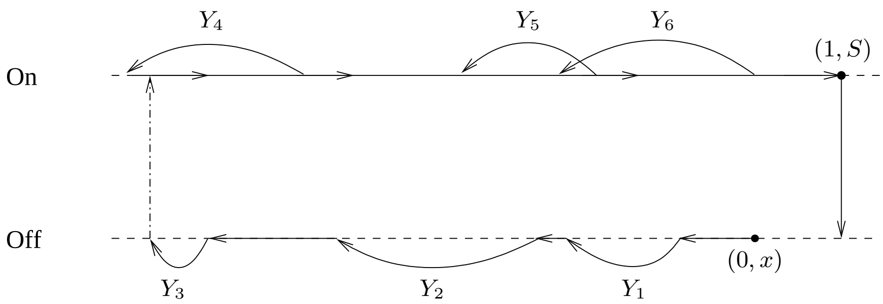


Figure 1: Representation of the state-space of the joint production-inventory process $(P_t, I_t)_{t \geq 0}$ starting in state $(0, x)$.

(iii) $h(x) \to \infty$ if $x \to \infty$;

(iv) The demand Y and $h(\cdot)$ to be such that $\mathrm{E}h(x - Y) < \infty$ for all x.

Note that the function $h(\cdot)$ is allowed to be quasi-convex.

We are concerned with the problem how to control the production-inventory process. To characterize the evolution of $(P_t, I_t)_{t \geq 0}$ under a policy π, it suffices to specify points in time $0 = \pi_0 < \pi_1 < \ldots < \pi_i < \ldots$ at which the policy π decides to switch P_t. Suppose $P_0 = 0$, then the π-controlled production process

$$P_t^\pi = \begin{cases} 0, & \text{if } t \in [\pi_{2i}, \pi_{2i+1}), \\ 1, & \text{if } t \in [\pi_{2i+1}, \pi_{2(i+1)}), \end{cases}$$

for $i = 0, 1, \ldots$. If $\mathbf{1}\{\cdot\}$ is the indicator function, the π-controlled inventory level at time t can be then written as

$$I_t^\pi = I_0 - X_t + r \int_0^t \mathbf{1}\{P_s^\pi = 1\} ds.$$

When no ambiguity results, we will suppress in the notation the dependency on π.

In the sequel we restrict our attention to the space of stationary policies $\mathcal{U}$, i.e., policies whose actions at time t deterministically depend on the state of the production-inventory process at time t. A stationary production policy π can be conveniently characterized by subsets of the off-line and on-line at which production switches state. For instance, a stationary (s, S)-policy is characterized by the subsets $(-\infty, S)$ and $[S, \infty)$ on the on-line such that for $I_t \in (-\infty, S)$ production remains on, while for $I_t \in [S, \infty)$ production switches off. On the off-line, the (s, S)-policy is characterized by the subsets $(-\infty, s]$ and (s, ∞) such that for $I_t \in (s, \infty)$ production remains off, while for $I_t \in (-\infty, s]$ production switches on.

For a policy $\pi \in \mathcal{U}$, we define the expected cost up to time T by

$$L^\pi(T) = \mathrm{E} \int_0^T h(I_s^\pi) ds + K \sum_{i=0}^{\infty} \mathbf{1}\{\pi_{2i+1} \leq T\}, \tag{2}$$

where $P_0 = 0, I_0 < S$. The *long-run average cost* is defined by

$$L^\pi = \limsup_{T \to \infty} \frac{1}{T} L^\pi(T).$$

We are concerned with two goals. The first is to identify general conditions such that a minimizing policy π^* exists, i.e., that there is a policy π^* such that the infimum

$$L^{\pi^*} = \inf_{\pi \in \mathcal{U}} L^\pi \tag{3}$$

is attained. The second goal is to identify a set of conditions when such a minimizing policy π^* has an (s, S)-structure. Since any (s, S)-policy with finite s, S, $s < S$, results in a finite long-run average cost, a policy π with infinite cost cannot be optimal. Without loss of generality we can therefore restrict $\mathcal{U}$ to the set of stationary policies with finite long-run average cost.

3 Existence of optimal policies

In this section we prove that a stationary policy exists such that the optimal control problem (3) has a solution under the conditions of Section 2. We proceed in a number of steps. In Section 3.1 we show that the optimal policy must be a 'produce-up-to' policy, i.e., a policy such that, if production on, it is optimal to keep on producing until some inventory level S is hit. Clearly this fixes the structure on the on-line, so that it remains to characterize the structure of the optimal policy on the off-line. As shown in Section 3.2, using the concept of g-revised costs, this problem turns out to be equivalent to solving an optimal stopping problem. In Section 3.3 we derive conditions to ensure that the optimal stopping problem on the off-line has a solution. Since we attack the existence proof by using tools of free-boundary theory, we rewrite in Section 3.4 the functional equation for the value function as a Lagrange functional. This form proves theoretically and practically highly useful. Then, in Section 3.5 we solve the associated free-boundary problem, and show by verification, that its solution is the value function, i.e., the solution for the optimal stopping problem on the off-line. The overall existence proof of an optimal policy follows then immediately.

3.1 Produce-up-to policies are optimal

We start the proof by showing that we can restrict the search for an optimal policy to the set $\mathcal{H} \subset \mathcal{U}$ of 'produce-up-to' policies, i.e., policies that, when production is on, switch production off whenever the inventory is above some level, and remain on when the inventory is below this level. To this end, let $\mathcal{H}_S$ contain all policies π that split the on-line into two sets $(-\infty, S)$ and $[S, \infty)$ so that production remains on until the inventory level hits the set $[S, \infty)$ and switches off immediately when the inventory hits $[S, \infty)$. Let $\mathcal{H} = \cup_S \mathcal{H}_S$

The following theorem of Van Foreest and Wijngaard (2013, Theorem 4) shows that no general stationary policy can improve the best policy in $\mathcal{H}$. We include a proof for the case that the demand distribution has infinite support for convenience.

Theorem 3.1. *The optimal produce-up-to policy, i.e., the optimal policy in $\mathcal{H}$, is also optimal in the class of stationary policies $\mathcal{U}$.*

Proof. Consider an arbitrary stationary policy $\pi \in \mathcal{U}$. As π, by assumption, is a finite cost policy it must specify that that production switches off at some point, S say. If the π-controlled inventory process ever becomes lower than S (which it must by the finiteness of the cost under π and the infinite support of the demand), the skip-freeness-to-the-right implies that the inventory process cannot exceed a afterward. Hence, the set (S, ∞) must be transient.

It is clear that the drift condition (1) implies that S is always hit when production is on and the inventory is smaller than S, provided there is not another points $\tilde{S} < S$ at which π also decides to switch off production.

When π specifies more than one such point, i.e. $S_1, S_2, \ldots$ simply take $S = \min\{S_i, i = 1, 2, \ldots\}$. Of course, S must be a finite number, for otherwise the inventory would drift to $-\infty$, resulting in infinite cost.

Then, under π, production always switches off at S, it never switches off in $(-\infty, S)$, and (S, ∞) is a transient set. □

Theorem 3.1 has three important consequences. The first is that an optimal policy is a 'produce-up-to' policy, that is, an optimal policy can be found in the class $\mathcal{H}$. Second, for a fixed produce-up-to level S, independent of the starting state, the production-inventory process will eventually hit the state such that $P_t = 0$ and $I_t = S$, and from then on perform *regenerative cycles* that start and stop at this state. Therefore, it suffices to limit the search for a long-run average optimal policy in $\mathcal{H}_S$ to a search for a policy that minimizes the expected cost of just *one* such regenerative cycle. Third, and most importantly, as the structure of any optimal policy is trivial on the on-line, i.e., 'produce-up-to some level S', the problem that remains is to characterize the structure of the optimal policy on the off-line.

3.2 Deciding when to switch on is an optimal stopping problem

To study the structure of optimal policies on the off-line, it is convenient to split the off-line into two disjoint subsets. The sets C^π and its complement $D^\pi = \mathbb{R} \setminus C^\pi$ are such that

$$P_t^\pi = \begin{cases} 0 & = \text{ when } I_t^\pi \in C^\pi, \\ 1 & = \text{ when } I_t^\pi \in D^\pi, \end{cases} \tag{4}$$

i.e., on C^π production is off and on D^π it switches on right away. In view of the optimal stopping problem formulation below, we refer to C^π as the *continuation set* and D^π as the *stopping set*. Obviously, a policy π uniquely characterizes its stopping set D^π, but vice versa, a set D can be used as a stopping set and hence characterize uniquely a policy. Thus, rather than letting the set D^π depend on the policy π, we will in the sequel identify a policy $\pi_D \in \mathcal{H}$ by means of its stopping set D on the off-line. As a notational consequence, we will often write

$$D \subset \mathcal{S} := (-\infty, S],$$

to denote a policy $\pi_D \in \mathcal{H}_S$.

