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7 $\mathcal{H}_\infty$-control in the behavioral framework

7.1 Introduction

In this chapter we will study the $\mathcal{H}_\infty$ control problem in a behavioral framework. Starting from a given to-be-controlled behavior, some components of the system variable are assumed to be free, again in the sense that they are not constrained by the model. These components are the disturbances acting on the system. Other components of the system variable are variables that we want to keep small. These are called the to-be-controlled variables. A third group of components are the interconnection variables (some of them are also free of course) as already explained before in this thesis. The control problem that we consider in this chapter, is to design a controller behavior, i.e. constraints on the interconnection variable, such that, roughly speaking, the to-be-controlled variables are "small" whatever the disturbance that occurs, provided of course the disturbance is bounded in magnitude. We want to stress that this set-up generalizes the "classical" approach to $\mathcal{H}_\infty$ control. In that context, for the interconnection variable one would take the composite vector (u,y), with u the control inputs and y the measured outputs.

The $\mathcal{H}_\infty$ control problem in the behavioral framework was studied before in Trentelman & Willems [41] and in Meinsma [24]. In a more general perspective, it can be considered as a special case of the problem of dissipativity synthesis (i.e. the problem of rendering a given plant dissipative by interconnection). This problem was studied extensively in Willems & Trentelman [50], Trentelman & Willems [42] and Belur & Trentelman [3].

In the present chapter we extend the behavioral $\mathcal{H}_\infty$ control problem that was studied and resolved in Trentelman & Willems [41]. This extended problem will be used in solving the robust stabilization problem in chapter 8. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba [[36], [37], [38]].

This chapter is structured as follows. In section 7.2 we formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework. To solve this problem, we use the theory of dissipative systems with respect to supply rates given by quadratic differential forms (QDF's). The concept of QDF and dissipative systems are elaborated in section 7.3. Finally, in section 7.4 we give a solution to our extended version of the behavioral $\mathcal{H}_\infty$-control problem.

7.2 Problem formulation

In this section, we will formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework.

We start with a system behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with system variable (w,c,v). The system variable has been partitioned into w, c and v. These variables represent the to-be-controlled variable, the interconnection variable, and an unknown disturbance, respectively. The interconnection variable c is the system variable through which we are allowed to interconnect $\mathcal{P}$ with a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Interconnection leads to the *interconnection of* $\mathcal{P}$ *and* $\mathcal{C}$ *through* c:

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w,c,v) \mid (w,c,v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \tag{7.1}$$

We recall (Proposition 3.2.2) that the interconnection in Equation (7.1) is *regular* if and only if

$$\mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C}).$$

Recall (see section 3.3.3) that in that case we call the controller $\mathcal{C}$ regular. In our context, the variable v represents an unknown disturbance. This is formalized by assuming v to be free in $\mathcal{P}$. As v is interpreted as unknown disturbance, it should remain free (see Definition 2.9.1) after interconnecting the plant with a controller. In order to highlight this, we recall the following definition of free-disturbance controller from chapter 5:

Definition 7.2.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *free-disturbance* if v is free in $\mathcal{P} \wedge_c \mathcal{C}$.

Following Trentelman & Willems [41], in the context of $\mathcal{H}_\infty$ synthesis a controller is called stabilizing if, whenever the disturbance v is zero, the to-be-controlled variable w tends to zero as time runs off to infinity:

Definition 7.2.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. A free-disturbance controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *stabilizing* if $[(w,0,c) \in \mathcal{P} \wedge_c \mathcal{C}] \Rightarrow [\lim_{t\to\infty} w(t) = 0]$.

Remark 7.2.3. We note that the concept of stabilizing controller defined above is different from the concept of stabilizing controller given in the context of asymptotic tracking and regulation in chapter 5 (see Definition 5.2.2). In contrast to the requirement that a stabilizing controller in the context of asymptotic tracking and regulation drives the variables w and c to zero as time tends to infinity, a stabilizing controller in the context of $\mathcal{H}_\infty$ synthesis is required to drive only the variable w to zero. Thus, every stabilizing controller in the sense of Definition 5.2.2 is stabilizing in the sense of Definition 7.2.2, but the converse does not hold.

The following result characterizes the property that a controller is free-disturbance, stabilizing and regular in terms of the matrices appearing in the kernel representations of the plant and the controller.

Proposition 7.2.4. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ and $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal rational kernel representation of $\mathcal{P}$ and a minimal polynomial kernel representation of $\mathcal{C}$, respectively. Assume that in $\mathcal{P}$ c is observable from (w,v). Then the following are equivalent:*

1. *$\mathcal{C}$ is a free-disturbance, stabilizing, regular controller for $\mathcal{P}$,*
2. *$\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$.*

Proof: From Proposition 2.4.2, c is observable from (w,v) in $\mathcal{P}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. From Corollary 6.4.4 we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) & R_3(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \end{bmatrix}\right). \tag{7.2}$$

Define $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) := \{(w,c) \mid (w,c,0) \in \mathcal{P} \wedge_c \mathcal{C}\}$. It is easy to see that

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker\left(\begin{bmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right). \tag{7.3}$$

From Equation (7.3), it is easy to see that c is observable from w in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ (use the fact that $\begin{bmatrix} R_2(\lambda) \\ C(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$).

