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Metastates, non-Gibbsianness and Phase Transitions

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a stroll through statistical mechanics

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Contents

1 Introduction

O sol che sani ogni vista turbata,
tu mi contenti sì quando tu solvi,
che, non men che saver, dubbiar m'aggrata.
Dante, Inferno XI, 91-93

Statistical mechanics addresses the problem of deducing macroscopic (observable) properties of matter from the atomic hypothesis. According to the hypothesis, matter is thought of as consisting of a very large number N of atoms (particles), subjected to the laws of classical mechanics [28]. It is the big value of $N \sim 6.02 * 10^{23}$ (Avogadro's number) that renders us unable to describe how the system (matter + interaction) behaves at the microscopic level. Nonetheless, we can see (measure) some global properties of the system like temperature, density and magnetization. This is exactly the point where probability starts to play its role. The many degrees of freedom of the system are treated as random variables distributed according to a probability measure, and the global properties are thought of as an average over the microscopic realizations (configurations) of the system. This concept is very well summarized — in my opinion — by two sentences:

- "The microscopic complexity can be overcome by a statistical approach; the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers".
  (*Georgii*, [29])
- "What looks still to our crude eyes is a wild and dynamic dance".
  (*Feynman*, [26])

The starting point of statistical mechanics is then to consider any observation of a physical system as a (joint) realization of many random variables associated to the system's components. Due to the presence of an interaction between the microscopic components, (as well as the interaction that components might have with external forces), the associated random variables will be dependent with respect to one another, allowing certain types of macroscopic orderings

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O sun that heals all sight that is perplexed,/ when I ask you, your answer so contents / that doubting pleases me as much as knowing.

(patterns). In mathematical terms, the interaction is described by the *Hamiltonian* function, which assigns an energy to each realization of the system. The equilibrium phases with respect to a given interaction are described as probability measures concentrated on the "most probable" realizations of the system. These probability measure are called *Gibbs measures*.

The word "equilibrium" that we used above, refers to the forces acting on the system. As a matter of fact, in the "real" world it is very rare to come across a pure equilibrium phase. Any system subjected to forces requires "some" time, usually (for human time-scale) long, to converge to an equilibrium phase. Over this time span, however, external conditions change requiring a new equilibrium to be sought. In [25], Feynman says: "When all fast things have happened and all slow things have not, then the system is in equilibrium". Nevertheless, although within the limits of equilibrium statistical mechanics, some physical phenomena can be described.

In this regard, the most famous and successful example is provided by the *Ising model*. This model was invented in the 1920's by Lenz and his student Ising in order to describe ferromagnetism, more specifically, ferromagnetic phase transitions. To briefly grasp an idea of what ferromagnetism is, let us consider a piece of iron (widely-known example!), and let us associate to each atom a magnetic moment. The interaction between the atoms favours the alignment of their magnetic moments (the iron is, indeed, a ferromagnet). However, the tendency of atoms to align themselves is hindered by the thermal motion. At high temperature the thermal motion is too strong and the ordering effect is suppressed. If the temperature is low, below a certain threshold T_c (*Curie temperature*), the interaction dominates the thermal chaos and a global order of the magnetic moments arises, inducing a macroscopic magnetic field, a phenomenon known as *spontaneous magnetization* (the term spontaneous refers to the absence of an external field). The Ising model is a very simple model in which each magnetic moment (*spin*) admits only two orientations, either up or down. Although it is quite a simple model, it displays a rich anthology of results, (see e.g. [29]). A natural generalization of the Ising model is provided by the so-called *Potts model*. Whereas in the Ising model spins are allowed to take only two values, in the Potts model each spin might take $q \geq 2$ different values. As in the Ising case, the ferromagnetic Potts interaction among the components of the system favours configurations where neighbouring spins tend to align themselves.

In this thesis, we shall present a few results obtained for the Ising and Potts model in particular settings: mean-field, Cayley trees and the lattice $\mathbb{Z}^2$.

1.1 Gibbs Measure

In order to get a better understanding of what a Gibbs measure is, we need to go to a more precise mathematical description of a physical system, while still remaining at a heuristic level. The reader who is greedy for mathematical technicalities should wait until Chapter 2.

First of all we must say, mathematically, how many particles we are dealing with and what their positions are. This is done by introducing a set V (finite), which labels the system's components. Secondly, we have to introduce a set Ω_0 which describes the possible configurations of each component. From this time on, we shall denote by σ_i the configuration of the component that has been marked with the label i. For a ferromagnet, for example, V consists of the sites of the crystal lattice formed by the positions of the atoms and Ω_0 is chosen to be the set of all possible orientations of the magnetic moment. This choice relies on the understanding that ferromagnetism is due to alignments of the spins of the atoms. In the Ising model, where only two orientations are allowed, the set Ω_0 is given by $\{-1, +1\}$. (We mentioned already in the previous section that the alignment persists, in a certain range of temperature ($T < T_c$), even when an external field is switched off). Once we have specified the sets V and Ω_0, we can now think of a particular configuration of the whole system to be an element $(\sigma_i)_{i \in V}$ of the product space $\Omega = \Omega_0^V$, which is called *configuration space*. Due to the complexity of the microscopic structure, it is not a single element of the configuration space that properly describes the state of a system which one observes (indeed, our knowledge of the system is affected by the fluctuations of any microscopic measurement), but rather a family $(\sigma_i)_{i \in V}$ of Ω_0-valued random variables, or better their joint distribution μ, which is a probability measure on Ω.

As we have seen previously, the number of particles in a system is huge, so V must be very large. It is mathematically reasonable to assume that the properties of a very large system can be made manifest by limiting procedures. In [5], this idea is formulated as an informal axiom: "A system composed of a very large number of degrees of freedom can be well approximated by an infinite system". It is therefore common in statistical mechanics, to take the limit $|V| \to \infty$, which is known as *thermodynamic limit*, where $|V|$ denotes the number of elements in V. This assumption represents the key step towards the mathematical comprehension of *phase transition*. Phase transitions may be thought of as the possibility that from the same microscopic laws (interaction), different macroscopic behaviours occur. Although the thermodynamic limit opened the door of phase transitions to mathematics, it also presented the mathematical bill of defining a Gibbs measure on an infinite space (like $\Omega =$

$\Omega_0^{\mathbb{Z}^d}$). Remembering a quote from Thoreau: "You never gain something but that you lose something". The problem is that the Hamiltonian of a system with infinitely many interacting components will be ill-defined. Think, as an example, of the Hamiltonian defining the Ising ferromagnet, $\Omega_0 = \{-1, +1\}$, on a finite subset Λ of $V = \mathbb{Z}^d$

$$H_\Lambda(\sigma_\Lambda) = -\sum_{\substack{i,j\in\Lambda \\ ||i-j||_1=1}} \sigma_i\sigma_j - h\sum_{i\in\Lambda} \sigma_i \, , \tag{1.1}$$

where $||i - j||_1$ denotes the lattice distance. The Hamiltonian is supposed to measure the energy of a configuration, but in the infinite-volume limit the previous formula makes no sense. What we can do is to interpret (1.1) as the energy of an infinite-volume configuration restricted to a finite volume Λ. To do that we need to slightly change (1.1), replacing $\displaystyle\sum_{\substack{i,j\in\Lambda \\ ||i-j||_1=1}}$ with $\displaystyle\sum_{\substack{i \text{ or } j\in\Lambda \\ ||i-j||_1=1}}$.

At first glance, this might look just as a little change in the notation, although its consequences are very important. The new Hamiltonian contains the energy corresponding to the interaction between spin inside Λ with those outside (how many depending on the range of the interaction). We could now consider the Hamiltonian as a function $H : (\Lambda, \sigma) \to H_\Lambda(\sigma)$ from the pairs consisting of finite subsets of $\mathbb{Z}^d$ and configuration in Ω to real numbers. This allows us to define for any fixed configuration of spin $\omega \in \Omega$, which we will be referring to as *boundary condition*, and for any finite subset $\Lambda \subset \mathbb{Z}^d$, a probability measure

$$\mu_\Lambda^\omega(d\sigma_\Lambda) = \frac{e^{-\beta H_\Lambda((\sigma_\Lambda\omega_{\Lambda^\complement}))}}{Z_{\beta,\Lambda}(\omega)}\alpha_\Lambda(d\sigma_\Lambda) \, , \tag{1.2}$$

where by $\sigma_\Lambda\omega_{\Lambda^\complement}$ we denote the configuration that agrees with ω outside Λ and with σ on Λ, and α is an a priori measure, which is supposed to describe the non-interacting system. Equation (1.2), via the ω-dependence, defines a family of measures on Λ. The idea, which dates back to the late 1960's and is due to Dobrushin, Lanford and Ruelle (DLR), is to consider this family of measures as a family of conditional (on ω) probabilities of some measure(s) μ, defined on the infinite volume. Starting from this observation, μ will be defined to be an infinite-volume Gibbs measure for the Hamiltonian H if its conditional distributions, given the configurations outside Λ, are given by (1.2). The family $(\mu_\Lambda^\omega)_{\omega,\Lambda}$ of all conditional distributions is known as *specification* for the measure μ, (see e.g. [18, 29]). Gibbs measures are often called *states* and, henceforward, we shall refer to them with either of these appellatives.

At this point, two questions arise: "Does such a measure exist?" and then "If it exists, is it unique?" The answer to the first question, as far as the systems we are concerned with in this thesis, is "yes" and it follows by a compactness argument. With regard to the second question, we will encounter situations where the infinite-volume measure is not uniquely determined, i.e., several infinite-volume measure exist for the same Hamiltonian and for the same inverse temperature β. This is exactly what statistical mechanics needs to account for phase transitions. If we then go back to the Ising ferromagnet example, in the absence of any external field, we know that if $d \geq 2$ and the temperature is low there has to be a measure describing a phase with positive magnetization and another one with negative magnetization: the system can, in fact, be in either of them. However, such a dichotomy ceases to exist when the temperature is high, which leads to the uniqueness of the Gibbs measure (at high temperature there is always a unique Gibbs measure).

Above we indicated how to infer that a measure μ, defined on the infinite volume, is a Gibbs measure for a given Hamiltonian. However, it is possible to determine whether an infinite-volume measure is Gibbs or not without any reference to a specific Hamiltonian. This is due to Kozlov [48], who found necessary and sufficient conditions on the family of conditional distributions for the corresponding measure to be Gibbs. One of the basic ideas behind these conditions which we shall focus the most on, is that the family of conditional distributions needs to satisfy a sort of "continuity" property, termed *quasilocality*. This represents the main tool to investigate the non-Gibbsian property of an infinite-volume measure, and we shall discuss this matter in more detail in Chapter 4.

1.2 Loss of Gibbsianness

In the previous section, we saw that the Gibbsianness of an infinite-volume measure requires some conditions to hold. Then, at least logically, we know about the possibility for an infinite-volume measure not to be Gibbs; we "simply" need to violate at least one of these conditions. It might be natural to ask whether these violations are just at the level of mathematical finesse, or if they admit a physical explanation. In [24], Fernández point of view is that the theory which has been developed so far is still mostly at the level of mathematical finesse. I find myself concurring with such a point of view.

The dilemma "to be or not to be Gibbs" first appeared within the framework of renormalization-group transformations (RG) [32, 33]. A renormalization-group transformation was initially defined as a map for generating a spin con-

figuration σ' (a.k.a. "block spins") given a configuration σ ("original spins").
This map is given by a *probability kernel* $T(\sigma|\sigma')$, and it can be easily extend
to a map from measures to measures, namely

$$\mu'(\sigma') = (\mu T)(\sigma') \equiv \sum_{\sigma} \mu(\sigma) T(\sigma|\sigma') \ . \tag{1.3}$$

We have seen how Hamiltonians are related to measures, it was done via the
concept of a specification (1.2). A question poses itself: how does the map T
act on the Hamiltonian? In other words, if μ is a Gibbs measure associated
with an Hamiltonian H, then one (often wrongly) assumes that μ' is a Gibbs
measure associated to some Hamiltonian H'. This induces a map $\mathcal{R}$, from
Hamiltonians to Hamiltonians, as sketched below,

$$\begin{array}{ccc}
\mu & \xrightarrow{\ \ T\ \ } & \mu' \equiv \mu T \\
\uparrow & & \downarrow \\
H & \xrightarrow{\ \ \mathcal{R}\ \ } & H'
\end{array}$$

A problem arises here: one peculiarity of the infinite-volume limit, is the pos-
sibility that corresponding to a Hamiltonian, there might be several Gibbs
measures. This phenomenon might cause the map $\mathcal{R}$ to be multivalued. It has
been shown that this pathology cannot occur on regular lattices. Yet, is the
image Hamiltonian H' well-defined? In other words, is the image measure μ'
Gibbs or not? If the Gibbsianness is preserved under the renormalization-group
transformations T, that's to say μ' is also Gibbs, then there is a unique Hamil-
tonian associated with it, which is H', if this is not the case, then H' is not well
defined [18]. As pointed out first by Israel [44], the loss of the Gibbsianness for
the renormalized measure μ' is often related to the violation of the quasilocal-
ity property. Quasilocality, as we shall see in the sequel, is a mathematically
well-defined property which is based upon the physical concept of an "isolated
system". A thoroughly isolated system is of course an idealization, but one
assumes the possibility to render a system as close to isolated as desired, by
moving it "away" from all other objects. This relies on a fundamental assump-
tion, which is often made in different parts of science, that is the "decay of
interactions".

The decay of interactions is often referred to as "locality"[1]. The violation of quasilocality for the transformed measure μ' was detected from the fact that the "original spins" may undergo a phase transition for some fixed "block spin" configuration $\sigma'_{special}$. What happens is that information can be transmitted from distant "block spins" to "block spins" in a particular region (*origin*) via the "original spins" in the intermediate region, even when the "block spins" in the intermediate region are fixed. This violates the quasilocality condition. It is peculiar, however, that a single "block spin" configuration $\sigma'_{special}$, which has probability zero, can cause the non-Gibbianness of the renormalized measure. What happens exactly is that the phenomenon of "far-away dependence" for the "original spins" manifest itself for "block spins" which are close (in the sense of product topology) to $\sigma'_{special}$. The configuration $\sigma'_{special}$ is called a *bad configuration* and we shall see in Chapter 4, that it represents a discontinuity point for a particular class of functions termed *conditional expectations*. After this discovery the importance of the quasilocality property for the renormalized measure was in the public eye.

In [19] a related issue was first addressed, namely the loss and possible recovery of the Gibbsianness of an initial Gibbs measure under a stochastic time-evolution. The authors studied the evolution, under high-temperature spin-flip Glauber dynamics, of a low-temperature Gibbs measure of an Ising model. They observed that, in case of an initial external field zero, after a certain time the Gibbsian property was lost and the time-evolved measure stays non-Gibbs forever. Moreover, at large times the Gibbs property was recovered, in the presence of a non-zero external field in the initial system. In Chapter 4, a very similar problem is addressed: we study homogeneous Gibbs measures on a *Cayley tree*, subjected to an infinite-temperature Glauber evolution, and consider their (non-)Gibbsian properties. We show that, in contrast to what happens on regular lattices such as $\mathbb{Z}^d$, the Gibbsian properties of evolved Gibbs measures for models on trees turn out to depend on which of the different low-temperature phases (the Gibbs measures $\mu^+, \mu^-, \mu^\sharp$) one considers. We remark that the existence of these low-temperature phases corresponds to

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[1]Newton, in a letter to Bentley from 1693, pronounce the following stark rejection of action at a distance:

It is inconceivable that inanimate brute matter should, without the mediation of something else which is not material, operate upon and affect other matter without mutual contact...That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it. (A. Janiak. *Isaac Newton: Philosophical writings.* Cambridge University Press, 2004).

the existence of three fixed points for a selfconsistency equation. Whilst $\mu^{\pm}$ are linked to the (two) stable fixed points, $\mu^{\sharp}$ is linked to the unstable fixed point, (see [29], Chapter 12). We shall see how this, so far tree-peculiar behaviour, is related to the different nature of the bad configurations. We mentioned above that bad configurations represent discontinuity points of the conditional expectations. These discontinuities may be of two sorts: *μ-essential discontinuity* and *strong discontinuity*. While the latter is a discontinuity that all the conditional expectations (they depend on the initial measure) will have, the former might be a removable discontinuity for some measures, and a nonremovable one for others. Another interesting aspect of non-Gibbsianness is the study of the set of discontinuity points. In [38] the fuzzy Potts model on trees was considered. There, the possibility of having a positive-measure set of discontinuity points was found. In Chapter 4, we show that after a certain time, the set of discontinuity points will be empty if we start with $\mu^{\pm}$, whereas it will have full measure if we start with $\mu^{\sharp}$.

1.3 Random Gibbs Measures and Metastates

The Gibbs measures serve to describe an equilibrium state of a system. However, from what has been said so far, it is glaring that there is no information at all on how the system attained such a state of matter (equilibrium state), nor on which the initial conditions were. The issue is sorted out by assuming that, on the time-scales on which the system is observed, the dynamics relaxes to equilibrium and loses any information whatsoever about the the initial conditions. Even if this assumption — and in general assumptions, rational or not, do — makes life easier, it also creates a dichotomy between systems which satisfy it and systems which do not. A typical example for which this assumption would not hold are solid alloys. Think of a material made of a mixture of iron and gold atoms, and imagine them to be allocated to some lattice sites. If we heat the system up the atoms will change their positions, however it is reasonable to assume that if we turn down the temperature the motion of atoms will slow down, and that at a sufficiently low temperature, their motion will be thoroughly suppressed. This could be rephrased by saying that, over large time-scales, the microscopic realization of the two-type atoms' mixture will not change. One says that the positions of the atoms are *frozen*. Let us now consider, for the very same system other degrees of freedom, namely the magnetic moments of the iron atoms in the alloy. These magnetic moments are not frozen and we might describe their behaviour via a Gibbs **m**easure. It comes to light that the Gibbsian description must take into account the posi-

tions of these atoms, the interaction between them depends on their positions indeed. Thus, if we denote by η_i the position of the iron atoms, we can write $H[\eta](\sigma)$ for the spin Hamiltonian associated to the system with iron atoms positions frozen at η. We would then write the (formal) Gibbs measure as

$$\mu_\beta[\eta](\sigma) = \frac{e^{-\beta H[\eta](\sigma)}}{Z_\beta[\eta]} \ , \tag{1.4}$$

where the partition function $Z_\beta[\eta]$ depends on the positions of the atoms as well. We will refer to such a system as *disordered*. What is the meaning of such a Gibbs measure? Well, the meaning is that of a measure describing the magnetic properties of the system when the position of the iron atoms is fixed to be η. Is it reasonable to do that? In other words, is such a description realistic? The answer to the latter is not easy. It would be certainly nice to have such a description for all possible arrangements of the η_i but this represents, from a complexity point of view, a similar problem to the one encountered in a deterministic description of a large interacting system. We could once again hope that the microscopic details (positions in this case) do not matter too much, so that we could extract enough information from the macroscopic observables. We could then treat the possible arrangements of the η_i as random variables and introduce some probability distribution $\mathbb{P}_\eta$, on the possible realizations of the iron positions, to measure their likelihood. We will refer to this randomness (η) as *quenched*, which is a term lent from metallurgy describing the fast cooling of an alloy via immersion in water. While dealing with two different types of randomness, namely the quenched disorder and the magnetic moments, we must be aware of the fact that they do not equilibrate on the same time-scale. This makes it inappropriate to talk about thermal equilibrium for the whole (positions and magnetic moments) system. What we might hope for is that certain properties of the material do not depend too much on the microscopic realizations, and are the same for almost all realizations of the disorder variables.

The first thing one would like to understand, for these models, is what the possible limiting measures (phases) are, and how they depend on the realization of the disorder. Secondly, even if we suppose that the phases are identified, it is not a priori clear what role they will play for the behaviour of a large but finite system. It is very reasonable, and at the same time instructive, to ask why this difficulty in determining the phases shows up in the presence of disorder and we do not face it, at least in many situations, for non-disordered systems. The reason lies in the fact that, in the absence of disorder, we usually know how to choose finite-volume boundary conditions so that the infinite-volume

limit is a pure phase, and we know the limiting mixture of pure phases for other boundary conditions as well. For disordered models the relation between boundary conditions and pure phases is unknown. This makes the meaning of the infinite-volume limit rather unclear. Indeed, for systems with a quenched disorder, in general, there will not be a single infinite-volume limit but many different limits along different subsequences of volumes. In other words, the only way to preselect a particular limiting measure is by looking at subsequences of volumes $\Lambda_{n[\eta]}$, which may in general depend on the disorder. This phenomenon is called *chaotic size dependence*. Despite the fact that we introduced it emphasizing the "ineffective role" of the boundary conditions, the phenomenon of chaotic size dependence manifests itself also in situations where the notion of boundary condition makes no sense, that is in mean-field models. To better understand the limitation of such an approach to the thermodynamic limit, one has to remember that an infinite-volume Gibbs measure is supposed to approximate a very large system under particular conditions. If this approximation is only valid for some particular volumes, which ones depending on the realization of the disorder, knowing the set of all infinite-volume measures is not enough. Indeed, when there are several Gibbs measures, it might be that a system find itself (approximately) in one of these states for one volume, and (approximately) in another Gibbs state for another volume, which one depending on the choice of the disorder.

To account for disordered systems where we face the problem of chaotic size dependence Newman and Stein [61, 63] proposed, in order to capture the asymptotic volume-dependence, to look at the empirical average

$$\kappa_N[\eta] := \frac{1}{N} \sum_{n=1}^{N} \delta_{\mu_n[\eta]} \;, \tag{1.5}$$

where $\mu_n[\eta]$ denotes the finite-volume Gibbs measure in the volume Λ_n, and δ is the Dirac measure. Note that κ_N is a random measure (through its η-dependence) on the states (Gibbs measures) of the system. For large N it will be supported by a subset of the infinite-volume Gibbs measures. Intuitively it means to look at the frequency of occurrence of a state along a volume sequence. The limit

$$\kappa[\eta] := \lim_{N\uparrow\infty} \kappa_N[\eta] \;, \tag{1.6}$$

is called *Newman-Stein metastate*, if it exists for $\mathbb{P}$-almost all realizations η. (There are general existence results about the convergence for $\mathbb{P}$-almost all realizations η that follow from compactness arguments but these are only for sparse enough sub-sequences of n's and N's). Another way to tackle the chaotic

size dependence was provided by Aizenman and Wehr [1]. They suggested to look at the probability distribution of the pair of the finite-volume random Gibbs measure and the disorder variable $(\mu_n[\eta], \eta)$. Suppose the limit exists in the sense of weak convergence, then we denote the resulting limiting distribution $K(d\mu, d\eta)$. If we now take the conditional distribution, obtained by conditioning on the disorder variable η, this provides us with a measure on the first variable, which we denote by $\kappa^{AW}[\eta](d\mu)$. This object is called *Aizenman-Wehr metastate*. Again, the existence of such a limit is guaranteed only for subsequences of n's. In [60] it was proved that for sufficiently sparse subsequences the two notions coincide. As a warning to the unaware reader, we point out that the metastate notion should be distinguished from the more standard decomposition of a Gibbs state into convex-combinations (mixture) of extremal Gibbs states. Indeed, the metastates are measures on all possible Gibbs states, including the non-extremal ones. A variety of situations might occur for the large N-behaviour of the metastate κ_N: it can be the Dirac measure on a mixture of states, it can be a mixture of Dirac measures on pure states or it can be a mixture of Dirac measures on mixtures. We mentioned above that the metastate is a probability measure on the infinite-volume Gibbs measures which gives the weights in the large-volume asymptotic to find the system close to one of the possible candidates among the Gibbs measure. In Chapter 3, we study very simplified disordered models, namely disordered mean-field models with finitely many values for both spin and disorder variables. We derive an explicit construction for the metastate, where the weights of the possible states are obtained by studying the distribution of the free energy fluctuations w.r.t. the disorder variables. We provide also a geometric picture to identify the *invisible states*, that is those Gibbs states whose corresponding weight is zero.

1.4 Potts Model and Random-cluster Representation

As already pointed out, statistical mechanics aims for a description of the macroscopic behaviour of a system in thermal equilibrium starting from the microscopic interaction which takes place between the very many components of the system. Depending on the type of interaction and on the temperature, realizations of the system's components display different geometric patterns. In the ferromagnetic Ising model, for example, due to the nearest-neighbour interaction, spins tend to align themselves in the same direction. However, at high temperature the thermal excitation dominates this tendency to a spin alignment, making the components to behave almost independently. On the

other hand, the low-temperature behaviour can be described as a random perturbation of a ground state configuration (either all plus or all minus spins). At low temperature, we expect a typical realization to be the ground state configuration with finite clusters where the spins disagree with the ground state. In 1936, Peierls had the idea to describe the phase transition in this model by looking at geometric objects named *contours*, which correspond to boundaries separating regions with plus spins from region with minus spins. The contour description confirmed the above heuristics. At low temperature large contours are improbable, due to the high energy cost, so a typical phase consists of an infinite connected "sea" in which the spins take a particular ground state value (plus or minus), with finite "islands" on which the spins take mostly different values. More specifically, the plus phase (where most of the spins take the value +1) is realized by an infinite "sea" of plus spin with finite "islands" of minus spins, which, in turn, may contain lakes of plus and so forth. If the temperature parameter is increased above the Curie temperature, we cannot expect the existence of an infinite path connecting nearest neighbours with the same spin value; the components will be in fact almost independent. In this respect, percolation theory plays an important role in equilibrium statistical mechanics and the study of phase transition.

A particular model of dependent percolation, invented by Fortuin and Kasteleyn in the late 1960's, which has been shown to be very successful in analysing the phase transition behaviour of Ising and Potts model is called the *Random-cluster model*. Its strength lies in the discovery that spontaneous magnetization in Ising and Potts models corresponds to percolation in their random-cluster representations. For a systematic account of the theory we refer to [30, 34, 37]. In [55], the authors showed that the Potts model undergoes a first-order phase transition in temperature, in any dimension $d \geq 2$, if q (number of allowed spin configurations) is large enough. They used the *Pirogov-Sinai theory* applied to a random-cluster representation of the model[2]. Pirogov-Sinai theory is a powerful extension of the above-mentioned Peierls contour argument. One might wonder why such an extension of the Peierls contour argument was needed? To answer this question, we have to go back to the original idea behind the Peierls argument. The idea was, that at low temperature, a typical realization of the system's components is nothing but a perturbation of the ground state configuration. In the Ising model, Peierls contour argument proved such a behaviour by proving that, at low temperature, the probability of a large contour is very small. However, this example spoilt our understanding. In fact, due

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[2] The original proof of the first-order phase transition (in temperature) for the high-q Potts model in any dimension $d \geq 2$ is due to Kotecký and Shlosman, see [46], and uses reflection positivity.

to the symmetry in the Ising interaction, computing the probability of appearance of a contour was rather easy. In more general situations, where there is no symmetry between the different phases, computing the probability of appearance of a contour is much more complicated, due to the lack of independence amongst nested contours. Pirogov-Sinai theory provides a way to estimate the probability of appearance of a contour.

In Chapter 5, we study an analogous problem: we show the existence of a first-order phase transition for a variant of the Potts model, termed *Potts model with invisible colours*. This model, which was introduced recently by Tanaka, Tamura and Kawashima [70, 72], consists in a ferromagnetic Potts interaction acting among q "visible" colours along with the presence of r "invisible" colours, which do not interact with their neighbours. The proof of the existence of a first-order phase transition for this model is based on an application of the Pirogov-Sinai theory to a random-cluster representation of the model. The random-cluster representation of the Potts model with r "invisible" colours, differs from the standard random-cluster model, and we named it the *r-biased random-cluster model*. The adjective biased refers to the fact that isolated vertices and non-singleton connected components are weighted differently. Whilst the latter have weights q, exactly as in the standard random-cluster model, the former have weights $q + r$; see Chapter 5 for an in-depth examination.

The Potts model with invisible colours provides an example that symmetry breaking and order of phase transition cannot be inferred one from another. In fact, although the number of ground states and low-temperature states equals q, and there is at low temperature spontaneous symmetry breaking of the q-fold permutation symmetry, just as in the standard q-colour Potts model, the transition for low $q = 2, 3, 4$ and high r is different from the second-order transition of the ordinary two-dimensional q-colour Potts model. In fact a first-order transition in the temperature-parameter appears.

1.5 Lattice Models, Tree Models and Mean-field Models: Heuristics and Differences[3]

In this thesis we shall encounter three basic and widely known models in statistical mechanics: lattice models, tree models and mean-field models. To be more precise, we will simply deal with two examples out of the first two categories: $\mathbb{Z}^2$ ($\mathbb{Z}^d$) as far as lattice models are concerned and $\mathfrak{CT}(d)$ (Cayley tree) for the tree-models' class. To convey a complete survey about these classes of models

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[3]Although a tree is but a special case of a lattice (if the latter is thought of as a graph), with perhaps "too much" freedom we shall distinguish between them.

would definitely require an effort comparable to writing up another thesis from scratch and the author is not willing and yet not prepared for such a bold and demanding task. Instead, we opt for a very concise introduction to the three above-mentioned classes, intending to provide the rationale behind, as well as some basic differences between them.

Lattice models are characterized by having a quite rich geometric structure. We shall think of a lattice as a connected graph, with a "non-simple" connectivity scheme, i.e., where loops are present. A further requirement for a lattice consists in being an *amenable* graph, which means that the size of the volume asymptotically dominates the size of the boundary. Lattice models represent the foundation-stone of the DLR formalism, which defines a Gibbs measure to be the distribution of a countably infinite family of random variables which admits some prescribed conditional probabilities. A remarkable example of the success of a lattice model is, beyond doubt, the ferromagnetic Ising model on $\mathbb{Z}^2$, for which a phase transition was shown.

Mean-field models are based on the quite radical assumption that all spins interact in the same way, regardless of their actual position, which corresponds to the abandonment of any underlying geometric structure. The Hamiltonian describing the interaction between Ising or Potts spins can thus be written as a function of a macroscopic quantity which is termed *empirical spin distribution*. This enables us to compute the free energy quite easily. The existence of several minima of the free energy, is usually a good alarm bell that a phase transition might display itself for models with a more reasonable geometric structure, either in high dimension or when the interaction is long-range. A main difference between lattice and mean-field models is that the notion of boundary conditions, which is crucial for the DLR formalism in lattice models, loses its meaning in mean-field models. In fact, in the latter models, the notion of boundary itself becomes inappropriate. The infinite-volume mean-field measure consists of the product measure of single-site marginals, which are described by a "effective" (mean) field. In other words, each site feels a field which embodies the information of the whole system. Such an "effective" field is given by the solution of a fixed-point equation. For example, in the mean-field Ising models (a.k.a. Curie-Weiss), the "effective" field corresponds to the solution of the famous mean-field equation $m = \tanh(\beta m)$.

Tree models lie somehow in the middle between lattice models and mean-field models. A tree is nothing but a connected graph with a stern connectivity requirement: between any two vertices there is a unique path joining them. Similarly to what happens in lattice models, a DLR formalism can be used to define an infinite-volume Gibbs measure for models on trees. In fact, tree-structures

admit a notion of boundary. Furthermore, due to the peculiar geometry (more precisely, the absence of loops), they also admit fixed-point equations whose solution(s) plays the role of an effective field acting on a single site, similarly to mean-field models. However, there are some differences between lattice models and tree models. First and foremost, while in the latter the boundary size of a volume is asymptotically of the same order of magnitude of the inside volume (*non-amenability*), in the former the size of the inside volume is "much bigger" than the boundary size. Second, the tree structure is sufficiently simple to admit the notion of a Markov chain (in fact, it is impossible to define a Markov chain on graphs containing loops, see [29], Chapter 12). Another peculiar behaviour for tree models is the existence of multiple transition points, whereas the corresponding lattice model has a single transition point. A typical example is the ferromagnetic Ising model. Whilst on a lattice this displays a single transition point, on a tree a second transition temperature, strictly below the ferromagnetic transition temperature, appears. In the interval of temperatures hemmed-in by these two transition temperatures, the free boundary condition Gibbs measure is extreme on a trees whereas it is not on a lattice, see [42].