Now consider one such regenerative cycle that starts and stops at $(P_0, I_0) = (0, S)$, and let us associate to the stopping set $D \subset \mathcal{S}$ the following two stopping times

$$\sigma_D = \inf\{t \geq 0; P_t = 0, I_t \in D\},$$
$$\tau_D = \inf\{t \geq \sigma_D; P_t = 1, I_t = S\}.$$

Thus, σ_D is the time to switch on, and τ_D is the time to switch off, given that the production-inventory process starts in $(0, S)$. As we restrict the search for optimal policies to the space of policies with finite long-run expected cost, it follows for any reasonable policy D that

$$0 < \mathrm{E}_S \sigma_D \leq \mathrm{E}_S \tau_D < \infty,$$

where E_S is the expectation of functionals of the production-inventory process starting at $(0, S)$.

Using the definition of τ_D, it follows from (2) that when starting in $(0, S)$, the expected cycle cost is $\mathrm{E}_S[L(\tau_D)]$ and the expected cycle time is $\mathrm{E}_S[\tau_D]$. From the theory of regenerative processes, the *long-run average cost* g^D of the policy D then satisfies

$$g^D = \frac{\mathrm{E}_S[L^D(\tau_D)]}{\mathrm{E}_S[\tau_D]}. \tag{5}$$

Consequently, the minimal long-run average cost in $\mathcal{H}_S$ is

$$g_S^* = \inf_{D \subset \mathcal{S}} g^D. \tag{6}$$

Thus, the problem of finding the optimal D is an *optimal stopping problem*. Then, combining this with Theorem 3.1, we find that the minimal long-run average cost g^* for the policies in $\mathcal{H} = \cup_S \mathcal{H}_S$ must equal

$$g^* = \inf_S g_S^* = \inf_{\pi \in \mathcal{U}} L^\pi = L^{\pi^*} \tag{7}$$

where L^{π^*} is defined in (3). In words, g^* is the minimal cost that can be achieved by any stationary policy.

In summary, to solve the optimization problem (3) we first solve the optimal stopping problem (6) and then solve in (7) for the minimizing policy in $\mathcal{H}$.

3.3 Conditions for the existence of an optimal stopping policy

In this section, we use the concept of g-revised cost as developed by Wijngaard and Stidham Jr. (1986) to reformulate the optimal stopping problem (5) into an equivalent optimal stopping problem that turns out to be much easier to analyze.

Define the *g-revised* inventory cost rate as $h(\cdot) - g$, $g > 0$, so that g has the interpretation as a reward rate per unit time to compensate for the inventory and switching cost incurred during one cycle. Define further the g-revised expected cost $V^D(S; g)$ for a cycle that starts and stops at $(0, S)$ as

$$V^D(S; g) := K + \mathrm{E}_S \int_0^{\tau_D} (h(I_t) - g) dt. \tag{8}$$

The revision rate g^D such that $V^D(S; g^D) = 0$, assuming that such g^D exists, is of particular importance. Using (2),

$$0 = V^D(S; g^D) = \mathrm{E}_S\left(K + \int_0^{\tau_D} h(I_t)dt - \int_0^{\tau_D} g^D dt\right) = \mathrm{E}_S[L(\tau_D) - g^D\tau_D]. \qquad (9)$$

Therefore

$$\mathrm{E}_S[L(\tau_D)] = g^D\,\mathrm{E}_S[\tau_D]$$

so that, by (5), g^D is equal to the long-run average cost of the D-controlled production-inventory cycle.

With these preliminary observations we state the main theorem of this section.

Theorem 3.2. *Suppose that the following holds.*

(i) For all S and g there exists a stopping set $D \subset \mathcal{S}$ that solves the optimal stopping problem

$$V(S; g) := \inf_{D\subset\mathcal{S}} V^D(S; g) = K + \inf_{D\subset\mathcal{S}} \mathrm{E}_S \int_0^{\tau_D} (h(I_t) - g)dt. \qquad (10)$$

In other words, we assume that the minimal g-revised cost $V(S; g)$ of cycles that start and stop at level S is well-defined.

(ii) For all $g \geq 0$ there exist an S_g that realizes the overall minimal g-revised cost to complete one production cycle, that is, S_g solves

$$V(g) := V(S_g; g) = \min_S V(S; g). \qquad (11)$$

In other words, for fixed g,

$$S_g = \operatorname*{argmin}_x\{V(x; g)\}. \qquad (12)$$

(iii) The function $g \mapsto V(g)$ is continuous.

Then there exists a minimal overall long-run average cost g^*, i.e., a g^* that solves (7)*, an optimal switch-off level S_{g^*}, and an optimal stopping set $D^* \subset \mathcal{S}_{S_{g^*}}$ such that*

$$0 = V(g^*) = V(S_{g^*}; g^*) = V^{D^*}(S_{g^*}; g^*). \qquad (13)$$

Proof. Assumption (i) ensures that a g-minimizing policy exists for all g and S. Assumption (ii) implies that for all g there is an optimal point S_g to switch off. Finally, from Assumption (iii) it follows right away that (13) has a solution if there exist $g_- < g^* < g_+$ such that $V(g_-) \geq 0 \geq V(g_+)$. Assuming this is the case, if g^* is such that $V(g^*) = 0$, then using (9), (10), (11), and (13), it follows that

$$0 = V^{D^*}(S_{g^*}; g^*) = \min_S \inf_{D\subset\mathcal{S}} \mathrm{E}_S[L^D(\tau_D) - g^*\tau_D].$$

Since this infimum over D and S is 0, it must be the case that $\mathrm{E}_S[L(\tau_D) - g^*\tau_D] \geq 0$ for any other S and any other policy D. Using (5), we can replace in this inequality $\mathrm{E}_S[L(\tau_D)]$ by $\mathrm{E}_S[g^D\tau_D]$ from which follows that $\mathrm{E}_S[(g^D - g^*)\tau_D] \geq 0$. Since $0 \leq \mathrm{E}_S\tau_D < \infty$, it must be that

$$g^D \geq g^*.$$

Thus, g^* is the minimal long-run average cost. Since the smallest cost is achieved at S_{g^*}, it must be optimal to switch off at S_{g^*}. Finally, g^* being the minimal cost, its associated policy $D^* \subset (-\infty, S_{g^*}]$ must be the optimal policy.

Thus it remains to establish bounds $g_- \leq g^* \leq g_+$ such that $V(g_-) \geq 0 \geq V(g_+)$. Take $g_- = g = 0$ in (10) and note that $h(\cdot) \geq 0$. Then Eq. (10) implies that $V(S; 0) \geq K$ for all S. Thus, it is optimal to switch on immediately everywhere, hence $V(S; 0) \equiv K$ in (10). As a consequence, $V(g_-) = V(0) = K \geq 0$. To find an upper bound on g^*, consider an arbitrary policy $\tilde{D} \subset \tilde{\mathcal{S}}$. Let the associated cost be given by $\tilde{g}$, so that by (9), $V^{\tilde{D}}(\tilde{S}; \tilde{g}) = 0$. By Assumption (i), for $\tilde{g}$ there exist an inventory level and policy such that $V(\tilde{g}) = \min_S V(S; \tilde{g}) = \min_S \inf_D V^D(S; \tilde{g})$. But this implies that

$$V(\tilde{g}) \leq V(\tilde{S}; \tilde{g}) \leq V^{\tilde{D}}(\tilde{S}; \tilde{g}).$$

Since by construction $V^{\tilde{D}}(\tilde{S}; \tilde{g}) = 0$, it follows that $V(\tilde{g}) \leq 0$. Thus, taking $g^+ = \tilde{g}$, we see that there exists $g_+ = \tilde{g} > 0 = g_-$ such that $V(g_+) \leq 0 \leq V(g_-)$. □

Thus, to prove the existence of an optimal revision cost g^* and an optimal stopping policy D^*, it suffices to show that the three assumptions of Theorem 3.2 can be satisfied. Proving Assumption (i), i.e., that a solution exists for the optimal stopping problem (10) on the off-line, requires the most work, and is established by Theorem 3.6 below. Assumptions (ii) and (iii) follow then from Lemma 3.7 below.

Remark 3.3. Besides providing conditions to prove the existence of an optimal stopping policy for (7), Theorem 3.2 also provides a (numerically efficient) method to actually find the optimal policy and the minimal long-run average cost g^*. In Section 3.4 we show that $V(S,g)$ can be cast into a nice form that is easy to solve (numerically) for any g. To identify the 'right' g, i.e. the revision cost that solves (13), we can use bisection as described in the proof of Theorem 3.2.

3.4 A useful functional equation

To solve the optimal stopping problem in (10), we need to find the stopping set $D \subset \mathcal{S}$ at which the Lagrange functional $\mathrm{E}_S \int_0^{\tau_D} (h(I_t) - g)dt$ attains its infimum. In other words, we have to evaluate the g-revised cost of the inventory process when production is off (i.e. for $t \in [0, \sigma_D)$) and when production is on (i.e. for $t \in [\sigma_D, \tau_D)$). In this section, we derive an expression $\gamma_g(\cdot)$ for the (negative) derivative of expected g-revised cost when production is on. We then use this expression to reformulate (10) into an equivalent optimal stopping problem that only requires the evaluation of a Lagrange functional of the inventory process when production is off. This new optimal stopping problem turns out to be easier to analyze than (10).