(1) ⇒ (2) Since $\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix}$ and C have full row rank, the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular if and only if $\begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix}$ has full row rank. Thus, by Proposition 6.2.1, v is free in $\mathcal{P} \wedge_c \mathcal{C}$ if and only if $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ has full row rank. We will now show that $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable. Since $\mathcal{C}$ is free-disturbance and stabilizing, $(w,c) \in \mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ implies $w(t) \to 0$ $(t \to \infty)$. This implies that the projection $(\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}))_w$ of $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ onto the variable w is stable. It is easily seen that in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$, $c(t) \to 0$ $(t \to \infty)$ (use the fact that c is observable from w in $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$). Hence $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable. Therefore from Equation (7.3) and Lemma 6.2.3, $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is square, nonsingular and has no zeros in $\bar{\mathbb{C}}^+$. The converse implication *(2) ⇒ (1)* is proven in a similar way. □

Definition 7.2.5. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w = (w_1,w_2)$. Let $\gamma > 0$. $\mathfrak{B}$ is called *γ-contractive* if for all $(w_1,w_2) \in \mathfrak{B} \cap \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})$ we have $\|w_1\|_2 \leq \gamma \|w_2\|_2$. It is called *strictly γ-contractive* if there exists $\epsilon > 0$ such that $\mathfrak{B}$ is $(\gamma - \epsilon)$-contractive.

Remark 7.2.6. Of course, by a density argument, $\mathfrak{B}$ is γ-contractive if and only if the contractivity condition $\|w_1\|_2 \leq \gamma\|w_2\|_2$ holds for all $(w_1,w_2) \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2})$, i.e. for all trajectories in $\mathfrak{B}$ of compact support.

Next, we characterize the property of strict contractiveness of a behavior in terms of the rational matrices appearing in a rational representation of the behavior:

Proposition 7.2.7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1,w_2). Let $\gamma > 0$. Let a minimal rational kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Assume that R_1 is square and nonsingular. Then $\mathfrak{B}$ is strictly γ-contractive if and only if $R_1^{-1}R_2$ is proper, has no poles on the imaginary axis, and $\|R_1^{-1}R_2\|_\infty < \gamma$.*

Proof: (Only if) Let $\begin{bmatrix} R_1 & R_2 \end{bmatrix} = P^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then $Q_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + Q(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$ is a minimal polynomial kernel representation, and Q_1 is square, nonsingular. Clearly,

$$\begin{aligned} G &:= R_1^{-1}R_2 \\ &= Q_1^{-1}Q_2. \end{aligned}$$

Let

$$G = -ND^{-1} \tag{7.4}$$

be a right coprime factorization over $\mathbb{R}[\xi]$. We have

$$Q_1N + Q_2D = 0.$$

Therefore

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} N(\frac{\mathrm{d}}{\mathrm{d}t}) \\ D(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}) \text{ for all } \ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1). \tag{7.5}$$

Thus, by assumption, there exists $\epsilon > 0$ such that

$$\|N(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|_2 \leq (\gamma - \epsilon)\|D(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|_2 \text{ for all } \ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1).$$

Taking Fourier transforms it follows from Parseval's theorem that

$$N^\top(-i\omega)N(i\omega) \leq (\gamma - \epsilon)D^\top(-i\omega)D(i\omega) \text{ for all } \omega \in \mathbb{R}.$$

Using that ND^{-1} is a right coprime factorization, this implies that $D(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$. Thus G has no poles on the imaginary axis and

$$G^\top(-i\omega)G(i\omega) \leq (\gamma - \epsilon)I \text{ for all } \omega.$$

This implies that G is proper and $\|G\|_\infty < \gamma$.
(If) Conversely, in $\mathfrak{B}$ w_1 is output and w_2 is input, and the transfer matrix from w_2 to w_1 is equal to $G = R_1^{-1}R_2$. Since G is proper and has no poles on the imaginary axis, the system $\mathfrak{B}$ induces a bounded operator that maps $w_2 \in \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ to $w_1 \in \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1})$. The norm of this operator is equal to $\|G\|_\infty < \gamma$, and therefore there exists $\epsilon > 0$ such that

$$\|w_1\|_2 \leq (\gamma - \epsilon)\|w_2\|_2 \text{ for all } (w_1,w_2) \in \mathfrak{B} \cap \mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}).$$

□

Using the above notion of γ-contractiveness we define the following.

Definition 7.2.8. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Let $\gamma > 0$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *strictly γ-contracting* if $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ is strictly γ-contractive.

Before we introduce the main problem studied in this chapter, we review the notion of *orthogonal complement* of a behavior (see Willems & Trentelman [49]). Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a controllable behavior. Then we define its orthogonal complement $\mathfrak{B}^\perp$ by

$$\mathfrak{B}^\perp := \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{\infty} w^\top w' dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})\}.$$

$\mathfrak{B}^\perp$ is again controllable. If $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal polynomial kernel representation of $\mathfrak{B}$, then $\tilde{w} = R^\top(-\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an observable polynomial image representation of $\mathfrak{B}^\perp$ (see Willems & Trentelman [49], Section 10).

Now we formulate the main problem studied in this chapter.

Problem: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with system variable (w,c,v). Assume that v is free in $\mathcal{P}$. Let $\gamma > 0$. Find necessary and sufficient conditions for the existence of a free-disturbance, stabilizing, regular and strictly γ-contracting controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$.

This problem was studied before in Trentelman & Willems [41] without the requirement of regular interconnection. The assumptions on the plant behavior that were made in Trentelman & Willems [41] are however too restrictive for our purposes, for example to solve the robust stabilization problem in chapter 8. We will therefore in this chapter extend the results from Trentelman & Willems [41] in order to make these applicable in chapter 8.