1.6 Overview of the Thesis

After this very brief introduction, which aimed to give a flavour of the field of statistical mechanics and, at the very same time, sketched the boundaries inside which we shall be constrained henceforth, we are ready to outline the core of this thesis.

In Chapter 2, we properly define many of the concepts we have encountered so far. In the first part of this chapter, the notions of specification, Gibbs measure, quasilocality and other related concepts typical of the Gibbsian formalism will be analysed. The second part will be devoted to the notion of metastate.

In Chapter 3, we consider a general class of disordered mean-field models where both the spin variables and disorder variables η take finitely many values. To investigate the size-dependence in the phase-transition regime (when several Gibbs states are available) we construct the metastate describing the probabilities to find a large system close to a particular convex combination of the pure infinite-volume states. We show that, under a non-degeneracy assumption, only pure states j are seen, with non-random probability weights w_j for which we derive explicit expressions in terms of interactions and distributions of the disorder variables. As a consequence we show that, in the case where precisely two pure states are available, these must necessarily occur with the same weight, even if the model has no obvious symmetry relating the two;

a phenomenon that we called *random restoration of symmetry*.

In Chapter 4, in the same spirit as [19], we study homogeneous Gibbs measures on a Cayley tree, subjected to an infinite-temperature Glauber evolution, and consider their (non)-Gibbsian properties. We show that the intermediate Gibbs state (which in zero field is the free-boundary-condition Gibbs state) behaves different from the plus and the minus state. For example, at large times, all configurations are bad for the intermediate state, whereas the plus configuration is never bad for the plus state. Moreover, we show that for each state there are two transitions. For the intermediate state there is a transition from a Gibbsian regime to a non-Gibbsian regime where some, but not all configurations are bad, and a second one to a regime where all configurations are bad. For the plus and minus state, the two transitions are from a Gibbsian regime to a non-Gibbsian one and then back to a Gibbsian regime again. We present our results both in zero and non-zero external fields.

In Chapter 5, we study a variant of the ferromagnetic Potts model, introduced by Tamura, Tanaka and Kawashima [70, 72], in which along with the q ordinary "visible" colours (the ordinary Potts colours), r "invisible" colours are possible. A ferromagnetic nearest-neighbour interaction acts exclusively among the visible colours. The invisible colours, on the other hand, are neutral, and have no interaction with their neighbours, regardless of the state of the neighbours. We introduce a random-cluster representation for the model, for which we prove the existence of a first-order transition for any $q > 0$, as long as r is large enough. When $q > 1$, the low-temperature regime displays a q-fold symmetry breaking. The proof involves a Pirogov-Sinai analysis applied to this random-cluster representation of the model.

2 General Formalism and Probability Structures

Our purpose in this chapter is to render many of the concepts we have encountered so far more rigorous and hopefully to make the reader more familiar with the Gibbsian formalism. In the first part we collect notations and definitions; we refer those who wish to delve further into the probabilistic formalism of statistical mechanics to [18, 29]. In the second part, the chaotic size-dependence phenomenon will be briefly discussed and the Aizenman-Wehr metastate will be defined, see [1, 5, 50] for an in-depth description.

2.1 Topological Preliminaries

This section is geared towards providing a bedrock for many probabilistic concepts as well as definitions which we shall use extensively in this thesis. We present them in considerable generality, without any particular assumption on the model used nor on the geometric structure laying behind the scenes. Sporadically, we shall refer to particular models though.

Let Ω_0 be the *single-spin space*. This is the space of possible realizations of a random variable aimed to model the microscopic realizations of a single particle. We shall assume Ω_0 to be equipped with a σ-algebra which we denote $\mathcal{F}_0$ of "measurable sets", to obtain a measure space $(\Omega_0, \mathcal{F}_0)$. In general Ω_0 is taken to be a complete separable metric space equipped with the Borel σ-algebra generated by the metric. In this thesis, we shall be concerned only with discrete single-spin spaces: the Ising single-spin space that is given by $\Omega_0 = \{-1, +1\}$ and the Potts single-spin space given by $\Omega_0 = \{1, \ldots, q\}$, with $q \in \mathbb{Z}^+$. To complete a probabilistic description of a single-spin space we need to introduce a probability measure α_0 (a priori distribution), being a function from $\mathcal{F}_0$ to $[0, 1]$. Then, by $(\Omega_0, \mathcal{F}_0, \alpha_0)$ we denote the single-spin probability space. Let V be a countably infinite vertex set of a graph and let $\mathcal{S}$ be the set of all finite subsets of V. We do not specify the edge set of the graph, so as to keep the framework as general as possible. In other words, for the moment,

we do not give to V any particular geometric structure. We denote by Ω the infinite-volume configuration space, which consists of the set of possible configurations of the whole system. We define Ω to be the Cartesian product $(\Omega_0)^V := \{\sigma = (\sigma_i)_{i \in V} | \sigma_i \in \Omega_0, \text{ for all } i \in V\}$, and we equip this with the product σ-algebra $\mathcal{F} = (\mathcal{F}_0)^V$ and with the product topology. The product topology basically means that a sequence of configurations σ^n converges to σ as n tends to infinity if, and only if, $\sigma_i^n \to \sigma_i$ for all $i \in V$. This allows us to introduce the notion of a *neighbourhood* of a configuration σ (which is a *cylinder set*), which in mathematical terms reads:

$$\mathcal{N}_{\Lambda,\varepsilon}(\sigma) := \{\tilde{\sigma} : \text{ dist}(\sigma_i, \tilde{\sigma}_i) < \varepsilon \text{ for all } i \in \Lambda\} ,\qquad (2.1)$$

where $\Lambda \in \mathcal{S}$ and $\varepsilon > 0$. Then, a typical neighbourhood in the product topology is the set of all configurations that are "close" to σ on some finite set Λ, and are arbitrary outside Λ. In case of a discrete single-spin space, due to the trivial metric used, a typical neighbourhood will be the set of all configurations that are equal to σ on some finite set Λ, and are arbitrary outside Λ. The notion of a neighbourhood will turn to be very important for the non-Gibbsianness analysis we shall discuss later in Chapter 4. For $\Lambda \in \mathcal{S}$ and $\sigma \in \Omega$ we denote by σ_Λ the restriction (projection) of σ to Λ, while Ω_Λ denotes the set of all such restrictions. Furthermore $\mathcal{F}_\Lambda \equiv \mathcal{F}_0^\Lambda$ denotes the σ-algebra of *local* events. We identify $\mathcal{F}_\Lambda \subset \mathcal{F}$ with the sub-σ-algebra of events depending only on the spins σ_Λ. Similarly, setting $\Lambda^{\complement} := V \setminus \Lambda$, we denote by $\mathcal{F}_{\Lambda^{\complement}}$ the σ-algebra of events which do not depend on the spins in the finite region Λ.

A real-valued function f defined on $(\Omega, \mathcal{F})$ is said to be *measurable* (with respect to $\mathcal{F}$) if, for any Borel set $A \subset \mathcal{B}(\mathbb{R})$ the preimage of A under f is contained in $\mathcal{F}$. In statistical mechanics one usually refers to such functions as *observables*. A real-valued function on Ω is said to be *local* if it is measurable w.r.t. $\mathcal{F}_\Lambda$ for some $\Lambda \in \mathcal{S}$, i.e. if it depends only on the value of the spins in some finite set Λ. A *quasilocal* function is defined to be the uniform limit of some sequence of local functions. Equivalently, a function f is quasilocal if

$$\lim_{\Lambda \uparrow V} \sup_{\substack{\sigma, \tilde{\sigma} \in \Omega \\ \sigma_\Lambda = \tilde{\sigma}_\Lambda}} |f(\sigma) - f(\tilde{\sigma})| = 0 .\qquad (2.2)$$

If the single-spin space Ω_0 is finite then quasilocality and continuity are equivalent.

Let $\mathcal{P}(\Omega)$ be the set of all probability measures on Ω. In order to properly account for what happens to measures along the thermodynamic limit, we need to define what it means for a sequence of measures μ_n to converge to a limiting measure μ. What is needed is to equip the space $\mathcal{P}(\Omega)$ with a topology. The

topology we will use is the *bounded quasilocal topology*. This means that we shall say that $\mu_n \to \mu$ if $\mu_n(f) \to \mu(f)$ for all bounded and quasilocal functions f. Here $\mu_n(f)$ denotes the expectation of f with respect to μ_n. In case Ω_0 is finite, then the bounded quasilocal topology coincides with the weak topology (on continuous functions).

2.2 From Interactions to Hamiltonians: Specifications and Gibbs Measures

We now introduce one of the primal concepts of statistical mechanics, which we have not stressed yet, and which is referred to as *interaction*. Its importance lies in the possibility to classify the Hamiltonian functions into classes: one-body terms, two-body terms and so forth.

Definition 2.2.1. *An* interaction *(a.k.a.* potential*) is a family* $\Phi = (\Phi_A)_{A \in \mathcal{S}}$ *of functions* $\Phi_A : \Omega \to \mathbb{R}$ *such that for each* $A \in \mathcal{S}$*, the function* Φ_A *is* $\mathcal{F}_A$*-measurable (i.e. a family of local functions)*

An interaction is called *regular* (or *absolutely summable*), if for all $i \in V$, there exists a constant c such that

$$\sum_{A \ni i} \|\Phi_A\|_\infty \leq c < \infty \ . \tag{2.3}$$

By means of an interaction Φ we define the corresponding *finite-volume Hamiltonian* H_Λ^Φ, for any finite volume Λ, as follows:

$$H_\Lambda^\Phi(\sigma) = \sum_{\substack{A \in \mathcal{S} \\ A \cap \Lambda \neq \emptyset}} \Phi_A(\sigma) \ . \tag{2.4}$$

The condition (2.3) assures that the latter sum converges to a finite limit for all $\sigma \in \Omega$. The absolutely summability might be characterized by saying that the interaction between one spin and the rest of the universe is finite. Note that this is a stronger requirement than the "decay of interactions".

In the introduction we mentioned the important role of boundary conditions. The idea was to define, for any finite volume $\Lambda \in \mathcal{S}$ and any fixed configuration $\omega_{\Lambda^\complement}$ outside Λ, a probability measure on Λ itself.

The mathematical object which can very well express such an idea is called *probability kernel* [1]. We can now introduce the concept of a specification which is meant to formalize the idea of conditioning on the exterior of any finite volume Λ.

Definition 2.2.2. *A* specification *is a family* $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathcal{S}}$ *of probability kernels from* $(\Omega, \mathcal{F})$ *to itself, satisfying the following conditions:*

  i) For all $\mathcal{A} \in \mathcal{F}$ *and for all* $\Lambda \in \mathcal{S}$, $\pi_\Lambda(\cdot, \mathcal{A})$ *is a* $\mathcal{F}_{\Lambda^c}$*-measurable function.*

  ii) For any Λ, π_Λ *is* $\mathcal{F}_{\Lambda^c}$*-proper, i.e. for any* $\mathcal{B} \in \mathcal{F}_{\Lambda^c}$, $\pi_\Lambda(\sigma, \mathcal{B}) = \chi_\mathcal{B}(\sigma)$.

  iii) For any two volumes, Λ *and* Λ', *with* $\Lambda \subset \Lambda'$, $\pi_{\Lambda'}\pi_\Lambda = \pi_{\Lambda'}$ [2].

The third condition is a compatibility condition for pairs of volumes $\Lambda \subset \Lambda'$: (it states that if a volume Λ' is in equilibrium with its exterior, then all subvolumes are in equilibrium with their exteriors). (A similar condition holds for regular conditional probabilities, although in that case it is not required to be satisfied for all conditioning but only almost surely w.r.t. the measure[3]).

Hereafter, we stress two main properties of a specification, which will turn out to play a fundamental role for the characterization of a particular type of specifications, namely the *Gibbsian specifications* corresponding to a regular interaction Φ.

Definition 2.2.3. *A specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathcal{S}}$ *is said to be* quasilocal *if for each* $\Lambda \in \mathcal{S}$ *and for any quasilocal function* f *on* Ω,
$(\pi_\Lambda f)(\omega) \equiv \int \pi_\Lambda(\omega, d\omega')f(\omega')$ *is quasilocal as a function of* ω.

Definition 2.2.4. *A specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathcal{S}}$ *is said to be* uniformly nonnull *(with respect to* α*) if for each* $\Lambda \in \mathcal{S}$ *there exist constants* $0 < u_\Lambda \leq v_\Lambda < \infty$ *such that*

$$u_\Lambda \alpha(\mathcal{A}) \leq \pi_\Lambda(\omega, \mathcal{A}) \leq v_\Lambda \alpha(\mathcal{A}) , \tag{2.5}$$

for all $\omega \in \Omega$ *and* $\mathcal{A} \in \mathcal{F}_\Lambda$ *with* $\alpha = \prod_{i \in V} \alpha_i$ *being an a priori (product) measure.*

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[1] A probability kernel π_Λ from $(\Omega_{\Lambda^c}, \mathcal{F}_{\Lambda^c})$ to $(\Omega_\Lambda, \mathcal{F}_\Lambda)$ is a map $\pi_\Lambda : \Omega_{\Lambda^c} \times \mathcal{F}_\Lambda \to [0,1]$ satisfying:

  a) For each fixed $\omega_{\Lambda^c} \in \Omega_{\Lambda^c}$, $\pi_\Lambda(\omega_{\Lambda^c}, \cdot)$ is a probability measure on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$.

  b) For each fixed $\mathcal{A} \in \mathcal{F}_\Lambda$, $\pi_\Lambda(\cdot, \mathcal{A})$ is an $\mathcal{F}_{\Lambda^c}$-measurable function on Ω_{Λ^c}.

[2] $\pi_{\Lambda'}\pi_\Lambda(\omega, \mathcal{A}) \equiv \int \pi_{\Lambda'}(\omega, d\omega')\pi_\Lambda(\omega', \mathcal{A})$.

[3] For regular conditional probabilities, the "infinite-volume" measure is known in advance, so that the notion of almost surely makes sense. However, for specifications the consistent measure is not given and it has to be sought, impoverishing the meaning of almost surely, see ([69]).

Let Φ be an absolutely summable interaction, and let H_Λ^Φ be the corresponding Hamiltonian. We pointed out already several times, that it is convenient to think of the configuration outside Λ as fixed. Therefore, for any fixed boundary condition $\omega \in \Omega$ and for every $\Lambda \in \mathcal{S}$, we define the Hamiltonian $H_{\Lambda,\omega}^\Phi$ as follows:

$$H_{\Lambda,\omega}^\Phi(\sigma) = H_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda^\complement}) \;, \tag{2.6}$$

where we recall that the notation $\sigma_\Lambda \omega_{\Lambda^\complement}$ stands for the configuration that agrees with σ in Λ and with ω outside Λ. Let $\alpha = \prod_{i \in V} \alpha_i$ be an a priori probability measure, then we define the conditional partition function

$$Z_{\beta,\Lambda}^\Phi(\omega) = \int e^{-\beta H_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda^\complement})} \prod_{i \in \Lambda} \alpha_i(d\sigma_i) \;, \tag{2.7}$$

where β is the inverse temperature. Note that Φ being absolutely summable implies $Z_\Lambda^\Phi(\omega) < \infty$ for all $\Lambda \in \mathcal{S}$ and for all $\omega \in \Omega$ (H_Λ^Φ is everywhere finite and bounded below). Then the probability measure

$$\mu_{\beta,\Lambda}^\omega(d\sigma_\Lambda) := \frac{e^{-\beta H_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda^\complement})}}{Z_{\beta,\Lambda}^\Phi(\omega)} \prod_{i \in \Lambda} \alpha_i(d\sigma_i) \tag{2.8}$$

is called the *Gibbs distribution* in the volume Λ with boundary conditions $\omega_{\Lambda^\complement}$. The family $M^\Phi = \{\mu_{\beta,\Lambda}^{(\cdot)}\}_{\Lambda \in \mathcal{S}}$ is a specification and it is called the *Gibbsian specification* for Φ. The Gibbsian specifications for absolutely summable interactions are completely characterized by two properties: a) quasilocality, and b) uniformly nonnullness, see Theorem 2.12 in [18] (Kozlov's theorem).

We are now ready to give the definition of an infinite-volume Gibbs measure.

Definition 2.2.5. *Let $M^\Phi = \{\mu_{\beta,\Lambda}^{(\cdot)}\}_{\Lambda \in \mathcal{S}}$ be a Gibbs specification (for the regular interaction Φ and a priori measure α). A measure μ_β on Ω is a* Gibbs measure *for M^Φ if and only if, for all $\Lambda \in \mathcal{S}$,*

$$\mu_\beta \mu_{\beta,\Lambda}^{(\cdot)} = \mu_\beta \;. \tag{2.9}$$

The latter equations are called the Dobrushin-Lanford-Ruelle (DLR) equations. Equivalently, we say that a measure μ_β on Ω is a *Gibbs measure* for M^Φ if and only if is *compatible* with M^Φ, that is, for all $\Lambda \in \mathcal{S}$ and all f bounded measurable functions

$$\mu_\beta(f|\mathcal{F}_{\Lambda^\complement})(\cdot) = \mu_{\beta,\Lambda}^{(\cdot)}(f), \quad \mu_\beta\text{-a.s.} \tag{2.10}$$

We denote by $\mathcal{G}(M^{\Phi})$ the set of Gibbs measures for the specification M^{Φ}. If $|\mathcal{G}(M^{\Phi})| > 1$, where $|\cdot|$ denotes the cardinality of the set, then the system, which is locally described by M^{Φ}, is said to have a *phase transition*, while it is said to be in the uniqueness regime whenever $|\mathcal{G}(M^{\Phi})| = 1$. The set $\mathcal{G}(\cdot)$ embodies the possibility that physical system exhibit one or more "phases", depending on the values of some parameters which we have control over (e.g. temperature, external field). Implicitly, we are assuming that associated to any system there exists a specification $M^{(\cdot)}$ whose corresponding space $\mathcal{G}(M^{(\cdot)})$ describes all the possible "phases" of the systems which are possible for a given choice of the parameters.

It is worthwhile to mention that if V were to be finite, then there would be a unique measure consistent with any specification Π, that is $\pi_V(\omega, \cdot)$, which must be independent of ω. This underlines the impossibility to describe mathematically (or to better say, statistical-mechanically) phase transitions in finite systems, (see [18]).

2.3 Non-Gibbsianness: Absence of Quasilocality

Kozlov's theorem (Theorem 2.12 in [18]) implies two main causes of non-Gibbsianness: lack of non-nullness and lack of quasilocality. As the title of the current section suggests, we will be focusing on the latter cause.

The main concept is the following: a measure μ is not quasilocal, if it is not consistent (in the sense of (2.10)) with any quasilocal specification. We saw, in the previous section, what it takes for a specification $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathcal{S}}$ to be quasilocal, namely that for any quasilocal function f and for any $\Lambda \in \mathcal{S}$, $\pi_\Lambda f$ is quasilocal. We emphasize that $(\pi_\Lambda f)(\omega)$ represents the mean value of f with respect to the measure $\pi_\Lambda(\omega, \cdot)$, which is a probability distribution on Λ obtained by fixing the configuration outside Λ to be $\omega_{\Lambda^{\mathsf{c}}}$. The quasilocality condition (2.2) for $\pi_\Lambda f$ reads

$$\lim_{\Lambda' \uparrow V} \sup_{\substack{\omega^1, \omega^2 \in \Omega \\ \omega^1_{\Lambda'} = \omega^2_{\Lambda'}}} \left| (\pi_\Lambda f)(\omega^1) - (\pi_\Lambda f)(\omega^2) \right| = 0 \ . \tag{2.11}$$

The latter says that the mean values of quasilocal functions depend "weakly" on the spins far from Λ, when the spins in the intermediate region $\Lambda' \setminus \Lambda$ are fixed. Henceforth, we shall be referring to the intermediate region above as "annulus".

Let us denote by

$$\mu_\Lambda(f|\omega) = \mathbb{E}_\mu(f|\mathcal{F}_{\Lambda^{\mathsf{c}}})(\omega) \ , \tag{2.12}$$

a realization for the conditional expectation of μ, for bounded f, $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$. The reader must be aware of the fact that conditional expectations admit an infinite number of versions, all differing on measure-zero sets. So the function in (2.12) has to be thought of as a family of $L^1(\mu)$ objects. To infer that the measure μ is not quasilocal, it is enough to find a single, non-removable, point of discontinuity for a single μ_Λ and for a single quasilocal f. From a mathematical point of view, a dichotomy for non-removable discontinuities is still possible, and we are driven to consider the two definitions below.

Definition 2.3.1. $\mu_\Lambda(f|\cdot)$ *is μ-essentially discontinuous at $\bar{\omega}$, iff there exists an $\varepsilon > 0$ such that for every $\Lambda' \subset \mathcal{S}$ there exists $\Lambda'' \supset \Lambda'$ and configurations ξ^1, ξ^2, such that*

$$\left| \mu_\Lambda(f|\bar{\omega}_{\Lambda'\setminus\Lambda}\xi^1_{\Lambda''\setminus\Lambda'}\omega) - \mu_\Lambda(f|\bar{\omega}_{\Lambda'\setminus\Lambda}\xi^2_{\Lambda''\setminus\Lambda'}\omega) \right| > \varepsilon \ ,$$

for all $\omega \in \mathcal{A}$, where $\mathcal{A} \in \mathcal{F}_{(\Lambda'')^c}$ is of positive μ-measure.

Definition 2.3.2. $\mu_\Lambda(f|\cdot)$ *is strongly discontinuous at $\bar{\omega}$, iff there exists an $\varepsilon > 0$ such that for every $\Lambda' \subset \mathcal{S}$ there exists $\Lambda'' \supset \Lambda'$ and configurations ξ^1, ξ^2, such that*

$$\left| \mu_\Lambda(f|\bar{\omega}_{\Lambda'\setminus\Lambda}\xi^1_{\Lambda''\setminus\Lambda'}\omega) - \mu_\Lambda(f|\bar{\omega}_{\Lambda'\setminus\Lambda}\xi^2_{\Lambda''\setminus\Lambda'}\omega) \right| > \varepsilon \ ,$$

for all $\omega \in \mathcal{A}$, where $\mathcal{A} \in \mathcal{F}_{(\Lambda'')^c}$ is open.

The point $\bar{\omega}$ is also called a *bad configuration*. Above we saw that the "badness" of such a point might in principle be of two types: μ-essential or strong (see Fernández [24]). The reason why we are bothered with such a detailed matter will be discussed in Chapter 4. There we will study the quasilocality property of homogeneous low-temperature Ising Gibbs measures on Cayley trees, subjected to an infinite-temperature Glauber evolution. We will show that at finite large times there can be transitions between regimes where the evolved measure is quasilocal and regimes where the evolved measure is not quasilocal. This will be done by finding a non-removable point of discontinuity for the evolved measure. However, in contrast to what happens on regular lattices such as $\mathbb{Z}^d$, the discontinuities we detect for the evolved measure on trees are of a μ-essential type. This implies that such a discontinuity might, in principle, be non-removable for a measure and removable for another measure (due to the fact that the measures are not absolutely continuous with respect to each other).

2.4 Disorder and Metastates

Let us consider a disordered spin system with quenched randomness η. We denote the probability space of the random variables η by $(E', \mathcal{B}, \mathbb{P})$, and E' will always be assumed to be a Polish space. By $\mathbb{E}$ we shall denote the expectation w.r.t. $\mathbb{P}$. A disordered spin model is defined by prescribing a Hamiltonian $H[\eta](\sigma)$, for each realization of η. Similarly to what was done in the absence of disorder, fixing a boundary condition ω, we can define the finite-volume Gibbs distribution $\mu_\Lambda^\omega[\eta]$ in the finite volume Λ. The family of these distributions, for all volumes Λ and all boundary conditions ω, is called *random Gibbs specification.*

The problem we face now is as follows: if there exist many pure states, given a sequence of volumes Λ_n, $\mu_{\Lambda_n}^\omega[\eta]$ will not have, in general, a single limit as $n \uparrow \infty$, but rather many different limits along different subsequences of n (by a compactness argument). We remark that such a situation might already occur in non-disordered systems, for particular choices of the boundary conditions (van Enter et. al. [20] refer to those as "incoherent boundary conditions"). Think, for example, of the Ising model on $\mathbb{Z}^d$ and take the boundary conditions $+$ on even-size cubes Λ_{2n} and the boundary conditions $-$ on the odd-size cubes Λ_{2n+1}. Although the sequence μ_{Λ_n} oscillates between μ^+ and μ^-, the subsequence $\mu_{\Lambda_{2n}}$ converges to the Gibbs measure μ^+ and the subsequence $\mu_{\Lambda_{2n+1}}$ converges to μ^-. Such an unclear — from a physical point of view — phenomenon of size-dependence can be circumvented by choosing boundary conditions whose effect on the systems does not depend on the (absorbing) sequence of volumes, for example the all $+$ and all $-$ boundary conditions. The phenomenon of size-dependence, however, is hard to avoid in disordered systems. In fact, even if the disorder is fixed (quenched), the effect of the boundary conditions changes randomly when we look at increasing sequences of volumes, due to the presence of the disorder. From what we have discussed in the introduction, as well as in the few lines above, the reader should retain that for a given realization η of the disorder variables, in order to obtain convergence of $\mu_{\Lambda_n}^\omega[\eta]$ to a particular limit point as $n \uparrow \infty$, we have to take a subsequence of volumes $\Lambda_{n[\eta]}$ which, in general, will depend on the realization of the disorder (in case it is possible to find a deterministic sequence along which the local specification converges, then the chaotic size dependence will not display itself!). We are then led to study the situation where the boundary conditions ω for the measure $\mu_{\Lambda_n}^\omega[\eta]$ do not preselect a particular infinite-volume Gibbs measure. In case when several Gibbs measures are available, similarly to the non-disordered example given above with the states being μ^+ and μ^-, it might be that the system can be approximated by one of these states (measures) for

one volume and by another state for another volume, which one depending on the choice of the disorder. The problem of size-dependence is to characterize the behaviour of $\mu_{\Lambda_n}^\omega[\eta]$ along the sequence $(\Lambda_n)_n$. To study disordered systems, a new approach to the thermodynamic limit has been developed, which takes into account the phenomenon of chaotic size dependence. This is done by means of the metastate $\kappa[\eta]$, being a probability measure on the possible limiting states for the given η. The main point behind the introduction of metastate, is the abandonment of the idea that a disordered system, in the thermodynamic limit, must be described by a single state, namely by a function from disorder configurations to single infinite-volume Gibbs state. While we have come to understand that for a given realization of the disorder variables there might be several infinite-volume states, the metastate aims to determine the probabilities (or frequencies) with which any of the possible limits states will appear along the infinite-volume limit.

In pragmatic terms, we imagine that, for large n, the system will be "close" to a mixture of random infinite-volume extremal Gibbs states $\mu_\infty^m[\eta]$:

$$\mu_{\Lambda_n}[\eta] \approx \sum_m P_n^m[\eta]\mu_\infty^m[\eta] \ . \tag{2.13}$$

In the decomposition (2.13) the important role is played by the coefficients P_n^m. They contain the information about the degeneracy between the phases in the finite-volume; a degeneracy which is random due to the chaotic size dependence. Relying on the former decomposition, the metastate will give us the relative weights of any of these η-dependent degeneracies.

As we already mentioned in the introduction, there are two ways to construct the metastate: one is based on the randomness of η, and the other is based on the chaotic size dependence for a fixed η. The approach based on the randomness of η is due to Aizenman and Wehr. They suggested to look at the probability distribution K_Λ^ω on the space $\mathcal{P}(\Omega) \times E'$, of the pairs of finite-volume Gibbs measures and the disorder variables $(\mu_{\Lambda_n}^\omega[\eta], \eta)$, defined in such a way that the marginal distribution of η is $\mathbb{P}$, while the conditional distribution given η is $\delta_{\mu_{\Lambda_n}^\omega[\eta]}$, the Dirac-measure concentrated on the corresponding local specification. It is remarkable that $\delta_{\mu_{\Lambda_n}^\omega[\eta]}$ is a random object living on $\mathcal{P}(\mathcal{P}(\Omega))$. It can be seen as a kernel from E' to $\mathcal{P}(\Omega)$. For the measure $K_{\Lambda_n}^\omega$ we borrow the notation $K_{\Lambda_n}^\omega \equiv \mathbb{P} \times \delta_{\mu_{\Lambda_n}^\omega[\eta]}$ from Bovier [5].

Then we have the following definition:

Definition 2.4.1. *Assume that, for every bounded continuous functions Ξ : $\mathcal{P}(\Omega) \times E' \to \mathbb{R}$ the limit*

$$\lim_{n\uparrow\infty} \int \Xi(\mu^\omega_{\Lambda_n}[\eta], \eta)\mathbb{P}(d\eta) = \int \Xi(\mu, \eta)K^\omega(d\mu, d\eta) , \qquad (2.14)$$

exists. Then the conditional distribution $\kappa^\omega[\eta](d\mu) := K^\omega(d\mu|\eta)$ is called the AW-metastate.

We remark that, if the limit exists, it defines a probability measure $K^\omega \in \mathcal{P}(\mathcal{P}(\Omega) \times E')$ whose regular conditional probability given η, is the AW-metastate $\kappa^\omega[\eta](d\mu)$. So the metastate is the measurable map $\kappa^\omega : E' \to \mathcal{P}(\mathcal{P}(\Omega))$ such that $\int \Xi(\mu, \eta)K^\omega(d\mu, d\eta) = \mathbb{E}[\Xi(\mu, \eta)\kappa^\omega[\eta](d\mu)]$. Although convergence has not been proved for any sequence, Aizenman and Wehr [1] showed that every subsequential limit K^ω has a conditional distribution $\kappa^\omega[\eta]$, given η, that for $\mathbb{P}$-a.s. η is supported on the infinite-volume Gibbs measure for that η.

The second approach is due to Newman and Stein [62]. They considered η to be fixed, and they suggested to capture the asymptotic volume dependence by looking at the empirical distribution of $\mu^\omega_{\Lambda_n}[\eta]$ as n varies, namely

$$\kappa^\omega_N[\eta] := \frac{1}{N} \sum_{n=1}^{N} \delta_{\mu^\omega_{\Lambda_n}[\eta]} , \qquad (2.15)$$

where $\mu^\omega_{\Lambda_n}[\eta]$ denotes the finite-volume Gibbs distribution in the volume Λ_n, with boundary condition ω. In general (2.15) does not converge for almost every fixed η, unless one takes sparse enough subsequences. However, a weaker type of convergence, e.g. convergence in law, might still be possible. An example of a such behaviour, to be found in [50], was observed for the Curie-Weiss random-field Ising model. Its finite-volume Gibbs distributions $\mu_n[\eta]$ in the finite volume $\{1, \cdots, n\}$, at inverse temperature β are given by

$$\mu_n[\eta](\sigma_1, \cdots, \sigma_n) = \frac{1}{Z_n[\eta]} \exp\left\{ \frac{\beta}{2n} \sum_{1 \le i,j \le n} \sigma_i\sigma_j + \beta \sum_{1 \le i \le n} \eta_i\sigma_i \right\} , \qquad (2.16)$$

where $\sigma_i = \pm 1$ are Ising spins, η_i are i.i.d Bernoulli random variables taking the values $\varepsilon, -\varepsilon$ with probability $\frac{1}{2}$, and $Z_n[\eta]$ is the disorder-dependent partition function. It is known that for β large and ε small the models behaves like a

ferromagnet with two "pure" phases, $\mu_\infty^+[\eta]$ and $\mu_\infty^-[\eta]$ [4]. It was shown that if one considers the sequence of volumes $\{1, \cdots, n\}$, obtained by adding one site at a time, the empirical metastate does not converge for a.e. realization of the disorder. However, it does converge in distribution. That is to say that for all bounded continuous functions $F : \mathcal{P}(\Omega) \to \mathbb{R}$

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} F(\mu_n[\eta]) \stackrel{d}{=} n_\infty F(\mu_\infty^+[\eta]) + (1 - n_\infty)F(\mu_\infty^-[\eta]) , \qquad (2.17)$$

where n_∞ is a random variable, independent of η, distributed according to $\mathbb{P}(n_\infty < x) = \frac{2}{\pi} \arcsin \sqrt{x}$. On the other hand, if a sufficiently sparse sequence of volumes is chosen, we have convergence of the l.h.s to $\frac{1}{2}F(\mu_\infty^+[\eta]) + \frac{1}{2}F(\mu_\infty^-[\eta])$, for $\mathbb{P}$-a.e η.