First we introduce some relevant notation. Let

$$I_t^0 := I_t, \quad \text{for } 0 \le t < \sigma_D,$$
$$I_t^1 := I_t, \quad \text{for } \sigma_D \le t < \tau_D,$$

so that I_t^0 is the inventory level as long as production is off, and I_t^1 is the inventory level when production is on. With these definitions, we define $V_1(x; g)$ as the g-revised expected cost to move from level x on the on-line until level S is reached, that is,

$$V_1(x; g) = \mathrm{E} \int_0^{\tau} (h(I_t^1) - g)dt,$$

where $I_0^1 = x \le S$ and

$$\tau = \inf\{t \ge 0; P_t = 1, I_t = S\}.$$

The g-revised expected cost to move from level x on the off-line to level S on the on-line becomes then

$$V_0^D(x; g) := \mathrm{E}_x \int_0^{\sigma_D} (h(I_t^0) - g)dt + K + \mathrm{E}_x V_1(I_{\sigma_D}^1; g). \tag{14}$$

Clearly, the g-revised cost $V^D(S; g)$, as defined in (8), of cycles that start and stop at S satisfies $V^D(S; g) = V_0^D(S; g)$. More generally, letting $V^D(x; g)$ be the cost to stop and start at level x, the above implies that for all $x \le S$,

$$V^D(x; g) = V_0^D(x; g) - V_1(x; g). \tag{15}$$

We now arrive at the main result of this section.

Theorem 3.4. *Provided* $\mathrm{E}_S \sigma_D < \infty$*, the optimization problem* $V(S; g) = \inf_{D \subset \mathcal{S}} V^D(S; g)$ *in Eq.*(10) *is equivalent to the continuous-time optimal stopping problem*

$$V(S; g) = K + \inf_{\sigma \ge 0} \mathrm{E}_S \int_0^{\sigma} r\, \gamma_g(I_t^0)dt, \tag{16}$$

where

$$\gamma_g(x) = -\frac{dV_1(x; g)}{dx},$$

and the infimum is taken over all stopping times σ *of* $(I_t^0)_{t \ge 0}$ *satisfying* $\mathrm{E}_S\, \sigma < \infty$*.*

Proof. See Appendix A. □

3.5 Analysis of the associated free-boundary problem

We split the analysis in a number of smaller steps.

1. In Peskir and Shiryaev (2006) it is shown that the optimal stopping problem (16) is equivalent to the problem of finding the largest subharmonic function $\hat{V}$ that is dominated by K on the state space. If such $\hat{V}$ is found, it follows that $V \equiv \hat{V}$ and

$$\sigma_D = \inf\{t \geq 0;\ I_t^0 \in D\} \tag{17}$$

is the optimal stopping time, where $D = \{x \mid \hat{V}(x;g) = K\}$ is the optimal stopping set, and $C = \{x \mid \hat{V}(x;g) < K\}$ is the continuation set. As a consequence, see Peskir and Shiryaev (2006, Section 8) for further background, $\hat{V}$ and C should solve the *free-boundary problem*:

$$\mathbb{L}_{I^0}\hat{V}(x;g) \geq -r\gamma_g \quad (\hat{V} \text{ maximal}), \tag{18a}$$

$$\hat{V} < K \quad \text{on } C, \tag{18b}$$

$$\hat{V} = K \quad \text{on } D. \tag{18c}$$

Here $\mathbb{L}_{I^0}$ is the infinitesimal operator of I_t^0 which, as can be shown from the standard theory of continuous-time Markov processes, operates on functions f in the set C^1 of continuously differentiable functions as

$$\begin{aligned}\mathbb{L}_{I^0} f(x) &= \lim_{t\downarrow 0} \frac{\mathrm{E}_{(I_0^0=x)} f(I_t^0) - f(x)}{t} \\ &= -qf'(x) + \lambda[\,\mathrm{E}f(x-Y) - f(x)],\end{aligned}$$

where $f'(x) = df(x)/dx$. Thus, to solve the optimization problem, both $\hat{V}$ and C (hence D) should be determined. For this purpose it is essential to study γ_g in more detail.

2. Let us collect here some useful properties of γ_g.

Lemma 3.5. *The function $x \mapsto \gamma_g(x)$ has the following properties:*

(i) γ_g *is the unique solution of the integral equation*

$$\gamma_g(x) = \frac{h(x) - g}{r - q} + \frac{\lambda}{r - q} \int_0^\infty \gamma_g(x - y) G(y) dy. \tag{19}$$

(ii) If h is convex then γ_g is also convex for all g.

(iii) $\lim_{x\to\pm\infty} \gamma_g(x) = \infty$.

(iv) For an arbitrary $\hat{g} \in \mathbb{R}$, γ_g can be written as

$$\gamma_g(x) = \gamma_{\hat{g}}(x) - \beta(g - \hat{g}), \tag{20}$$

where $\beta = (r - \lambda\mu - q)^{-1}$.

Proof. See Appendix B. □

To illustrate these properties, we plot in Figure 2 the functions h, γ_g for two revision cost ($g = 3$ and $g = 4$), and $\gamma_3 - \gamma_4$ for a system with production rate $r = 1$, inventory cost

$$h(x) = -4\min\{0, x\} + \max\{0, x\},$$

constant demand rate $q = 0.3$, arrival rate of stochastic demand $\lambda = 0.5$, and uniformly distributed demand on $[0, 2]$ such that $\mu = 1$.

The figure shows that γ_3 and γ_4 preserve the convexity of the inventory cost function h. The plot of $\gamma_3 - \gamma_4$ confirms that the difference between γ_3 and γ_4 is constant and equal to $\beta(g - \hat{g}) = 5(4 - 3) = 5$. From (20), it is clear that when g is sufficiently large, $\gamma_g < 0$ at some interval. This will turn out to be necessary to ensure that an interesting stopping problem remains, as we will show next.

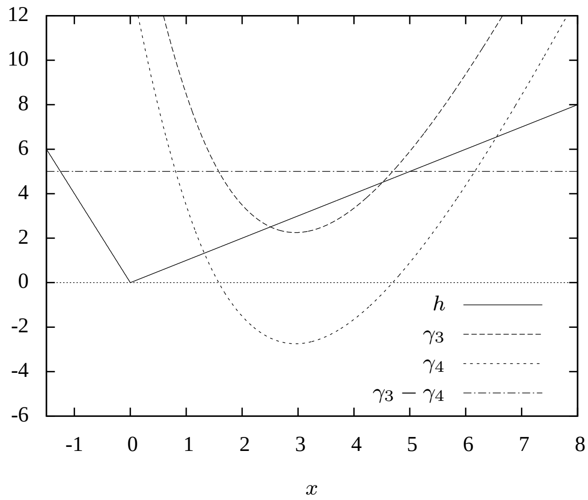


Figure 2: Graph of h, γ_3, γ_4, and $\gamma_3 - \gamma_4$ as a function of the inventory level x.

3. The above properties of γ_g allow us to make some simple observations about the form of the optimal stopping set D. Suppose first that $\gamma_g(x) \geq 0$ for all x. Then it is evident from (16) that it is optimal to stop right away, i.e., when $\sigma = 0$. Thus, assume henceforth that g is such that $\gamma_g < 0$ on at least one interval, and let s_g denote its left-most root. Next, note that $(I_t^0)_{t\geq 0}$ is a (Markov) process with continuous drift q and jumps to the left (see Figure 1). This implies that $\gamma_g(I_t^0) > 0$ when $I_0^0 = x \leq s_g$. Hence, it must be that $(-\infty, s_g] \subset D$. In general the continuation set C can be a collection of disjoint open sets, so that $C = \cup_{i=1}^n (s_g^i, t_g^i)$. Observe, once again, that the boundary points $\{s_g^1, t_g^1, \ldots, t_g^n\}$ are to be determined.

The above considerations leads us to reformulate (18) into the following free-boundary problem

$$\mathbb{L}_{I^0}\hat{V} = -r\,\gamma_g \quad \text{on } C, \tag{21a}$$

$$\hat{V} < K \quad \text{on } C, \tag{21b}$$

$$\hat{V} = K \quad \text{on } D, \tag{21c}$$

$$\hat{V}(s_g^i+; g) = \hat{V}(t_g^i-; g) = K, \quad \text{for } i = 1, \ldots, n \quad (\textit{continuous fit}) \tag{21d}$$

$$q\hat{V}'(s_g^i+; g) = 0, \quad \text{for } i = 1, \ldots, n \quad (\textit{smooth fit}) \tag{21e}$$

Let us comment on these equations. Eq. (21a) follows from the general requirement that $\hat{V}$ has to satisfy the Lagrange problem (16). Conditions (21b) and (21c) are evident from (18). Conditions (21d) and (21e) are not part of the general theory. As made plausible by Peskir and Shiryaev (2006, Section 9), the solution $\hat{V}$ should be continuous, hence satisfy the continuous fit condition (21d). When $q > 0$, the process $(I_t^0)_{t\geq 0}$ enters the interior of D immediately if it starts at s_g^i. This results in the smooth fit condition at s_g^i. When $q = 0$, this condition is void.

4. The free-boundary problem (21) has a unique solution which can be constructed from left to right. Observe that this construction not only yields $\hat{V}$, but also the continuation set C and the stopping set D. (We assume that $q > 0$, and remark that the case $q = 0$ follows the same line of reasoning.)