7.3 Two-variable polynomial matrices, QDF's and dissipative systems

A major role in our study of the $\mathcal{H}_\infty$ control problem in this chapter and our forthcoming study of the robust stabilization problem in chapter 8 will be played by the notions of dissipativeness, strict dissipativeness and storage function in a behavioral context. These notions have been studied before in Willems [44], Willems and Trentelman [[49], [50]] and Trentelman and Willems [42]. In this section we review these notions. An important role is played by two-variable polynomial matrices and quadratic differential forms. An extensive treatment can be found in Willems and Trentelman [49]. We will give a brief review here.

An $\mathtt{l}_1 \times \mathtt{l}_2$ two-variable polynomial matrix in the indeterminates ζ and η is an expression of the form

$$\Phi(\zeta,\eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k \tag{7.6}$$

where $\Phi_{h,k}$ are real $\mathtt{l}_1\times\mathtt{l}_2$ matrices, and where $N \geq 0$ is an integer. With any such two-variable polynomial matrix we can associate a bilinear functional

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}_2}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \tag{7.7}$$

by defining

$$L_\Phi(\ell_1,\ell_2) := \sum_{h,k=0}^{N} (\frac{d^h\ell_1}{dt^h})^T\Phi_{h,k}\frac{d^k\ell_2}{dt^k}. \tag{7.8}$$

The two-variable polynomial matrix $\Phi(\zeta,\eta)$ is called *symmetric* if $\Phi_{h,k} = \Phi^T_{k,h}$ for all h,k. In that case we also associate with $\Phi(\zeta,\eta)$ the *quadratic differential form* (QDF)

$$Q_\Phi(\ell) := L_\Phi(\ell,\ell). \tag{7.9}$$

The properties of the two-variable polynomial matrix $\Phi(\zeta,\eta)$ are completely determined by the real constant $(N+1)\mathtt{l}\times$ $(N+1)$ $\mathtt{l}$ matrix $\tilde{\Phi}$ whose $(h,k)^{\text{th}}$ block is equal to $\Phi_{h,k}$. This matrix will be called the *coefficient matrix* associated with $\Phi(\zeta,\eta)$. Factorizations of the coefficient matrix immediately give rise to corresponding factorizations of the associated two-variable polynomial matrix and quadratic differential form.

The QDF Q_Φ is called *non-negative* if

$$Q_\Phi(\ell) \geq 0,$$

in the sense that $Q_\Phi(\ell)(t) \geq 0$ for all $t \in \mathbb{R}$. It is easily seen that Q_Φ is non-negative if and only if the coefficient matrix $\tilde{\Phi}$ satisfies $\tilde{\Phi} \geq 0$.

7.3.1 Dissipativity

Consider, in general, a controllable linear differential system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, represented by the observable polynomial image representation

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell \tag{7.10}$$

with $W \in \mathbb{R}^{\mathtt{w}\times \mathtt{l}}[\xi]$. In addition, let Q_Φ be the QDF associated with the symmetric two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ will be called the *supply rate*. The system $\mathfrak{B}$ will be called *dissipative* with respect to the supply rate Q_Φ if for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ we have

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0. \tag{7.11}$$

$\mathfrak{B}$ is called *strictly dissipative* with respect to the supply rate Q_Φ if there exists $\epsilon > 0$ such that for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|w(t)\|^2 dt. \tag{7.12}$$

Given a polynomial image representation as in Equation (7.10) together with a two-variable polynomial matrix $\Phi(\zeta,\eta)$ we can define a new two-variable polynomial matrix $\Phi' \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ by

$$\Phi'(\zeta,\eta) := W^\top(\zeta)\Phi(\zeta,\eta)W(\eta). \tag{7.13}$$

It is easily verified that, if w and ℓ are related by Equation (7.10), then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. Therefore, the system is dissipative if and only if for all $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)dt \geq 0,$$

and strictly dissipative if and only if there exists $\epsilon > 0$ such that, for all $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\|^2 dt.$$

These conditions are equivalent to

$$\Phi'(-i\omega,i\omega) \geq 0 \ \text{ for all } \omega \in \mathbb{R} \tag{7.14}$$

and

$$\Phi'(-i\omega,i\omega) \geq \epsilon^2 W^\top(-i\omega)W(i\omega) \text{ for all } \omega \in \mathbb{R} \tag{7.15}$$

respectively (see Willems & Trentelman [49]). It is well known (see Callier [4], Coppel [5], Ran & Rodman [28], and Kwakernaak & Sebek [22]) that, if Equation (7.14) holds then we can factorize

$$\partial\Phi'(\xi) := \Phi'(-\xi,\xi) = F^\top(-\xi)F(\xi),$$

with $F \in \mathbb{R}^{1\times 1}[\xi]$. If Equation (7.15) holds, then F can be chosen Hurwitz, and also anti-Hurwitz. Introduce now the two-variable polynomial Δ, defined by

$$\Delta(\zeta,\eta) := \Phi'(\zeta,\eta) - F^\top(\zeta)F(\eta). \tag{7.16}$$

Since $\Delta(-\xi,\xi) = 0$, the two-variable polynomial Δ must contain a factor $\zeta + \eta$ (see Willems & Trentelman [49], Theorem 3.1), and therefore we can define the new two-variable polynomial Ψ by

$$\Psi(\zeta,\eta) := (\zeta+\eta)^{-1}\Delta(\zeta,\eta). \tag{7.17}$$

Consider now the QDF's Q_Ψ and Q_Δ associated with Ψ and Δ, respectively. We have

$$Q_\Delta(\ell) = Q_{\Phi'}(\ell) - \|F(\tfrac{\mathrm{d}}{\mathrm{dt}})\ell)\|^2. \tag{7.18}$$

Furthermore, Equation (7.17) is equivalent to:

$$\frac{dQ_\Psi(\ell)}{dt} = Q_\Delta(\ell) \text{ for all } \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1). \tag{7.19}$$

Thus we obtain

$$\frac{dQ_\Psi(\ell)}{dt}(t) \leq Q_{\Phi'}(\ell)(t), \tag{7.20}$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1)$, for all $t \in \mathbb{R}$.