Roughly speaking, having convergence means that the fraction of Λ_n's in which a given thermodynamic state μ_∞ appears along a sequence of volumes, converges.

---

[4] The random-field Ising model on the lattice $\mathbb{Z}^d$ is a very instructive model. In fact, for the symmetric version of this model, it was shown in [1] that there is a unique Gibbs measure in 2-dimensions at any temperature, for almost all realizations of the disorder variables. This displays a difference with the related non-disordered model, which it is well known to exhibit a phase transition. The moral of such a different behaviour is that an arbitrarily small random perturbation of the model can destroy the phase transition. In [8], Bricmont and Kupiainen showed that in 3 or higher dimensions, the random field Ising model at small temperature and for small disorder undergoes a phase transition. In other words, the ferromagnetic ordering survives the disorder.

3 Metastates in Finite-type Mean-field Models*

3.1 Introduction

Dealing with phase transitions in the theory of Gibbs measures of disordered systems is usually not an easy task. Indeed, in a regime where there are competing extremal phases (say a plus and a minus phase in a random field model) it may depend on the realization of the disorder variables which of the convex combinations the system in equilibrium will be close to. Some of the possible infinite-volume equilibrium states might not even show up in a typical large volume. To make sense of these questions, the concept of a metastate has been invented by Aizenman and Wehr [1], Newman and Stein [60, 61, 63], being a probability measure which gives the weights in the large-volume asymptotic to find a system close to one of the possible candidates among the Gibbs measures.

Explicit constructions for lattice models are difficult (see, however, van Enter et. al. [20] where the influence of random boundary conditions on an Ising model was analysed), but possible in mean-field models. Previously treated examples are given in very specific models, namely the symmetric random field Ising model and Hopfield model with a finite or a growing number of patterns [6, 7, 49, 50, 51].

In this chapter, we aim for completeness in a particular direction, namely disordered mean-field models with finitely many values for both spin and disorder variables. Such models include in particular the random-field Curie-Weiss Ising model and Potts-type Curie-Weiss random-field models, with or without symmetries in Hamiltonians or random field distributions. What we aim for is the abstract construction of the phase diagram, embellished with probability weights giving us the appearance of the candidate states. That is, we first say *which* states are available. This, for disordered mean-field models comes from an investigation of the corresponding free energy (resp. rate functions) and is a standard thing. Next and new in this chapter is the additional information on the *weights* with which they occur, and the proof of the validity of a corresponding approximate extreme decomposition, asymptotically for large

---

volumes. This is then cast in the metastate formulation. The weights are obtained by studying the distribution of the free energy fluctuations w.r.t. the disorder variables entering. Will the same type of results be true for corresponding lattice models at low temperatures at phase coexistence? We believe yes, but a proof will have to be built around sophisticated expansion techniques and be technically rather challenging. One would need to first show the coexistence of states (as was done for the random field Ising model in Bricmont and Kupiainen [8]), and then the dominance of one of the available states over the others for typical realizations of the disorder. The mean-field results should provide guidance for that, and moreover we believe that they are a rather nice complete example of a limit theorem in statistical mechanics.

3.1.1 The Models: Mean-Field Models with Local Disorder

These are the models we consider. At each site $i = 1, \ldots, n$ there is a spin variable $\sigma(i)$ taking values in a finite set E and a disorder variable $\eta(i)$ taking values in the finite (possibly different) set E'. We write $\mathcal{P}(E)$ for the set of probability measures on E, and use similar notation for other spaces. We write $L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\sigma(i)} \in \mathcal{P}(E)$ for the (total) empirical measure of the spins and consider a twice continuously differentiable function F on $\mathcal{P}(E)$. The influence of the disorder variables on the Gibbs measures for the spins is through the local a priori measures $\alpha[b] \in \mathcal{P}(E)$, for any possible type of the disorder $b \in E'$. Hence, the present analysis excludes models with disorder entering the interaction such as e.g. the Hopfield model treated in Bovier and Gayrard [6], Külske [50].

Definition 3.1.1. *The mean-field model with Hamiltonian $nF(\nu)$ and a priori measures $\alpha[b] \in \mathcal{P}(E)$, for all $b \in E'$, is given by the disorder-dependent finite-volume Gibbs distribution*

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](\sigma(1) = \omega(1), \ldots, \sigma(n) = \omega(n))$$

$$= \frac{1}{Z_{F,n}[\eta(1), \ldots, \eta(n)]} \exp\left(-nF\left(L_n^\omega\right)\right) \prod_{i=1}^{n} \alpha[\eta_i](\omega_i) , \tag{3.1}$$

together with the prescription of a probability distribution $\pi \in \mathcal{P}(E')$ for the disorder variables, according to which they are chosen independently over the sites. We assume $\pi(b) > 0$ for all $b \in E'$.

To summarize, our model depends on the triple of parameters (F, α, π): mean-field interaction F, a priori measures $\alpha = (\alpha[b])_{b \in E'}$ and disorder distribution π.

We need to introduce more notations. Given η, we write

$$\Lambda_n(b) = \{i \in \{1, 2, \ldots, n\}; \eta(i) = b\} \ ,$$

for all $b \in E'$, for the b-like sites. Furthermore, we denote by

$$\hat{\pi}_n(b) = \frac{|\Lambda_n(b)|}{n} \ ,$$

the frequency of the b-like sites (empirical distribution of random field types) and by

$$\hat{L}_n(b) = \frac{1}{|\Lambda_n(b)|} \sum_{i \in \Lambda_n(b)} \delta_{\sigma(i)} \ ,$$

the empirical spin-distribution on the b-like sites. We shall also write $\hat{L}_n = (\hat{L}_n(b))_{b \in E'}$ for the vector of empirical distributions. The total empirical distribution is then the scalar product of $\hat{\pi}_n$ with the vector of empirical spin distributions

$$L_n = \sum_{b \in E'} \hat{\pi}_n(b) \hat{L}_n(b) \ . \tag{3.2}$$

3.1.2 The Metastate on the Level of the States

Let us jump into the following definition of a metastate, obtained by a conditioning procedure, which was given first by [1]. There are different constructions of a metastate, but the present one will be the only one considered here. This construction, which is due to Aizenman and Wehr, is related to a different and more intuitive construction as empirical averages of Gibbs measures along volume-(sub-)sequences by Newman and Stein. We refer to the monographs Bovier [5], Newman [60].

Definition 3.1.2. *Assume that, for every bounded continuous* $\Xi : \mathcal{P}(E^\infty) \times (E')^\infty \to \mathbb{R}$ *the limit*

$$\lim_{n \uparrow \infty} \int \mathbb{P}(d\eta) \Xi(\mu_n[\eta], \eta) = \int J(d\mu, d\eta) \Xi(\mu, \eta) \ , \tag{3.3}$$

exists. Then the conditional distribution $\kappa[\eta](d\mu) := J(d\mu|\eta)$ *is called the AW-metastate on the level of the states.*

As is common, continuity is meant in the following sense: A function on an infinite product of a finite space is continuous (w.r.t. local topology) if it is a

uniform limit of local functions. For probability measures on $\mathcal{P}(E^\infty)$ we use the weak topology (according to which a sequence of measures converges iff it converges on continuous test-functions), and for $\mathcal{P}(E^\infty) \times (E')^\infty$, we use the product topology.

3.1.3 Main Theorem

How do we get the possible equilibrium states of the system? They are obtained as solutions to the following minimization problem.

Definition 3.1.3. *Consider the free energy minimization problem*

$$\hat{\nu} \mapsto \Phi[\pi](\hat{\nu}) \,, \tag{3.4}$$

on $\mathcal{P}(E)^{E'}$, with the free energy functional

$$\Phi : \mathcal{P}(E') \times \mathcal{P}(E)^{E'} \mapsto \mathbb{R} \,,$$

$$\Phi[\hat{\pi}](\hat{\nu}) = F\left(\sum_{b \in E'} \hat{\pi}(b)\hat{\nu}(b)\right) + \sum_b \hat{\pi}(b)S(\hat{\nu}(b)|\alpha[b]) \,, \tag{3.5}$$

where $S(p_1|p_2) = \sum_{a \in E} p_1(a) \log \frac{p_1(a)}{p_2(a)}$ is the relative entropy. We say that the random mean-field system obeys the non-degeneracy condition 1) *if $\hat{\nu} \mapsto \Phi[\pi](\hat{\nu})$ has a finite set of minimizers $M^* = M^*(F, \alpha, \pi)$ where all the eigenvalues of the Hessian are strictly positive.*

It is very hard for a system not to satisfy this condition and we will assume in the following that it is satisfied. If it is true the vector of the empirical spin distributions of the system, $\hat{L}_n$, will concentrate around the set M^*. **M**ore than that, it may even concentrate on a smaller set. The following theorem about the metastate will tell us how this concentration will take place and get the weights w_j.

Let $\hat{\nu}_j$ be a fixed element in M^*. Let us consider the linearization of the free energy functional at the fixed minimizers as a function of π, which reads

$$\Phi[\tilde{\pi}](\hat{\nu}_j) - \Phi[\pi](\hat{\nu}_j) = -B_j[\tilde{\pi} - \pi] + o(\|\tilde{\pi} - \pi\|) \,, \tag{3.6}$$

where

$$B_j[\tilde{\pi} - \pi] =$$
$$-\left(dF_{\pi \cdot \hat{\nu}_j}\left(\sum_b (\tilde{\pi}(b) - \pi(b))\hat{\nu}_j(b)\right) + \sum_b [\tilde{\pi}(b) - \pi(b)]S(\hat{\nu}_j(b)|\alpha[b])\right) \,. \tag{3.7}$$

This defines an affine function on the tangent space of field type measures $T\mathcal{P}(E')$ (i.e. vectors which sum up to zero), for any j.

Definition 3.1.4. *We call B_j the stability vector of $\hat{\nu}_j$. We call*

$$R_j := \{x \in T\mathcal{P}(E'), \langle x, B_j \rangle > \max_{k \neq j} \langle x, B_k \rangle \} , \tag{3.8}$$

the stability region of $\hat{\nu}_j$.

Now comes our second condition.

Definition 3.1.5. *We say the vector $B = (B_1, \ldots, B_k)$ satisfies the non-degeneracy condition 2) if no different minimizers j, j' have the same $B_j = B_{j'}$.*

In other words the randomness lifts all symmetries. The Hopfield model for instance (which was excluded already before because of the structure of the randomness) would also be excluded at this state since it has spin-flip symmetry for all realizations, and so minimizers in the free energy would come in pairs.

Note that the condition in the definition implies that $\left(\bigcup_{j=1,\ldots,k} R_j \right)^{\complement}$ has zero Lebesgue measure in $T\mathcal{P}(E')$. Indeed, if the map $j \mapsto \langle x, B_j \rangle$ has no unique maximizer for fixed x, then, for some pair $j \neq k$ we have that $\langle x, B_j - B_k \rangle = 0$. For fixed j, k this set of x's is a hyperplane (hence a measure zero set) since, by assumption, $B_j \neq B_k$. We note the following simple but important geometric lemma.

Lemma 3.1.6. $R_j \neq \emptyset \Leftrightarrow B_j \in ex(\mathcal{H}_{conv}\{B_1, \ldots, B_k\})$.

Here, for a subset $A \subset \mathbb{R}^d$, $\mathcal{H}_{conv}(A)$ denotes the *convex hull* of A, that is the smallest convex set which contains A. $ex(C)$, for a convex set C, denotes the extremal points of C, that is those points which can not be written as a non-trivial convex combination with points from C. In our case $\mathcal{H}_{conv}\{B_1, \ldots, B_k\}$ is a convex polyhedron and $ex(\mathcal{H}_{conv}\{B_1, \ldots, B_k\})$ is the smallest set of points which generates it.

Proof. We prove the implication " $\Rightarrow$ " by contradiction. Suppose that B_j is not an extremal point. Then it can be written as a non-trivial convex combination $B_j = \sum_i \alpha_i B_i$ with $\sum_{i=1}^{k} \alpha_i = 1$, where $\alpha_i \geq 0$ and non-zero only for $B_i \in ex(\mathcal{H}_{conv}\{B_1, \ldots, B_k\})$. Any vector $x \in R_j$ satisfies $\langle x, B_j \rangle > \langle x, B_i \rangle$ for all $i \neq j$ and hence $\langle x, B_j \rangle = \sum_i \alpha_i \langle x, B_j \rangle > \sum_{i=1}^{k} \alpha_i \langle x, B_i \rangle = \langle x, B_j \rangle$. This is a contradiction and hence $R_j = \emptyset$.

To prove the opposite implication " $\Leftarrow$ " let us consider an extremal point B_j and note the following: If $B_j \notin \mathcal{H}_{conv}\{B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_k\}$ then,

after a suitable translation and rotation, we can find coordinates such that the vectors take the form $B_j = (0, \ldots, 0, B_{j,d})$ and $B_i = (B'_i, B_{i,d})$ with $B_{j,d} > 0$ and $B_{i,d} \leq 0$ for $i \neq j$. (The latter statement follows from the fact that there is a separating hyperplane between $\mathcal{H}_{\text{conv}}\{B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_k\}$ and the point B_j. This finite-dimensional version of the Hahn-Banach theorem is a classical result in geometry, see Theorem 1.2.4 in Matoušek [59]. Having this separating hyperplane we choose the origin as the orthogonal projection of B_j to this plane, the first coordinates as orthogonal coordinates inside the plane, and the last coordinate axis pointing in the direction of B_j.) The proof relies on the last two inequalities. Indeed we have, with the general notation $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ that

$$
\begin{aligned}
R_j &= \{x \in \mathbb{R}^d : \forall i \neq j \text{ holds } \langle x, B_j - B_i \rangle > 0\} \\
&= \{x \in \mathbb{R}^d : \forall i \neq j \text{ holds } \langle x', B'_j - B'_i \rangle + x_d(B_{j,d} - B_{i,d}) > 0\} \\
&= \left\{ x \in \mathbb{R}^d : x_d > \max_{i:i \neq j} \frac{\langle x', B'_i - B'_j \rangle}{B_{j,d} - B_{i,d}} \right\} \neq \emptyset \; .
\end{aligned}
\tag{3.9}
$$

$\square$

Before we state our theorem let us introduce the kernels

$$
\gamma[b](a|\nu) = \frac{e^{-dF_\nu(a)}\alpha[b](a)}{\sum_{\bar{a} \in E} e^{-dF_\nu(\bar{a})}\alpha[b](\bar{a})} \; ,
\tag{3.10}
$$

with $\nu \in \mathcal{P}(E)$. These are the limiting local distributions of a spin at a site with a disorder variable in the state b if the empirical spin-average of the rest of the system is given by the measure ν. The products over all sites of these quantities, for $\nu = \pi\hat{\nu}_j$, will play the role of pure measures. We are now in the position to give our main result.

Theorem 3.1.7. *Assume that the model satisfies the* non-degeneracy *assumptions* 1) *and* 2). *Define the weights*

$$
w_j := \mathbb{P}_\pi(G \in R_j) \; ,
\tag{3.11}
$$

where $G \in T\mathcal{P}(E')$ is a centred Gaussian variable with the same covariance as $\sqrt{n}(\hat{\pi}_n - \pi)$ which is given by the expression $C_\pi(b, b') = \pi(b)1_{b=b'} - \pi(b)\pi(b')$.

Then $\sum_{j=1}^k w_j = 1$ and the metastate on the level of the states equals

$$
\kappa[\eta](d\mu) = \sum_{j=1}^k w_j \delta_{\mu_j[\eta]}(d\mu) \; ,
\tag{3.12}
$$

where $\mu_j[\eta] := \prod_{i=1}^{\infty} \gamma[\eta(i)](\,\cdot\,|\pi\hat{\nu}_j)$.

Comment. We like to reformulate our result on the visibility or invisibility of the phases in the following way. Let us denote by $M^{**} = \{\hat{\nu} \in M^* : w_{\hat{\nu}} > 0\}$ the subset of *visible pure phases* among the pure phases M^*. Let us use the symbol $B.$ for the injective map (under our hypothesis)

$$B. : M^* \to T\mathcal{P}(E') \,,$$
$$\hat{\nu} \mapsto B_{\hat{\nu}} \,.$$

Then we can write in short

$$M^{**} = (B.)^{-1}\Big(\mathrm{ex}(\mathcal{H}_{\mathrm{conv}}(B.(M^*)))\Big) \,.$$

Let us derive the following immediate consequence which provides a symmetry, due to the randomness (the symmetry of the Gaussian, obtained via the CLT).

Corollary 3.1.8. *Suppose that the system admits precisely two pure phases, i.e. $|M^*| = 2$. Then the metastate is the symmetric mixture between the two, i.e.*

$$\kappa[\eta](d\mu) = \frac{1}{2}\delta_{\mu_1[\eta]}(d\mu) + \frac{1}{2}\delta_{\mu_2[\eta]}(d\mu) \,. \tag{3.13}$$

The corollary is clear from the theorem since in that case $R_1 = -R_2$ and this implies by the *non-degeneracy assumption 2)* that $w_1 = w_2$.

Corollary 3.1.9. *Suppose that the random-field is two-valued, i.e. $|E'| = 2$, and the number of pure phases $|M^*| \geq 2$ arbitrary. Then the set of visible states has two elements and $w(\hat{\nu}) = \frac{1}{2}$ for both elements $\hat{\nu} \in M^{**}$.*

The corollary is clear from the theorem since any convex polyhedron in one dimension has only two extremal points.

Exploiting the Mean-field Equation

Using variational calculus and assuming differentiability of F, one sees that the minimizers of the variational problem (3.4) must satisfy the consistency (mean-field) equations

$$\hat{\nu}[b](a) = \gamma[b](a|\pi \cdot \hat{\nu}) \,, \tag{3.14}$$

which are coupled over $b \in E'$. Summing over these indices one gets the mean-field equation for the total empirical mean $\nu = \pi \cdot \hat{\nu}$ of the form

$$\nu(a) = \sum_{b \in E'} \pi(b)\gamma[b](a|\nu) \; . \tag{3.15}$$

We note the following Lemma.

Lemma 3.1.10. *Define the function $\hat{\Gamma} : \mathcal{P}(E) \to \mathcal{P}(E)^{E'}$ by the r.h.s. of the mean field equation, namely*

$$\hat{\Gamma}(\nu) = \Big(\gamma[b](\cdot|\nu)\Big)_{b \in E'} \; . \tag{3.16}$$

Define the function $\hat{B} : \mathcal{P}(E) \to T\mathcal{P}(E')$ by

$$\hat{B}_\nu[b] = \log\left(\sum_{a \in E} e^{-dF_\nu(a)}\alpha[b](a)\right) - C \; ,$$
$$C = \frac{1}{|E'|}\sum_{b \in E'} \log \sum_{a \in E} e^{-dF_\nu(a)}\alpha[b](a) \; . \tag{3.17}$$

Then, for all $\hat{\nu} \in M^*$,

$$\hat{\nu} = \hat{\Gamma}(\pi\hat{\nu}) \; ,$$
$$B_{\hat{\nu}} = \hat{B}_{\pi\hat{\nu}} \; . \tag{3.18}$$

For all $\nu \in \pi M^*$ the free energy can be written as

$$\Phi[\pi](\hat{\Gamma}(\nu)) = F(\nu) - \langle dF_\nu, \nu \rangle - \langle \hat{B}_\nu, \pi \rangle + C \; . \tag{3.19}$$

The first statement is just a rephrasing of the mean-field equation. It serves us to see that there is a bijection between $\pi M^* = \{\pi\hat{\nu}|\hat{\nu} \in M^*\} \subset \mathcal{P}(E)$ (a subset in a space of measures with dimension $|E| - 1$) and M^* (a subset in a space of measures with dimension $(|E| - 1)^{|E'|}$). The second part means that the logarithm of the normalization factor ("little partition function") of the mean-field kernels in the total empirical distribution ν of type b, produces the b'th component of the stability vector corresponding to the minimizer with total empirical mean ν. The interesting feature is that the form of π does not enter at all into this formula (it enters however through the question which minimizer $\hat{\nu}$, and hence also ν, appears.)

Proof. The first part is obvious. To prove the second part, for $\hat{\nu} \in M^*$, we write, with a constant C' to be determined

$$
\begin{aligned}
-B_{\hat{\nu}}[b] &= \sum_{a \in E} dF_{\pi \cdot \hat{\nu}}(a) \hat{\nu}[b](a) + S(\hat{\nu}[b] | \alpha[b]) - C' \\
&\equiv \hat{\nu}[b](dF_{\pi \cdot \hat{\nu}}(\cdot)) + S(\hat{\nu}[b] | \alpha[b]) - C' \\
&= -\log \sum_{a \in E} e^{-dF_{\pi \hat{\nu}}(a)} \alpha[b](a) - C' \ .
\end{aligned}
\tag{3.20}
$$

The last equality follows from the mean-field equation (3.14) and from (3.10). This proves the second claim. The last claim follows from the first equality of the last display multiplying with $\pi(b)$ and summing over $b \in E'$. $\qquad\square$

3.1.4 Ising Random-field Examples

Let us take the Ising model with $F(\nu) = -\beta(\nu(+)^2 + \nu(-)^2)$. Any possible local single-site measure α can be described as an $\alpha[h](\sigma_i) = \frac{e^{h\sigma_i}}{2 \cosh h}$. Any $\nu = \nu_m$ can be described in terms of its mean value $\nu_m(+) - \nu_m(-) = m$. By means of the above, we can write

$$
\begin{aligned}
\hat{B}_{\nu_m}[h] \equiv \hat{B}_{\nu_m}[\alpha[h]] &= \log \frac{e^{\beta 2 \frac{1+m}{2} + h} + e^{\beta 2 \frac{1-m}{2} - h}}{2 \cosh h} - C \\
&= \beta + \log \frac{\cosh(\beta m + h)}{\cosh h} - C \ .
\end{aligned}
\tag{3.21}
$$

Let us now fix $E' = \text{supp}(\pi) = \{\alpha_h : h \in \{h_1, h_2, \ldots, h_L\}\}$ as the set of allowed local measures. This gives us the normalized vector in the tangent space $T\mathcal{P}(E')$ with entries

$$
\hat{B}_{\nu_m}[h_i] := \log \frac{\cosh(\beta m + h_i)}{\cosh h_i} - \frac{1}{L} \sum_{j=1}^{L} \log \frac{\cosh(\beta m + h_j)}{\cosh h_j} \ .
\tag{3.22}
$$

Writing a vector with $L = |E'|$ components we have

$$
\hat{B}_{\nu_m} = \begin{pmatrix} \log \frac{\cosh(\beta m + h_1)}{\cosh h_1} \\ \ldots \\ \log \frac{\cosh(\beta m + h_L)}{\cosh h_L} \end{pmatrix} - \frac{1}{L} \sum_{j=1}^{L} \log \frac{\cosh(\beta m + h_j)}{\cosh h_j} \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix} \ .
$$

Lemma 3.1.11. *Let $E' \subset \mathbb{R}$, $2 \le |E'| < \infty$. Then the map $m \mapsto \hat{B}_{\nu_m}$ is injective.*

Proof. We have at least two elements, $h_1 < h_2$ (after possible change of indices) in E'. Let $\nu_m, \nu_{\tilde{m}}$ be given with $\hat{B}_{\nu_m} = \hat{B}_{\nu_{\tilde{m}}}$. By easy manipulations looking at the first two components of B the latter implies that

$$\frac{\cosh(\beta m + h_1)}{\cosh(\beta m + h_2)} = \frac{\cosh(\beta \tilde{m} + h_1)}{\cosh(\beta \tilde{m} + h_2)} \ . \tag{3.23}$$

From this follows $m = \tilde{m}$ by injectivity of the function $x \mapsto \frac{\cosh x}{\cosh(x+1)}$.

$\square$

Let us extend the random-field Ising model to a non-quadratic Hamiltonian $F(\nu) = G(\nu(+) - \nu(-))$ and general local measures $\alpha = (\alpha[h])_{h \in E'}$ with a finite set E' just as above in the quadratic case. Then the mean field equation becomes

$$m = \sum_{i=1}^{L} \pi(h_i) \tanh(-G'(m) + h_i) \ . \tag{3.24}$$

The stability vector becomes

$$\hat{B}_{\nu_m}[h_i] := \log \frac{\cosh(-G'(m) + h_i)}{\cosh h_i} - \frac{1}{L} \sum_{j=1}^{L} \log \frac{\cosh(-G'(m) + h_j)}{\cosh h_j} \ . \tag{3.25}$$

Then the injectivity of the map $m \mapsto \hat{B}_{\nu_m}$ holds under the assumption that $m \mapsto G'(m)$ is injective, by the same proof, replacing m by $-G'(m)$ in (3.23). We have thus proven the following statement.

Proposition 3.1.12. *For a random-field Ising model with Hamiltonian $F(\nu) = G(\nu(+) - \nu(-))$ and G' injective the second non-degeneracy assumption is automatically satisfied, for any distribution of random fields with finite support.*

It is easy to create a two-minima situation where there is no symmetry, by looking at the equal-depth condition for the free energy

$$\Phi[\pi](\hat{\Gamma}(\nu_m)) =$$
$$= F(\nu_m) - \sum_{a \in E} dF_{\nu_m}(a)\nu_m(a) - \sum_{b \in E'} \pi(b) \log \sum_{a \in E} e^{-dF_{\nu_m}(a)} \alpha[b](a)$$
$$= G(m) - mG'(m) - \sum_{i=1}^{L} \pi(h_i) \log \frac{\cosh(-G'(m) + h_i)}{\cosh h_i} \ ,$$

where both minima would get the same weight in the metastate necessarily. In fact, a situation with precisely two minimizers not related by symmetry was proven to occur (even) for the (symmetric) model $G(m) = -\frac{\beta m^2}{2}$, $E = E' = \{1, -1\}$, $\pi(1) = \frac{1+\alpha}{2} = 1 - \pi(-1)$, $\alpha[b](a) = \frac{e^{\beta \varepsilon ab}}{\cosh \beta \varepsilon}$, in a region of the $(\beta^{-1}, \varepsilon)$-plane, see [52].

3.1.5 Potts Random-field Examples

Let us take the Potts model with quadratic interaction

$$F(\nu) = -\frac{\beta}{2}(\nu(1)^2 + \cdots + \nu(q)^2) \,,$$

in the presence of the local single-site measures $\alpha[b](\sigma_i)$ (specified below) where we write

$$E' = \text{supp}(\pi) = \{\alpha[b] : b \in \{b_1, b_2, \ldots, b_L\}\} \,.$$

Then we have for the stability vector

$$\hat{B}_\nu = \begin{pmatrix} \log \sum_{a=1}^q e^{\beta\nu(a)}\alpha[b_1](a) \\ \ldots \\ \log \sum_{a=1}^q e^{\beta\nu(a)}\alpha[b_L](a) \end{pmatrix} - \frac{1}{L}\sum_{j=1}^L \log \sum_{a=1}^q e^{\beta\nu(a)}\alpha[b_j](a) \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix} \,.$$

Remark 3.1.13. *The map $\hat{B}. : \mathcal{P}(E) \to T\mathcal{P}(E')$ is a map between spaces of dimension $|E| - 1$ and $|E'| - 1$. It has a chance to be injective as such (on the whole space $\mathcal{P}(E)$) only when $|E'| \geq |E|$.*

Let us take $E \equiv E'$ and π to be the equidistribution and switch to the specific case $\alpha[b](a) = \frac{e^{B1_{b=a}}}{e^B + q - 1}$ (random field with homogeneous intensity). The kernels become

$$\gamma[b](a|\nu) = \frac{e^{\beta\nu(a) + B1_{a=b}}}{\sum_{\bar{a}\in E} e^{\beta\nu(\bar{a}) + B1_{\bar{a}=b}}} \,.$$

We will be looking at measures in $\nu_{j,u} \in \mathcal{P}(E)$ of the form $\nu_{j,u}(j) = \frac{1+u(q-1)}{q}$, $\nu_{j,u}(i) = \frac{1-u}{q}$ for $i \neq j$. The stability vector for $\nu_{1,u}$ is given by

$$\hat{B}_{\nu_{1,u}} = \begin{pmatrix} \frac{q-1}{q}\log\frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \\ -\frac{1}{q}\log\frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \\ \ldots \\ -\frac{1}{q}\log\frac{e^{\beta u+B}+q-1}{e^{\beta u}+e^B+q-2} \end{pmatrix} \,,$$

the other ones are related by symmetry. We note that the first entry is strictly positive while the other entries are negative (for $B > 0$ and $u > 0$). The mean-field equation in the u variable takes the following form:

$$u = \frac{e^{\beta u}}{e^{\beta u} + e^{B} + (q - 2)} - \frac{1}{e^{\beta u + B} + (q - 1)} \ . \tag{3.26}$$

We notice that $u = 0$ is always a solution, and for $B = 0$ we obtain exactly the known mean-field equation for Potts without disorder. The latter model shows a first-order transition as a function of temperature at critical temperature $\beta_c = \frac{2(q-1)}{q-2} \log(q - 1)$ (see Ellis and Wang [13]). The r.h.s. of (3.26) is always positive, as a computation shows. This gives rise to a non-trivial solution u, in a certain range of parameters. Note that this non-trivial solution is not always the one to be chosen. It is to be chosen iff $\Phi[\pi](\hat{\Gamma}(\nu_{j,u})) < \Phi[\pi](\hat{\Gamma}(\nu_{j,u=0}))$. So, the first order transition point is given by equality of the last equation. Forgetting a u-independent term we have, independently of the direction j,

$$\Phi[\pi](\hat{\Gamma}(\nu_{j,u})) =$$
$$\log \frac{e^{B} + q - 1}{e^{\beta u} + e^{B} + q - 2} + \frac{\beta(q - 1)}{2q} u^2 + \frac{\beta}{q} u - \frac{1}{q} \log \frac{e^{\beta u + B} + q - 1}{e^{\beta u} + e^{B} + q - 2} \ , \tag{3.27}$$

with the property that $\Phi[\pi](\hat{\Gamma}(\nu_{j,u=0})) = 0$. For illustrational purposes let us focus on the case $q = 3$. We don't provide a complete bifurcation analysis here, but just outline the picture. The case $B = 0$ is perfectly understood and we know that there is a first order transition at the critical inverse temperature $\beta = 4 \log 2$. The nature of the transition stays the same when B takes small enough positive values and there is a line in the space of temperature and coupling strength B of an equal-depth minimum at $u = 0$ and a positive value of $u = u^*(\beta, q)$. (See Fig. 1 for a numerical example.) Along this line the set of Gibbs measures is strictly bigger than the set of states which are seen under the metastate.