On the set $(-\infty, s_g] \subset D$, $\hat{V}$ is well-defined and equals K. Let $s_g^1 = s_g$. Conditions (21d) and (21e) together with the continuity of γ_g and the fact that the integration $\mathrm{E}\hat{V}(x - Y; g)$ only requires $\hat{V}$ to the left of x, ensure that the integro-differential equation (21a)

$$q\hat{V}'(x; g) - \lambda[\mathrm{E}\hat{V}(x - Y; g) - \hat{V}(x; g)] = r\gamma_g(x) \tag{22}$$

can be uniquely integrated from s_g^1 onwards. The integration can be continued up to the point that $\hat{V}(x; g)$, $x > s_g^1$, hits K again. This point is uniquely determined. Denote it by t_g^1. From this point onwards, take $\hat{V} = K$ as long as the $\hat{V}$ that solves (22) is greater or equal to K, i.e. $r\gamma(x)/\lambda + \mathrm{E}\hat{V}(x -$

$Y; g) \geq K$. Take, if it exists, s_g^2 as the smallest root of $r\gamma(x)/\lambda + \mathrm{E}\hat{V}(x - Y; g) = K$ such that $s_g^2 > t_g^1$ and $r\gamma(x)/\lambda + \mathrm{E}\hat{V}(x - Y; g) < K$ for $x \in (s_g^2, s_g^2 + \epsilon)$ for sufficiently $\epsilon > 0$. Then, at s_g^2, we invoke Conditions (21d) and (21e) again as boundary conditions and integrate (22) from the left to the right until $\hat{V}$ hits K again. This point determines t_g^2 uniquely. Continuing like this, we obtain the solution on the remaining parts of C and D. Finally, since $\lim_{x\to\infty} \gamma_g(x) = \infty$, while $\hat{V}(x; g) \leq K$ for all x, the possibility that $t_g^n = \infty$ is ruled out. Thus, there is an $n \geq 1$ such $[t_g^n, \infty) \subset D$. Note that from this derivation, it follows that $x \mapsto \hat{V}(x; g)$ is C^1 on $\mathbb{R} \setminus \{t_g^1, \dots, t_g^n\}$ but only continuous on $\{t_g^1, \dots, t_g^n\}$.

Combining the above construction with (33) shows that

$$\hat{V}(x; g) = \begin{cases} K + \mathrm{E}_x \int_0^{\sigma_D} r\gamma_g(I_t^0)dt & \text{if } x \in C, \\ K & \text{if } x \in D, \end{cases} \tag{23}$$

where σ_D, defined by (17), is the unique solution of the free-boundary problem (21).

5. It remains to prove that the value function V of the optimal stopping problem (16) may be identified with $\hat{V}$, thereby proving Assumption (i) of Theorem 3.2.

Theorem 3.6. *The function $\hat{V}$ defined by* (23) *solves also* (16)*, hence $\hat{V}$ is equal to the value function of the stopping problem, and the stopping time $\sigma_D = \inf\{t > 0; I_t^0 \in D\}$ of* (17) *is optimal in* (16).

Proof. See Appendix C. □

6. Now that we know that the optimal stopping problem (16) has a solution, we can also prove that Assumptions (ii) and (iii) of Theorem 3.2 can be satisfied, which thereby implies the existence of an optimal production-inventory policy.

Lemma 3.7. *Let $x \mapsto V(x; g)$ solve the optimal stopping problem* (16)*. Then*

- *(i) For all $g \geq 0$ the function $x \mapsto V(x; g)$ is continuous for all $g \geq 0$, hence attains its minimum at S_g, i.e., $V(g) = V(S_g, g)$.*
- *(ii) The function $g \mapsto V(g)$ continuous.*

Proof. See Appendix D. □

4 Structure of optimal policies

In the previous section we established the existence of an optimal policy. In this section we focus on the structural properties of the optimal policy. We first derive bounds on the switching sets as defined in (4), i.e., on the stopping set D_g, i.e., the set at which it is optimal to switch on for a given g, and the continuation set $C_g = \mathbb{R} \setminus D_g$, i.e., the set at which it is optimal to remain off. Once we have these bounds we derive conditions that guarantee that D_g, and consequently C_g, is a half-bounded interval so that necessarily the optimal policy has an (s, S)-structure.

4.1 Bounds on the switching sets

It is easy to bound C_g from below. As explained above (21) it is optimal to switch on at the set $(-\infty, s_g]$, where s_g is the left-most root of γ_g, hence $(-\infty, s_g] \subset D_g$. However, it may be the case that D_g is larger than $(-\infty, s_g]$, c.f. Section 3.5.3. Therefore it is only possible to guarantee that s_g is a lower bound for C_g.

A natural upper bound on D_g is given by S_g as defined by (12), since, by Theorem 3.1, after a possibly transient phase the inventory level will never increase beyond S_g. Hence it can be assumed without loss of generality that $[S_g, \infty) \cap D_g = \varnothing$.

With the lemma below we provide an upper bound of S_g in terms of the right-most root, t_g say, of γ_g. This not only has important structural consequences, c.f. Section 4.2, but it also limits the search region for S_g: for the numerical procedure it suffices to integrate (22) from s_g to t_g. It is then guaranteed that the function $x \to V(x; g)$ has attained its minimum somewhere in $[s_g, t_g]$. Furthermore, this implies that S_g is finite whenever all roots of γ_g are finite.

Lemma 4.1. *For any fixed $g > 0$, the left-most minimizer S_g of $V(S; g)$, as defined in* (12)*, is bounded from above by the right most root t_g of γ_g, that is $S_g \leq t_g$.*

Proof. The assumption that $S_g > t_g$ leads to a contradiction, as follows. Since S_g is a minimizer it must be that $V(S_g; g) < K$, hence $S_g \in C_g$. From (22) we see that for all $x \in C_g$, and in particular for S_g,

$$V(S_g; g) = \frac{r}{\lambda}\gamma_g(S_g) - \frac{q}{\lambda}V'(S_g; g) + \mathrm{E}V(S_g - Y; g).$$

Suppose first that $V'(S_g; g) \leq 0$. Since, by assumption $S_g > t_g$, it must be that $\gamma_g(S_g) > 0$. The above equality then implies that $V(S_g; g) > \mathrm{E}V(S_g - Y; g)$. On the other hand, for any x it must be that $\mathrm{E}V(x - Y; g) \geq \inf_{y \leq x}\{V(y; g)\}$. These two inequalities together then imply that $V(S_g; g) > \mathrm{E}V(S_g - Y; g) \geq \inf_{y \leq S_g}\{V(y; g)\} = V(S_g; g)$, where the last equality follows since $V(\cdot; g)$ attains its minimum at S_g. Contradiction. Now assume, on the other hand, that $V'(S_g; g) > 0$. This implies that V is increasing at S_g; hence $V(\cdot; g)$ cannot attain its minimum at S_g. □

4.2 Optimality of (s, S)-policies

The above bounds allow us to prove the main *structural* result of our paper.

Theorem 4.2. *Let g^* be the minimal long-run average cost rate so that it is the solution of* (13)*. Let s_{g^*} be the left root of $\gamma_{g^*}(\cdot)$ and $S_{g^*} = \operatorname{argmin}_x\{V(x; g^*)\}$ as in Eq.* (12)*. If, and only if, the solution $V(x; g^*)$ of* (22)*, is such that*

$$\begin{aligned} &V(x; g^*) = K, \text{ for } x \leq s_{g^*}, \\ &V(x; g^*) < K, \text{ for } x \in (s_{g^*}, S_{g^*}], \end{aligned} \tag{24}$$

then the optimal policy associated with g^* has an (s, S)-structure with $s = s_{g^*}$ and $S = S_{g^*}$.

Proof. As $V(x; g^*) = K$ for $x \leq s_{g^*}$, the stopping set $D_{g^*} = (-\infty, s_{g^*}]$. Thus it is optimal to switch on once the inventory hits D_{g^*}. Next, as $V(x; g^*) < K$ for $x \in (s_{g^*}, S_{g^*}]$, the continuation set $C_{g^*} = (s_{g^*}, S_{g^*}]$. Thus, it is optimal to not switch on when the inventory is in $(s_{g^*}, S_{g^*}]$. Finally, as $V(S_{g^*}; g^*) = 0$, it is optimal to produce up to S_{g^*}. Hence, the inventory will never exceed S_{g^*} once it has reached a level below S_{g^*}. □

Corollary 4.3. *When γ_{g^*} has two roots, the optimal policy has an (s, S)-structure.*

Proof. We prove that the condition on γ_{g^*} implies that Conditions (24) are satisfied.