If we interpret $Q_\Psi(\ell)(t)$ as the amount of supply (e.g., energy) stored inside the system at time t, then Equation (7.20) expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. The inequality in Equation (7.20) is called the *dissipation inequality.* Any quadratic differential form $Q_\Psi(\ell)$ that satisfies this inequality is called a *storage function* for $\mathfrak{B}$. It can be shown that $\mathfrak{B}$ is dissipative if and only if there exists a symmetric two-variable polynomial matrix $\Psi(\zeta,\eta)$ such that the corresponding QDF Q_Ψ satisfies Equation (7.20). In general, storage functions are not unique. In fact, we quote Willems & Trentelman [49], Theorem 5.7:

Proposition 7.3.1. *Let $\mathfrak{B}$ be represented by the observable image representation (7.10). Assume $\mathfrak{B}$ is dissipative with respect to Q_Φ. Then there exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that any other storage function Q_Ψ satisfies*

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}.$$

If $\mathfrak{B}$ is strictly dissipative then Ψ_- and Ψ_+ may be constructed as follows. Let H and A be respectively Hurwitz and anti-Hurwitz factorizations of $\partial\Phi'$. Then

$$\Psi_+(\zeta,\eta) = \frac{\Phi'(\zeta,\eta) - A^\top(\zeta)A(\eta)}{\zeta + \eta}$$

and

$$\Psi_-(\zeta,\eta) = \frac{\Phi'(\zeta,\eta) - H^\top(\zeta)H(\eta)}{\zeta + \eta}.$$

In this thesis the supply rate will always be given by a constant real symmetric matrix, say Σ. In that case we have $Q_\Sigma(w) = w^\top \Sigma w$. We say that the system $\mathfrak{B}$ is (strictly) Σ-*dissipative* if it is (strictly) dissipative with respect to the supply rate $Q_\Sigma(w)$.

The following proposition obtained in Willems & Trentelman [49] (also see Trentelman and Willems [34]) gives the relation between storage functions and states.

Proposition 7.3.2. *Let $\mathfrak{B}$ be represented by the observable image representation (7.10). Assume $\mathfrak{B}$ is Σ-dissipative, where $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and let $Q_\Psi(\ell)$ be a storage function. Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{l}}[\xi]$ define a minimal state map of $\mathfrak{B}$. Then there exists a real symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $\Psi(\zeta,\eta) = X^\top(\zeta)KX(\eta)$, equivalently, $Q_\Psi(\ell) = (X(\frac{\mathrm{d}}{\mathrm{d}t})\ell)^\top KX(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.*

Finally, we define positive and negative definiteness of storage functions of behaviors.

Definition 7.3.3. A storage function Q_Ψ for $\mathfrak{B}$ is called *positive (negative) definite* if there exists a minimal state map X for $\mathfrak{B}$ and a real symmetric matrix $K > 0$ ($K < 0$) such that $Q_\Psi(\ell) = (X(\frac{\mathrm{d}}{\mathrm{d}t})\ell)^\top KX(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.

7.4 A solution to the $\mathcal{H}_\infty$ control problem

The $\mathcal{H}_\infty$ control problem in the behavioral frame work was originally formulated and solved in Trentelman & Willems [41]. In Trentelman & Willems [41], only the full-information case was considered, i.e. the special case in which the entire system variable is determined uniquely by knowledge of the interconnection variable, equivalently, (w,v) is observable from c in $\mathcal{P}$. In this chapter we generalize this to the case where (w,v) is only detectable from c in $\mathcal{P}$. In contrast to Trentelman & Willems [41], we also require that the interconnection of the plant and controller is regular, which plays an important role in stabilization.

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ be controllable. Let $\gamma > 0$. It is well known that strict contractiveness and strict dissipativeness are equivalent, in the sense that a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is strictly $\frac{1}{\gamma}$-contracting if and only if $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ is strictly Σ_γ-dissipative, where

$$\Sigma_\gamma := \begin{bmatrix} -I_{\mathtt{w}} & 0 \\ 0 & \frac{1}{\gamma^2} I_{\mathtt{v}} \end{bmatrix}. \tag{7.21}$$

Note that

$$-\Sigma_\gamma^{-1} = \begin{bmatrix} I_{\mathtt{w}} & 0 \\ 0 & -\gamma^2 I_{\mathtt{v}} \end{bmatrix}. \tag{7.22}$$

In Trentelman & Willems [41], necessary and sufficient conditions for the existence of a free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting controller (however, without regularity condition) for $\mathcal{P}$ were established in terms of $-\Sigma_\gamma^{-1}$-dissipativeness of an orthogonal behavior associated with $\mathcal{P}$. We summarize the relevant results here as propositions. Recall from section 6.2 that $(\mathcal{P})_{(w,v)}$ denotes the projection of $\mathcal{P}$ onto the variable (w,v), while $(\mathcal{P})_{(w,v)}^\perp$ denotes its orthogonal complement. Our first proposition is a restatement of Lemma 9.2 from Trentelman & Willems [41]:

Proposition 7.4.1. *Let $\mathcal{P} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Let $\gamma > 0$. If there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ then $(\mathcal{P})_{(w,v)}^\perp$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.*

The next proposition can be found as Theorem 9.1 in Trentelman & Willems [41]. It states that if our synthesis problem is a full information problem (i.e. (w,v) observable from c), then the conditions in Proposition 7.4.1 are also sufficient:

Proposition 7.4.2. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\text{cont}}$. Assume v is free $\mathcal{P}$, and*

1. *(w,v) is observable from c in $\mathcal{P}$,*
2. *c is observable from (w,v) in $\mathcal{P}$.*

Let $\gamma > 0$. Then there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ if and only if $(\mathcal{P})^{\perp}_{(w,v)}$ is $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.

In this chapter we extend the above proposition in *two directions*. In the first place, we relax condition 1) of the proposition to the condition that (w,v) is only *detectable* from c in $\mathcal{P}$. Secondly, we establish conditions under which the controller in the statement of the proposition, in addition, can be taken *regular*. The following theorem is the main result of this chapter. It states that the necessary and sufficient conditions of Proposition 7.4.2 remain valid:

Theorem 7.4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\text{cont}}$. Assume v is free $\mathcal{P}$, and*

1. *(w,v) is detectable from c in $\mathcal{P}$,*
2. *c is observable from (w,v) in $\mathcal{P}$.*

Let $\gamma > 0$. Then there exists a free-disturbance, stabilizing, regular, and strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$ if and only if $(\mathcal{P})^{\perp}_{(w,v)}$ is $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function.

In the remainder of this section we will give a proof of Theorem 7.4.3. The idea is, starting from $\mathcal{P}$, to construct a new full plant behavior $\mathcal{P}'$ that satisfies the conditions of Proposition 7.4.2. We then apply this proposition to $\mathcal{P}'$, and finally translate back to $\mathcal{P}$ to obtain a proof of Theorem 7.4.3.

In order to proceed we need the following lemma:

Lemma 7.4.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\text{cont}}$ with system variable (w_1,w_2) be given by the image representation*

$$\mathfrak{B} = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{1}) \text{ such that } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}, \tag{7.23}$$

where $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, i.e. the image representation is observable (see Proposition 2.5.2). Then

1. *w_1 is observable from w_2 in $\mathfrak{B}$ if and only if $B(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ (equivalently, the behavior* $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t})) = \{0\}$*).*

2. *w_1 is detectable from w_2 in $\mathfrak{B}$ if and only if $B(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (equivalently, the behavior* $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ *is stable).*

Proof: 1. If w_1 is observable from w_2, from Definition 2.4.1 for all $(w_1,0) \in \mathfrak{B}$ we have $w_1 = 0$. From Equation (7.23) this implies that for all $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ we have $\ell \in \ker(A(\frac{\mathrm{d}}{\mathrm{d}t}))$. In other words $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t})) \subseteq \ker(A(\frac{\mathrm{d}}{\mathrm{d}t}))$. Consequently, we have

$$\begin{aligned}\ker(B(\tfrac{\mathrm{d}}{\mathrm{d}t})) &= \ker(A(\tfrac{\mathrm{d}}{\mathrm{d}t})) \cap \ker(B(\tfrac{\mathrm{d}}{\mathrm{d}t})) \\ &= \ker\left(\begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}\right) \\ &= \{0\},\end{aligned}$$

by observability. The converse implication is easy to prove, we skip the details.

2. From Definition 2.4.3, if w_1 is detectable from w_2, for all $(w_1,0) \in \mathfrak{B}$ we have $\lim_{t\to\infty} w_1(t) = 0$. From Equation (7.23), this implies that if $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ then $\lim_{t\to\infty} w_1(t) = 0$. Therefore w_1 is a stable Bohl function. Also we have

$$\begin{bmatrix} w_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A(\frac{\mathrm{d}}{\mathrm{d}t}) \\ B(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell. \tag{7.24}$$

As $\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, there exists a polynomial matrix $\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ such that $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = I$. From Equation (7.24), we have

$$\ell = F_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) w_1. \tag{7.25}$$

As w_1 is stable Bohl, by Equation (7.25) ℓ is a stable Bohl. Therefore for all $\ell \in \ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$, we have ℓ stable Bohl. Hence $\ker(B(\frac{\mathrm{d}}{\mathrm{d}t}))$ is stable. The converse implication is easy to prove, again we skip the details here. □

Going back to the proof of Theorem 7.4.3, as $\mathcal{P}$ is controllable it admits an observable polynomial image representation (see Proposition 6.2.2):

$$\mathcal{P} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \text{ such that } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}.$$

(7.26)

From Lemma 7.4.4, (w,v) is detectable from c in $\mathcal{P}$ if and only if the matrix $C(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore we can factorize C as $C = C'L$, with L and C' polynomial matrices such that L is Hurwitz and $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Define now a new behavior $\mathcal{P}' \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ as follows:

$$\mathcal{P}' := \left\{ \begin{bmatrix} w \\ c' \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \text{ such that } \begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}. \tag{7.27}$$