The Plot shows the graph of $u \mapsto \Phi[\pi](\hat{\Gamma}(\nu_{j,u}))$ for $B = 0.3, q = 3, \beta = 4\log 2 + 0.03203$ at which there is the first order transition.
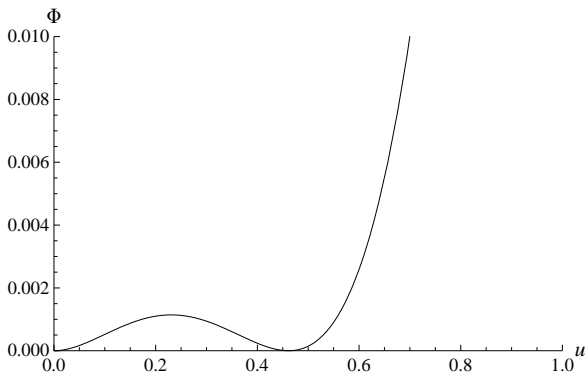
The metastate becomes $\kappa[\eta](d\mu) = \frac{1}{3}\sum_{j=1}^{3}\delta_{\mu_j[\eta]}$ with $\mu_j[\eta] = \prod_{i=1}^{\infty}\gamma[\eta(i)](\cdot|\nu_{j,u=u^*(\beta,q)})$. This follows from the form of the stability vector using that $\hat{B}_{\nu_{1,u=0}} = 0$ and hence lies in the convex hull of the three others.

3.1.6 Strategy of the Proof, Non-degeneracy Assumption and Concentration

The outline of the remainder of this chapter is as follows. We begin in Section 3.2 with a discussion of another related result, namely the metastate on the level of the empirical spin-distributions. The theorem is quite analogous, the same weights w_j appear, and the proof is slightly easier than that of the full theorem. To arrive at the proof of this theorem we will discuss the concentration property of the vector of the empirical distributions for good realizations of the disorder which will force the system to be in one definite state. In particular, it will show how the non-degeneracy assumptions 1) and 2) are naturally used in that argument and this will explain how the CLT for empirical distributions of disorder variables translates into the form of the weights w_j. Then in Section 3.3 we will turn to the proof of the metastate theorem on the level of states and conclude.

3.2 The Metastate on the Level of the Empirical Spin-distribution

Two ways of looking at the spin-distributions of disordered mean-field systems are natural. In the first one, described in the introduction, we focus on measures on the spins themselves, and evaluate them on local observables. In the second one, we focus on aggregate properties of the system, and look at functions of the empirical spin-distribution of the whole system. From the second point of view it is natural to make the following definition of a metastate on the level of the empirical spin-distribution.

Denote by $\rho[\eta](n) := \mu_{F,n}[\eta](L_n)$ the image of the finite-volume Gibbs-measure under the empirical distribution. This defines a disorder-dependent element in $\mathcal{P}(\mathcal{P}(E))$. Under our assumptions these measures will concentrate on the finite set $\pi M^* = \{\pi\hat{\nu}_j, j = 1, \ldots, k\}$. It is useful to introduce a metastate which tells us more precisely how this concentration takes place. This is the reason for the following definition.

Definition 3.2.1. *Assume that, for every bounded continuous* $\Psi : \mathcal{P}(\mathcal{P}(E)) \times (E')^\infty$ *the limit*

$$\lim_{n\uparrow\infty} \int \mathbb{P}(d\eta)\Psi(\rho[\eta](n), \eta) = \int K(d\rho, d\eta)\Psi(\rho, \eta) \;, \qquad (3.28)$$

exists. Then the conditional distribution $\bar{\kappa}[\eta](d\rho) := K(d\rho|\eta)$ *is called the metastate on the level of the empirical spin-distribution.*

Believing in the first theorem it is not surprising that this metastate takes the following form.

Theorem 3.2.2. *Under the* non-degeneracy *assumptions* 1) *and* 2),

$$\bar{\kappa}[\eta](d\rho) = \sum_{j=1}^{k} w_j \delta_{\delta_{\pi\hat{\nu}_j}}(d\rho) \;, \qquad (3.29)$$

for $\mathbb{P}_\pi$*-a.e.* η.

As a difference with respect to the first theorem let us point out that in this case the dependence on the disorder has vanished on the r.h.s.

Proof of Theorem 3.2.2. For $n_1 < n_2$ integers, let's define

$$X_{[n_1, n_2]}[\eta] = \frac{1}{\sqrt{n_2 - n_1 + 1}} \sum_{i=n_1}^{n_2} \delta_{\eta_i} - \sqrt{n_2 - n_1 + 1}\, \pi \;. \qquad (3.30)$$

Define n, l-dependent *good-sets* $\mathcal{H}_{n,l}^{\tau}$ of the realization of the randomness as follows

$$
\begin{aligned}
\mathcal{H}_{i,n,l}^{\tau} &:= \left\{ \eta \in (E')^{n-l} : X_{[l+1,n]}[\eta] \in R_{i,n}^{\tau} \right\} , \\
\mathcal{H}_{n,l}^{\tau} &:= \bigcup_{i=1}^{k} \mathcal{H}_{i,n,l}^{\tau} ,
\end{aligned}
\tag{3.31}
$$

where $R_{i,n}^{\tau} := \{ x \in T\mathcal{P}(E') : \langle x, B_i \rangle - \max_{k \neq i} \langle x, B_k \rangle > n^{-\frac{1}{2}+\tau}, \|x\| \leq n^{\frac{\tau}{4}} \}$, where $0 < \tau < \frac{1}{2}$. For the sake of clarity set $\delta_n = n^{-\frac{1}{2}+\tau}$. The chosen range of τ ensures that $\delta_n \downarrow 0$, but not too fast, namely in such a way that $\sqrt{n}\, \delta_n \uparrow \infty$. $\mathcal{H}_{i,n,l}^{\tau}$ is a region of the disorder random variables which allows us to deduce that the measure on the empirical distribution will be with large probability inside a ball around $\pi\hat{\nu}_i^*$.

Remark 3.2.3. *We need $\delta_n \downarrow 0$ because we want to cover all of the corresponding stability-region R_i, (3.8), in the large-n limit. The condition regarding the velocity with which δ_n is going to 0 ensures the concentration of the measure around a particular minimizer, in other words it will enable us to see the breaking of the degeneracy of the minimizers caused by the fluctuations of $\hat{\pi}_n$. The relevance of the cut-off $\|x\| \leq n^{\frac{\tau}{4}}$ will be seen later.*

Lemma 3.2.4. *Let us assume that $\eta \in \mathcal{H}_{i,n,0}^{\tau}$. Then*

$$
\mu_{F,n}[\eta(1), \ldots, \eta(n)](d(L_n, \pi\hat{\nu}_i^*) \leq \varepsilon) \geq 1 - \bar{r}(\varepsilon, n) ,
\tag{3.32}
$$

where $\lim_{n\uparrow\infty} \bar{r}(\varepsilon, n) = 0$ for all $\varepsilon > 0$.

Proof of Lemma 3.2.4. Call $M_n := \{ \nu \in \mathcal{P}(E) : \exists \omega \in E^n \text{ such that } L_n^{\omega} = \nu \}$. To every element $\nu \in M_n$ correspond several possible values of the empirical distribution vectors $\hat{L}_n \in \mathcal{P}(E)^{E'}$, given $\hat{\pi}_n$. We call this set $\hat{M}_n := \{ \hat{\nu} \in \mathcal{P}(E)^{E'} : \exists \omega \in E^n \text{ such that } \hat{\nu} = \hat{L}_n \}$. Let's define $\rho^{\varepsilon}[\eta](n) \in \mathcal{P}(\pi M^*)$ assigning probability weights to the ε-balls by

$$
\rho^{\varepsilon}[\eta](n)(\pi\hat{\nu}_i^*) := \frac{\mu_{F,n}[\eta(1), \ldots, \eta(n)](L_n \in B(\varepsilon, \pi\hat{\nu}_i^*))}{\displaystyle\sum_{j=1}^{k} \mu_{F,n}[\eta(1), \ldots, \eta(n)](L_n \in B(\varepsilon, \pi\hat{\nu}_j^*))} .
\tag{3.33}
$$

At this stage the measures appearing in the former definition involve a sum over $\nu \in M_n \bigcap B(\varepsilon, \pi\hat{\nu}_i^*)$ and for the correspondence formerly mentioned we

can write

$$\rho^\varepsilon[\eta](n)(\pi\hat{\nu}_i^*) = \frac{\displaystyle\sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_i^*)} \mu_{F,n}[\eta(1),\ldots,\eta(n)](\hat{L}_n = \hat{\nu})}{\displaystyle\sum_{j=1}^k \sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)} \mu_{F,n}[\eta(1),\ldots,\eta(n)](\hat{L}_n = \hat{\nu})} \tag{3.34}$$

$$= \frac{\displaystyle\sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_i^*)} \frac{\displaystyle\sum_{\sigma\in E^n:\hat{L}_n^\sigma=\hat{\nu}} e^{-nF(\hat{\pi}_n\hat{\nu})}\prod_{i=1}^n \alpha[\eta_i](\sigma_i)}{\displaystyle\sum_{\bar{\nu}\in\hat{M}_n}\sum_{\sigma\in E^n:\hat{L}_n^\sigma=\bar{\nu}} e^{-nF(\hat{\pi}_n\bar{\nu})}\prod_{i=1}^n \alpha[\eta_i](\bar{\sigma}_i)}}{\displaystyle\sum_{j=1}^k \sum_{\substack{\hat{\nu}: \\ \hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)}} \frac{\displaystyle\sum_{\sigma\in E^n:\hat{L}_n^\sigma=\hat{\nu}} e^{-nF(\hat{\pi}_n\hat{\nu})}\prod_{i=1}^n \alpha[\eta_i](\sigma_i)}{\displaystyle\sum_{\bar{\nu}\in\hat{M}_n}\sum_{\sigma\in E^n:\hat{L}_n^\sigma=\bar{\nu}} e^{-nF(\hat{\pi}_n\bar{\nu})}\prod_{i=1}^n \alpha[\eta_i](\bar{\sigma}_i)}} . \tag{3.35}$$

Decomposing the spin-sums into sums over possible values of the vector of empirical distributions on the b-like sites we can rewrite the last expression as

$$\frac{\displaystyle\sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_i^*)} e^{-nF(\hat{\pi}_n\hat{\nu})}\prod_{b=1}^{|E'|} \alpha[b]^{|\Lambda_n(b)|}(\Omega_{|\Lambda_n(b)|}(\hat{\nu}(b)))}{\displaystyle\sum_{j=1}^k \sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)} e^{-nF(\hat{\pi}_n\hat{\nu})}\prod_{b=1}^{|E'|} \alpha[b]^{|\Lambda_n(b)|}(\Omega_{|\Lambda_n(b)|}(\hat{\nu}(b)))} , \tag{3.36}$$

where $\Omega_{|\Lambda_n(b)|}(\hat{\nu}(b)) = \{\sigma \in E^{|\Lambda_n(b)|} : \hat{L}_n^\sigma(b) = \hat{\nu}(b)\}$, and $\alpha[b]^{|\Lambda_n(b)|}(\cdot)$ is the product measure on the b-like sites. For sake of clarity let us recall the finite-volume finite-alphabet version of Sanov's theorem which is stated as Lemma 2.1.8 in Dembo and Zeitouni [10], which we will make use of in the next step.

Lemma 3.2.5. *Let ν be a probability measure on a finite state space E. For fixed n define the set of* microstates *compatible with ν by*

$$\Omega(\nu) := \{\omega \in E^n | L_n^\omega = \nu\} . \tag{3.37}$$

Then, if $n\nu(x)$ is integer-valued for all $x \in E$ we have the upper and lower large deviation bounds

$$(n + 1)^{-|E|} e^{-nS(\nu|\mu)} \le \mu(\Omega(\nu)) = \mu(\{\omega : L_n^\omega = \nu\}) \le e^{-nS(\nu|\mu)} . \qquad (3.38)$$

Using (3.38) we get a lower bound to (3.36) of the form

$$\cfrac{1}{1 + \displaystyle\sum_{j \neq i}^{k} \cfrac{\displaystyle\sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu} \in B(\varepsilon,\pi\hat{\nu}_j^*)} e^{-n\Phi[\hat{\pi}_n](\hat{\nu})}}{\displaystyle\sum_{\hat{\nu}:\hat{\pi}_n\hat{\nu} \in B(\varepsilon,\pi\hat{\nu}_i^*)} e^{-n\Phi[\hat{\pi}_n](\hat{\nu})} \displaystyle\prod_{b \in E'} (|\Lambda_n(b)| + 1)^{-|E'|}}} . \qquad (3.39)$$

Let us notice that, in the last expression, the free energy (3.5) has appeared. However it does not involve yet the minimizer $\hat{\nu}_i^*$ in an explicit way. What we would like to do next, is to understand the $\hat{\pi}_n$-dependence of the minima in the different balls. Differences in the depths of the minima would not be present for $\hat{\pi}_n = \pi$ but will be created by the fluctuations of $\hat{\pi}_n$.

In order to achieve this, we first need to compare the values that the $\hat{\pi}_n$-dependent free energy takes on the ball with the one corresponding to the centre. As we will see in Proposition 3.2.6 this can be done uniformly with respect to the centres (π-minimizers). Secondly we will compare, for any fixed minimizer, the difference between the $\hat{\pi}_n$-dependent free energy and the π-dependent one; this will be done using the linearization procedure (3.6). Let us emphasize the fact that the definition of the good-sets $\mathcal{H}_{i,n,l}^\tau$ has been chosen *ad hoc* to guarantee, in the limit $n \uparrow \infty$, that the i-th stability vector will "dominate" the others, and thus the concentration around $\hat{\nu}_i^*$ will take place. We also need an upper bound on $\|\hat{\pi}_n - \pi\|$ for that procedure to work which is the reason for the cut-off in the definition of the good-sets. The next proposition formalizes the first step.

Proposition 3.2.6. *Under the non-degeneracy assumption 1) there exists an $\varepsilon_0 > 0$ and a positive constant K such that for all $\varepsilon \le \varepsilon_0$ and for n sufficiently large*

$$\frac{-K\|\hat{\pi}_n - \pi\|^2}{2} \le \inf_{\hat{\nu} \in \hat{M}_n \bigcap B(\varepsilon,\pi\hat{\nu}_j^*)} (\Phi[\hat{\pi}_n](\hat{\nu}) - \Phi[\hat{\pi}_n](\hat{\nu}_j^*)) , \qquad (3.40)$$

for all minimizers $\hat{\nu}_j^*$.

Proof. We will show that there exists a positive K such that

$$\frac{-K||\hat{\pi}_n - \pi||^2}{2} \leq \inf_{\hat{\nu} \in B(\varepsilon, \pi\hat{\nu}_j^*)} (\Phi[\hat{\pi}_n](\hat{\nu}) - \Phi[\hat{\pi}_n](\hat{\nu}_j^*)) \tag{3.41}$$

holds, hence the proposition will follow using the simple inequality
$\inf_{\hat{\nu} \in \hat{M}_n \bigcap B(\varepsilon, \pi\hat{\nu}_j^*)} \geq \inf_{\hat{\nu} \in \hat{B}(\varepsilon, \pi\hat{\nu}_j^*)}$. Let us take a Taylor expansion of $\Phi[\hat{\pi}_n](\hat{\nu})$ around
$\hat{\nu}_j^*$, namely

$$\begin{aligned}
\Phi[\hat{\pi}_n](\hat{\nu}) = \Phi[\hat{\pi}_n](\hat{\nu}_j^*) + \langle \nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*), \hat{\nu} - \hat{\nu}_j^* \rangle \\
+ \frac{1}{2}\langle \hat{\nu} - \hat{\nu}_j^*, H\Phi[\hat{\pi}_n](\hat{\nu}_j^*)(\hat{\nu} - \hat{\nu}_j^*) \rangle + ||\hat{\nu} - \hat{\nu}_j^*||^2 R(\hat{\nu}, \hat{\nu}_j^*) ,
\end{aligned} \tag{3.42}$$

where $R(\hat{\nu}, \hat{\nu}_j^*)$ is a continuous function at $\hat{\nu} = \hat{\nu}_j^*$ with $R(\hat{\nu}_j^*, \hat{\nu}_j^*) = 0$, and H
is the Hessian. So we obtain

$$\begin{aligned}
\Phi[\hat{\pi}_n](\hat{\nu}) - \Phi[\hat{\pi}_n](\hat{\nu}_j^*) \geq \langle \nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*), \hat{\nu} - \hat{\nu}_j^* \rangle \\
+ \frac{1}{2}\langle \hat{\nu} - \hat{\nu}_j^*, (H\Phi[\hat{\pi}_n](\hat{\nu}_j^*) - C_1)(\hat{\nu} - \hat{\nu}_j^*) \rangle ,
\end{aligned} \tag{3.43}$$

where C_1 is a non-negative constant which can be chosen arbitrarily close to zero when we restrict to balls with sufficiently small radii ε. The inf for the previous r.h.s. is obtained at the point
$\hat{\nu}^* = -(H\Phi[\hat{\pi}_n](\hat{\nu}_j^*) - C_1)^{-1}\nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*) + \hat{\nu}_j^*$, which leads to

$$\begin{aligned}
\inf_{\hat{\nu} \in B(\varepsilon, \pi\hat{\nu}_j^*)} (\Phi[\hat{\pi}_n](\hat{\nu}) - \Phi[\hat{\pi}_n](\hat{\nu}_j^*)) \\
\geq -\frac{1}{2}\langle \nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*), (H\Phi[\hat{\pi}_n](\hat{\nu}_j^*) - C_1)^{-1}\nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*) \rangle .
\end{aligned} \tag{3.44}$$

Non-degeneracy assumption 1) implies, together with the twice continuous differentiability of F, that there exists a positive constant $\tilde{K}$ such that

$$\langle x, H\Phi[\xi](\hat{\nu}_j^*)x \rangle \geq \tilde{K}||x||^2 , \tag{3.45}$$

for all ξ in a neighbourhood of π. Noticing that $||\nabla\Phi[\hat{\pi}_n](\hat{\nu}_j^*)|| \leq c||\hat{\pi}_n - \pi||$ we have

$$-\frac{K||\hat{\pi}_n - \pi||^2}{2} \leq \inf_{\hat{\nu} \in B(\varepsilon, \pi\hat{\nu}_j^*)} (\Phi[\hat{\pi}_n](\hat{\nu}) - \Phi[\hat{\pi}_n](\hat{\nu}_j^*)) \leq 0 , \tag{3.46}$$

with $K = \frac{c^2}{\tilde{K} - C_1}$, which is positive for ε_0 sufficiently small. $\square$

From the last right-hand side of (3.39) we have

$$
\cfrac{1}{1+\displaystyle\sum_{j\neq i}^{k}\cfrac{\displaystyle\sum_{\substack{\hat{\nu}:\\ \hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)}} e^{-n(\Phi[\hat{\pi}_n](\hat{\nu})-\Phi[\hat{\pi}_n](\hat{\nu}_j^*))}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_j^*)}}{\displaystyle\sum_{\substack{\hat{\nu}:\\ \hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_i^*)}} e^{-n(\Phi[\hat{\pi}_n](\hat{\nu})-\Phi[\hat{\pi}_n](\hat{\nu}_i^*))}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_i^*)}\displaystyle\prod_{b\in E'}(\mid\Lambda_n(b)\mid+1)^{-|E'|}}}
$$

$$
\geq \cfrac{1}{1+\displaystyle\sum_{j\neq i}^{k}\cfrac{\displaystyle\sum_{\substack{\hat{\nu}:\\ \hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)}} e^{-n\left(\displaystyle\inf_{\hat{\nu}\in\hat{M}_n\cap B(\varepsilon,\pi\hat{\nu}_j^*)}(\Phi[\hat{\pi}_n](\hat{\nu})-\Phi[\hat{\pi}_n](\hat{\nu}_j^*))\right)}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_j^*)}}{e^{-n(\Phi[\hat{\pi}_n](\tilde{\nu})-\Phi[\hat{\pi}_n](\hat{\nu}_i^*)))}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_i^*)}\prod_{b\in E'}(\mid\Lambda_n(b)\mid+1)^{-|E'|}}}
$$

$$
\geq \cfrac{1}{1+\displaystyle\sum_{j\neq i}^{k}\cfrac{\displaystyle\sum_{\substack{\hat{\nu}:\\ \hat{\pi}_n\hat{\nu}\in B(\varepsilon,\pi\hat{\nu}_j^*)}} e^{\frac{Kn\|\hat{\pi}_n-\pi\|^2}{2}}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_j^*)}}{e^{-C}e^{-n\Phi[\hat{\pi}_n](\hat{\nu}_i^*)}\prod_{b\in E'}(\mid\Lambda_n(b)\mid+1)^{-|E'|}}} \,.
$$

$$
\tag{3.47}
$$

In the first inequality we have chosen $\tilde{\nu}$ as a best-approximation of $\hat{\nu}_i^*$ in $\hat{M}_n$ to get rid of the sum in the denominator of the denominator. In the second inequality we have used Proposition 3.2.6, and moreover the bound on the corresponding discretization error of the order $1/n$ and the uniform boundedness of the first derivative of Φ. The sums over measures in balls only give rise to polynomial constants which are swallowed by the terms in the exponential (as we will see, because the random terms lifting the degeneracy between the minimizers will be of order squareroot.)

Now to the lowest order in $\hat{\pi}_n-\pi$, we have

$$
\Phi[\hat{\pi}_n](\hat{\nu}_i^*) = \Phi[\pi](\hat{\nu}_i^*) + \Phi_\pi[\pi](\hat{\nu}_i^*)(\hat{\pi}_n-\pi) + o(\|\hat{\pi}_n-\pi\|) \,. \tag{3.48}
$$

So the last right-hand side of (3.47) becomes

$$\geq \frac{1}{1 + \sum\limits_{j \neq i}^{k} e^{\frac{Kn\|\hat{\pi}_n - \pi\|^2}{2}} + C\, e^{-n\langle B_{\hat{\nu}_i^*} - B_{\hat{\nu}_j^*},\, \hat{\pi}_n - \pi\rangle} e^{-n \cdot o(\|\hat{\pi}_n - \pi\|)} \prod\limits_{b \in E'} (\mid \Lambda_n(b) \mid +1)^{2|E|}} \, .$$

$$(3.49)$$

We are considering n sufficiently large such that there is at least one element in $\{\hat{\nu} : \hat{\pi}_n \hat{\nu} \in B(\varepsilon, \pi \hat{\nu}_i^*)\}$. For $\eta \in \mathcal{H}_{i,n,0}^{\tau}$ we have that

$$\rho^{\varepsilon}[\eta](n)(\pi \hat{\nu}_i^*) > 1 - r(n) \, , \qquad (3.50)$$

with $\lim_{n \uparrow \infty} r(n) = 0$. Indeed, we defined the good-set $\mathcal{H}_{i,n,0}^{\tau}$ in such a way that $n\|\hat{\pi}_n - \pi\|^2 \leq n^{\frac{\tau}{2}}$ and $n\langle B_{\hat{\nu}_i^*} - B_{\hat{\nu}_j^*}, \hat{\pi}_n - \pi\rangle \geq n^{\tau}$. Here we see the reason for the choice of the cutoff.

In order to prove Lemma 3.2.4, let us write

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](L_n \in B(\varepsilon, \pi \hat{\nu}_i^*))$$
$$= \rho^{\varepsilon}[\eta](n)(\pi \hat{\nu}_i^*)(1 - \mu_{F,n}[\eta(1), \ldots, \eta(n)](d(L_n, \pi M^*) \geq \varepsilon)) \, . \qquad (3.51)$$

Now we can use the concentration property for the empirical distribution saying that $\forall \varepsilon > 0$ and for all $\eta \in \mathcal{H}_{i,n,0}^{\tau}$ we have

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](d(L_n, \pi M^*) \geq \varepsilon) \leq \hat{r}(n, \varepsilon) \, , \qquad (3.52)$$

with $\lim_{n \uparrow \infty} \hat{r}(n, \varepsilon) = 0$ for all positive ε. This concentration property is a consequence of the bound

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](d(L_n, \pi M^*) \geq \varepsilon)$$

$$\leq \prod_{b \in E'} (n\hat{\pi}_n(b) + 1)^{2|E|} \exp\left(-n \inf_{\substack{\hat{\nu} \in \hat{M}_n \\ d(\hat{\pi}_n \hat{\nu}, \pi M^*) \geq \varepsilon}} \Phi[\hat{\pi}_n](\hat{\nu}) + n \inf_{\hat{\nu}' \in \hat{M}_n} \Phi[\hat{\pi}_n](\hat{\nu}') \right)$$

$$\leq \prod_{b \in E'} (n\hat{\pi}_n(b) + 1)^{2|E|} e^{K_2 n\|\hat{\pi}_n - \pi\| + C_2} \times$$

$$\times \exp\left(-n \inf_{\substack{\hat{\nu} \in \mathcal{P}(E)^{|E'|} \\ d(\pi \hat{\nu}, \pi M^*) \geq \varepsilon}} \Phi[\pi](\hat{\nu}) + n \inf_{\hat{\nu}' \in \mathcal{P}(E)^{|E'|}} \Phi[\pi](\hat{\nu}') \right) \, ,$$

$$(3.53)$$

where in the second inequality we have used the Lipschitz property of Φ w.r.t. π and the control of the discretization error. On the good-sets we have $n\|\hat{\pi}_n - \pi\| \leq n^{\frac{1}{2}+\frac{\tau}{4}}$, while the quadratic nature of the minima gives us a term of exponential decay in n from the rightmost exponential, for any fixed $\varepsilon > 0$. This proves the concentration property. So the Lemma 3.2.4 follows from (3.50), (3.51) and (3.52).

Having proven, for a particular choice of the disorder variables, the concentration of the empirical distribution around a given minimizer, the following lemma represents the natural extension to averages.

Lemma 3.2.7. *For any real-valued continuous function g on $\mathcal{P}(E)$ the following holds:*

$$|\rho[\eta](n)(g) - g(\pi\hat{\nu}_j^*)| \leq \tilde{r}(n), \quad \forall \eta \in \mathcal{H}_{j,n,0}^\tau , \tag{3.54}$$

where $\lim_{n\uparrow\infty} \tilde{r}(n) = 0$.

Proof. Let $B(\varepsilon, \pi\hat{\nu}_j^*)$ be an ε-ball around the measure $\pi\hat{\nu}_j^*$. Then for any $\varepsilon > 0$ and integer n,

$$\begin{aligned}
&|\rho[\eta](n)(g) - g(\pi\hat{\nu}_j^*)| \\
&= \left| \rho[\eta](n)\left(1_{B(\varepsilon,\pi\hat{\nu}_j^*)}(g - g(\pi\hat{\nu}_j^*))\right) + \rho[\eta](n)\left(1_{B^{\complement}(\varepsilon,\pi\hat{\nu}_j^*)}(g - g(\pi\hat{\nu}_j^*))\right) \right| \\
&\leq \sup_{\nu \in B(\varepsilon,\pi\hat{\nu}_j^*)} |g(\nu) - g(\pi\hat{\nu}_j^*)| + 2\|g\|_\infty \, \rho[\eta](n)\left(B^{\complement}(\varepsilon, \pi\hat{\nu}_j^*)\right)
\end{aligned}$$

holds. Choosing first ε sufficiently small and then n sufficiently large proves the lemma. $\qquad\square$

Now comes the study of how the probability of the good-sets $\mathcal{H}_{j,n,l}^\tau$ behaves in the limit $n \uparrow \infty$. Out of this analysis the weights (3.11) will arise. The fundamental step is that the limit will not depend on any finite number l of coordinates η, while the corresponding tail will provide, using CLT, the longed-for weights. This together with the Stone-Weierstrass theorem and Lemma 3.2.7 are the overriding tools for proving Theorem 3.2.2.

Let us start the analysis looking at the n,l-dependent good-sets $\mathcal{H}_{i,n,l}^\tau$ in a slightly different way. For any $l < n$, we have

$$\mathcal{H}_{i,n,0}^\tau = \left\{ \eta \in (E')^n : \frac{\sqrt{nl}}{n} X_{[1,l]}[\eta] + \frac{\sqrt{n(n-l)}}{n} X_{[l+1,n]}[\eta] \in R_{i,n}^\tau \right\} . \tag{3.55}$$

Saying that $X_{[1,n]}[\eta] \in R_{i,n}^\tau$ means

$$a_n \langle X_{[1,l]}[\eta], B_i \rangle + b_n \langle X_{[l+1,n]}[\eta], B_i \rangle$$
$$- \max_{k \neq i} \left(a_n \langle X_{[1,l]}[\eta], B_k \rangle + b_n \langle X_{[l+1,n]}[\eta], B_k \rangle \right) > \delta_n , \qquad (3.56)$$

and $\|X_{[1,n]}[\eta]\| \leq n^{\frac{\tau}{4}}$, where $a_n = \frac{\sqrt{nl}}{n}$ and $b_n = \frac{\sqrt{n(n-l)}}{n}$ Let us further define a subregion of $\mathcal{H}_{i,n,0}^\tau$, namely $\mathcal{H}_{i,n,0}^\tau(l)$ as follows

$$\mathcal{H}_{i,n,0}^\tau(l) := \left\{ \eta \in (E')^n : a_n \langle X_{[1,l]}[\eta], B_i \rangle + b_n \langle X_{[l+1,n]}[\eta], B_i \rangle \right.$$
$$- \max_{k \neq i} \left(a_n \langle X_{[1,l]}[\eta], B_k \rangle \right) - \max_{k \neq i} \left(b_n \langle X_{[l+1,n]}[\eta], B_k \rangle \right) > \delta_n,$$
$$\left. \text{and } \|X_{[1,n]}[\eta]\| \leq n^{\frac{\tau}{4}} \right\} .$$

$$(3.57)$$

Remark 3.2.8. *While $\mathcal{H}_{i,n,0}^\tau$ does not depend on l , $\mathcal{H}_{i,n,0}^\tau(l)$ does, indeed the partitioning might change the* max-value.

It is worthwhile mentioning the following results.

Lemma 3.2.9. *For any integer l, $\mathbb{P}(\mathcal{H}_{i,n,0}^\tau \setminus \mathcal{H}_{i,n,0}^\tau(l))$ goes to zero in the limit $n \uparrow \infty$.*

Proof. Note that

$$\mathcal{H}_{i,n,0}^\tau \setminus \mathcal{H}_{i,n,0}^\tau(l) \subseteq \left\{ \eta \in (E')^n : \right.$$
$$\max_{k \neq i} \langle a_n X_{[1,l]}[\eta], B_k \rangle + \max_{k \neq i} \langle b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n$$
$$\geq \langle a_n X_{[1,l]}[\eta] + b_n X_{[l+1,n]}[\eta], B_i \rangle$$
$$\left. > \max_{k \neq i} \langle a_n X_{[1,l]}[\eta] + b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n \right\} ,$$

$$(3.58)$$

and furthermore,

$$\mathcal{H}_{i,n,0}^\tau \setminus \mathcal{H}_{i,n,0}^\tau(l) \subseteq \left\{ \eta : \frac{C(l)}{\sqrt{n}} + \max_{k \neq i} \langle b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n \right.$$
$$\geq \langle a_n X_{[1,l]}[\eta] + b_n X_{[l+1,n]}[\eta], B_i \rangle \qquad (3.59)$$
$$\left. > -\frac{C(l)}{\sqrt{n}} + \max_{k \neq i} \langle b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n \right\} ,$$

where $C(l) = \sqrt{l} \max_\eta \max_k |\langle X_{[1,l]}[\eta], B_k \rangle| \leq \sqrt{l} \max_k \|B_k\|_\infty$.