The first condition, i.e., $V(x; g^*) = K$ on $x \leq s_{g^*}$, holds by construction. To prove the second part of Condition (24) we show that $(s_{g^*}, t_{g^*}) \subset C_{g^*}$. We have from (22) that $qV'(x; g^*) - \lambda[\mathrm{E}V(x - Y; g^*) - V(x; g^*)] = r\,\gamma_{g^*}(x)$. As $\gamma_{g^*}(x) < 0$ for all $x \in (s_{g^*}, t_{g^*})$, it must be that $qV'(x; g^*) - \lambda[\mathrm{E}V(x - Y; g^*) - V(x; g^*)] < 0$ on (s_{g^*}, t_{g^*}). Therefore, on (s_{g^*}, t_{g^*}),

$$\begin{aligned} V(x; g^*) &< \mathrm{E}V(x - Y; g^*) - \frac{q}{\lambda}V'(x; g^*) \\ &\leq K - \frac{q}{\lambda}V'(x; g^*), \end{aligned}$$

where the second inequality follows from the domination of $V \leq K$. Suppose now that x is such that $V'(x; g^*) \geq 0$, then the above inequality implies that $V(x; g^*) < K$. On the other hand, if $V'(x; g^*) < 0$, V is decreasing at x, so that necessarily $V(x; g^*) < K$. Thus, in either case $V(x; g^*) < K$ when $x \in (s_{g^*}, t_{g^*})$. Finally, using Lemma 4.1 it follows that $S_{g^*} \leq t_{g^*}$. Since, clearly, $S_{g^*} > s_{g^*}$, we conclude that $(s_{g^*}, S_{g^*}) \in C_{g^*}$. □

Corollary 4.4. *If $h(\cdot)$ is such that $\gamma_{g^*}(\cdot)$ has two roots then the optimal policy has an (s, S)-structure. This holds in particular when $h(\cdot)$ is convex.*

Proof. The first claim follows immediately from Corollary 4.3. The second part follows from Lemma 3.5.ii: The convexity of $h(\cdot)$ implies the convexity of $\gamma_g(\cdot)$. As any suitable g, and in particular g^*, is such that γ_g has a left root, convexity (and the implied continuity) implies that γ_g must have one right root. □

Thus, the convexity condition on h is just an easy sufficiency condition to prove the existence of an optimal (s, S)-policy, but is by no means necessary. In general, whenever the conditions of Theorem 3.2 are shown to hold (numerically perhaps), the numerical integration of (22) will yield an optimal policy, whether this policy has an (s, S)-structure or not. Moreover, Theorem 4.2 provides if-and-only-if conditions for the optimality of (s, S)-policy; convexity of h is a much stronger condition.

5 Analytical examples

The analytical tools developed for the existence proof in Section 3 allow us to handle a number of related inventory problems. We illustrate this by applying our approach to two related models.

5.1 The lost-sales model

Let us show how to find optimal policies for models with lost sales. In particular, we consider a loss model with a *partial acceptance* issuing policy, c.f. Germs and Van Foreest (2013a). Under the partial acceptance policy, if the demand y is larger than the on-hand stock x, the order is partially satisfied and $y-x$ is lost at the expense of a cost $l(y-x)$, where l is an increasing non-negative function with $l(0)=0$. We refer to Germs and Van Foreest (2013b) for a more detailed discussion on how our approach can be applied to other issuing policies such as *complete rejection* and *complete acceptance*.

We assume that the average cost rate of rejecting all orders altogether is higher than the average holding and setup cost, for otherwise it is simply optimal to reject all orders and to never switch the machine on. Because of this assumption, we can restrict the search for an optimal policy to the set of policies in $\mathcal{H}$ such that $0 \in D_g$, that is, when the production is off it is at least optimal to switch on when the inventory is empty.

In the following, we discuss the parts of Section 3 that need to be modified to incorporate lost sales. First of all, we include the rejection cost in the g-revised cost functions defined in Section 3.4. To do so, observe that the rejection cost rate $R(x)$ for a given inventory level x under the partial acceptance issuing policy is

$$R(x) = \lambda \operatorname{E}[l(Y-x)\mathbf{1}\{Y > x\}],$$

as the expected rejection cost per customer arrival is $\operatorname{E}[l(Y-x)\mathbf{1}\{Y > x\}]$ and customer arrivals occur at a rate λ. The g-revised expected cost on the on-line $V_1(x;g)$ becomes

$$V_1(x;g) = \operatorname{E} \int_0^\tau (h(I_t^1) + R(I_t^1) - g) dt.$$

The g-revised expected cost $V_0^D(x;g)$ and $V^D(x;g)$ change in a similar way.

Second, the inventory process on the off-line I^0 and on-line I^1 is different in the lost-sales model due to rejection. Therefore, the infinitesimal operators $\mathbb{L}_{I^0}$ and $\mathbb{L}_{I^1}$ change as follows

$$\begin{aligned} \mathbb{L}_{I^1} f(x) &= (r-q)f'(x) + \lambda \operatorname{E}[(f(x-Y) - f(x))\mathbf{1}\{Y < x\}] + \lambda \mathrm{P}(Y > x)(f(0) - f(x)) \\ &= rf'(x) + \mathbb{L}_{I^0} f(x). \end{aligned} \tag{25}$$

Similar derivations as in Appendix A yield that for the lost-sales model

$$\mathbb{L}_{I^1} V_1(x;g) = -h(x) - R(x) + g. \tag{26}$$

From (25) and (26) it then follows that $\gamma_g(x) = -V_1'(x;g)$ is the solution of the equation

$$\begin{aligned} (r-q)\gamma_g(x) = h(x) + R(x) - g + \lambda \operatorname{E}[(V_1(x-Y;g) - V_1(x;g))\mathbf{1}\{Y < x\}] \\ + \lambda \mathrm{P}(Y > x)(V_1(0;g) - V_1(x;g)). \end{aligned}$$

This can be rewritten to the integral equation

$$(r-q)\gamma_g(x) = h(x) + R(x) - g + \lambda \int_0^x \gamma_g(x-y) G(y) dy.$$

The rest of the equations in Section 3 also hold for the lost-sales model. Now suppose that l and h are such that γ_g is a convex function, then after some simple modifications of the relevant results of Section 4 we can show that the optimal policy for the lost-sales model is an (s,S)-policy.

5.2 Inventory models with infinite replenishment rates

In the classical (s,S)-inventory problem with compound Poisson demand, and more generally, the mixture of deterministic and compound Poisson demand as considered by Hordijk and Van der Duyn Schouten (1986) and Presman and Sethi (2006), it is assumed that the replenishment rate is infinite. As such we expect that these models can obtained as the limiting case of a sequence of production-inventory models with ever larger production rates. Here we show how to apply the analysis of Sections 3 and 4 to prove this conjecture. We note in passing that this implies the existence of an optimal policy for these stochastic inventory systems with infinite production rate. Moreover, with the numerical approach, to be developed in Section 6, it is possible to efficiently compute an optimal policy.

The crux of the proof is to show that when the production rate $r \to \infty$, the free-boundary problem (21) reduces to the set of quasi-variational inequalities (QVIs) as considered by Presman and Sethi (2006, Eqs. 38–40). In our notation, these QVIs can be stated as

$$0 = h(x) - g^* + \mathbb{L}_{I_0} V(x; g^*), \quad \text{for } x > s, \tag{27a}$$

$$0 \le h(x) - g^* + \mathbb{L}_{I_0} V(x; g^*), \quad \text{for } x \in \mathbb{R}, \tag{27b}$$

$$V(x; g^*) = K + V(S_{g^*}; g^*), \quad \text{for } x \le s. \tag{27c}$$

$$V(x; g^*) \le K + V(x + u; g^*), \quad \text{for } x \in \mathbb{R} \text{ and } u > 0. \tag{27d}$$

Note that, as we consider average-cost optimality, the killing rate used by Presman and Sethi (2006) is here 0. We also do not take the purchasing cost c into account, as these cost have no influence on the average-cost optimal policy.

Except for the Lemma below, it is a basic exercise to check that (21) reduces to (27) if $r \to \infty$. Specifically, since $V(S_{g^*}; g^*) = 0$, we see that (21c) is equivalent to (27c). Since $V(x + u; g^*) \ge 0$ for $x \in \mathbb{R}$ and $u > 0$, Eqs. (21b) and (21c) imply (27d). Finally, in Lemma 5.1 below we prove that $r\gamma_g(x) \to h(x) - g$ as $r \to \infty$, so that (21a) and (18a) reduce to (27a) and (27b), thereby completing the proof.

Lemma 5.1. *Provided h is an element of a suitable Banach space, $r\gamma_g$ converges to $h - g$ as $r \to \infty$.*

Proof. See Appendix E. □

6 Numerical examples

In Section 6.1 we present an efficient numerical method that can be used to compute the optimal policy parameters for the models considered in this paper. In Section 6.2 and 6.3 we apply this method to analyze numerically the influence of the system parameters on the optimal policy.

6.1 Numerical procedure

In general it is not be possible to obtain closed form expressions for the solutions of the Equations (11) and (13). In this section, we therefore describe an efficient method for solving these equations numerically.