Clearly $(\mathcal{P})_{(w,v)} = (\mathcal{P}')_{(w,v)}$. Most important, from Lemma 7.4.4, in $\mathcal{P}'$, (w,v) is observable from c' (use the fact that $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$). We now first prove the following lemma which states that, due to the fact that in $\mathcal{P}'$ (w,v) is observable from c', the full controlled behavior corresponding to a given controller can also be implemented by a controller of the form $c' = C'(\frac{\mathrm{d}}{\mathrm{d}t})\ell'$, $K(\frac{\mathrm{d}}{\mathrm{d}t})\ell' = 0$:

Lemma 7.4.5. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}_{\mathrm{cont}}$ be given by*

$$\mathcal{P} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \textit{ such that } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \right\}.$$

Assume (w,v) is observable from c in $\mathcal{P}$. Let $\mathcal{C}_1 \in \mathfrak{L}^{\mathtt{c}}$. There exists a full row rank polynomial matrix K such that

$$\mathcal{C}_2 := \{c \mid \exists \ell' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1) \textit{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell',\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0\}$$

satisfies $\mathcal{P} \wedge_c \mathcal{C}_1 = \mathcal{P} \wedge_c \mathcal{C}_2$.

Proof: Let $\mathcal{C}_1$ be represented by, say, $S(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. Let K be a full row rank polynomial matrix such that

$$\ker(SC(\tfrac{\mathrm{d}}{\mathrm{d}t})) = \ker(K(\tfrac{\mathrm{d}}{\mathrm{d}t})).$$

We claim that, with such K, the statement of the lemma holds. Indeed, let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_1$. Then there exists ℓ such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \text{ and } S(\tfrac{\mathrm{d}}{\mathrm{d}t})c = 0.$$

Since $S(\frac{\mathrm{d}}{\mathrm{d}t})C(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$, we get $K(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$. Thus $c \in \mathcal{C}_2$, so $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_2$. Conversely, let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_2$. Then there exist ℓ and ℓ' such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' \text{ and } K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0.$$

Since $C(\lambda)$ has full column rank for all λ, this implies that $\ell = \ell'$. Thus, $S(\frac{\mathrm{d}}{\mathrm{d}t})c = S(\frac{\mathrm{d}}{\mathrm{d}t})C(\frac{\mathrm{d}}{\mathrm{d}t})\ell' = 0$ since $K(\frac{\mathrm{d}}{\mathrm{d}t})\ell' = 0$. We conclude that $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}_1$. □

Next, we formulate and prove the following theorem:

Theorem 7.4.6. *Let $\mathcal{P}$, $\mathcal{P}' \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ be as given in Equations (7.26) and (7.27), respectively. Let $\gamma > 0$. If there exists a free-disturbance stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$ then there exists a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$.*

Proof: The proof of this theorem will make use of a series of lemmas, Lemmas 7.4.7 to 7.4.11, to be formulated and proved in the sequel.

First, using Lemma 7.4.5, let

$$\mathcal{C}' = \{c' \mid \exists \ell' \text{ such that } c' = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell',\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0\} \tag{7.28}$$

be a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$. Using observability of (w,v) from c' in $\mathcal{P}'$ we can prove the following:

Lemma 7.4.7. *Let $\mathcal{P}'$ and $\mathcal{C}'$ be as given in Equations (7.27) and (7.28), respectively. Then*

$$\mathcal{P}' \wedge_{c'} \mathcal{C}' = \left\{ \begin{bmatrix} w \\ c' \\ v \end{bmatrix} \middle| \ \exists \ell' \ s.\ t \begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell',\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0 \right\}. \tag{7.29}$$

Proof: Let $(w,c',v) \in \mathcal{P}' \wedge_{c'} \mathcal{C}'$. Then $(w,c',v) \in \mathcal{P}'$ and $c' \in \mathcal{C}'$. From Equations (7.27) and (7.28), there exists ℓ_1 and ℓ_2 such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell_1,\ v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell_1,\ c' = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell_1,\ c' = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell_2 \text{ and } K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell_2 = 0.$$

Since $C'(\lambda)$ has full column rank for all λ, $c' = C'(\frac{\mathrm{d}}{\mathrm{d}t})\ell_1$ and $c' = C'(\frac{\mathrm{d}}{\mathrm{d}t})\ell_2$ implies that $\ell_1 = \ell_2$. Therefore

$$\begin{bmatrix} w \\ c' \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C'(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell',\ \ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0.$$

The converse inclusion is trivial. □

Define $\mathcal{K}_1 := (\mathcal{P} \wedge_{c'} \mathcal{C}')_{(w,v)}$. Then from Equation (7.29), we have

$$\mathcal{K}_1 := \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \;\middle|\; \exists \ell' \text{ such that } \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell',\, K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell' = 0 \right\}. \tag{7.30}$$

Let L be the Hurwitz polynomial matrix obtained from the factorization $C = C'L$ above. As L is Hurwitz, the matrix KL^{-1} is a stable rational matrix. Factorize KL^{-1} as $KL^{-1} = {P_1}^{-1}Q_1$, where P_1 is Hurwitz. Then we have

$$P_1 K = Q_1 L. \tag{7.31}$$

Define

$$\mathcal{C} := \{c \mid \exists \ell \text{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\, P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}. \tag{7.32}$$

Lemma 7.4.8. *Let $\mathcal{P}$ and $\mathcal{C}$ be given by Equations (7.26) and (7.32), respectively. Then*

$$\mathcal{P} \wedge_c \mathcal{C} = \left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \text{ s. t. } \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\, P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}. \tag{7.33}$$