The set on the right-hand side of (3.59) can be written as

$$\left\{ \eta : \frac{C(l)}{\sqrt{n}} + \max_{k \neq i} \langle b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n \geq a_n \langle X_{[1,l]}[\eta], B_i \rangle + b_n \langle X_{[l+1,n]}[\eta], B_i \rangle \right\}$$

$$\cap$$

$$\left\{ \eta : -\frac{C(l)}{\sqrt{n}} + \max_{k \neq i} \langle b_n X_{[l+1,n]}[\eta], B_k \rangle + \delta_n < a_n \langle X_{[1,l]}[\eta], B_i \rangle + b_n \langle X_{[l+1,n]}[\eta], B_i \rangle \right\}$$

$$=$$

$$\left\{ \eta : (\frac{C(l)}{\sqrt{n}} - a_n \langle X_{[1,l]}[\eta], B_i \rangle + \delta_n) b_n^{-1} \geq \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \neq i} \langle X_{[l+1,n]}[\eta], B_k \rangle \right\}$$

$$\cap \left\{ \eta : (-\frac{C(l)}{\sqrt{n}} - a_n \langle X_{[1,l]}[\eta], B_i \rangle + \delta_n) b_n^{-1} \right.$$

$$\left. < \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \neq i} \langle X_{[l+1,n]}[\eta], B_k \rangle \right\}$$

$$\subset \left\{ \eta : (\frac{2C(l)}{\sqrt{n}} + \delta_n) b_n^{-1} \geq \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \neq i} \langle X_{[l+1,n]}[\eta], B_k \rangle \right\}$$

$$\cap \left\{ \eta : (-\frac{2C(l)}{\sqrt{n}} + \delta_n) b_n^{-1} < \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \neq i} \langle X_{[l+1,n]}[\eta], B_k \rangle \right\}.$$

$$(3.60)$$

If we define

$$\varphi^i(X_{[l+1,n]}[\eta]) := \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \neq i} \langle X_{[l+1,n]}[\eta], B_k \rangle , \qquad (3.61)$$

then

$$\mathbb{P}(\mathcal{H}^\tau_{i,n,0} \setminus \mathcal{H}^\tau_{i,n,0}(l))$$

$$\leq \mathbb{P} \left(\left\{ \eta : \varphi^i(X_{[l+1,n]}[\eta]) \in b_n^{-1} \left(\delta_n - \frac{2C(l)}{\sqrt{n}}, \delta_n + \frac{2C(l)}{\sqrt{n}} \right) \right\} \right) .$$

$$(3.62)$$

To take care of the n-dependence of the interval it's enough to notice that, $\forall \varepsilon > 0 \, \exists \, \bar{n}(\varepsilon)$ such that, for all $n > \bar{n}(\varepsilon)$ the following holds

$$b_n^{-1} \left(\delta_n - \frac{2C(l)}{\sqrt{n}}, \delta_n + \frac{2C(l)}{\sqrt{n}} \right) \subset (-\varepsilon, \varepsilon) . \qquad (3.63)$$

So

$$
\lim_{n\uparrow\infty} \mathbb{P}\left(\left\{\eta : \varphi^i(X_{[l+1,n]}[\eta]) \in b_n^{-1}\left(\delta_n - \frac{2C(l)}{\sqrt{n}}, \delta_n + \frac{2C(l)}{\sqrt{n}}\right)\right\}\right)
$$
$$
\leq \lim_{n\uparrow\infty} \mathbb{P}\left(\left\{\eta : \varphi^i(X_{[l+1,n]}[\eta]) \in (-\varepsilon, \varepsilon)\right\}\right) . \tag{3.64}
$$

By the multidimensional CLT we have

$$
\lim_{n\uparrow\infty} \mathbb{P}\left(\left\{\eta : \varphi^i(X_{[l+1,n]}[\eta]) \in (-\varepsilon, \varepsilon)\right\}\right) = \mathbb{P}_\pi\left(\varphi^i(G) \in (-\varepsilon, \varepsilon)\right) , \tag{3.65}
$$

where G is a centred Gaussian variable. Taking the limit $\varepsilon \downarrow 0$ and using the *non-degeneracy assumption 2)*, the lemma is proven. $\square$

We have just seen that, for any fixed integer l, there is a subregion of the good-set which will not play any role in the limit $n \uparrow \infty$. We focus now on the probability of the main part of the good-set, especially on how its limit does not depend on any finite number of η-coordinates. Let us formalize the previous heuristics.

The condition (3.57) defining $\mathcal{H}_{i,n,0}^\tau(l)$ can also be written as

$$
\langle X_{[l+1,n]}[\eta], B_i\rangle - \max_{k\neq i}\langle X_{[l+1,n]}[\eta], B_k\rangle
$$
$$
> \frac{a_n}{b_n}\left(\max_{k\neq i}\langle X_{[1,l]}[\eta], B_k\rangle - \langle X_{[1,l]}[\eta], B_i\rangle\right) + b_n^{-1}\delta_n , \tag{3.66}
$$
$$
\text{and } \|X_{[1,n]}[\eta]\| \leq n^{\frac{\tau}{4}} .
$$

Define the following sets

$$
A_{i,n}^\tau(l) := \left\{\eta : \langle X_{[l+1,n]}[\eta], B_i\rangle - \max_{k\neq i}\langle X_{[l+1,n]}[\eta], B_k\rangle\right.
$$
$$
\left.> -\frac{a_n}{b_n}C_2(l) + b_n^{-1}\delta_n, \|X_{[l+1,n]}[\eta]\| \leq b_n^{-1}n^{\frac{\tau}{4}}\right\} ,
$$
$$
\tag{3.67}
$$
$$
B_{i,n}^\tau(l) := \left\{\eta : \langle X_{[l+1,n]}[\eta], B_i\rangle - \max_{k\neq i}\langle X_{[l+1,n]}[\eta], B_k\rangle\right.
$$
$$
\left.> \frac{a_n}{b_n}C_2(l) + b_n^{-1}\delta_n, \|X_{[l+1,n]}[\eta]\| \leq b_n^{-1}(n^{\frac{\tau}{4}} - \tilde{C}_2(l))\right\} ,
$$

where $C_2(l) = \max_\eta |\max_{k\neq i}\langle X_{[1,l]}[\eta], B_k\rangle - \langle X_{[1,l]}[\eta], B_i\rangle |\leq 2\max_k \|B_k\|_\infty$, and $\tilde{C}_2(l) = \max_\eta \|X_{[1,l]}[\eta]\|$. These maxima give us the intended independence

of the set from $\eta \in (E')^l$ and we have

$$B_{i,n}^{\tau}(l) = (E')^l \times \mathcal{H}_{i,n,l}^{1,\tau} \ ,$$
$$A_{i,n}^{\tau}(l) = (E')^l \times \mathcal{H}_{i,n,l}^{2,\tau} \ ,$$

(3.68)

where

$$\mathcal{H}_{i,n,l}^{1,\tau} = \left\{ \eta \in (E')^{n-l} : \varphi^i(X_{[l+1,n]}[\eta]) > b_n^{-1} \left(\delta_n + a_n C_2(l) \right) \ , \right.$$

$$\left. \text{and } \|X_{[l+1,n]}[\eta]\| \le b_n^{-1}(n^{\frac{\tau}{4}} - \tilde{C}_2(l)) \right\} \ ,$$

$$\mathcal{H}_{i,n,l}^{2,\tau} = \left\{ \eta \in (E')^{n-l} : \varphi^i(X_{[l+1,n]}[\eta]) > b_n^{-1} \left(\delta_n - a_n C_2(l) \right) \ , \right.$$

$$\left. \text{and } \|X_{[l+1,n]}[\eta]\| \le b_n^{-1} n^{\frac{\tau}{4}} \right\} \ .$$

(3.69)

The following holds

$$A_{i,n}^{\tau}(l) \supseteq \mathcal{H}_{i,n,1}^{\tau}(l) \supseteq B_{i,n}^{\tau}(l) \ ,$$
$$\mathcal{H}_{i,n,l}^{1,\tau} \subseteq \mathcal{H}_{i,n,l}^{2,\tau} \ .$$

(3.70)

Lemma 3.2.10. *For any integer l, $\mathbb{P}(\mathcal{H}_{i,n,l}^{2,\tau} \setminus \mathcal{H}_{i,n,l}^{1,\tau})$ goes to zero in the limit $n \uparrow \infty$.*

Proof.

$$\mathcal{H}_{i,n,l}^{2,\tau} \setminus \mathcal{H}_{i,n,l}^{1,\tau} \subseteq \left\{ \eta : \frac{a_n}{b_n} C_2(l) + b_n^{-1}\delta_n \ge \langle X_{[l+1,n]}[\eta], B_i \rangle - \max_{k \ne i} \langle X_{[l+1,n]}[\eta], B_k \rangle \right.$$

$$\left. > -\frac{a_n}{b_n} C_2(l) + b_n^{-1}\delta_n \right\}$$

$$= \left\{ \eta : b_n^{-1} \left(\delta_n + a_n C_2(l) \right) \ge \varphi^i(X_{[l+1,n]}[\eta]) > b_n^{-1} \left(\delta_n - a_n C_2(l) \right) \right\} \ .$$

(3.71)

By the same argument we have used in Lemma 3.2.9, we have $\mathbb{P}(\mathcal{H}_{i,n,l}^{2,\tau} \setminus \mathcal{H}_{i,n,l}^{1,\tau}) \longrightarrow 0$ in the limit $n \uparrow \infty$. $\qquad \square$

Lemma 3.2.11. *For any integer l, $\lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}_{i,n,l}^{1,\tau}) = \mathbb{P}_\pi(G \in R_i)$ where $G \sim \mathcal{N}(0, \Sigma)$.*

Proof. From the previous lemma we know that

$$\lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}^{\tau}_{i,n,1}(l)) = \lim_{n\uparrow\infty} \mathbb{P}(B^{\tau}_{i,n}(l)) \quad \text{and}$$
$$\lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}^{2,\tau}_{i,n,l}) = \lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}^{1,\tau}_{i,n,l}) \ . \tag{3.72}$$

Now $\forall \varepsilon > 0 \ \exists \ n_0(\varepsilon)$ such that for all $n > n_0(\varepsilon)$ the following holds

$$\gamma^n_{-\varepsilon} \supset \mathcal{H}^{1,\tau}_{i,n,l} \supset \gamma^n_{\varepsilon} \ , \tag{3.73}$$

where $\gamma^n_{\varepsilon} = \{\eta : \varphi^i(X_{[l+1,n]}[\eta]) > \varepsilon\}$. Therefore

$$\lim_{n\uparrow\infty} \mathbb{P}(\gamma^n_{-\varepsilon}) \geq \lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}^{1,\tau}_{i,n,l}) \geq \lim_{n\uparrow\infty} \mathbb{P}(\gamma^n_{\varepsilon}) \ . \tag{3.74}$$

Applying the CLT to both the right and the left-hand side and taking the limit for $\varepsilon \downarrow 0$ we have

$$\lim_{n\uparrow\infty} \mathbb{P}(\mathcal{H}^{1,\tau}_{i,n,l}) = \mathbb{P}_{\pi}(G \in R_i) \ . \tag{3.75}$$

where $G \sim \mathcal{N}(0, \Sigma)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us now summarize what we have done above for the decompositions of the various regions of the η-configuration space.

$$1_{\mathcal{H}^{\tau}_{i,n,0}} = 1_{\mathcal{H}^{\tau}_{i,n,0}(l)} + 1_{\mathcal{H}^{\tau}_{i,n,0}\backslash\mathcal{H}^{\tau}_{i,n,0}(l)} \ ,$$
$$1_{B^{\tau}_{i,n}(l)} = 1_{(E')^l} 1_{\mathcal{H}^{1,\tau}_{i,n,l}} \ , \tag{3.76}$$
$$1_{A^{\tau}_{i,n}(l)} = 1_{(E')^l} 1_{\mathcal{H}^{2,\tau}_{i,n,l}} \ .$$

To state our next result let us fix one more notation. We let Ψ be a continuous real-valued function on $\mathcal{P}(\mathcal{P}(E)) \times (E')^m$, for some positive integer m.

Lemma 3.2.12. *Suppose Ψ is as above. Then under the* non-degeneracy *assumptions 1) and 2) the following holds:*

$$\lim_{n\uparrow\infty} \int_{\mathcal{H}^{\tau}_{i,n,0}} \mathbb{P}_{\pi}(d\eta)\Psi(\rho[\eta](n),\eta) = w_i \int_{(E')^m} \pi^{\otimes m}(d\eta)\Psi(\delta_{\pi\hat{\nu}_i},\eta), \tag{3.77}$$

where $\pi^{\otimes m}(d\eta) = \prod_{k=1}^m \pi(d\eta_k)$, $w_i = \mathbb{P}(G \in R_i)$ with $G \sim \mathcal{N}(0, \Sigma)$

Proof. Set $l = m$,

$$\int_{\mathcal{H}^\tau_{i,n,0}} \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n), \eta) = \int_{\mathcal{H}^\tau_{i,n,0}} \mathbb{P}_\pi(d\eta)(\Psi(\rho[\eta](n), \eta) - \Psi(\delta_{\pi\hat\nu_i}, \eta))$$
$$+ \int_{\mathcal{H}^\tau_{i,n,0}} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \ . \tag{3.78}$$

We can assume that Ψ is of the form $\Psi(\rho, \eta) = \tilde\Psi(\rho(g_1), \dots, \rho(g_l), \eta_{[1,m]})$ for a finite l with continuous and bounded g_i's, and continuous $\tilde\Psi$. So, together with the Lemma 3.2.7 we have that the first term in the left-hand side is going to 0 in the limit $n \uparrow \infty$. Now from the first equality of (3.76)

$$\int_{\mathcal{H}^\tau_{i,n,0}} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) = \int_{\mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta)$$
$$+ \int_{\mathcal{H}^\tau_{i,n,0} \setminus \mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \ .$$

Under the *non-degeneracy assumption 2)* the second term on the right-hand side of the above equation plays no role in the limit, indeed

$$\int_{\mathcal{H}^\tau_{i,n,0} \setminus \mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \le \parallel \Psi \parallel_\infty \mathbb{P}(\mathcal{H}^\tau_{i,n,0} \setminus \mathcal{H}^\tau_{i,n,0}(l)) \ , \tag{3.79}$$

and from the Lemma 3.2.9 $\mathbb{P}(\mathcal{H}^\tau_{i,n,0} \setminus \mathcal{H}^\tau_{i,n,0}(l))$ goes to zero in the limit $n \uparrow \infty$.

Observe from the first inclusion relation of (3.70) that

$$\int_{B^\tau_{i,n}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \le \int_{\mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \le \int_{A^\tau_{i,n}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \tag{3.80}$$

Next, observe from (3.68) that

$$\int_{(E')^l} \pi^{\otimes l}(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \int_{\mathcal{H}^{1,\tau}_{i,n,l}} \pi^{\otimes n-l}(d\eta) \le \int_{\mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \ ,$$
$$\int_{\mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \le \int_{(E')^l} \pi^{\otimes l}(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \int_{\mathcal{H}^{2,\tau}_{i,n,l}} \pi^{\otimes n-l}(d\eta) \ . \tag{3.81}$$

Taking the limit $n \uparrow \infty$ we obtain

$$\lim_{n\uparrow\infty} \int_{\mathcal{H}^\tau_{i,n,0}(l)} \mathbb{P}_\pi(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) = \int_{(E')^l} \pi^{\otimes l}(d\eta)\Psi(\delta_{\pi\hat\nu_i}, \eta) \lim_{n\uparrow\infty} \int_{\mathcal{H}^{2,\tau}_{i,n,l}} \pi^{\otimes n-l}(d\eta) \ , \tag{3.82}$$

and using Lemma 3.2.11 we are done.   □

Now we have provided all the ingredients, and so the proof of the Theorem 3.2.2 is straightforward.

$$\int \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta)$$
$$= \sum_{i=1}^{k} \int \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta)1_{\mathcal{H}^\tau_{i,n,0}}(\eta) + \int \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta)1_{(\mathcal{H}^\tau_{n,0})^{\complement}}(\eta) .$$
$$(3.83)$$

Clearly for bounded Ψ one has

$$\left| \int \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta)1_{(\mathcal{H}^\tau_{n,0})^{\complement}}(\eta) \right| \leq \|\Psi\|_\infty \mathbb{P}_\pi((\mathcal{H}^\tau_{n,0})^{\complement}(\eta)) , \qquad (3.84)$$

and the *non-degeneracy assumption 2)* tells us that this term will not play any role in the limit $n \uparrow \infty$. For every summand of the first term, by Lemma 3.2.12 we have

$$\lim_{n\uparrow\infty} \int_{\mathcal{H}^\tau_{i,n,0}} \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta) = w_i \int_{(E')^m} \pi^{\otimes m}(d\eta)\Psi(\delta_{\pi\hat{\nu}_i},\eta), \qquad (3.85)$$

where $w_i = \mathbb{P}(G \in R_i)$ with $G \sim \mathcal{N}(0,\Sigma)$.
Therefore

$$\lim_{n\uparrow\infty} \int \mathbb{P}_\pi(d\eta)\Psi(\rho[\eta](n),\eta) = \sum_{i=1}^{k} \int \mathbb{P}_\pi(d\eta)\Psi(\rho,\eta)w_i\delta_{\delta_{\pi\hat{\nu}_i}}(d\rho) . \qquad (3.86)$$

Looking now at the definition of the AW-metastate, we can identify the joint distribution K we are interested in as

$$K(d\rho, d\eta) = \sum_{i=1}^{k} \mathbb{P}_\pi(d\eta)w_i\delta_{\delta_{\pi\hat{\nu}_i}}(d\rho)$$
$$\Longrightarrow K(d\rho|\eta) = \sum_{i=1}^{k} w_i\delta_{\delta_{\pi\hat{\nu}_i}}(d\rho) . \qquad (3.87)$$

□

3.3 The Metastate on the Level of States

Let us go from the global perspective (talking about the empirical mean) to the local view (talking about finitely many variables $\sigma_1, \ldots, \sigma_k$). In different words, we are fixing a sub-population of finite size, and we are asking how it will behave when we couple it to a large system whose size n will be sent to infinity. Let us introduce a metric on the space of probability measures $\mu, \mu' \in \mathcal{P}(E^\infty)$ by

$$d(\mu, \mu') = \sum_{i=1}^{\infty} 2^{-i} ||\mu - \mu'||_i \, , \tag{3.88}$$

where

$$||\mu - \mu'||_i := \frac{1}{2} \sum_{\omega_1, \ldots, \omega_i} |\mu(\omega_1, \ldots, \omega_i) - \mu'(\omega_1, \ldots, \omega_i)| \, , \tag{3.89}$$

is the total variation norm of the restriction of the measure to the first i coordinates.

The statement about the metastate promised in the main theorem implies in particular that, for all $\varepsilon > 0$

$$\lim_{n \uparrow \infty} \mathbb{P}(d(\mu_{F,n}[\eta], ext(\mathcal{G}[\eta])) > \varepsilon) = 0 \, , \tag{3.90}$$

where

$$\mathcal{G}[\eta] = \{ \sum_{\hat{\nu} \in M^*} p_{\hat{\nu}} \mu_{\hat{\nu}}[\eta], \ p_{\hat{\nu}} \in \mathcal{P}(\pi M^*) \} \, , \tag{3.91}$$

and $\mu_{\hat{\nu}}[\eta](\cdot) = \prod_{i=1}^{\infty} \gamma[\eta(i)](\cdot | \pi \hat{\nu})$. Throughout this chapter we identify $\mu_{F,n}[\eta]$ with the infinite-volume measure which is obtained by tensorization with the equidistribution for sites outside of $\{1, \ldots, n\}$.

We will in fact prove that

$$\lim_{n \uparrow \infty} \sup_{\eta \in \mathcal{H}_{i,n,0}^\tau} d(\mu_{F,n}[\eta], \mu_{\hat{\nu}_i^*}[\eta]) = 0 \, , \tag{3.92}$$

where $\mathcal{H}_{i,n,0}^\tau$ are the disorder sets ensuring the dominance of the i-th minimizer. Let us remark that it can not be expected in general that the limit $\lim_{n \uparrow \infty} d(\mu_{F,n}[\eta], ext(\mathcal{G}[\eta])) = 0$ for $\mathbb{P}$-a.e. η, as already the example of the random field Ising model discussed in [50] shows, due to the empirical distribution $\hat{\pi}_n$ passing regions of "ties" outside of the good sets infinitely often.

We are about to prove that the possible limiting distributions will be product measures of a particular sort. These limiting measures will depend on which

region of the disorder variables we are restricting ourselves to. Let us look at the k-marginal

$$
\mu_{F,n}[\eta](\sigma_1, \ldots, \sigma_k) = \sum_{\omega_{k+1}, \ldots, \omega_n} \mu_{F,n}[\eta](\sigma_1, \ldots, \sigma_k, \omega_{k+1}, \ldots, \omega_n)
$$

$$
= \sum_{\omega_{k+1}, \ldots, \omega_n} \frac{e^{-nF(L_n^{\sigma_{[1,k]}, \omega_{[k+1,n]}})} \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \prod_{j=k+1}^{n} \alpha[\eta_j](\omega_j)}{\sum_{\bar{\sigma} \in E^n} e^{-nF(L_n^{\bar{\sigma}})} \prod_{i=1}^{n} \alpha[\eta_i](\bar{\sigma}_i)} \ .
$$

$$(3.93)$$

Let us now introduce the suitable decomposition of the empirical distribution, obtained by dividing the volume $\{1, \ldots, n\}$ in two subvolumes $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, where k is the size of the marginal we are considering; then we focus on the respective b-like sites for both of the subvolumes.

$$
L_n^{\sigma_{[1,k]}, \omega_{[k+1,n]}} = \frac{k}{n} \frac{1}{k} \sum_{i=1}^{k} \delta_{\sigma_i} + \frac{n-k}{n} \frac{1}{n-k} \sum_{i=k+1}^{n} \delta_{\omega_i}
$$

$$
= \frac{k}{n} \sum_{b \in E'} \hat{\pi}_{[1,k]}(b) \hat{L}_{[1,k]}(b) + \frac{n-k}{n} \sum_{b \in E'} \hat{\pi}_{[k+1,n]}(b) \hat{L}_{[k+1,n]}(b)
$$

$$(3.94)$$

In the process to carry out (3.94), we have also made use of the following definitions:

$$
\begin{aligned}
\Lambda_{[1,k]}(b) &= \{i \in \{1, \ldots, k\} : \eta(i) = b\} \ , \\
\Lambda_{[k+1,n]}(b) &= \{i \in \{k+1, \ldots, n\} : \eta(i) = b\} \ , \\
\hat{\pi}_{[1,k]}(b) &= \frac{|\Lambda_{[1,k]}(b)|}{k} \ , \quad \hat{\pi}_{[k+1,n]}(b) = \frac{|\Lambda_{[k+1,n]}(b)|}{n-k} \ .
\end{aligned}
$$

$$(3.95)$$

Proof of Theorem 3.1.7. Let us start providing the key result, namely the *weak convergence* of the measure $\mu_{F,n}[\eta]$ to $\mu_j[\eta] = \prod_{i=1}^{\infty} \gamma[\eta(i)](\cdot|\pi\hat{\nu}_j)$ conditional on the suitable region of the disorder.
The following Lemma is the short view (local topology) version of Lemma 3.2.7.

Lemma 3.3.1. *For any event A which depends only on the first k coordinates the following holds*

$$
| \mu_{F,n}[\eta](A) - \mu_j[\eta](A) | \leq \tilde{r}(n) \quad \forall \, \eta \in \mathcal{H}_{j,n,k}^{\tau} \ ,
$$

$$(3.96)$$

where $\lim_{n\uparrow\infty} \tilde{r}(n) = 0$.

Proof. It suffices to consider the event A which fixes the first k coordinates and write

$$\mu_{F,n}[\eta](\sigma_1,\ldots,\sigma_k) =$$

$$= \sum_{\omega_{k+1},\ldots,\omega_n} \frac{e^{-nF\left(\frac{k}{n}\sum_{b\in E'}\hat{\pi}_{[1,k]}(b)\hat{L}^{\sigma}_{[1,k]}(b) + \frac{n-k}{n}\sum_{b\in E'}\hat{\pi}_{[k+1,n]}(b)\hat{L}^{\omega}_{[k+1,n]}(b)\right)}}{\sum_{\bar{\sigma}\in E^n} e^{-nF(L^{\bar{\sigma}}_n)} \prod_{i=1}^{n}\alpha[\eta_i](\bar{\sigma}_i)} \times$$

$$\times \prod_{i=1}^{k}\alpha[\eta_i](\sigma_i) \prod_{j=k+1}^{n}\alpha[\eta_j](\omega_j)$$

$$= \frac{\displaystyle\sum_{\hat{\nu}\in\hat{M}_{[k+1,n]}} \sum_{\substack{\omega_{k+1},\ldots,\omega_n:\\ \hat{L}^{\tilde{\omega}}_{[k+1,n]}(\cdot)=\hat{\nu}(\cdot)}} e^{-nF\left(\frac{k}{n}\left\langle\hat{\pi}_{[1,k]},\hat{L}^{\sigma}_{[1,k]}\right\rangle + \frac{n-k}{n}\left\langle\hat{\pi}_{[k+1,n]},\hat{\nu}\right\rangle\right)}}{\displaystyle\sum_{\bar{\nu}\in\hat{M}_n} \sum_{\substack{\bar{\sigma}\in E^n:\\ \hat{L}^{\bar{\sigma}}_n=\bar{\nu}}} e^{-nF(L^{\bar{\sigma}}_n)} \prod_{i=1}^{n}\alpha[\eta_i](\bar{\sigma}_i)} \times$$

$$\times \prod_{i=1}^{k}\alpha[\eta_i](\sigma_i) \prod_{j=k+1}^{n}\alpha[\eta_j](\omega_j) \, ,$$

$$(3.97)$$

where we have introduced the following space

$$\hat{M}_{[k+1,n]} :=$$
$$\left\{\hat{\nu}\in\mathcal{P}(E)^{E'} : \exists\,\tilde{\omega}=(\tilde{\omega}_{k+1},\ldots,\tilde{\omega}_n) : \hat{L}^{\tilde{\omega}}_{[k+1,n]}(b)=\hat{\nu}(b), \forall\, b\in E'\right\} \, . \quad (3.98)$$

Using the partition induced by the disorder variables η on the sub-volume $\{k+1,\ldots,n\}$, we have

$$\sum_{\substack{\omega_{k+1},\ldots,\omega_n:\\ \hat{L}^{\omega}_{[k+1,n]}(\cdot)=\hat{\nu}(\cdot)}} \prod_{j=k+1}^{n}\alpha[\eta_j](\omega_j) = \prod_{b\in E'}\alpha[b]^{|\Lambda_{[k+1,n]}(b)|}\left(\Omega_{|\Lambda_{[k+1,n]}(b)|}(\hat{\nu}(b))\right) \, .$$

$$(3.99)$$

To avoid heavy notation, with shall "improperly" refer to the above by $\alpha(\Omega)$.

Then we have

$$\mu_{F,n}[\eta](\sigma_1,\ldots,\sigma_k) =$$

$$\frac{\displaystyle\prod_{i=1}^{k}\alpha[\eta_i](\sigma_i)\sum_{\hat\nu\in\hat M_{[k+1,n]}}e^{-nF\left(\frac{k}{n}\left\langle\hat\pi_{[1,k]},\hat L_{[1,k]}^{\sigma}\right\rangle+\frac{n-k}{n}\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)}\cdot\alpha(\Omega)}{\displaystyle\sum_{\nu'\in\hat M_{[1,k]}}\sum_{\hat\nu\in\hat M_{[k+1,n]}}\sum_{\substack{\bar\sigma\in E^k:\\\hat L_{[1,k]}^{\bar\sigma}=\nu'}}e^{-nF\left(\frac{k}{n}\left\langle\hat\pi_{[1,k]},\hat L_{[1,k]}^{\bar\sigma}\right\rangle+\frac{n-k}{n}\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)}\prod_{i=1}^{k}\alpha[\eta_i](\bar\sigma_i)\cdot\alpha(\Omega)}\,.$$

$$(3.100)$$

Now multiplying and dividing, both numerator and denominator, by $e^{-nF(\langle\hat\pi_{[k+1,n]},\hat\nu\rangle)}$ we arrive at:

$$\mu_{F,n}[\eta](\sigma_1,\ldots,\sigma_k) =$$

$$\frac{\displaystyle\prod_{i=1}^{k}\alpha[\eta_i](\sigma_i)\sum_{\hat\nu\in\hat M_{[k+1,n]}}e^{-n\left[F\left(\frac{k}{n}\left\langle\hat\pi_{[1,k]},\hat L_{[1,k]}^{\sigma}\right\rangle+\frac{n-k}{n}\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)-F\left(\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)\right]}\rho_{n,k}^{F}[\eta](\hat\nu)}{\displaystyle\sum_{\substack{\nu'\in\\\hat M_{[1,k]}}}\sum_{\substack{\hat\nu\in\\\hat M_{[k+1,n]}}}\sum_{\substack{\bar\sigma\in E^k:\\\hat L_{[1,k]}^{\bar\sigma}=\nu'}}e^{-n\left[F\left(\frac{k}{n}\left\langle\hat\pi_{[1,k]},\hat L_{[1,k]}^{\bar\sigma}\right\rangle+\frac{n-k}{n}\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)-F\left(\left\langle\hat\pi_{[k+1,n]},\hat\nu\right\rangle\right)\right]}\prod_{i=1}^{k}\alpha[\eta_i](\bar\sigma_i)\rho_{n,k}^{F}[\eta](\hat\nu)}$$

$$(3.101)$$

where $\rho_{n,k}^{F}[\eta]\in\mathcal{P}(\hat M_{[k+1,n]})$ is defined as

$$\rho_{n,k}^{F}[\eta](\hat\nu) := \frac{e^{-nF(\langle\hat\pi_{[k+1,n]},\hat\nu\rangle)}\prod_{b\in E'}\alpha[b]^{|\Lambda_{[k+1,n]}(b)|}\left(\Omega_{|\Lambda_{[k+1,n]}(b)|}(\hat\nu(b))\right)}{\displaystyle\sum_{\tilde\nu\in\hat M_{[k+1,n]}}e^{-nF(\langle\hat\pi_{[k+1,n]},\hat\nu\rangle)}\prod_{b\in E'}\alpha[b]^{|\Lambda_{[k+1,n]}(b)|}\left(\Omega_{|\Lambda_{[k+1,n]}(b)|}(\tilde\nu(b))\right)}\,.$$

$$(3.102)$$

Note that the measure $\rho_{n,k}^{F}[\eta]$ depends on the disorder variables just in the subvolume $\{k+1,\ldots,n\}$.