From the free-boundary formulation (21), it follows that we have to solve (numerically) the integro-differential equation

$$qV'(x) - \lambda\, \mathrm{E} V(x - Y) + \lambda V(x) = r\gamma(x), \tag{28}$$

where we dropped, for ease of notation, the dependence on g. For this purpose, we approximate the value function V over the continuous domain $\mathbb{R}$ by a discrete set of function values at a discrete set of points in the domain. That is, we discretize the state space by reducing the real numbers to the grid $\{\dots, k\delta, (k+1)\delta, \dots\}$ for $k \in \mathbb{Z}$, where $\delta > 0$ denotes the grid size. Writing $V_k = V(k\delta)$, $\gamma_k = \gamma(k\delta)$, $F_k = F(k\delta)$, $V_k'(g) = \delta^{-1}(V_k - V_{k-1})$ and $\mathrm{E} V(k\delta - Y) = \sum_i V_{k-i} f_i$, with $f_i = F_i - F_{i-1}$, Eq. (28) reduces to

$$(q\delta^{-1} - \lambda f_0 + \lambda) V_k - (q\delta^{-1} + \lambda f_1) V_{k-1} - \lambda \sum_{i=2}^{\infty} V_{k-i} f_i = r\gamma_k.$$

Clearly, since V_k is expressed in terms of $V_i, i < k$, this leads to a simple recursion for V_k. The initial values for V_k that start the recursion can be obtained right away from the fact that $V(x) = K$ for $x < s_g$.

To determine γ we reason similarly. From (19) it follows that

$$(r - q)\gamma_k = h_k - g + \lambda\delta \sum_{i=0}^{\infty} \gamma_{k-i} G_i.$$

Taking the term $\gamma_k G_0$ out of the summation and to the left, we obtain the recursion

$$(r - q - \lambda\delta G_0)\gamma_k = h_k - g + \lambda\delta \sum_{i=1}^{\infty} \gamma_{k-i} G_i.$$

To obtain a suitable set of initial conditions for γ_k we use some elements of the proof of Lemma 5.1. There we prove that $\gamma = (r - q - A)^{-1}h$, where $(Ah)(x) = \lambda \int_0^\infty h(x-y)G(y)dy$. If we approximate $(Ah)(x)$ by $\lambda\mu h(x)$ for $x \le z$ for some given $z \ll 0$, then we can set $\gamma_k = h_k/(r-q-\lambda\mu)$ for $k\delta \le z$ as initial values for γ. Van Foreest and Wijngaard (2013) prove that the influence of the initial values of γ reduces exponentially fast as $z \to -\infty$.

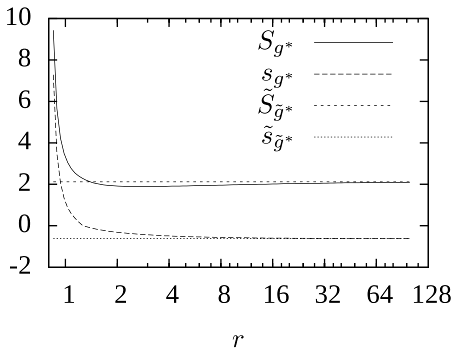
Finally, it remains to find a proper lower and upper bound for g that can act as starting values for the bisection in g. As described in the proof of Theorem 3.2 this is trivial: as a lower bound simply take 0; as an upper bound take the average cost rate of some arbitrary policy, for instance the average cost related to the (s, S)-policy with $s = 0$ and $S = 1$. (Of course, the optimal policy should perform at least as well as this simple policy.) In the actual implementation of our computer program we use a conceptually even simpler method. Just start with $g = 1$, and keep on doubling g as long as $V(g) > 0$. Once g is such that $V(g) < 0$, it must be that $g^* \in [g/2, g)$.

6.2 Analysis of the optimal values of the (s, S)-policy

In this section, we analyze how s_{g^*}, S_{g^*} and g^* are influenced by the production rate r, the deterministic demand rate q, switching cost K, and arrival rate λ of the compound Poisson process. We assume that demand arrives as single units. The inventory cost is fixed to be

$$h(x) = -4\min\{0, x\} + \max\{0, x\}, \tag{29}$$

In Figure 3 we vary the production rate r and keep the other parameters fixed such that the arrival rate $\lambda = 1/2$, the constant demand rate $q = 3/10$, and the setup cost $K = 5$. As is apparent from the figure, S_{g^*} and g^* first decrease and then increase as a function of r, while s_{g^*} decreases monotonically. We see that as $r \to \infty$ the values converge to those of the model of Presman and Sethi (2006). As a further consequence, $r\gamma(x) \to h(x) - g$, c.f. Lemma 5.1, so that in the numerical computations we replace $r\gamma(x)$ by $h(x) - g$.



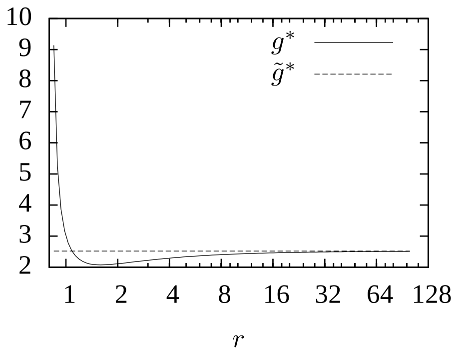


Figure 3: The optimal values of switching levels and the revision cost as functions of the production rate r. Here $\tilde{s}_{\tilde{g}^*}$, $\tilde{S}_{\tilde{g}^*}$, and $\tilde{g}^*$ refer to the optimal values for the model of Presman and Sethi (2006).

In the left graph of Figure 4, we consider the influence of q and λ for the above system and fix $r = 1$. We vary λ from 0 to 0.8, but keep the load constant by taking $q = 0.8 - \lambda\mu$. Observe that due to the Poisson character of the demand process, the inventory may decrease to quite low levels when $\lambda > 0$. Hence, we expect that S_{g^*} and s_{g^*} should increase as a function of λ. Moreover, the fluctuations in the inventory process should increase as λ increases, hence, g^* should also increase. Finally, since the setup cost K remains equal, $S_{g^*} - s_{g^*}$ should remain roughly the same. Indeed, the graphs in Figure 4 support this reasoning. Of particular interest is the graph of g^*. It increases, apparently linearly, quite quickly. Thus it is, from a managerial point of view, interesting to negotiate (long-term) contracts with customers to shift part of the stochastic demand to deterministic demand, if possible.

Finally, the right graph of Figure 4 shows the influence of the setup cost K for the system with $r = 1$. Clearly g^* is increasing in K as we see from the figure. Also the difference between S_{g^*} and s_{g^*} is increasing in K which is intuitive since the production cycle length should be an increasing function of K.

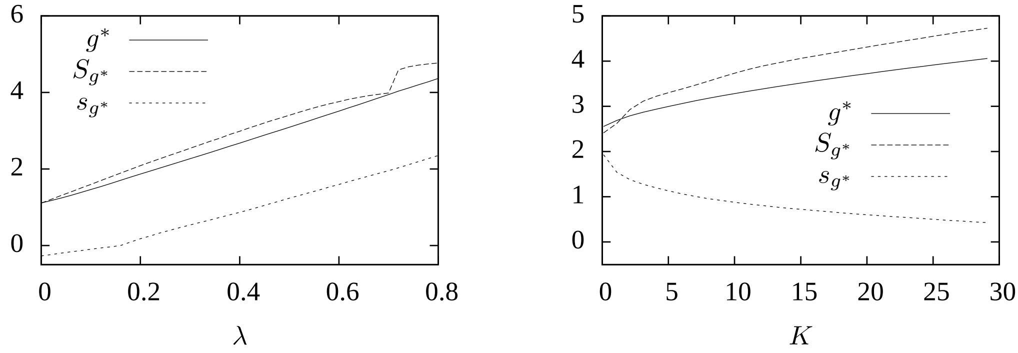


Figure 4: The left figure shows the optimal values of s, S and g as functions of the arrival rate λ. The right figure shows the optimal values of s, S and g as functions of the setup cost K.

6.3 Analysis of the structure of the optimal policy

In Figure 5 we present four cases to show that the value function and the structure of the optimal policy depends non-trivially on the system parameters. The system of interest is nearly the same as in Section 6.2, except that now λ and q are as indicated in the title of each of the figures.

In the upper-left figure the graph of the value function has a small cusp at $x \approx 2$. This indicates that just below this cusp it is optimal to switch on, but when $x \approx 2$ ('in this cusp') it is not optimal to switch on. This seems to show that the optimal policy is not (s, S). However, $S_{g^*} \approx 1$, so that once the inventory level is below S_{g^*}, it will never reach the cusp. We note that such cusps do not always occur. When $\lambda = 0.5$ the cusp in V disappears, as is apparent from the upper right panel in Figure 5.

For the lower two panels we take a quasi-convex inventory cost of the form

$$h(x) = -4\lfloor x \rfloor \cdot \mathbf{1}\{x \leq 0\} + \lfloor x \rfloor \mathbf{1}\{x > 0\}, \tag{30}$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Now the situation is drastically different. There is a cusp in the graph of V around $x = 3$, but now $S_{g^*} \approx 5$. The presence of this cusp below S_{g^*} implies that the optimal policy is not (s, S). Interestingly, by slightly increasing q to 0.1, the cusp disappears, and the optimal policy is again (s, S), even though h is quasi-convex. Thus, the convexity assumption for h is not necessary to ensure that (s, S)-policies are optimal.