Proof: Let $(w,c,v) \in \mathcal{P} \wedge_c \mathcal{C}$. From the representations of $\mathcal{P}$ and $\mathcal{C}$ it is evident that there exists an ℓ such that

$$w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\; v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \text{ and } c = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,$$

and there exists an $\hat{\ell}$ such that

$$c = C'(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell},\; P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell} = 0.$$

As $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ we get

$$L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = L(\tfrac{\mathrm{d}}{\mathrm{d}t})\hat{\ell}. \tag{7.34}$$

Using Equations (7.31) and (7.34) we have

$$\begin{aligned} P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell &= Q_1(\tfrac{\mathrm{d}}{\mathrm{d}t})L(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell \\ &= 0. \end{aligned}$$

Therefore

$$\begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0.$$

The converse inclusion is trivial. □

Define $\mathcal{K}_2 := (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$. From the above we have

$$\mathcal{K}_2 := \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \;\middle|\; \exists \ell \text{ such that } \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}. \tag{7.35}$$

In order to proceed we need the following lemma.

Lemma 7.4.9. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w,c,v) be given by*

$$\left\{ \begin{bmatrix} w \\ c \\ v \end{bmatrix} \;\middle|\; \exists \ell \ s.t \ \begin{bmatrix} w \\ c \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell,\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0 \right\}, \tag{7.36}$$

where K has full row rank and $\begin{bmatrix} W(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Define $\mathcal{S} := \{w \mid \exists c \text{ such that } (w,c,0) \in \mathfrak{B}\}$. Then

1. *v is free in $\mathfrak{B}$ if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ has full row rank.*
2. *$\mathcal{S}$ is stable if and only if $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.*

Proof: 1. We have

$$(\mathfrak{B})_v = \{v \mid \exists \ell \text{ s. t. } v = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ K(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}. \tag{7.37}$$

It is easy to check that

$$\begin{aligned} \mathtt{p}((\mathfrak{B})_v) &= \operatorname{rank}\left(\begin{bmatrix} I & V \\ 0 & K \end{bmatrix}\right) - \operatorname{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right) \\ &= \operatorname{rowdim}(V) + \operatorname{rowdim}(K) - \operatorname{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right). \end{aligned} \tag{7.38}$$

Recall that v is free in $\mathfrak{B}$ if and only if $(\mathfrak{B})_v = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}})$, equivalently, $\mathtt{p}((\mathfrak{B})_v) = 0$. Therefore from Equation (7.38), v is free in $\mathfrak{B}$ if and only if $\operatorname{rank}\left(\begin{bmatrix} V \\ K \end{bmatrix}\right) = \operatorname{rowdim}(V) + \operatorname{rowdim}(K)$, equivalently, $\begin{bmatrix} V \\ K \end{bmatrix}$ has full row rank.

2. From Equation (7.36), we have

$$\mathcal{S} = \left\{ w \;\middle|\; w = W(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell, \begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0 \right\}. \tag{7.39}$$

If $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, $\begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0$ implies that ℓ is stable Bohl, which in turn implies that $w = W(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is stable Bohl. Hence $\mathcal{S}$ is stable. Conversely, if $\mathcal{S}$ is stable then for all ℓ such that $\begin{bmatrix} V(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell = 0$, we have

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} W(\frac{\mathrm{d}}{\mathrm{d}t}) \\ V(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \tag{7.40}$$

is stable Bohl. As $\begin{bmatrix} W(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$ there exists $\begin{bmatrix} F_1 & F_2 \end{pmatrix}$ such that $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix} = I$. Therefore, from Equation (7.40), we have

$$\ell = F_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w. \tag{7.41}$$

As w is stable Bohl, from the above ℓ is a stable Bohl. Hence $\begin{bmatrix} V(\lambda) \\ K(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.

□

Lemma 7.4.10. *Let the controllers $\mathcal{C}'$ and $\mathcal{C}$ be given by Equations (7.28) and (7.32), respectively. Then $\mathcal{C}$ is a free-disturbance, stabilizing controller for $\mathcal{P}$ if and only if $\mathcal{C}'$ is a free-disturbance, stabilizing controller for $\mathcal{P}'$.*

Proof: Using Equation (7.33) along with Definition 7.2.2 and Lemma 7.4.9, $\mathcal{C}$ is free-disturbance and stabilizing controller for $\mathcal{P}$ if and only if $\begin{bmatrix} V \\ P_1 K \end{bmatrix}$ is square, nonsingular and Hurwitz. In the same way, using Equation (7.29),

$\mathcal{C}'$ is free-disturbance and stabilizing controller for $\mathcal{P}'$ if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ is square, nonsingular and Hurwitz. The proof is then completed by noting that $\begin{bmatrix} V \\ P_1 K \end{bmatrix}$ is square, nonsingular and Hurwitz if and only if $\begin{bmatrix} V \\ K \end{bmatrix}$ is square, nonsingular and Hurwitz (use the fact that P_1 is Hurwitz). □

In the following, recall that $\mathfrak{D}(\mathbb{R},\mathbb{R}^\bullet)$ denotes the space of compact support functions from $\mathbb{R}$ to $\mathbb{R}^\bullet$.