Recall that a function $F:\mathcal{P}(E)\to\mathbb{R}$ is differentiable if, for all $\alpha\in\mathcal{P}(E)$ there is a linear map $dF_\alpha:T(\mathcal{P}(E))\to\mathbb{R}$ on the tangent space such that

$$F(\alpha') = F(\alpha) + dF_\alpha(\alpha'-\alpha) + \|\alpha'-\alpha\|r(\alpha',\alpha)\,,\qquad(3.103)$$

where $\alpha'\to r(\alpha',\alpha)$ is continuous at $\alpha'=\alpha$ with $r(\alpha,\alpha)=0$. Then, uniformly in α,α',

$$\sup_{\alpha,\alpha'}|F(\alpha+p(\alpha'-\alpha))-F(\alpha)-pdF_\alpha(\alpha'-\alpha)|\leq Cpr(p)\,,\qquad(3.104)$$

where $r(p) \downarrow 0$ as p tends to 0. The uniformity in α, α' follows by the compactness of $\mathcal{P}(E)$. In our set up, we have

$$|F\left(\frac{k}{n}L_{[1,k]}(\sigma) + \frac{n-k}{n}\nu\right) - F(\nu) - dF_\nu \frac{k}{n}(L_{[1,k]}(\sigma) - \nu)| \leq C\frac{k}{n}r(\frac{k}{n}) \ , \quad (3.105)$$

where we have set $\nu = \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle$. This gives, recognizing that $\left\langle \hat{\pi}_{[1,k]}, \hat{L}^\sigma_{[1,k]} \right\rangle$ and ν are both elements in $\mathcal{P}(E)$, the upper bound

$$\mu_{F,n}[\eta](\sigma_1, \ldots, \sigma_k)$$

$$\leq \frac{e^{2Ckr(\frac{k}{n})} \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \sum_{\hat{\nu} \in \hat{M}_{[k+1,n]}} \prod_{i=1}^{k} e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\sigma_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)} \rho^F_{n,k}[\eta](\hat{\nu})}{\sum_{\hat{\nu} \in \hat{M}_{[k+1,n]}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \sum_{\bar{\sigma}_i \in E} \alpha[\eta_i](\bar{\sigma}_i) e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\bar{\sigma}_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)}} ,$$

$$(3.106)$$

and the corresponding lower bound which is obtained from the last r.h.s. by replacing $C > 0$ with $-C$. The measure $\rho^F_{n,k}[\eta]$ can be written in the form

$$\rho^F_{n,k}[\eta](\hat{\nu}) = \frac{\rho^F_{n-k}[\eta](\hat{\nu})e^{-kF(\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)}}{\sum_{\bar{\nu} \in \hat{M}_{[k+1,n]}} \rho^F_{n-k}[\eta](\bar{\nu})e^{-kF(\left\langle \hat{\pi}_{[k+1,n]}, \bar{\nu} \right\rangle)}} , \quad (3.107)$$

where we have recovered the proper random mean-field measure on the empirical distribution of size $n - k$.

Note once again that $\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle \in \mathcal{P}(E)$, and when $\hat{\nu}$ moves in $\hat{M}_{[k+1,n]}$, the corresponding measure $\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle$ is moving among the possible empirical measures of size $n - k$.

Let us write $C(A, \varepsilon) := \{\nu \in \mathcal{P}(E)^{E'} : d(\nu, A) \leq \varepsilon\}$, for the ε-ball of a set A. By definition, $\rho^F_n[\eta]$ is said to concentrate on the set A iff $\rho^F_n[\eta](C(A, \varepsilon)^\complement) \downarrow 0$, $\forall \varepsilon > 0$. So whenever $\rho^F_n[\eta]$ concentrates on a finite set, so does $\rho^F_{n,k}[\eta]$, by the boundedness of F.

We remark that $\rho^F_{n-k}[\eta](\hat{\nu}) = \mu_{F,n-k}[\eta](L_{[k+1,n]} = \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)$ has the property of concentrating around the minimizers, as we know from the analysis in the previous chapter.

To study the bound (3.106) and the corresponding lower bound, let's introduce the quantities

$$
\xi_N = \sum_{\hat{\nu} \in \hat{M}_{[k+1,n]}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \prod_{i=1}^{k} e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\sigma_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)} \,,
$$

$$
\xi_D = \sum_{\hat{\nu} \in \hat{M}_{[k+1,n]}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \sum_{\tilde{\sigma}_i \in E} \alpha[\eta_i](\tilde{\sigma}_i) \prod_{i=1}^{k} e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\tilde{\sigma}_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)} \,.
$$

$$(3.108)$$

Let us decompose the $\sum_{\hat{\nu} \in \hat{M}_{[k+1,n]}}$ over $C(M^*, \varepsilon)$ and its complement and compare the terms with their values at the midpoints:

$$
\xi_N = \sum_{\hat{\nu}^* \in M^*} \sum_{\substack{\hat{\nu} \in \\ B(\varepsilon, \hat{\nu}^*) \cap \hat{M}_{[k+1,n]}}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \prod_{i=1}^{k} e^{-dF_{\pi\hat{\nu}^*}(\delta_{\sigma_i} - \pi\hat{\nu}^*)}
$$

$$
+ \sum_{\hat{\nu}^* \in M^*} \sum_{\substack{\hat{\nu} \in \\ B(\varepsilon, \hat{\nu}^*) \cap \hat{M}_{k+1,n}}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \times
$$

$$
\times \left[\prod_{i=1}^{k} e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\sigma_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)} - \prod_{i=1}^{k} e^{-dF_{\pi\hat{\nu}^*}(\delta_{\sigma_i} - \pi\hat{\nu}^*)} \right]
$$

$$
+ \sum_{\substack{\hat{\nu} \in \\ C(M^*, \varepsilon)^{\complement} \cap \hat{M}_{[k+1,n]}}} \rho^F_{n,k}[\eta](\hat{\nu}) \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) \prod_{i=1}^{k} e^{-dF\left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle (\delta_{\sigma_i} - \left\langle \hat{\pi}_{[k+1,n]}, \hat{\nu} \right\rangle)} \,.
$$

$$(3.109)$$

The sum in the last line is bounded by a function $r_{\mathrm{cp}}(\varepsilon, n)$, where $\lim_{n \uparrow \infty} r_{\mathrm{cp}}(\varepsilon, n) = 0$ when η is in the union of the good-sets. This holds by the concentration property of the empirical distribution given in (3.52) applied to the measure for sites $\geq k$, using the boundedness of the first derivative of F.

The second line is bounded in modulus by a function $\gamma(\varepsilon)$ where $\lim_{\varepsilon \downarrow 0} \gamma(\varepsilon) = 0$ by the twice continuous differentiability of F.

This implies the bounds

$$
\left| \xi_N - \sum_{\hat{\nu}^* \in M^*} \rho^F_{n,k}[\eta](\tilde{B}(\varepsilon, \hat{\nu}^*)) \prod_{i=1}^{k} \alpha[\eta_i](\sigma_i) e^{-dF_{\pi\hat{\nu}^*}(\delta_{\sigma_i} - \pi\hat{\nu}^*)} \right|
$$

$$
\leq \gamma(\varepsilon) + r_{cp}(\varepsilon, n) \,,
$$

$$(3.110)$$

under the assumption that η is in the union of the good-sets.

Summing over the finitely many values of $\sigma_1, \ldots, \sigma_k$ we obtain the same type of bounds (with possibly worse functions $\gamma(\varepsilon), r_{cp}(\varepsilon, n)$) for ξ_D.

Recall the definition of the kernels (3.10) and choose the disorder variable $\eta \in \mathcal{H}^{\tau}_{i,n,k}$ in the part of the good-set which ensures the dominance of the i-th minimizer. This gives that

$$| \mu_{F,n}[\eta](\sigma_1, \ldots, \sigma_k) - \prod_{j=1}^{k} \gamma[\eta_j](\sigma_j | \pi \hat{\nu}^*_i) | \leq \ \zeta(\varepsilon) + \ \chi(\varepsilon, n) \ , \qquad (3.111)$$

where $\lim_{n\uparrow\infty} \chi(\varepsilon, n) = 0$ and $\lim_{\varepsilon\downarrow 0} \zeta(\varepsilon) = 0$. This proves the lemma, as well as the statement (3.92). $\qquad\square$

Lemma 3.3.2. *Let Ξ be a continuous real-valued function on $\mathcal{P}(E^\infty) \times (E')^\infty$. Then under the* non-degeneracy *assumptions 1) and 2) the following holds:*

$$\lim_{n\uparrow\infty} \int_{\mathcal{H}^{\delta_n}_{i,n}} \mathbb{P}_\pi(d\eta)\Xi(\mu_{F,n}[\eta], \eta) = w_i \int \mathbb{P}_\pi(d\eta)(d\eta)\Xi(\mu_i[\eta], \eta) \ , \qquad (3.112)$$

where $w_i = \mathbb{P}_\pi(G \in R_i)$.

The proof of this lemma, thanks to the continuity of Ξ which allows finite dimensional approximation and to the Lemma 3.3.1, follows the trail drawn by the proof of the Lemma 3.2.12.

Using all the tools we have provided, we find

$$\lim_{n\uparrow\infty} \int \mathbb{P}_\pi(\eta)\Xi(\mu_{F,n}[\eta], \eta) = \sum_{j=1}^{k} \int \mathbb{P}_\pi(\eta)\Xi(\mu, \eta)w_j\delta_{\mu_j[\eta]}(d\mu) \ , \qquad (3.113)$$

where we can identify

$$J(d\mu, d\eta) = \sum_{j=1}^{k} \mathbb{P}_\pi(d\eta)w_j\delta_{\mu_j[\eta]}(d\mu)$$

$$\Longrightarrow J(d\mu|\eta) = \sum_{j=1}^{k} w_j\delta_{\mu_j[\eta]}(d\mu) \ . \tag{3.114}$$

This finishes the proof of the Main Theorem 3.1.7 and concludes the chapter. $\square$

4 Gibbs-non-Gibbs Properties for Evolving Ising Models on Trees[*]

4.1 Introduction

In this chapter, we consider the Gibbsian properties of homogeneous low-temperature Ising Gibbs measures on trees, subjected to an infinite-temperature Glauber evolution. This problem has been considered before on regular lattices see e.g. [11, 16, 17, 19, 22, 53, 54, 57, 64, 66], for Ising spins, n-vector and unbounded spins, and for various finite- or infinite-temperature dynamics, of Glauber, Kawasaki, or diffusion type, and even for non-Markovian evolutions.

At high initial temperatures, or for sufficiently short times, standard methods can be used to prove Gibbsianness, also in our situation. Thus the interesting case is to find out what happens for low initial temperatures. As usual (but see the mean-field analysis of [23]), low-temperature dynamics are beyond reach so far. For simplicity, we will consider infinite-temperature dynamics, but high-temperature evolutions are expected to behave qualitatively similarly.

In contrast to what happens on regular lattices such as $\mathbb{Z}^d$, the Gibbsian properties of evolved Gibbs measures for models on trees turn out to depend on which of the different Gibbs measures (plus or minus, versus intermediate) one considers. In all cases there are two transition times: for the intermediate measure, after the first transition time it becomes non-Gibbsian in the familiar sense that some, but not all, configurations are " bad" (that is, they are points of discontinuity), while it turns out that after a certain later time the evolved intermediate Gibbs measure becomes "totally bad"; thereafter it has the surprising property that *all* spin configurations are discontinuity points.

This last property is something which will not happen for the other two extremal invariant Gibbs measures. For those measures, although after a first transition time they also become non-Gibbsian, after the second transition time they become Gibbsian again.

We will provide proofs for these results on the Cayley tree, and don't aim for the greatest generality here, but we will indicate why these results should be expected to hold more generally. We present our results both in zero and non-zero external fields.

Our analysis illustrates (again) how different models on trees are as compared to models on regular (amenable) lattices.

Non-Gibbsian properties of some other measures of statistical mechanical origin on trees have been considered before, [36, 38, 56]. For FK measures as well as fuzzy Potts models on trees, the possibility of having a positive-measure set of "bad" discontinuity points was found.

4.2 Background Facts and Notation

4.2.1 The Ising Model on a Cayley Tree

Let $\mathfrak{CT}(d)$ be a Cayley tree for some $d \geq 1$, that is the unique connected tree with $|\partial i| = d + 1$ for all $i \in \mathfrak{CT}(d)$. Let $\Omega = \{-1, +1\}^{\mathfrak{CT}(d)}$, endowed with the product topology. Elements in Ω are denoted by σ. A configuration σ assigns to each vertex $x \in \mathfrak{CT}(d)$ a spin value $\sigma(x) = \pm 1$. Denote by $\mathbb{S}$ the set of all finite subtrees of $\mathfrak{CT}(d)$. For $\Lambda \in \mathbb{S}$ and $\sigma \in \Omega$ we denote by σ_Λ the restriction of σ to Λ, while Ω_Λ denotes the set of all such restrictions. Let $\Lambda \in \mathfrak{CT}(d)$ be any set, finite or infinite. We denote by E_Λ its set of edges and by V_Λ its set of vertices.

Let now $\Lambda \in \mathbb{S}$, hence finite. We will consider the nearest-neighbour Ising model on the tree. The finite-volume Gibbs distribution on any finite subtree Λ for an Ising model in an inhomogeneous external field, given by fields h_i at sites i, boundary condition ω, at inverse temperature β, is defined by the following Boltzmann-Gibbs distribution

$$\mu_\Lambda^\omega(\sigma_\Lambda) =$$

$$\frac{1}{Z_\Lambda^\omega(\beta, \{h_i\}_{i \in V_\Lambda})} \exp\left\{ \beta \sum_{\{i,j\} \in E_\Lambda} \sigma_i \sigma_j + \sum_{i \in V_\Lambda} h_i \sigma_i + \sum_{\substack{\{i,j\} \\ i \in V_\Lambda, j \in V_{\Lambda^\complement}}} \sigma_i \omega_i \right\} . \tag{4.1}$$

Infinite-volume Gibbs measures are defined by having their conditional probabilities of finite-volume configurations, conditioned on the configurations outside the volume, of this Gibbsian form, see e.g. [29] and [18]. In equation form

we require that for all volumes Λ and configurations σ_Λ μ satisfies

$$\mu(\sigma_\Lambda) = \int \mu_\Lambda^\omega(\sigma_\Lambda)\mu(d\omega) . \tag{4.2}$$

The infinite-volume Gibbs measures are parametrized by the external magnetic fields (in most of what follows we will consider a homogeneous field h_0), and by the inverse temperature $\beta \geq 0$. This will lead us to consider finite-volume Gibbs distributions with this same homogeneous field plus a possibly different boundary field. We put $\beta(1) = \infty$ and, for $d > 1$,

$$\beta(d) = \operatorname{arccoth} d = \frac{1}{2} \ln \frac{d+1}{d-1} ,$$
$$h(\beta, d) = \left[d \operatorname{arctanh} \left(\frac{dw-1}{d\bar{w}-1} \right)^{\frac{1}{2}} - \operatorname{arctanh} \left(\frac{d-\bar{w}}{d-w} \right)^{\frac{1}{2}} \right] \mathbb{I}_{\beta > \beta(d)} , \tag{4.3}$$

where $w = \tanh \beta = \bar{w}^{-1}$.

It is known (see [29]), that if $\beta > \beta(d)$ and $|h_0| \leq h(\beta, d)$, then the system exhibits a phase transition. Throughout this chapter we will assume $|h_0| < h(\beta, d)$, $\beta > \beta(d)$, and $d > 1$, whenever the opposite is not indicated. This condition ensures the existence of three homogeneous phases which, in accordance with Georgii, we denote μ^-, $\mu^\sharp$, μ^+. The plus and minus measures μ^- and μ^+ are obtainable via the standard procedure of taking plus and minus boundary conditions and then taking the thermodynamic limit, but there exists also an intermediate measure $\mu^\sharp$ (which in the case of zero external field can be obtained by imposing free boundary conditions). These phases are extremal in the set of invariant infinite-volume Gibbs measures; μ^+ and μ^- are also extremal in the set of all infinite-volume Gibbs measures, whereas $\mu^\sharp$ becomes non-extremal in this set below a certain temperature strictly smaller than the phase transition temperature [3, 43]; however, this second transition will not concern us here.

Let Γ_n be the Cayley tree with n generations and $\Gamma_{n-1} = \Gamma_n \setminus \partial\Gamma_n$ the (sub-) Cayley tree with $n-1$ generations, where $\partial\Gamma_n$ stands for the inner boundary of Γ_n. It is a known result for the Ising model on trees that the marginal on Γ_{n-1} of the finite-volume Gibbs measure on Γ_n is a finite-volume Gibbs measure on Γ_{n-1}, with a possibly different external magnetic field at the boundary. See Appendix A for how this works out in marginalizing infinite-volume Gibbs measures by using boundary laws.

Marginalizing on Γ_{n-1}, that is to a tree of one generation less, leaves us with a finite-volume Gibbs measure on Γ_{n-1}, parametrized by the following external

fields

$$i \in \partial\Gamma_{n-1}, \ h_i = h_0 + d\varphi(h_n) \ ,$$
$$i \in \Gamma_{n-2}, \quad h_i = h_0 \ ,$$

(4.4)

where $\varphi(x) = \operatorname{arctanh}(\tanh\beta \tanh x)$.

Thus, summarising, taking the marginal of an Ising model Gibbs measure on a tree with n generations with homogeneous boundary field h_n results in an Ising model on an $(n-1)$-generation tree with a homogeneous boundary field h_{n-1}. The map from h_n to h_{n-1}, (4.4), has three fixed points $h^+, h^\sharp$ and h^-. (Equivalently, one could consider the map from the magnetisation at generation n to the magnetisation at generation $n-1$, which again has the corresponding three fixed points $m^+, m^\sharp$ and m^-.) Whereas h^+ and h^- are stable, $h^\sharp$ is an unstable fixed point which implies that weak positive boundary conditions will result in a plus state, once one is far enough from the boundary. In other words, the phase transition is *robust* [65].

These three fixed points determine the three homogeneous extremal invariant infinite-tree Gibbs measures mentioned above.

4.2.2 Dynamics, non-Gibbsian Measures, Main Questions

Let $\mathcal{P}(\Omega, \mathcal{F})$ be the set of all probability measures on Ω and $\mathcal{G}(\beta, h_0)$ be the set of all Gibbs measures of the Ising model with an inverse temperature β and external field h_0. Let $\mathcal{P}_{I(B)}(\Omega, \mathfrak{F})$ denote the set of all $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ which are invariant under all the graph automorphisms (translations, rotations, reflections etc). Let $\mu \in \mathcal{G}_{I(B)}(\beta, h_0)$, where $\mathcal{G}_{I(B)}(\beta, h_0) = \mathcal{G}(\beta, h_0) \bigcap \mathcal{P}_{I(B)}(\Omega, \mathcal{F})$.

We aim to study here the time-dependence of the Gibbsian property of the three Gibbs measures $\mu^\star$, for $\star \in \{+, -, \sharp\}$, under an infinite-temperature Glauber dynamics. This is the stochastic evolution $S(t)$ which is obtained by having independent spin flips at each vertex at a certain given rate. In other words, we want to investigate whether or not $\mu^\star S(t) =: \hat{\mu}$ is a Gibbs measure at a given time $t > 0$.

By assumption the initial measure μ is a Gibbs measure. This immediately guarantees the non-nullness of the measure $\mu^\star S(t)$ for all t (including $t = 0$). It will thus suffice to study whether the transformed measure is quasi-local or not.

Define $\hat{\mu}_\Lambda(f|\omega) = \mathbb{E}_{\hat{\mu}}(f|\mathcal{F}_{\Lambda^\complement})(\omega)$ to be a realization of the corresponding conditional expectation for bounded f, finite $\Lambda \subset \mathbb{S}$, $\omega \in \Omega$. We also use the notation $\hat{\mu}_\Lambda(f|\omega_W) = \mathbb{E}_{\hat{\mu}}(f|\mathcal{F}_W)(\omega)$ when we condition only on configurations on a finite subset of sites $W \subset \Lambda^\complement$. With this notation we have, e.g.,

$\hat{\mu}_\Lambda(f|\omega_{\Lambda'\setminus\Lambda}) = \int \hat{\mu}_\Lambda(f|\omega)\hat{\mu}(d\omega_{(\Lambda')^c})$, for volumes $\Lambda' \supset \Lambda$ where $\hat{\mu}(d\omega_{(\Lambda')^c})$ denotes integration over the variables outside Λ'.

The measure $\hat{\mu}$ is not quasilocal, if it is not consistent with any quasilocal specification. To prove this, it is enough to find a single, nonremovable, point of discontinuity (in the product topology) for a single $\hat{\mu}_\Lambda$ for a single (quasi)local function f [18, 24]. The definition of the non-quasilocality for the transformed measure can be refined, see in particular [24]. The relevant definitions read as follows:

Definition 4.2.1. *The measure $\hat{\mu}$ is not quasilocal at $\bar{\eta} \in \Omega$ if there exists $\Lambda_0 \in \mathbb{S}$ and f local such that no realization of $\hat{\mu}_{\Lambda_0}(f|\cdot)$ is quasilocal at $\bar{\eta}$.*

In other words, any realization of $\hat{\mu}_{\Lambda_0}(f|\cdot)$ must exhibit an essential discontinuity at $\bar{\eta}$; one that survives zero-measure modifications. (Remember that conditional probabilities are only defined up to measure-zero sets.)

Definition 4.2.2. *For a local function f as above, $\hat{\mu}_{\Lambda_0}(f|\cdot)$ is $\hat{\mu}$-essentially discontinuous at $\bar{\eta}$, if there exists an $\varepsilon > 0$ such that*

$$\limsup_{\Lambda\uparrow\infty} \sup_{\substack{\xi^1,\xi^2 \\ \Lambda'\supset\Lambda \\ |\Lambda'|<\infty}} |\hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}) - \hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda})| > \varepsilon . \tag{4.5}$$

If $\hat{\mu}_{\Lambda_0}(f|\cdot)$ is $\hat{\mu}$-essentially discontinuous at $\bar{\eta}$, informally it means that there exists an $\varepsilon > 0$ such that for every $\Lambda \in \mathbb{S}$ there exists $\Lambda' \supset \Lambda$ and configurations ξ^1, ξ^2, such that

$$\left|\hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}\eta) - \hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda}\eta)\right| > \varepsilon , \tag{4.6}$$

for $\eta \in A$, where $A \in \mathcal{F}_{(\Lambda')^c}$ is of positive $\hat{\mu}$-measure.

Definition 4.2.3. *$\hat{\mu}_{\Lambda_0}(f|\cdot)$ is strongly discontinuous at $\bar{\eta}$, iff there exists an $\varepsilon > 0$ such that*

$$\limsup_{\Lambda\uparrow\infty} \sup_{\substack{\xi^1,\xi^2 \\ \Lambda'\supset\Lambda \\ |\Lambda'|<\infty}} \inf_{\substack{\eta^1,\eta^2 \\ \Lambda''\supset\Lambda' \\ |\Lambda''|<\infty}} |\hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}\eta^1_{\Lambda''\setminus\Lambda'}) - \hat{\mu}_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda}\eta^2_{\Lambda''\setminus\Lambda'})| > \varepsilon .$$
$$\tag{4.7}$$

Remark 4.2.4. *Intuitively the difference is that whereas for $\hat{\mu}$-essential discontinuity one needs to estimate a difference on two measurable sets of positive measure, for a strong discontinuity one needs an estimate of a difference on open sets; however, because of the impossibility of conditioning on individual configurations, we get the somewhat unwieldy definitions above.*

A useful tool to study the question whether $\hat{\mu}$ stays Gibbs is to consider the joint two-time distribution ν on (σ, η), where the initial spins σ are distributed according to μ, and the evolved spins η according to $\hat{\mu}$. This joint distribution will be denoted by either ν or ν^t. It can be viewed as a Gibbs measure on $\{-1, +1\}^L$ with $L = \mathfrak{CT}(d) \cup \mathfrak{CT}(d)$ consisting of two "layers" of $\mathfrak{CT}(d)$. Formally, the Hamiltonian of ν^t is

$$H_t(\sigma, \eta) = H_\mu(\sigma) - \ln p_t(\sigma, \eta) , \tag{4.8}$$

where $p_t(\sigma, \eta)$ is the transition kernel of the dynamics. We consider independent spin-flip dynamics, so

$$\ln p_t(\sigma, \eta) = \sum_{x \in \mathfrak{CT}(d)} \frac{1}{2} \ln \frac{1 + e^{-t}}{1 - e^{-t}} \sigma(x) \eta(x) \tag{4.9}$$

Let us denote

$$h^t = \frac{1}{2} \ln \frac{1 + e^{-t}}{1 - e^{-t}} . \tag{4.10}$$

This approach to study the evolved measure as the marginal of a two-layer Gibbs measure was introduced in [19], and has been applied repeatedly since.

Remark 4.2.5. *Here we will find for $\mu^\sharp S(t)$, by making the choices $\xi_1 = +1$, $\xi_2 = -1$, that in any open neighbourhood of $\bar{\eta}$ two positive-measure sets exist, on which the limits differ, however, in contrast to amenable graphs, these sets are not open (which allows different behaviour between different evolved Gibbs measures $\mu^\sharp$ and μ^+ with regards to their Gibbsianness, something which is excluded on amenable graphs such as $\mathbb{Z}^d$). In other words we will show a $\hat{\mu}$-essential, although non-strong, discontinuity.*

As explained in the Appendix A we have the representation of the conditional probabilities of the time-evolved measure μ_t of the form

$$\hat{\mu}_t(\eta_0 | \eta_{\Lambda \setminus 0}) = \int \mu[\eta_{\Lambda \setminus 0}](d\sigma_0) P_t(\sigma_0, \eta_0) , \tag{4.11}$$

with the perturbed η-dependent measure on spin configurations $\mu[\eta_{\Lambda \setminus 0}](d\sigma) \equiv \mu[\eta_{\Lambda \setminus 0}, \eta_0 = 0](d\sigma)$, whose finite-volume marginals look like

$$\mu[\eta_{\Lambda'}](\sigma_{\Lambda'}) = C \exp \left\{ \beta \sum_{(i,j) \in \Lambda'} \sigma_i \sigma_j + \sum_{i \in \Lambda' \setminus \partial \Lambda'} h_i \sigma_i + \sum_{i \in \partial \Lambda'} \tilde{h}_i \sigma_i \right\} , \tag{4.12}$$

where

$$h_i = h_0 + \eta_i h^t \ ,$$
$$\tilde{h}_i = h_0 + \eta_i h^t + h^\star \ ,$$

(4.13)

where the external fields at the boundaries are given in terms of $h^\star$. This value represents the fixed point of the recursion relation with homogeneous field h_0, (4.4), and is bijectively related with the starting measure $\mu^\star$. More generally, such a representation is always valid if the initial measure is a Markov chain on the tree. Markov chains can be described by boundary laws, and conditional probabilities of infinite-temperature time evolutions, are, for finite-volume conditionings, described by boundary laws obeying recursions which are local perturbations of those of the initial measure, see the Appendix A and [29].

In what follows we choose $\xi^1 = (+)$ and $\xi^2 = (-)$. With this notation, for non-Gibbsianness it is enough to prove that, at $\bar{\eta}$, there exists an $\varepsilon > 0$ such that, for all Λ, there exists $\Lambda' \supset \Lambda$ such that

$$\left| \mu[\bar{\eta}_{\Lambda\backslash 0}, \xi^1_{\Lambda'\backslash\Lambda}](\sigma_0) - \mu[\bar{\eta}_{\Lambda\backslash 0}, \xi^2_{\Lambda'\backslash\Lambda}](\sigma_0) \right| > \varepsilon \ .$$

(4.14)

4.3 Absence of the Initial Field, ($h_0 = 0$)

4.3.1 Marginals and η-dependent Fields

To prove the non-Gibbsianness of $\hat{\mu}$, we will have to consider the phase transition behaviour of the Gibbs measures on the first layer in various external fields. These external fields are determined by the various conditionings, as well as by the choice of the initial Gibbs measure.

Let k, m be integers with $k < m$, let us denote $\Lambda' = \Gamma_m$ and $\Lambda = \Gamma_k$. We shall consider first the case $h_0 = 0$. Marginalizing on Γ_m leaves us with a finite-volume Gibbs measure on Γ_m denoted by $\nu^{h^\star}_{\Gamma_m}$ and parametrized by the following external fields

$$i \in \partial\Gamma_m, \ h_i = \eta_i h^t + d\varphi(h^\star) \ ,$$
$$i \in \Gamma_{m-1}, \ h_i = \eta_i h^t \ .$$

(4.15)

In order to apply the (marginalisation) procedure to the η-dependent finite-volume Gibbs measure $\nu^{h^\star}_{\Gamma_m}$ on Γ_m, we need to identify the role played by η. It can be shown that taking the marginal on Γ_{m-1} of the finite-volume Gibbs

measure on Γ_m (summing out the spin $\sigma \in \partial\Gamma_m$) gives us a finite-volume Gibbs measure on Γ_{m-1} with an external field at the boundary equal to

$$h_i = \eta_i h^t + \sum_{l \sim i} \varphi(\eta_l h^t) \, . \tag{4.16}$$

Here the sum is over the nearest neighbours $l \in \partial\Gamma_m$.

The equation (4.16) tells us how the configurations $\eta_{\partial\Gamma_m}$ will affect the field acting on $i \in \partial\Gamma_{m-1}$ after having taken a one-generation marginal.

The configuration $\eta_{\Gamma_m \backslash \Gamma_k}$ will govern the value of the fields at $\partial\Gamma_k$, when the marginal on Γ_k is taken. Let us see how:

- $\eta_{\Gamma_m \backslash \Gamma_k} = +$

$$
\begin{aligned}
&i \in \partial\Gamma_m, \ h_i^{(0)} = h^t + d\varphi(h^\star) \, , \\
&\text{after summing out the m-th generation we have} \\
&i \in \partial\Gamma_{m-1}, \ h_i^{(1)} = h^t + d\varphi(h_i^{(0)}) \, , \\
&i \in \partial\Gamma_j, k < j < m-1, \ h_i^{(j)} = h^t + d\varphi(h_i^{(j-1)}) \, .
\end{aligned}
\tag{4.17}
$$

- $\eta_{\Gamma_m \backslash \Gamma_k} = -$

$$
\begin{aligned}
&i \in \partial\Gamma_m, \ h_i^{(0)} = -h^t + d\varphi(h^\star) \, , \\
&\text{after summing out the m-th generation we have} \\
&i \in \partial\Gamma_{m-1}, \ h_i^{(1)} = -h^t + d\varphi(h_i^{(0)}) \, , \\
&i \in \partial\Gamma_j, k < j < m-1, \ h_i^{(j)} = -h^t + d\varphi(h_i^{(j-1)}) \, .
\end{aligned}
\tag{4.18}
$$

Note that the above-chosen η-conditioning on the annulus makes the recursion homogeneous. Choosing m big enough guarantees that the recursions (4.17), (4.18) approach their time-dependent fixed points; we denote them respectively by $H_t^\pm$, $H_t^\sharp$ and $h_t^\pm$, $h_t^\sharp$, (see Figure 4.1).

Assume that we start at time $t = 0$ with the measure $\mu^\sharp$, then $h^\star = h^\sharp = 0$. It ensures that the recursions (4.17), (4.18) will approach, respectively, $H_t^+ > 0$ and $h_t^- = -H_t^+ < 0$. H_t^+ represents the biggest stable fixed point for the ($\eta = +$) recursion (4.17), and h_t^- the smallest stable fixed point for the ($\eta = -$) recursion (4.18). The fact that both recursions have as a starting point the unstable fixed point $h^\sharp = 0$ guarantees that the plus conditioning will drag the field towards H_t^+ and the minus one towards h_t^-. This will not be the case for μ^+ and μ^- as we will see later.
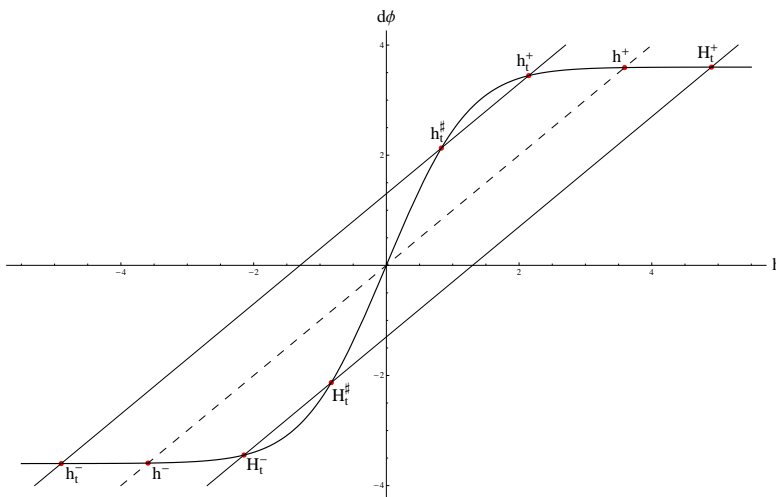
Figure 4.1: Fixed points[1]

The $(\eta_{\Gamma_m \setminus \Gamma_k} = \pm)$-dependent marginals on Γ_k, of the measure on Γ_m, are finite-volume Gibbs measures parametrized by the following fields: for the case $(\eta_{\Gamma_m \setminus \Gamma_k} = +)$

$$
\begin{aligned}
& i \in \partial \Gamma_k, \ h_i^{+,(0)} = \eta_i h^t + d\varphi(H_t^+) \ , \\
& i \in \Gamma_{k-1}, \ h_i^{(0)} = \eta_i h^t \ ,
\end{aligned}
\tag{4.19}
$$

and in the case $(\eta_{\Gamma_m \setminus \Gamma_k} = -)$

$$
\begin{aligned}
& i \in \partial \Gamma_k, \ h_i^{-,(0)} = \eta_i h^t + d\varphi(h_t^-) \ , \\
& i \in \Gamma_{k-1}, \ h_i^{(0)} = \eta_i h^t \ .
\end{aligned}
\tag{4.20}
$$

Remark 4.3.1. *Notice that only the fields at $\partial \Gamma_k$ depend on $\eta_{\Gamma_m \setminus \Gamma_k}$ and not the ones acting on the interior. We emphasize that the broadcasting is absorbed by the boundary and has no direct influence on the interior.*

Now we investigate how the recursion relation $h_i^{(j)} = \eta_i h^t + \sum_{l \sim i} \varphi(h_l^{(j-1)})$, obtaining by summing out generations in Γ_k, will depend on the fixed configurations $\eta_{\Gamma_m \setminus \Gamma_k} = \pm$, namely on the fields H_t^+, h_t^- acting on the generation

---

[1] "Longum est iter per praecepta, breve et efficax per exempla", Seneca.