7 Conclusions and suggestions for further research

In this paper, we study a continuous-review production-inventory model with demand consisting of a compound Poisson process and a constant demand rate. Demand is met from on-hand inventory or otherwise backlogged. We establish conditions on the inventory costs under which an average-cost optimal (s, S)-policy exists. Our approach starts with the problem to minimize the g-revised cost to complete one production cycle. We reformulate this problem as an optimal stopping problem, and then reduce the optimal stopping problem to a free-boundary problem. We identify a solution of the latter problem and prove by verification that this solution is also a solution for the optimal stopping problem. By bisection on the revised cost parameter g, we can prove the existence of an optimal policy for the production-inventory system. An interesting byproduct of our approach is that it provides a fast and efficient numerical scheme to compute the optimal policy, not only for our model, but for a wide class of inventory systems. In particular, we show that the scheme can also be applied to continuous-review systems with unrestricted capacity and production-inventory systems with lost sales.

Our approach appears to be a powerful way to address other related inventory problems. For instance, our model can be easily extended to state-dependent production rate $r(x)$ and demand rates $\lambda(x)$ and $q(x)$; it is straightforward to adapt the related integro-differential equations. This extension is of interest to model inventory systems that are subject to deterioration: e.g., if $q(x) = \alpha x$, $\alpha > 0$, for $x > 0$ and $q(x) = 0$ for $x \leq 0$ the rate of deterioration is proportional to the inventory level. Another interesting problem would be to consider the (maximal) cost of using an (s, S)-policy for a system for which an (s, S)-policy is non-optimal, the motivation being that when the best (s, S)-policy leads to minor extra

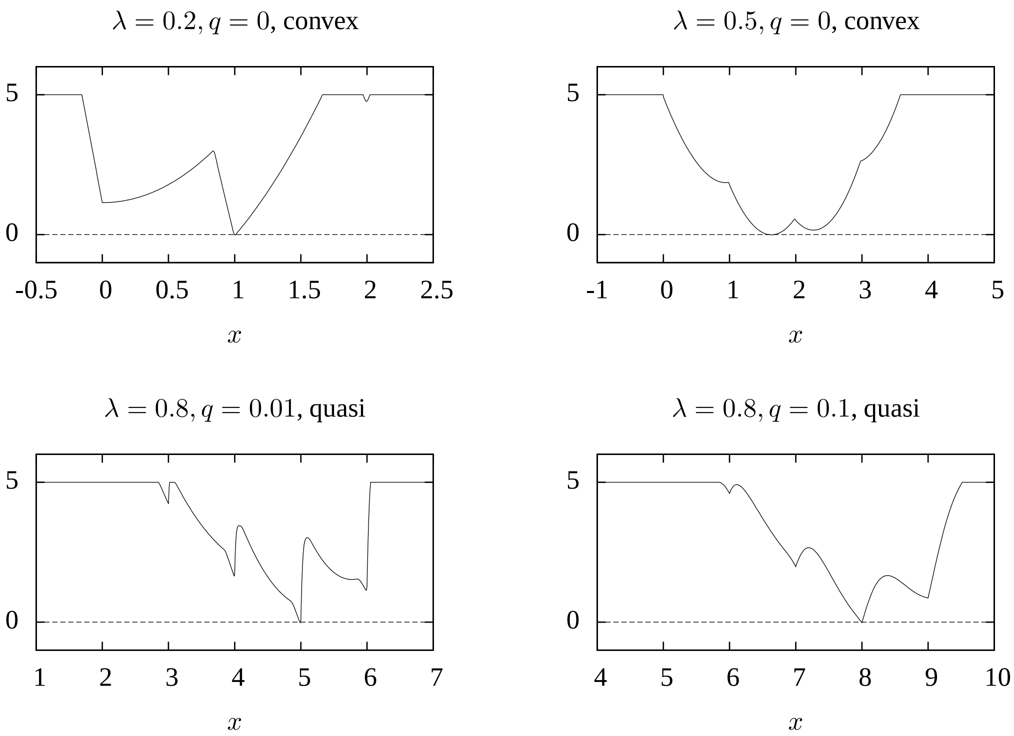


Figure 5: Graphs of the value functions of the cases as indicated by the figure titles; 'convex' denotes that h is given by (29), while 'quasi' corresponds to h given by (30).

costs compared to the optimal policy it may be, from a practical point of view, preferable to use the (s, S)-policy, not withstanding its non-optimality. It would also be interesting to apply our approach to production-inventory models with a demand process that is the sum of Brownian motion and a compound Poisson process. For inventory system with infinite ordering/production capacity, Bensoussan et al. (2005) prove that an (s, S)-policy is optimal in the special case when the jumps of the compound Poisson process are exponentially distributed. The general case when the demand is the sum of a diffusion process and a compound Poisson process is still open. Yet another interesting extension would be to develop a method to find the optimal policy when service constraints have to be met, similar to the problems addressed in De Kok et al. (1984). Also, as already mentioned in the Introduction, our approach can be used to study optimal clearing policies for production-inventory systems, c.f., Germs and Van Foreest (2013b).

Acknowledgments

We are indebted to J. Wijngaard for sharing with us his intuition about the manifold uses of g-revised cost, and to J.W. Nieuwenhuis for his insistence on also proving the 'obvious'. The present work greatly benefited from their ideas and insights.

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A Proof of Theorem 3.4

Let $\mathbb{L}_{I^0}$ and $\mathbb{L}_{I^1}$ denote the infinitesimal operators of I_t^0 and I_t^1, respectively. Then, from the standard theory of continuous-time Markov processes, it can be shown that for functions f in the set C^1 of continuously differentiable functions

$$\begin{aligned} \mathbb{L}_{I^1} f(x) &= \lim_{t\downarrow 0} \frac{\mathrm{E}_{(I_0^1=x)} f(I_t^1) - f(x)}{t} \\ &= (r-q)f'(x) + \lambda[\,\mathrm{E} f(x-Y) - f(x)] \\ &= r f'(x) + \mathbb{L}_{I^0} f(x), \end{aligned} \tag{31}$$

where $f'(x) = df(x)/dx$, and where the last equality follows from applying the definition of the infinitesimal operator of $(I_t^0)_{t\geq 0}$ to f.

Applying Dynkin's formula, $\mathrm{E}_x f(I_\sigma^0) = f(x) + \mathrm{E}_x \int_0^\sigma \mathbb{L}_{I^0} f(I_s^0) ds$, to $V_0^D(I_t; g)$ at the stopping time σ_D we obtain

$$V_0^D(x;g) = \mathrm{E}_x V_0^D(I_{\sigma_D}^0; g) - \mathrm{E}_x \int_0^{\sigma_D} \mathbb{L}_{I^0} V_0^D(I_s^0; g) ds.$$

Computing (14) at the point $I_{\sigma_D}^0$, i.e., at the level at which production switches on, gives that $V_0^D(I_{\sigma_D}^0; g) = K + V_1(I_{\sigma_D}^0; g)$. Therefore,

$$V_0^D(x;g) = K + \mathrm{E}_x V_1(I_{\sigma_D}^0; g) - \mathrm{E}_x \int_0^{\sigma_D} \mathbb{L}_{I^0} V_0^D(I_s^0; g) ds.$$

Combining this with (14), we see that

$$\mathrm{E}_x \int_0^{\sigma_D} \mathbb{L}_{I^0} V_0^D(I_s^0; g) ds = -\,\mathrm{E}_x \int_0^{\sigma_D} (h(I_t^0) - g) dt,$$

from which

$$\mathbb{L}_{I^0} V_0^D(x;g) = -h(x) + g.$$

Similar derivations for $(I_t^1)_{t\geq 0}$ yields that

$$\mathbb{L}_{I^1} V_1(x;g) = -h(x) + g.$$

Finally, we apply the infinitesimal operator $\mathbb{L}_{I^0}$ to V^D to write V^D, as defined in (15), in a particularly useful form. From (15), (31) and (32), it follows that

$$\begin{aligned} \mathbb{L}_{I^0} V^D(x;g) &= \mathbb{L}_{I^0} V_0^D(x;g) - \mathbb{L}_{I^0} V_1(x;g) \\ &= \mathbb{L}_{I^0} V_0^D(x;g) - \mathbb{L}_{I^1} V_1(x;g) + r\frac{d}{dx} V_1(x;g) \\ &= r V_1'(x;g). \end{aligned}$$

Defining $\gamma_g(x) = -V_1'(x;g)$ and applying Dynkin's formula to $V^D(x;g)$ at time $I^0(\sigma_D)$, we can formulate $V^D(x;g)$ as the Lagrange functional

$$\begin{aligned} V^D(x;g) &= \mathrm{E}_x V^D(I_{\sigma_D}^0; g) - \mathrm{E}_x \int_0^{\sigma_D} \mathbb{L}_{I^0} V^D(I_s^0; g) ds \\ &= K + \mathrm{E}_x \int_0^{\sigma_D} r\, \gamma_g(I_s^0) ds, \end{aligned} \tag{33}$$

where we use that $\mathrm{E}_x V^D(I_{\sigma_D}^0; g) = K$.