Lemma 7.4.11. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be given by Equations (7.30) and (7.35), respectively. Then $\mathcal{K}_1 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}}) = \mathcal{K}_2 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Consequently, for any $\gamma > 0$, $\mathcal{K}_1$ is strictly $\frac{1}{\gamma}$-contractive if and only if $\mathcal{K}_2$ is strictly $\frac{1}{\gamma}$-contractive.*

Proof: We first prove that $\mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt})) = \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$. The implication $\mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt})) \subseteq \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$ is obvious. To show the converse inclusion, assume that $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(P_1 K(\frac{d}{dt}))$. Define $y := K(\frac{d}{dt})\ell$. Then $y \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{y}}) \cap \ker(P_1(\frac{d}{dt}))$. As $\ker(P_1(\frac{d}{dt})) \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{y}}) = 0$ (since P_1 is nonsingular) we have $y = 0$. Hence $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1) \cap \ker(K(\frac{d}{dt}))$.

Since $\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W(\frac{d}{dt}) \\ V(\frac{d}{dt}) \end{bmatrix} \ell$ is an observable representation (due to the assumption that c is observable from (w,v) in $\mathcal{P}$), $\ell \in \mathfrak{D}(\mathbb{R},\mathbb{R}^1)$ if and only if $\begin{bmatrix} w \\ v \end{bmatrix} \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Then, from the definitions of $\mathcal{K}_1$ and $\mathcal{K}_2$ we have the equality $\mathcal{K}_1 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}}) = \mathcal{K}_2 \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{v}})$. Therefore, immediately from Definition 7.2.5, $\mathcal{K}_1$ is strictly $\frac{1}{\gamma}$-contractive if and only if $\mathcal{K}_2$ is strictly $\frac{1}{\gamma}$-contractive. □

Applying the previous lemmas, we can now complete the proof of Theorem 7.4.6: from Lemmas 7.4.8 to 7.4.11 we conclude that, starting with the free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller $\mathcal{C}'$ for $\mathcal{P}'$, the controller $\mathcal{C}$ is a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$. □

We are now in a position to give a proof of Theorem 7.4.3:

Proof of Theorem 7.4.3: Starting with $\mathcal{P}$, introduce the new behavior $\mathcal{P}'$ as above. We have $(\mathcal{P})_{(w,v)} = (\mathcal{P}')_{(w,v)}$. Thus, if $(\mathcal{P})^\perp_{(w,v)}$ is strictly $-\Sigma_\gamma^{-1}$-dissipative and has a negative definite storage function, then the same holds for $(\mathcal{P}')^\perp_{(w,v)}$. By Proposition 7.4.2 there exists a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}'$. Then there also exists such controller for the original $\mathcal{P}$. Finally, we should prove that also a *regular* controller for $\mathcal{P}$ exists with these properties. Again, note that $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} = \mathcal{K}_2$ (see Lemma 7.4.8). Now, $\mathcal{K}_2$ is obviously implementable with respect to $\mathcal{P}$. Since $\mathcal{P}$ is assumed to be controllable, Proposition 3.3.10 then asserts that $\mathcal{K}_2$ is also regularly implementable. Any regular controller that implements $\mathcal{K}_2$ is then of course a free-disturbance, stabilizing strictly $\frac{1}{\gamma}$-contracting controller for $\mathcal{P}$.

The converse implication follows immediately from Proposition 7.4.1. □

Remark 7.4.12. Without going into the details, in this remark we will outline how to actually compute a free-disturbance, stabilizing, strictly $\frac{1}{\gamma}$-contracting, regular controller for $\mathcal{P}$ from the polynomial matrices W, V and C appearing in its image representation (7.26) (see also Trentelman & Willems [41]). In the following, let $\mathtt{l}$ denote the number of columns of W (i.e. the dimension of the latent variable ℓ). Let Σ_γ be given by (7.21). Denote $R^{\sim}(\xi) := R^\top(-\xi)$.

1. Factorize:
$$\begin{bmatrix} W \\ V \end{bmatrix}^{\sim} \Sigma_\gamma \begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} F_+ \\ F_- \end{bmatrix}^{\sim} \begin{bmatrix} I_{\mathtt{l}-\mathtt{v}} & 0 \\ 0 & -I_{\mathtt{v}} \end{bmatrix} \begin{bmatrix} F_+ \\ F_- \end{bmatrix}$$
   such that
   (a) $\begin{bmatrix} F_+ \\ F_- \end{bmatrix}$ is a Hurwitz polynomial matrix,
   (b) $\begin{bmatrix} W \\ V \end{bmatrix} \begin{bmatrix} F_+ \\ F_- \end{bmatrix}^{-1}$ is proper,
   (c) $\begin{bmatrix} V \\ F_+ \end{bmatrix}$ is Hurwitz.
2. Factorize: $C = C'L$ with C' and L polynomial matrices such that $C'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and L Hurwitz.
3. Factorize: $F_+L^{-1} = P_1^{-1}Q_1$ with P_1,Q_1 polynomial matrices, P_1 Hurwitz.

Define then a controller $\mathcal{C}$ for $\mathcal{P}$ by:

$$\mathcal{C} := \{c \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \text{ such that } c = C(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell,\ P_1(\tfrac{\mathrm{d}}{\mathrm{d}t})F_+(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}, \quad (7.42)$$

The controller $\mathcal{C}$ is then free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting. It can be shown that if c is free in $\mathcal{P}$ then the above controller $\mathcal{C}$ is also regular. If c is not free in $\mathcal{P}$ then, starting with $\mathcal{C}$ given by the Equation (7.42), a *regular*, free-disturbance, stabilizing and strictly $\frac{1}{\gamma}$-contracting controller can be constructed using ideas from Belur [1].