$\partial\Gamma_{k+1}$. We emphasize that the annulus configurations determine the starting point of the recursion. We will also show how the aforementioned recursion relation can be bounded from below if we are coming from $\eta_{\Gamma_m\setminus\Gamma_k} = +$, and from above for $\eta_{\Gamma_m\setminus\Gamma_k} = -$. Furthermore these bounds will turn out to be uniform with respect to η_{Γ_k} and with respect to j (number of iterations).

Lemma 4.3.2. *Given the recursion relation $h_i^{(j)} = \eta_i h^t + \sum_{l\sim i} \varphi(h_l^{(j-1)})$ we have : $h_i^{(j)} \geq h_t^+ > 0$, for all i and j, if $h_i^{(0)} = H_t^+$; and $h_i^{(j)} \leq H_t^- = -h_t^+$, for all i and j, if $h_i^{(0)} = h_t^-$.*
  Here h_t^+ is the fixed point for the homogeneous recursion
$h^{(j)} = -h^t + d\varphi(h^{(j-1)})$ *with $h^{(0)} = H_t^+$.*

Proof. Fixed points of the discussed recursion relation are given in the Figure 4.1. The proof follows by induction. Take first the case $h_i^{(0)} = H_t^+$. Naturally $H_t^+ > h_t^+$, so $h_i^{(0)} > h_t^+$ for all i. If we now assume $h_i^{(j)} > h_t^+$ for all i, then $h_i^{(j+1)} = \eta_i h^t + \sum_{l\sim i} \varphi(h_l^{(j)}) > -h^t + d\varphi(h_t^+) = h_t^+$. The case $h_i^{(0)} = h_t^-$ follows by symmetry; the corresponding recursion relation will be bounded from above by H_t^-.   □

4.3.2 Results: Total Badness of the Evolved $\mu^\sharp$; Difference Between Different Phases

Let t_2 be defined by
$$h^{t_2} = h(\beta, d) . \tag{4.21}$$
We remark that the monotonicity of the continuous function h^t, (4.10), together with the fact that $h(\beta, d) > 0$, assures the existence of t_2.

Theorem 4.3.3. *If σ is distributed according to $\mu^\sharp$, then after time t_2 all configurations η are bad configurations (points of essential discontinuity) for the transformed measure $\mu^\sharp S(t)$.*

Remark 4.3.4. *The main idea is as follows: If the plus configuration is bad (and by symmetry the same is true for the minus configuration), then all configurations $\bar{\eta}$ will be bad. This is because if minus boundary conditions give a minus magnetisation for the conditioned σ-spin at the origin, and plus boundary conditions a positive one, the same holds for all $\bar{\eta}$ (due to FKG e.g.). So take $\bar{\eta}$ to be plus. Choosing ξ to be plus in a large enough annulus $\Lambda' \setminus \Lambda$ and integrating the outside with $\mu^\sharp$ will lead to an effective plus boundary condition at Λ. The reason is that the positive magnetisation m^+ is an attractive fixed point for the recursive relation, and any positively magnetised field in Λ' will*

lead into its domain of attraction. The same is true for the negative magneti-sation. As there are different magnetisations with plus and minus boundary conditions, even in the presence of a weak plus field (the field is plus due the $\bar{\eta}$ being plus), the choice of plus or minus in the annulus influences the expected magnetisation at the origin, however big Λ is.

Proof. The definition of t_2, (4.21), will assure that we are in the phase-transition regime for the transformed system (for $t \geq t_2$). Making use of Lemma 4.3.2, the value of ε we are after, in order to prove the essential discontinuity, is given by $\varepsilon = 2\tanh(h_t^+)$. This value corresponds to taking, for the measure coming from $\eta_{\Gamma_m \setminus \Gamma_k} = +$, the smallest positive field along all the $k-1$ iterations, namely h_t^+. The field at the origin is given by $h^{(k)} = \eta_0 h^t + (d+1)\varphi(h^{(k-1)})$ and could be roughly bounded from below

$$h^{(k)} = \eta_0 h^t + (d+1)\varphi(h^{(k-1)}) \geq -h^t + d\varphi(h_t^+) = h_t^+ \ .$$

Thus the corresponding single-site measure is given by $\nu_+(\sigma_0) = \frac{e^{h_t^+ \sigma_0}}{e^{h_t^+} + e^{-h_t^+}}$, and

$$\mu[\bar{\eta}_{\Gamma_k}(+)_{\Gamma_m \setminus \Gamma_k}](\sigma_0) \geq \tanh(h_t^+) \ .$$

Analogously for the measure coming from $\eta_{\Gamma_m \setminus \Gamma_k} = -$, we take the biggest negative value along all the $k-1$ iterations, that is $H_t^- = -h_t^+$, therefore $\nu_-(\sigma_0) = \frac{e^{-h_t^+ \sigma_0}}{e^{h_t^+} + e^{-h_t^+}}$ and

$$\mu[\bar{\eta}_{\Gamma_k}(-)_{\Gamma_m \setminus \Gamma_k}](\sigma_0) \leq \tanh(-h_t^+) \ .$$

For $\varepsilon = 2\tanh(h_t^+)$ the inequality (4.14) holds. Let us notice that ε is chosen independently of η, thanks to the uniform bounds appearing in Lemma 4.3.2. This ensures the $\hat{\mu}$-essential discontinuity at any point. $\qquad \square$

As we mentioned before, the previous argument does not hold for μ^+ and μ^-. We treat here only the μ^+ case, the μ^- case is completely symmetrical. So, in case we start with the plus measure, even conditioning on a minus configuration in the annulus, due to the plus influence from the boundary will lead to a measure on Γ_k that looks like the plus measure in a negative field.

Lemma 4.3.5. *Given the starting measure μ^+, the fields acting on $\partial\Gamma_m$ for the marginal measure on Γ_m, which are given by $h_i^{(0)} = \eta_i h^t + d\varphi(h^+)$, $i \in \partial\Gamma_m$, satisfy the following inequality*

$$\eta_i h^t + d\varphi_\beta(h^+) > h_t^\sharp(d, \beta) \ , \tag{4.22}$$

for all $d > 1$, $\beta > \beta(d)$ and for all $t \in [t_2, \infty)$.

Proof. Let t_2 be as in (4.21). It suffices to show that $d\varphi_\beta(h^+(d,\beta)) > h_t^\sharp(d,\beta) + h^t$ in the aforementioned region of parameters. First of all we note that the expression on the right-hand side is zero in the limit $t \uparrow \infty$, and it is a decreasing function of t. So in order to prove the lemma it is enough to show

$$d\varphi_\beta(h^+(d,\beta)) > h_{t_2}^\sharp(d,\beta) + h^{t_2} \ . \tag{4.23}$$

Using that $h_{t_2}^\sharp(d,\beta)$ is a fixed point for the $(-)$ recursion at $t = t_2$, we arrive at

$$d\varphi_\beta(h^+(d,\beta)) > d\varphi_\beta(h_{t_2}^\sharp(d,\beta)) \ , \tag{4.24}$$

Note that $h_{t_2}^\sharp(d,\beta) = h_c(d,\beta) > 0$, where $h_c(d,\beta)$ is a tangent point to $d\varphi(x)$ such that $d\varphi'(h_c(d,\beta)) = 1$. We show that $h^+ > h_c(d,\beta)$. In fact we know that $d\varphi(h^+) - h^+ = 0$. Using the mean-value theorem together with the fact that $d\varphi(0) = 0$, we write $d\varphi'(\xi)h^+ - h^+ = 0$. It implies that ξ is such that $d\varphi'(\xi) = 1$. Using then that $d\varphi'$ is a decreasing function it follows that the domain of ξ, namely $(0, h^+)$ has to contain $h_c(d,\beta)$; so $h^+ > h_c(d,\beta)$. Using then the monotonicity of the functions φ_β the claim is proved. $\qquad\square$

Theorem 4.3.6. *If σ is distributed according to μ^+, then after time t_2 all configurations η are good configurations for the transformed measure $\mu^+ S(t)$.*

Proof. Based on Lemma 4.3.5, choosing Γ_m big enough we make sure that the recursion relation coming from the fixed "+"-annulus $\Gamma_m \setminus \Gamma_k$ will approach its fixed value H_t^+, so do we for the fixed "−"-annulus to approach its fixed value h_t^+. Then the magnetic fields for the finite-volume Gibbs measure on Γ_k are respectively given by

$$\begin{aligned}
&i \in \partial\Gamma_k, \ h_i^{+,(0)} = \eta_i h^t + d\varphi(H_t^+) \ , \\
&i \in \Gamma_{k-1}, \ h_i^{(0)} = \eta_i h^t \ ,
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
&i \in \partial\Gamma_k, \ h_i^{-,(0)} = \eta_i h^t + d\varphi(h_t^+) \ , \\
&i \in \Gamma_{k-1}, \ h_i^{(0)} = \eta_i h^t \ .
\end{aligned} \tag{4.26}$$

Define $\Delta^j = \max_i(h_i^{+,(j)} - h_i^{-,(j)})$. This maximum is always positive, as an inductive argument shows. We are about to prove that $\exists \delta \in (0,1)$ such that

$(1 - \delta)\Delta^j \geq \Delta^{j+1}$; this implies that $\lim_{j \uparrow \infty} \Delta_j = 0$.

$$\Delta^{j+1} = \max_i (h_i^{+,(j+1)} - h_i^{-,(j+1)}) = \max_i \left[\frac{1}{d} \sum_{l \sim i} \left(d\varphi(h_l^{+,(j)}) - d\varphi(h_l^{-,(j)}) \right) \right]$$

$$= \max_i \left[\sum_{l \sim i} \frac{d\varphi'(c_l)}{d} \left(h_l^{+,(j)} - h_l^{-,(j)} \right) \right] \leq (1 - \delta) \max_i \left[\frac{1}{d} \sum_{l \sim i} \left(h_l^{+,(j)} - h_l^{-,(j)} \right) \right]$$

$$= (1 - \delta) \max_i \max_{l \sim i} \left((h_l^{+,(j)} - h_l^{-,(j)}) \sum_{l \sim i} \frac{1}{d} \right) = (1 - \delta) \max_i \max_{l \sim i} \left(h_l^{+,(j)} - h_l^{-,(j)} \right)$$

$$= (1 - \delta)\Delta^j \ .$$

$$(4.27)$$

We used the mean-value theorem together with the fact that $d\varphi'(x) < 1$ for $x > h_c(d, \beta)$. $\qquad \square$

For σ distributed according to $\mu^\sharp$, we will show the existence of an intermediate time interval, where some, but not all, configurations are bad for $\hat\mu$. Theorem 4.3.11 will express this. We will show that the all plus and all minus configurations are good for $\mu^\sharp S(t)$ at all times in $(0, t_2)$. Moreover we will impose a condition on the field h^t (therefore on t itself), such that it guarantees the existence of at least one bad configuration for $\mu^\sharp S(t)$.

We will find a t_1, which is larger than the minimal value of time for which this condition is satisfied. This value t_1 will turn out to be strictly less than t_2. This will guarantee that t_1 is small enough so that the transformed measure, conditioned on an all plus or all minus η will not exhibit a phase transition.

Remark 4.3.7. *Note that this implies that at the same time t_2 the intermediate state has a transition to a totally non-Gibbsian regime, where all spin configurations are discontinuity points, whereas the plus and minus state have a transition to a Gibbsian regime, without discontinuity points.*

Lemma 4.3.8. *If σ is distributed according to $\mu^\sharp$ then for all $t \in (0, t_2)$ the $\eta = +$ and $\eta = -$ configurations are good configurations for the transformed measure $\mu^\sharp S(t)$.*

Proof. As was shown before, the recursions (4.17), (4.18) (related to the annuli) give us respectively H_t^+ and h_t^-. Let first η be the plus configuration. In this case $h_i^{+,(j)} = H_t^+$ for all i and j. In other words the field will stick to the fixed point value along the iterations. Using an inductive argument we show that

$h_i^{-,(j)} = h^{-,(j)}$; that's to say that it does not depend on i. Based on that, it is straightforward to get a monotonicity property for $h^{-,(j)}$, namely that $h^{-,(j+1)} > h^{-,(j)}$ for all j. Indeed $h^{-,(j+1)} = h^t + d\varphi(h^{-,(j)}) > h^{-,(j)}$. The last inequality follows from the fact that $d\varphi(x) > x - h^t$ for all $x \in [h_t^-, H_t^+)$, due to the chosen range of t. Recalling that for $t \in (0, t_2)$ the recursion relation $h^{-,(j+1)} = h^t + d\varphi(h^{-,(j)})$ has only one fixed point, namely H_t^+, the lemma is proven for $\eta = +$. The $\eta = -$ case follows by symmetry. □

Remark 4.3.9. *The chosen range of times enables the existence of a unique fixed point for each of the recursions (4.17), (4.18), independently of $h^\star$. This means that the fields we obtain at $\partial\Gamma_k$ depend on the annuli, but they do not depend on the exterior $\Gamma_m^\complement$. For this reason Lemma 4.3.8 applies to σ's distributed according to μ^+ and μ^- too.*

For the sake of clarity, let us recall that h^+ indicates the positive stable fixed point for the recursion (4.4) with $h_0 = 0$.

Lemma 4.3.10. *Let t_1 be given by*

$$h^{t_1} = h^+ \,, \tag{4.28}$$

then $t_1 \in (0, t_2)$

Proof. Recalling equation (4.10), the fact that t_1 lies in the interval $(0, t_2)$ is guaranteed by the truth of the inequality $h(d, \beta) < d\varphi(h^+)$, for $\beta > \beta(d)$ and $d > 1$. Indeed

$$h(d, \beta) < d \operatorname{arctanh}\left(w\left(\frac{d - \bar{w}}{d - w}\right)^{\frac{1}{2}}\right)$$
$$= d \operatorname{arctanh}\left(w \tanh(h_c)\right) = d\varphi(h_c) \,. \tag{4.29}$$

Knowing that $h_c < h^+$, the monotonicity of the function φ concludes the proof. □

Define the "alternating" configuration η^A to be $\eta_i^A = (-1)^n$ for $i \in \partial\Gamma_n$ and $n \in \mathbb{N}$, i.e. all vertices at each generation have the same sign different from the sign of the previous and the next generations. Naturally the configuration for which $-\eta_i^A = (-1)^n$ is also an "alternating" one. Let us call $h_i^{\pm,(j)}$ the field at the vertex $i \in \partial\Gamma_{k-j}$ after $(j + 1)$ applications of the recursion formula (4.16), starting respectively at H_t^+ or h_t^-. The particular structure of the "alternating" configuration makes the fields homogeneous at each generation; i.e., $h_i^{\pm,(j)} = h^{\pm,(j)}$, for all $i \in \partial\Gamma_{k-j}$.

Theorem 4.3.11. *If σ is distributed according to $\mu^\sharp$, and t_1 is given by (4.28), then for all $t \in [t_1, t_2)$ some, but not all, configurations η are bad for the transformed measure $\mu^\sharp S(t)$.*

Proof. Making use of Lemma 4.3.8, Lemma 4.3.10, to prove the theorem it is enough to find a particular configuration η that will be bad for all $t \in [t_1, t_2)$. The "alternating" configurations will be shown to be bad for all $t \geq t_1$, in other words they transmit the influence of the annulus to the origin, no matter how "distant" the annulus and the origin are. As remarked before, $h_i^{\pm,(j)}$ associated to the η^A configurations depend only on j, and we call the corresponding values $h^{\pm,(j)}$. Without loss of generality let us assume $\eta_i^A = +$, for $i \in \partial\Gamma_k$. By an inductive argument, based on the hypothesis $t \in [t_1, t_2)$ (which in terms of fields means $h^t \leq h^+$), and on the particular structure of the configuration η^A, we show that $h^{+,(j)} \geq h^+$ and $h^{-,(j)} \leq 0$, for all j even, namely for those j which relate to generations at which η^A is set to be $+$, and that $h^{-,(j)} \leq -h^+$ and $h^{+,(j)} \geq 0$ for j odd. This will imply $h^{+,(j)} - h^{-,(j)} \geq h^+$ for all j. Consider the case j even.

For $j = 0$ we have:
$$h^{+,(0)} = H_t^+ \geq h^+, \qquad\qquad h^{-,(0)} = h^t + d\varphi(h_t^-) \leq 0.$$
Both inequalities hold, because H_t^+ is a decreasing function of t whose lower bound is given by h^+.

Assuming the statement is true for j, let us see that it holds for $j+2$. We focus first on $h^{+,(j+2)}$.
$$h^{+,(j+2)} = h^t + d\varphi(h^{+,(j+1)}) = h^t + d\varphi(-h^t + d\varphi(h^{+,(j)})) , \qquad (4.30)$$

where the second equality is justified by the particular structure of the alternating configuration. Using the assumption $h^{+,(j)} \geq h^+$ and the monotonicity of φ we arrive at
$$h^{+,(j+2)} \geq h^t + d\varphi(-h^t + d\varphi(h^+)) . \qquad (4.31)$$

The fact that $0 \leq -h^t + h^+ \leq h^+$ ensures that $d\varphi(-h^t + h^+) \geq -h^t + h^+$. This concludes the proof for $h^{+,(j)}$.
For $h^{-,(j+2)}$ we have:
$$h^{-,(j+2)} = h^t + d\varphi(h^{-,(j+1)}) = h^t + d\varphi(-h^t + d\varphi(h^{-,(j)})) . \qquad (4.32)$$

Using always the assumption $h^{-,(j)} \leq 0$, the monotonicity of φ, and the assumption $h^t \leq h^+$, which guarantees $h^t \leq d\varphi(h^t)$, we obtain
$$h^{-,(j+2)} \leq h^t + d\varphi(-h^t) \leq 0 . \qquad (4.33)$$

The case j odd is analogous. $\qquad\qquad\square$

Remark 4.3.12. *The above result also applies to the evolved plus and minus measures. Indeed the alternating configuration displays a strong discontinuity here, whereas the above analysis shows that for large times all configurations display a $\mu^{\sharp}S(t)$-essential but nonstrong discontinuity. Whether the t_1 used above is optimal in any sense is not known. We conjecture that it may be for the intermediate state, but not for the plus or minus states.*

4.4 Non-zero Initial Field, $(h_0 \neq 0)$

Recall that $|h_0| < h(d, \beta)$, $\beta > \beta(d)$ and $d > 1$; these conditions guarantee the existence of three homogeneous phases for the original measure; we denote them, even if not fully consistent with the notation we have been using so far, $\mu^+_{h_0}$, $\mu^-_{h_0}$, and $\mu^{\sharp}_{h_0}$, just to emphasize their dependence on h_0. We show that the previous results, found for the case $h_0 = 0$, will also apply to the case $h_0 \neq 0$ but for different time values. Let $t_+(h_0), t_-(h_0)$ be given by the following equations:

$$h_0 + h^{t_+} = h(d, \beta) \; ,$$
$$h_0 - h^{t_-} = -h(d, \beta) \; . \tag{4.34}$$

Call

$$t_2(h_0) = \min \{ t_+(h_0), t_-(h_0) \} \; ,$$
$$t_3(h_0) = \max \{ t_+(h_0), t_-(h_0) \} \; . \tag{4.35}$$

Depending on the sign of the initial field, $t_+(h_0)$ might be either bigger or smaller than $t_-(h_0)$, as follows from (4.10). Nevertheless the definitions of $t_2(h_0)$, and $t_3(h_0)$ will always assure $t_2(h_0) < t_3(h_0)$ (e.g., for $h_0 < 0$ the order is $t_2(h_0) = t_+ < t_- = t_3(h_0)$).

The time $t_2(h_0)$ indicates the time value for which the dynamic field h^t, taken in the opposite direction to h_0, will first reach a value which guarantees the existence of a phase transition for the conditioned transformed measure. The time $t_3(h_0)$ refers to the analogous value, but for h^t taken with the same sign as h_0.

Suppose, w.l.o.g., that $h_0 < 0$. Note that for h_0 negative the magnetization corresponding to $\mu^{\sharp}_{h_0}$ is positive, see [29], Chapter 12. For $t > t_3(h_0)$ there exist three fixed points for the $(-)$-recursion $h^{(k+1)} = h_0 - h^t + d\varphi(h^{(k)})$, namely two stable ones $h^-_t(h_0)$, $h^+_t(h_0)$, and an unstable $h^{\sharp}_t(h_0)$. The existence of several fixed points makes the convergence to them be dependent on the starting point.

In particular the recursion will take us to $h_t^+(h_0)$ iff the starting point, $h^{(k=0)}$, lies to the right of the unstable one, that is when $h^{(k=0)} > h_t^\sharp(h_0)$; it will take us to $h_t^-(h_0)$ iff $h^{(k=0)} < h_t^\sharp(h_0)$, and will stick to $h_t^\sharp(h_0)$ iff $h^{(k=0)} = h_t^\sharp(h_0)$.

Given that $t_3(h_0) > t_2(h_0)$, the assumption $t > t_3(h_0)$ ensures the existence of three fixed points also for the $(+)$-recursion $h^{(k+1)} = h_0 + h^t + d\varphi(h^{(k)})$; they are denoted by $H_t^\pm(h_0)$, and $H_t^\sharp(h_0)$.

Assume that we start at time $t = 0$ with the measure $\mu_{h_0}^\sharp$, then the starting point for the $(\pm)$-recursions is $h^\star = h^\sharp(h_0) > 0$. However, for the chosen range of time, $t > t_3(h_0)$, it can be shown that $h^\sharp(h_0)$ will always lie to the right of $H_t^\sharp(h_0)$ and always to the left of $h_t^\sharp(h_0)$. So the next theorem reads:

Theorem 4.4.1. *If σ is distributed according to $\mu_{h_0}^\sharp$, then after time $t_3(h_0)$ all configurations η are bad configurations for the transformed measure $\mu_{h_0}^\sharp S(t)$.*

□

Analogously to the analysis for $h_0 = 0$, the former result will not hold for σ distributed according to $\mu_{h_0}^\pm$.

Two other results, obtained in the previous section, have equivalents for non-zero external field.

Lemma 4.4.2. *If σ is distributed according to $\mu_{h_0}^\sharp$, then for all $t \in (0, t_2(h_0))$ the $\eta = +$ and $\eta = -$ configurations are good configurations for the transformed measure $\mu_{h_0}^\sharp S(t)$.*

□

Theorem 4.4.3. *If σ is distributed according to $\mu_{h_0}^\pm$, then after time $t_3(h_0)$ all configurations η are good configurations for the transformed measure $\mu_{h_0}^\pm S(t)$.*

□

Remark 4.4.4. *It is worth remarking that the strict inequality $t_2(h_0) < t_3(h_0)$, always holding for $h_0 \neq 0$, implies the non-emptiness of the interval of times $[t_2(h_0), t_3(h_0))$. A similar result to the one given in Theorem 4.3.11 holds in the case $h_0 \neq 0$, namely that for $t \in [t_2(h_0), t_3(h_0))$ some, but not all, configurations are bad. In fact, it can be shown, for example in case $h_0 < 0$, that the time $t_2(h_0)$ corresponds to the time for which the plus configuration becomes bad, while for all times $t < t_3(h_0)$ the minus configuration will remain good. In case $h_0 > 0$, as symmetry may suggest, the time $t_2(h_0)$ will be the threshold for the minus configuration to become bad, while the plus configuration will be good till the time $t = t_3(h_0)$.*

Encouraged by the many analogies between the $h_0 = 0$ case and the $h_0 \neq 0$ case, one might ask what one can say about the $(h_0 \neq 0)$-equivalent of the time t_1, (4.28). Pursuing the former, let us define the values of times $\hat{t}_+$, $\hat{t}_-$ by the following equalities

$$
\begin{aligned}
h^{\hat{t}_+} &= h_0 + d\varphi(h^+(h_0)) - h^\sharp(h_0) \ , \\
-h^{\hat{t}_-} &= h_0 + d\varphi(h^-(h_0)) - h^\sharp(h_0) \ ,
\end{aligned}
\tag{4.36}
$$

and define further

$$
t_1(h_0) = \max\left\{\hat{t}_+, \hat{t}_-\right\} \ .
\tag{4.37}
$$
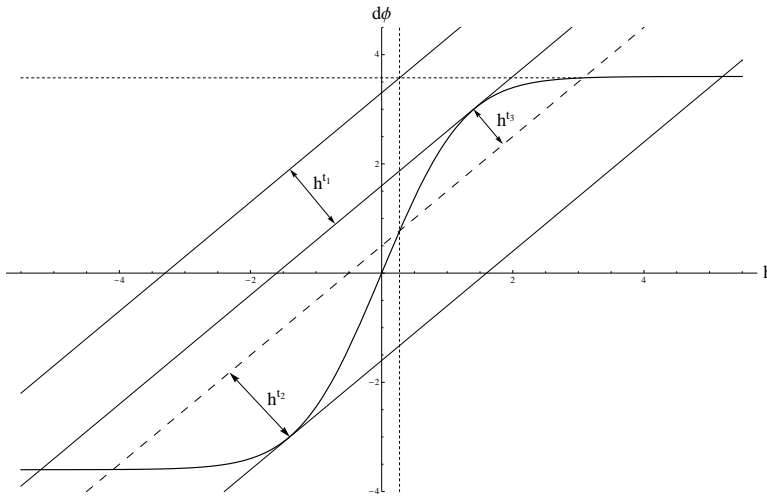
Figure 4.2: *"Some" times:* t_1, t_2, t_3.

The Figure 4.2 helps to understand the role played by the different times so far defined.

It can be shown that $t_1(h_0) < t_3(h_0)$ for $|h_0| < h(d, \beta)$. Nonetheless the relation between time $t_1(h_0)$ and $t_2(h_0)$ is not so trivial as we will show. The next lemma formalizes that for all time $t \geq t_1(h_0)$ the "alternating" configurations are bad for σ distributed according to $\mu^\sharp_{h_0}$.

Lemma 4.4.5. *If σ is distributed according to $\mu^\sharp_{h_0}$, and $t_1(h_0)$ is given by (4.37), then for all $t > t_1(h_0)$ "alternating" configurations are bad for the transformed measure $\mu^\sharp_{h_0} S(t)$.*

Proof. The proof follows the same route taken in the proof of Theorem 4.3.11 with some modifications on the bounds. Nonetheless, we reckon it is instructive to sketch the main points at least for $h_0 < 0$. For $t > t_1(h_0)$ an inductive argument leads to the following bounds:

$$\text{for even } j, \qquad h^{+,(j)} \geq h^+(h_0) \text{ and } h^{-,(j)} \leq h^\sharp(h_0) ,$$

$$\text{for odd } j, \qquad h^{+,(j)} \geq -h^\sharp(h_0) \text{ and } h^{-,(j)} \leq -h^+(h_0) ,$$

therefore $h^{+,(j)} - h^{-,(j)} \geq h^+(h_0) - h^\sharp(h_0)$ for all j. □

The previous lemma together with Remark 4.4.4 shows that if σ is distributed according to $\mu_{h_0}^\sharp$, then for all $t \in [t_1(h_0), t_3(h_0))$ some, but not all, configurations are bad. There are then two different time intervals where some, but not all, configurations are bad. We will not leave the reader wondering how these two intervals relate. We will show the existence of a critical value h_0^c such that for $|h_0| > h_0^c$ we have $[t_1(h_0), t_3(h_0)) \subset [t_2(h_0), t_3(h_0))$, for $|h_0| < h_0^c$ the inclusion is reversed, namely $[t_1(h_0), t_3(h_0)) \supset [t_2(h_0), t_3(h_0))$, and for $|h_0| = h_0^c$ the two intervals coincide.

Remark 4.4.6. *In the small-field regime $|h_0| < h_0^c$ we have that the "alternating" configuration becomes bad before the all plus and the all minus configurations. In that case, the dominant effect is that the alternating character of the conditioning provides some cancellations, just as in the zero-field case.*

In the other regimes we can just say what follows from $t_1(h_0) < t_3(h_0)$, i.e. that the "alternating" configurations become bad before the homogeneous configuration with all η's aligned with h_0, that is $\eta = \text{sign}(h_0)$. The impossibility to state something more in the other regimes is due to the fact that $t_1(h_0)$ is not a "sharp" threshold for the "alternating" configurations to become bad. However, in this case having a "bad" configuration, one may need to counteract the effect of the field, thus in a positive external field, the minus configuration becomes bad at an earlier time than the alternating one.

To explore the latter inclusions we need to compare the values $t_1(h_0)$ and $t_2(h_0)$, or equivalently $h^{t_1(h_0)}$ and $h^{t_2(h_0)}$. Consider the difference between the fields

$$f(h_0) := h^{t_1(h_0)} - h^{t_2(h_0)} . \tag{4.38}$$

Based on the definitions of the times, (4.37), (4.35) it turns out that the function f is even. So we might focus on its behaviour only for negative values of the initial field h_0. For such values of the field the function has the following form

$$f(h_0) = h^+(h_0) - h^\sharp(h_0) + h_0 - h(d, \beta) . \tag{4.39}$$

First of all the limit values of f in the interval $(-h(d,\beta),0)$ are given by

$$\lim_{h_0\downarrow -h(d,\beta)} f(h_0) = -2h(d,\beta) \ ,$$

$$\lim_{h_0\uparrow 0} f(h_0) = h^+ - h(d,\beta) \ .$$

Note that the second limit value is positive, as has been explained in the proof of Lemma 4.3.5, while the first one is negative by the definition of $h(d,\beta)$, and by (4.3). Taking now the derivative of f with respect to h_0 we obtain

$$f'(h_0) = (h^+(h_0))' - (h^\sharp(h_0))' + 1 \ . \tag{4.40}$$

Using the only thing we know about $h^+(h_0)$, $h^\sharp(h_0)$, namely that they are fixed points for the recursion $h^{k+1} = h_0 + d\varphi(h^k)$, the following equalities turn out to hold

$$(h^+(h_0))' = \frac{1}{1-d\varphi'(h^+(h_0))} \ ,$$

$$(h^\sharp(h_0))' = \frac{1}{1-d\varphi'(h^\sharp(h_0))} \ .$$

Because $h^\sharp(h_0) < h_c(d,\beta)$ and $h^+(h_0) > h_c(d,\beta)$, the monotonicity of $d\varphi'$ assures that $f'(h_0) > 0$ for all $h_0 \in (-h(d,\beta),0)$. Therefore the existence and uniqueness of h_0^c is guaranteed by an application of the intermediate-value theorem. We point out that the function f is not differentiable in $h_0 = 0$. Indeed, being f an even function and $\lim_{h_0\uparrow 0} f'(h_0) > 0$ clarify the discontinuity.