B Proof of Lemma 3.5

(i) From (31) and (32) it follows that $\gamma_g(x) = -V_1'(x;g)$ is the solution of the equation

$$\gamma_g(x) = \frac{h(x)-g}{r-q} + \frac{\lambda}{r-q}\,\mathrm{E}[V_1(x-Y;g) - V_1(x;g)].$$

This can be rewritten to the integral equation

$$\begin{aligned}\gamma_g(x) &= \frac{h(x)-g}{r-q} + \frac{\lambda}{r-q}\,\mathrm{E}\left[\int_{x-Y}^{x} \gamma_g(z)\,dz\right] \\ &= \frac{h(x)-g}{r-q} + \frac{\lambda}{r-q}\int_0^\infty \int_{x-y}^{x} \gamma_g(z)\,dz\,dF(y) \\ &= \frac{h(x)-g}{r-q} + \frac{\lambda}{r-q}\int_0^\infty \gamma_g(x-y)G(y)dy.\end{aligned}$$

The reversal of the integrals is allowed by the assumptions in Section 2. It is proven in Van Foreest and Wijngaard (2013, Lemma 4.1) that this integral equation has a unique solution.

(ii) The convexity of γ_g follows from Van Foreest and Wijngaard (2013, Theorem 4.5).

(iii) Since $h(x) \to \infty$ as $x \to \pm\infty$ by assumption, $\lim_{x\to\pm\infty} \gamma_g(x) = \infty$.

(iv) By inserting the relation $\gamma_g(x) = \gamma_{\hat{g}}(x) + \alpha$ into both sides of (19) and solving for α, it follows that for an arbitrary $\hat{g} \in \mathbb{R}$, γ_g can be written as (20).

C Proof of Theorem 3.6

We prove the claim by means of verification for the case $q > 0$. The proof for $q = 0$ follows the same line of reasoning.

The properties of $\hat{V}$ derived in Step 4 of Section 3.5 show that Itô's formula can be applied to $\hat{V}(I_t^0; g)$ in its standard form. This gives

$$\hat{V}(I_t^0; g) = \hat{V}(x; g) + \int_0^t \mathbb{L}_{I^0}\hat{V}(I_s^0; g)\,ds, \tag{34}$$

where, without loss of generality, we take $\mathbb{L}_{I^0}\hat{V}(x;g) = 0$ on $\{t_g^1, \dots, t_g^n\}$. Recalling that $\hat{V}(x;g) = K$ and $\gamma_g(x) \geq 0$ for $x \in D$, and using that $\hat{V}$ satisfies (21a) for $x \in C$, we see that

$$\mathbb{L}_{I^0}\hat{V}(x;g) \geq -r\,\gamma_g(x)$$

everywhere on $\mathbb{R}$ but $\{t_g^1, \dots, t_g^n\}$. Combining this with the above and the inequality $K \geq \hat{V}$ results in

$$K \geq \hat{V}(I_t^0; g) \geq \hat{V}(x;g) - \int_0^t r\gamma_g(I_s^0)ds.$$

As, clearly, in (16) it suffices to take the infimum only over stopping times σ satisfying $\mathrm{E}_x\sigma < \infty$, we may insert a stopping time σ for t, take E_x on both sides, and conclude that

$$K + \mathrm{E}_x \int_0^\sigma r\gamma_g(I_s^0)ds \geq \hat{V}(x;g) \tag{35}$$

for all $x \in \mathbb{R}$. Since this holds for all σ, we conclude from (16) that $V \geq \hat{V}$.

On the other hand, using (21a) and (21d) and the definition of σ_D, we see from (34) that

$$K = \hat{V}(I^0(\sigma_D); g) = \hat{V}(x;g) - r\int_0^{\sigma_D} \gamma_g(I_s^0)ds. \tag{36}$$

Since $\mathrm{E}_x\sigma_D < \infty$, we see by taking E_x on both sides of (36) that equality in (35) is attained at $\sigma = \sigma_D$, and thus $V = \hat{V}$. Combining this with the conclusions on the existence and uniqueness of the optimal stopping boundary $\{s_g^1, t_g^1, \dots, s_g^n, t_g^n\}$ derived in Step 4 of Section 3.5 completes the proof.

D Proof of Lemma 3.7

(i) The continuity follows immediately from the construction of $\hat{V}$, which is identified with V in Theorem 3.6. On D we have that $V = K$. On C, V solves the differential equation (21a). The continuous-fit conditions (21d) imply that the parts of V on D and C are continuously connected. Since s_g^1 and t_g^n are finite and V is continuous on the compact set $[s_g^1, t_g^n]$, V attains its minimum and maximum.

(ii) Assume that $\hat{g} > g$. Observe from (20) that $\gamma_g(x) \geq \gamma_{\hat{g}}(x)$. From (23) it then follows for arbitrary but fixed S that

$$V(S; g) \geq V(S; \hat{g}). \tag{37}$$

We next show that $V(S; g) - V(S; \hat{g}) \to 0$ when $\hat{g} \downarrow g$. Let τ_D and $\tau_{\hat{D}}$ denote the optimal stopping times associated with the value functions $V(S; g)$ and $V(S; \hat{g})$, respectively. Using (20) again, it follows that

$$\begin{aligned} 0 &\leq V(S; g) - V(S; \hat{g}) \\ &= \mathrm{E}_S \left[\int_0^{\tau_D} r\, \gamma_g(I_s^0) ds \right] - \mathrm{E}_S \left[\int_0^{\tau_{\hat{D}}} r\, \gamma_g(I_s^0) ds - \beta(\hat{g} - g)\tau_{\hat{D}} \right] \\ &\leq \beta(\hat{g} - g)\, \mathrm{E}_S \tau_{\hat{D}}, \end{aligned} \tag{38}$$

where the last inequality follows because

$$\mathrm{E}_S \int_0^{\tau_D} r\, \gamma_g(I_s^0) ds = \inf_{\tau \geq 0} \mathrm{E}_S \int_0^{\tau} r\, \gamma_g(I_s^0) ds \leq \mathrm{E}_S \int_0^{\tau_{\hat{D}}} r\, \gamma_g(I_s^0) ds.$$

Observe that $\beta(\hat{g} - g)\, \mathrm{E}_S \tau_{\hat{D}} \to 0$ as $\hat{g} \downarrow g$ because $\mathrm{E}_S \tau_{\hat{D}} < \infty$.

We are now in the position to prove that $V(g) \to V(\hat{g})$ as $\hat{g} \downarrow g$. First we need to establish that $V(g) \geq V(\hat{g})$ if $g < \hat{g}$. To see this, let $S_g = \mathrm{argmin}_S V(S; g)$ and $S_{\hat{g}} = \mathrm{argmin}_S V(S; \hat{g})$. Then, by (37),

$$V(g) = V(S_g; g) = \inf_S V(S; g) \geq \inf_S V(S; \hat{g}) = V(S_{\hat{g}}; \hat{g}) = V(\hat{g}).$$

As a consequence of this,

$$\begin{aligned} 0 &\leq V(g) - V(\hat{g}) \\ &= V(S_g; g) - V(S_{\hat{g}}; \hat{g}) \\ &= [V(S_g; g) - V(S_{\hat{g}}; g)] + [V(S_{\hat{g}}; g) - V(S_{\hat{g}}; \hat{g})] \\ &\leq V(S_{\hat{g}}; g) - V(S_{\hat{g}}; \hat{g}). \end{aligned}$$

The last inequality follows from the fact that $V(S_{\hat{g}}; g) \geq \inf_S V(S; g) = V(S_g; g)$ implying that the first term between brackets is non-positive. Finally, by (38), the first term becomes small as $\hat{g} \downarrow g$. Hence, $V(\hat{g}) \to V(g)$ when $\hat{g} \downarrow g$.

By the same token, $V(\hat{g}) \to V(g)$ when $\hat{g} \uparrow g$, thereby completing the proof that $V(g)$ is continuous.

E Proof of Lemma 5.1

From (19) follows that

$$(r - q)\gamma_r(x) = h(x) + \lambda \int_0^\infty \gamma_r(x - y) G(y) dy,$$

where we set $g = 0$ without loss of generality, and replace the dependency of γ on g by the dependency on r.

For ease, define the linear operator $(Af)(x) = \lambda \int_0^\infty f(x - y) G(y) dy$, and write the above as $(r - q)\gamma_r = h + A\gamma_r$.

To turn A into a well-defined operator, we need some technical assumptions. Suppose that $h \in \mathcal{B}$, where $\mathcal{B}$ is the Banach space of real-valued continuous functions f such that the weighted supremum norm $||f|| = \sup\{f(x)e^{\beta x}; x \leq R\} = M$ is finite for some suitable R and $\beta > 0$. If the demand is light-tailed distributed, there exist an $N < \infty$ such$(Af)(x) \leq MN$ for $f \in \mathcal{B}$. This implies in particular that A is a bounded operator. Therefore, whenever $\alpha > ||A||$, the resolvent $R(\alpha, A) = (\alpha - A)^{-1}$ is well-defined and bounded, hence

$$\alpha R(\alpha, A) = \left(1 - \frac{A}{\alpha}\right)^{-1} = \sum_{i=0}^\infty \left(\alpha^{-1} A\right)^i$$

also exists and is bounded. This in turn implies that $\alpha R(\alpha, A) f \to f$ for all $f \in \mathcal{B}$ as $\alpha \to \infty$.

To finish the proof, observe that from $(r-q)\gamma_r = h + A\gamma_r$ it follows that $\gamma_r = ((r-q) - A)^{-1} h = R(r-q, A)h$. From the above, we conclude that $(r-q)\gamma_r = (r-q)R(r-q, A)h \to h$ as $r \to \infty$.

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