We would like to remark that the case $h_0 = 0$ might be obtained from the previous analysis by taking the limit $h_0 \uparrow 0$. Indeed $\lim_{h_0\uparrow 0} t_1(h_0) = t_1$, $\lim_{h_0\uparrow 0} t_2(h_0) = \lim_{h_0\uparrow 0} t_3(h_0) = t_2$.

4.5 Conclusion and Final Remarks

We have shown that the Gibbs-non-Gibbs transition on trees has a number of different aspects, as compared to that on regular lattices. In particular, we have shown that different evolved Gibbs measures can have different Gibbsian properties. For the evolved intermediate state there are two transitions, one from being Gibbsian to being "standard non-Gibbsian" (having some, but not all configurations bad) and a second transition to a "totally non-Gibbsian" regime where *all* configurations are bad. Both these properties do not occur in the more familiar lattice and mean-field situations.

For the plus and minus measure there are also two transitions, namely one after which the evolved measure becomes non-Gibbsian, and some, but not all, configurations become discontinuity points and a second one after which the measure becomes Gibbsian again; this is the behaviour which on the lattice occurs for an initial Gibbs measure in an external field.

Although we have worked out the case of Cayley trees, we expect our results to hold for a wider class of trees. The instability of the fixed point $h^\sharp$ for example corresponds with the phase transition being robust, which is true in general for Ising models on trees [65]. Also, the property of plus boundary conditions in a not too strong minus field inducing a positively magnetised state, which was used in the proof that the plus configuration was good for the plus state holds quite generally. The choice of bad configuration in the intermediate regime may be somewhat tree-dependent. Moreover, it seems problematical to identify a unique measure $\mu^\sharp$ in a field (on random Galton-Watson trees for example).

5 Potts Model with Invisible Colours: Random-cluster Representation and Pirogov-Sinai analysis*

5.1 Introduction

Recently, in a series of papers [70, 71, 72], Tamura, Tanaka and Kawashima introduced a variant of the ferromagnetic Potts model to study the relation between symmetry breaking and the order of the phase transition. The model consists of a ferromagnetic Potts interaction taking place between q "visible" colours along with the presence of r "invisible" colours without any interaction. They observed, through numerical simulations, that in two dimensions with $q = 2, 3, 4$ and r large, the model undergoes a first-order phase transition with q-fold symmetry breaking. This is in contrast with the ordinary two-dimensional q-colour Potts model with $q = 2, 3, 4$, in which the transition accompanied by the q-fold symmetry breaking is known to be of second order [2]. This provides a simple example that a q-fold symmetry breaking in two dimensions does not universally identify the order of the transition.

The transition in this model (as well as in the standard q-colour Potts model) occurs from an *ordered* state (which has a favoured direction among q possibilities) to a *disordered* state (which has no favoured direction) when the temperature is increased. In the standard q-colour Potts model with q small, this transition is of *second* order (no latent heat at the transition point) whereas in the Tamura-Tanaka-Kawashima version of the Potts model, for the same values of q but r chosen sufficiently large, the transition is of *first* order (the system absorbs heat during the transition, without changing its temperature).

For the standard q-colour Potts model, when q is large enough, there is a variety of different rigorous proofs that the transition is of first order [9, 46, 55, 58] (see also [34, 47]). In this chapter, we prove, by minor adaptations of the proofs in [47, 55], that when $q + r$ is large enough, the Potts model with q visible colours and r invisible colours undergoes a first-order phase transition.

---

*This chapter is based on [14] and [15].

The proof is based on an application of the Pirogov-Sinai method to a random-cluster representation of the model.

The phase transition could be better understood if one thinks of a state of the system as a possible resolution of the conflict between order and disorder. The conflict should be resolved locally and in every region. In an ordered region, the neighbouring sites tend to take the same colour so as to minimize the energy, while in a disordered region, the neighbouring sites take their colours independently to maximize the entropy. To establish the resolution of the order-disorder conflict, one needs to take into account the disturbance present at the interface between ordered and disordered regions (the *contours*).

For the standard q-colour Potts model, the order-disorder conflict is niftily depicted in the Fortuin-Kasteleyn (a.k.a. random-cluster) representation of the model [27, 30, 34, 37]. In this representation, order is associated with the presence of bonds between neighbouring sites and disorder with the absence of bonds. In the same spirit, we introduce a variant of the Fortuin-Kasteleyn representation for the Potts model with invisible colours. The advantage of this new formulation is that it admits a neat definition of the interface between ordered and disordered regions. Now, having two reference configurations describing complete order and complete disorder — the one with every bond present, and the one with every bond absent — as in [55], we can apply the Pirogov-Sinai method [4, 47, 68, 73].

In Section 5.2, we describe the model and recall the formulation of first-order phase transition in the Gibbsian setup. Section 5.3 is dedicated to the introduction of a variant of the random-cluster model and its connection with the Potts model with invisible colours. In Section 5.4.1, formal definitions for contours are provided, and it is shown how to rewrite the partition functions of the model in terms of contours. These contour representations are then reduced, in Section 5.4.2, to two abstract contour models, on which standard techniques can be applied. In Section 5.5, starting from the two contour models, we obtain two approximations for the free energy of the Potts model with invisible colours. If $q + r$ is large, each of these approximations turns out to be accurate in an interval of temperatures, one whenever order prevails and the other when disorder is dominant. The two intervals exhaust all the temperatures and have a unique common point, which is the transition point of the system. Finally, the above two approximations are used in Section 5.6 to prove a first-order transition at the transition point. The occurrence of the symmetry breaking at the same transition point then follows, using standard properties of the random-cluster representation, which are reviewed in Appendix C.

5.2 Potts Model with Invisible Colours

5.2.1 The Model

Let $\mathbb{L}$ denote the two-dimensional square *lattice*, which we think of as a graph $(\mathbb{S}, \mathbb{B})$, where $\mathbb{S}$ denotes the set of sites (identified by $\mathbb{Z}^2$) and $\mathbb{B}$ the set of nearest neighbour bonds. In the (q, r)-*Potts model*, each site $i \in \mathbb{S}$ is in one of $(q + r)$ colours $1, 2, \ldots, q, q + 1, \ldots, q + r$. Therefore, a *configuration* of the model is an assignment of values from the set $\{1, 2, \ldots, q, q + 1, \ldots, q + r\}$ to the sites in $\mathbb{S}$. The (q, r)-Potts model [70, 71, 72] is expressed by the formal Hamiltonian

$$H(\sigma) = - \sum_{\{i,j\} \in \mathbb{B}} \delta(\sigma_i = \sigma_j \leq q) \,, \tag{5.1}$$

where $\delta(\sigma_i = \sigma_j \leq q)$ is 1 if $\sigma_i = \sigma_j \leq q$ and 0 otherwise. Each pair of neighbouring sites that have the same colour $\alpha \leq q$ contributes with energy -1, while sites with colours $\alpha > q$ do not contribute to the energy. The first q colours are hence called the *visible* colours, and the rest the *invisible* colours. If there are no invisible colours (i.e, if $r = 0$), the model reduces to the ordinary Potts model with q colours. As in the ordinary q-colour Potts model, the (q, r)-Potts model has precisely q periodic ground-state configurations, in which every site has the same visible colour.

Following the usual approach, we describe the system in thermal equilibrium via probability distributions on the space of all possible configurations of the model. The *Boltzmann distribution* on a finite volume $\Lambda \subseteq \mathbb{L}$ with boundary condition ω at inverse temperature β is defined by

$$\mu_{\beta,\Lambda}^{\omega}(\sigma_\Lambda) = \frac{1}{Z_\beta^\omega(\Lambda)} e^{-\beta H_\Lambda(\sigma_\Lambda \omega_{\Lambda^{\complement}})} \,, \tag{5.2}$$

where $H_\Lambda(\sigma_\Lambda \omega_{\Lambda^{\complement}})$ consists of a finite number of terms in the formal Hamiltonian (5.1) corresponding to the energy of σ_Λ and its interaction with the boundary condition ω. Namely,

$$H_\Lambda(\sigma) = - \sum_{\substack{\{i,j\} \in \mathbb{B} \\ i \in S(\Lambda) \text{ or } j \in S(\Lambda)}} \delta(\sigma_i = \sigma_j \leq q) \,, \tag{5.3}$$

which can be decomposed as a sum

$$H_\Lambda(\sigma_\Lambda \omega_{\Lambda^{\complement}}) = H_\Lambda^{\mathrm{int}}(\sigma_\Lambda) + H_\Lambda^{\mathrm{bound}}(\sigma_\Lambda \omega_{\Lambda^{\complement}}) \,, \tag{5.4}$$

where $H_\Lambda^{\mathrm{int}}(\sigma_\Lambda)$ involves the interaction terms within Λ, and $H_\Lambda^{\mathrm{bound}}(\sigma_\Lambda \omega_{\Lambda^\complement})$ represents the terms corresponding to the interaction of Λ with its boundary. The factor $Z_\beta^\omega(\Lambda)$ is a normalizing constant — the *partition function* — making $\mu_{\beta,\Lambda}^\omega$ a probability distribution. More specifically, the partition function of volume Λ with boundary condition ω is given by

$$Z_\beta^\omega(\Lambda) = \sum_{\sigma_\Lambda} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{int}}(\sigma_\Lambda) - \beta H_\Lambda^{\mathrm{bound}}(\sigma_\Lambda \omega_{\Lambda^\complement})} \ . \tag{5.5}$$

If we ignore the boundary term, then we obtain the *free-boundary* partition function of volume Λ:

$$Z_\beta^{\mathrm{free}}(\Lambda) = \sum_{\sigma_\Lambda} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{int}}(\sigma_\Lambda)} \ . \tag{5.6}$$

This is the normalizing factor for the free-boundary Boltzmann distribution on Λ. A *Gibbs measure* on the space of all configurations of the infinite lattice system, at inverse temperature β, is a probability measure μ whose conditional probabilities for every finite volume Λ, given the configuration ω outside Λ, are given by the Boltzmann distribution $\mu_{\beta,\Lambda}^\omega$. More specifically,

$$\mu\,(A \text{ and } B) = \int_B \mu_{\beta,\Lambda}^\omega(A)\mu(\mathrm{d}\omega) \ , \tag{5.7}$$

for every event A not depending on the colours of the sites outside Λ and every event B not depending on the colours of the sites in Λ (see e.g. [29]). It follows from a compactness argument that such measures exist at every temperature. However, when the temperature is sufficiently low, it is possible to have several distinct Gibbs measures. The multiplicity of Gibbs measures is then interpreted as the possibility of co-existence of distinguishable phases of the physical system (in this case, the possibility of spontaneous magnetization in q different directions). We refer to [29] for details.

One way to obtain Gibbs measures consists of taking the thermodynamic limit of the Boltzmann distribution with or without a fixed boundary condition. For a visible colour k, let ω^k denote the configuration of the lattice in which every site has colour k. Let μ_β^k denote a Gibbs measure obtained by taking a weak limit of finite-volume Boltzmann distributions with boundary condition ω^k at inverse temperature β, when the finite-volume grows to the whole lattice. Similarly, we obtain a Gibbs measure $\mu_\beta^{\mathrm{free}}$ by taking a weak limit of free-boundary Boltzmann distributions.

For every $n > 0$, let Λ_n denote the $(2n+1) \times (2n+1)$ central square in the lattice, which we see as the subgraph of the lattice induced by the sites in

$[-n, n]^2$. The *pressure* of the model is defined by

$$f(\beta) = \lim_{n \to \infty} \frac{1}{|S(\Lambda_n)|} \log Z_\beta^\omega(\Lambda_n) \,, \tag{5.8}$$

in which $S(\Lambda_n)$ denotes the set of sites in Λ_n. The function $-\frac{1}{\beta} f(\beta)$ is the *free energy* per site. The limit exists and is independent of the boundary condition ω (see e.g. [45, 67]). We would also get the same limit as in (5.8) if we used the free boundary partition function. The particular choice of volumes used above is not crucial, and can be replaced by any sequence satisfying the van Hove property (see [45]).

5.2.2 First-Order Phase Transition

A *first-order phase transition* in temperature is characterized by the presence of *latent heat* at the transition point [12]. This means that at the transition point, the system absorbs or gives out heat without a change in temperature. The presence of latent heat, therefore, corresponds to a jump in the internal energy.

In the Gibbsian setup, the state of a system in thermal equilibrium is represented by a Gibbs measure (see [29]). If the Gibbs measure is translation-invariant, the *internal energy* density of the system is described by the expected value of energy per site. The presence of latent heat at a temperature means that the limits of the internal energy from above and below the transition temperature are different. This implies, by continuity, the existence of two translation-invariant Gibbs measures at that temperature having different internal energy.

If the pressure function $f(\beta)$ is differentiable at a point β, its derivative at β coincides with the internal energy with respect to every translation-invariant Gibbs measure at β (see [45, 67]). (This, however, does not rule out the possibility of the existence of several translation-invariant Gibbs measures at β.) If, on the other hand, the pressure function $f(\beta)$ is non-differentiable at a point β, its left and right derivatives at β (which exist due to convexity) are different and coincide with the internal energy with respect to two different translation-invariant Gibbs measures at β. The difference between these two derivatives corresponds to a latent heat at β, implying that the system undergoes a first-order phase transition at β.

We show the existence of a first-order phase transition in the (q, r)-Potts model by proving that the pressure function $f(\beta)$ has a unique non-differentiable point β_c. Above β_c, the system admits q "ordered" translation-invariant Gibbs

measures. Each ordered measure can be thought of as a perturbation of one of the q ground-state configurations, in the sense that with probability 1, the configuration of the model consists of a unique infinite sea of one of the visible colours with finite islands of disturbance. Below β_c, the system has a "disordered" translation-invariant Gibbs measure, which can be seen as a perturbation of the uniform Bernoulli measure: there is a unique infinite sea of independent colours with finite islands of disturbance. At β_c, the q ordered measures co-exist with the disordered one.

5.3 Biased Random-Cluster Representation

In analogy with the standard Potts model [27, 30, 34], it is possible to rewrite the partition function for the (q, r)-Potts model in terms of the partition function for a variant of the random-cluster model. While the former is a model defined on sites, the latter will be a model defined on bonds. The random-cluster representation of the Potts model allows for an elegant formulation of the intuitive concepts of "order" and "disorder": the presence of a bond in the random-cluster representation is interpreted as "order", while the absence of a bond as "disorder" [55].

Although for the purpose of our problem, it suffices to present the connection for squares Λ_n in the lattice, we elucidate the connection for an arbitrary finite graph, where there is no boundary condition. Later, we explain how the boundary conditions affect this connection.

Let $\mathbb{G} = (S, B)$ be a finite graph. The *r-biased random-cluster* model on $\mathbb{G}$ is given by a probability distribution on the sets $X \subseteq B$. The distribution has three parameters $0 \leq p \leq 1$, $q > 0$ and $r \geq 0$, and is defined by

$$\varphi_{p,q,r}(X) = \frac{1}{Z^{\mathsf{RC}}_{p,q,r}(\mathbb{G})} \left[\prod_{b \in B} p^{\delta(b \in X)} (1 - p)^{\delta(b \notin X)} \right] (q + r)^{\kappa_0(S,X)} q^{\kappa_1(S,X)} ,$$

(5.9)

in which $\kappa_0(S, X)$ denotes the number of isolated sites of the graph (S, X) and $\kappa_1(S, X)$ the number of non-singleton connected components of (S, X) and $Z^{\mathsf{RC}}_{p,q,r}(\mathbb{G})$ the partition function. Notice that for $r = 0$, the model reduces to the standard random-cluster model, in which both singleton and non-singleton connected components have weight q. For $r > 0$, the above model induces a bias towards singleton connected components. Namely, the singleton connected components have weight $(q + r)$ whereas the non-singleton connected components have weight q.

Let us now see how the (q, r)-Potts model is related to the r-biased random-cluster model. This is a mere generalization of the standard relation between the Potts and random-cluster models [30, 34]. Let Ω be the set of (q, r)-Potts configurations on $\mathbb{G}$. The partition function of this model can be rewritten as

$$
\begin{aligned}
Z_\beta(\mathbb{G}) &= \sum_{\sigma \in \Omega} e^{\beta \sum_{\{i,j\} \in B} \delta(\sigma_i = \sigma_j \leq q)} \\
&= \sum_{\sigma \in \Omega} \prod_{\{i,j\} \in B} e^{\beta \delta(\sigma_i = \sigma_j \leq q)} \\
&= \sum_{\sigma \in \Omega} \prod_{\{i,j\} \in B} \left[1 + \delta(\sigma_i = \sigma_j \leq q)(e^\beta - 1) \right] \\
&= \sum_{\sigma \in \Omega} \sum_{X \subseteq B} (e^\beta - 1)^{|X|} \prod_{\{i,j\} \in X} \delta(\sigma_i = \sigma_j \leq q) \\
&= \sum_{\sigma \in \Omega} \sum_{X \subseteq B} \pi(\sigma, X) \,,
\end{aligned}
\tag{5.10}
$$

where

$$
\pi(\sigma, X) = \\
e^{\beta |B|} \prod_{\{i,j\} \in B} \left[\delta(\{i,j\} \in X)\delta(\sigma_i = \sigma_j \leq q)(1 - e^{-\beta}) + \delta(\{i,j\} \notin X)e^{-\beta} \right] \,.
\tag{5.11}
$$

The latter expression can be seen as a coupling of the (q, r)-Potts distribution on Ω and a probability distribution on the space $\{0, 1\}^B$. The marginal of this coupling on $\{0, 1\}^B$ is simply the r-biased random-cluster distribution $\varphi_{p_\beta, q, r}$ with $p_\beta = 1 - e^{-\beta}$. In particular, the weight $\pi(\sigma, X)$ can also be expressed as

$$
\pi(\sigma, X) = e^{\beta |B|} \cdot 1_{F_r}(\sigma, X) \cdot \prod_{\{i,j\} \in B} \left[p_\beta \, \delta(\{i,j\} \in X) + (1 - p_\beta) \, \delta(\{i,j\} \notin X) \right] \,,
\tag{5.12}
$$

where

$$
F_r \triangleq \{(\sigma, X) : \sigma_i = \sigma_j \leq q \text{ for all } \{i,j\} \in X \} \,.
\tag{5.13}
$$

The effect of the bias in the r-biased random-cluster model reduces to an increase in the number of compatible configurations with a given X, which is

driven by a larger number of choices for the colour of those sites constituting the singleton connected components. In short, for each $X \subseteq B$, we have

$$\sum_{\sigma \in \Omega} 1_{F_r}(\sigma, X) = q^{\kappa_1(S,X)}(q+r)^{\kappa_0(S,X)} . \tag{5.14}$$

The above coupling could be interpreted in either of the following ways [30, 34]:

I. We first sample σ according to the (q,r)-Potts distribution. Then, we choose the elements of X from B, randomly and independently, as follows: for each bond $\{i,j\} \in B$ with $\sigma_i = \sigma_j$, we put $\{i,j\}$ in X with probability p_β; for each bond $\{i,j\} \in B$ with $\sigma_i \neq \sigma_j$, we do not put $\{i,j\}$ in X.

II. We first sample X according to the r-biased random-cluster distribution $\varphi_{p_\beta,q,r}$. Then, for each non-singleton connected component of (S,X), we pick a random colour uniformly among the visible colours, and colour every site in the component with that colour. Last, for every isolated site in (S,X), we choose a random colour uniformly among all the possible colours. (The choices of colours ought to be independent of each other.)

We can now use (5.10) to obtain

$$Z_\beta(\mathbb{G}) = \mathrm{e}^{\beta|B|} Z_{p_\beta,q,r}^{\mathsf{RC}}(\mathbb{G}) \tag{5.15}$$

with $p_\beta = 1 - \mathrm{e}^{-\beta}$.

For finite subgraphs of the infinite lattice, we will be using only two types of partition functions for the (q,r)-Potts model, namely the one with free boundary and the ones with homogeneous boundary conditions. In the following, we see how the above two types of boundary conditions translate into the so-called *disordered* and *ordered* boundary conditions for the r-biased random-cluster model. Although, setting $\mathbb{G} = \Lambda_n$, equation (5.15) already provides a relation between the free-boundary partition functions of the two models, we will work with a slightly different relation, connecting the free-boundary partition function of the (q,r)-Potts model to a partition function for the r-biased random-cluster model that involves a boundary condition. This new relation will turn out to be more convenient in the sequel.

The free-boundary partition function for the (q,r)-Potts model can be written as

$$Z_\beta(\Lambda_n) = (q+r)^{-|S(\Lambda_{n+1})\setminus S(\Lambda_n))|} \cdot \mathrm{e}^{\beta|B(\Lambda_{n+1})|} \cdot Z_{p_\beta,q,r}^{\mathsf{RC.disord}}(\Lambda_{n+1}) , \tag{5.16}$$

where $Z^{\mathsf{RC.disord}}_{p_\beta,q,r}(\Lambda_{n+1})$ is the partition function with *disordered boundary condition* for the r-biased random-cluster model. The latter is defined by

$$Z^{\mathsf{RC.disord}}_{p_\beta,q,r}(\Lambda_{n+1}) = \sum_{X \in \mathcal{X}^{\mathrm{disord}}_{\Lambda_{n+1}}} p_\beta^{|X|}(1-p_\beta)^{|B(\Lambda_{n+1})\setminus X|}(q+r)^{\kappa_0(S(\Lambda_{n+1}),X)}q^{\kappa_1(S(\Lambda_{n+1}),X)} ,$$

(5.17)

where

$$\mathcal{X}^{\mathrm{disord}}_{\Lambda_{n+1}} = \{X \subseteq B(\Lambda_{n+1}) : X \cap (B(\Lambda_{n+1}) \setminus B(\Lambda_n)) = \varnothing\} . \qquad (5.18)$$

Similarly, for the boundary condition ω^k we get

$$Z^{\omega^k}_\beta(\Lambda_n) = q^{-1} \cdot (e^\beta - 1)^{-|B(\Lambda_{n+1}\setminus\Lambda_n)|} \cdot e^{\beta|B(\Lambda_{n+1})|} \cdot Z^{\mathsf{RC.ord}}_{p_\beta,q,r}(\Lambda_{n+1}) , \quad (5.19)$$

where $Z^{\mathsf{RC.ord}}_{p_\beta,q,r}(\Lambda_{n+1})$ is the partition function with *ordered boundary condition* for the r-biased random-cluster model, which is defined by

$$Z^{\mathsf{RC.ord}}_{p_\beta,q,r}(\Lambda_{n+1}) = \sum_{X \in \mathcal{X}^{\mathrm{ord}}_{\Lambda_{n+1}}} p_\beta^{|X|}(1-p_\beta)^{|B(\Lambda_{n+1})\setminus X|}(q+r)^{\kappa_0(S(\Lambda_{n+1}),X)}q^{\kappa_1(S(\Lambda_{n+1}),X)} ,$$

(5.20)

where

$$\mathcal{X}^{\mathrm{ord}}_{\Lambda_{n+1}} = \{X \subseteq B(\Lambda_{n+1}) : X \supseteq B(\Lambda_{n+1} \setminus \Lambda_n)\} . \qquad (5.21)$$

By $\Lambda_{n+1} \setminus \Lambda_n$ we mean the graph obtained from Λ_{n+1} by removing all the sites in Λ_n and the bonds attached to them. Let us remark that although mathematically $\mathcal{X}^{\mathrm{disord}}_{\Lambda_{n+1}}$ is simply the collection of all subsets $X \subseteq B(\Lambda_n)$, we wrote it as above to emphasize that the elements of $\mathcal{X}^{\mathrm{disord}}_{\Lambda_{n+1}}$ are configurations of $B(\Lambda_{n+1})$. See Figure 5.1 for typical examples of elements in $\mathcal{X}^{\mathrm{disord}}_{\Lambda_{n+1}}$ and $\mathcal{X}^{\mathrm{ord}}_{\Lambda_{n+1}}$.

In the following section, we will extend the definition of $Z^{\mathsf{RC.disord}}_{p_\beta,q,r}$ and $Z^{\mathsf{RC.ord}}_{p_\beta,q,r}$ to arbitrary subgraphs of the lattice.

Using the above relationships, we obtain that the pressure of the (q,r)-Potts model can be written as

$$f(\beta) = 2 \left(\beta + \lim_{n\to\infty} \frac{\log Z^{\mathsf{RC.disord}}_{p_\beta,q,r}(\Lambda_n)}{|B(\Lambda_n)|} \right) = 2 \left(\beta + \lim_{n\to\infty} \frac{\log Z^{\mathsf{RC.ord}}_{p_\beta,q,r}(\Lambda_n)}{|B(\Lambda_n)|} \right) .$$

(5.22)
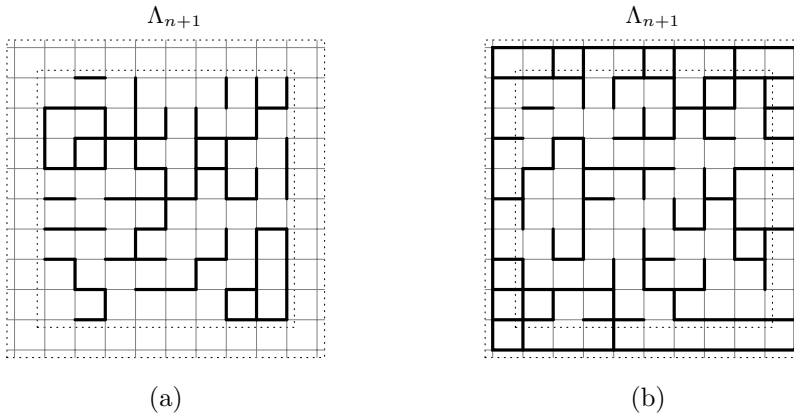
Figure 5.1: (a) A configuration in $\mathcal{X}_{\Lambda_{n+1}}^{\mathrm{disord}}$. (b) A configuration in $\mathcal{X}_{\Lambda_{n+1}}^{\mathrm{ord}}$.

We call the limit

$$f^{\mathsf{RC}}(\beta) = \lim_{n\to\infty} \frac{\log Z_{p_\beta,q,r}^{\mathsf{RC.disord}}(\Lambda_n)}{|B(\Lambda_n)|} = \lim_{n\to\infty} \frac{\log Z_{p_\beta,q,r}^{\mathsf{RC.ord}}(\Lambda_n)}{|B(\Lambda_n)|} \qquad (5.23)$$

the *pressure* (per bond) of the r-biased random-cluster representation. Note that the singularities of the (q,r)-Potts pressure function $f(\beta)$ can be detected by studying the pressure function $f^{\mathsf{RC}}(\beta)$. One advantage of this random-cluster representation is that it has a more transparent expression in terms of "contours", which helps us study the function $f^{\mathsf{RC}}(\beta)$.

5.4 Reduction to Contour Model

5.4.1 Contour Representation

Any configuration of the r-biased random-cluster model in a volume is a subset X of bonds in the volume. We interpret each bond in X as "ordered" and each bond outside X as "disordered". Any configuration X can then be seen as clusters of ordered and disordered bonds. Whether an equilibrium state is ordered or disordered can be seen as the result of a competition between ordered and disordered regions. The selection criterion for this competition is "energy". The term "energy" refers to an abstract notion of energy for the

r-biased random-cluster model, which in analogy with the Boltzmann distribution, corresponds to minus logarithm of probability.

Let us define the "energy" of an ordered bond as the "energy" per bond of the fully ordered configuration; that is,

$$e(\mathbb{B}) \triangleq \lim_{n \to \infty} \frac{-\log \left[q(1 - \mathrm{e}^{-\beta})^{|B(\Lambda_n)|} \right]}{|B(\Lambda_n)|} = -\log(1 - \mathrm{e}^{-\beta}) \ . \qquad (5.24)$$

Similarly, define the "energy" of a disordered bond as the "energy" per bond of the fully disordered configuration:

$$e(\varnothing) \triangleq \lim_{n \to \infty} \frac{-\log \left[\mathrm{e}^{-\beta|B(\Lambda_n)|}(q + r)^{|S(\Lambda_n)|} \right]}{|B(\Lambda_n)|} = -\log \left[\mathrm{e}^{-\beta} \sqrt{q + r} \right] \ . \qquad (5.25)$$

The "energy" of the ordered and the disordered regions can now be expressed as $|R^{\mathrm{o}}| \cdot e(\mathbb{B})$ and $|R^{\mathrm{d}}| \cdot e(\varnothing)$, respectively, where $|R^{\mathrm{o}}|$ and $|R^{\mathrm{d}}|$ denote the size of the ordered and disordered regions. The "energy" of X, in turn, can be written in terms of the "energy" of ordered and disordered regions plus a correction term due to the effect of the boundaries separating them. If the effect of these boundaries is negligible (which will turn out to be the case whenever $q + r$ is large), the selection criterion for the competition between order and disorder boils down to determining which of $e(\mathbb{B})$ and $e(\varnothing)$ is minimal. This is the starting point of the Pirogov-Sinai approach to study phase transitions (see e.g. [47]).

The presence of the correction term at the boundaries can be explained as follows: In the probability weight of a configuration X, each isolated site contributes with a factor $(q+r)$. To express the "energy" of the disordered regions purely in terms of bonds, we evenly distribute the contribution of the isolated sites among the 4 incident bonds. Doing so, every disorder bond acquires zero, one or two "energy"-shares, depending on the number of isolated sites it is incident to. Since in the fully disordered configuration $\varnothing$ there is no ordered region, every bond is incident to precisely two isolated sites and receives two "energy"-shares, leading to the factor $(q + r)^{\frac{2}{4}}$ in the expression of $e(\varnothing)$. In an arbitrary configuration, however, the disordered bonds on the borderline between the ordered and disordered regions, receive one or no "energy"-share, hence the need for a correction term.

It is possible to define a suitable notion of boundary between ordered and disordered regions, so that each configuration X is uniquely identified by its boundary (see below). We could then rewrite the partition functions as sums running over "admissible" boundaries, that is, those corresponding to configurations of bonds. Each admissible boundary is split into "primary" objects termed contours whose "energy" add up to the corresponding boundary effect.

In the following, we specify rigorously the above heuristic notions of "boundary" and "contours". We define the *boundary* of a configuration $X \subseteq \mathbb{B}$ as the set

$$\overline{\partial} X \triangleq \{(i, b) \in \mathbb{S} \times \mathbb{B} : i \sim b \text{ and } i \in S(X) \text{ and } b \notin X\}, \qquad (5.26)$$

where $i \sim b$ means site i and bond b are incident, and $S(X)$ is the set of sites incident to bonds in X. The set $\overline{\partial} X$ uniquely determines X. We say that two bonds b and b' in the lattice are *co-adjacent* if they belong to the same unit square. More intuitively, co-adjacency is equivalent to adjacency in the dual lattice. A set of bonds X is *co-connected* if for every two bonds $b, b' \in X$, there is a sequence $b = b_1, b_2, \ldots, b_n = b'$ of bonds in X such that b_i and b_{i+1} are co-adjacent. A *contour* is a set $\gamma \subseteq \mathbb{S} \times \mathbb{B}$ such that

i) the set of bonds appearing in γ, denoted by $B(\gamma)$, is co-connected, and
ii) there exists a configuration X such that $(S(X), X)$ is connected and $\gamma = \overline{\partial} X$.

We shall denote by Γ the set of all finite contours in $\mathbb{L}$. If γ is a contour, then removing the bonds $B(\gamma)$ breaks the lattice $\mathbb{L}$ into connected subgraphs. If γ is finite, the graph $\mathbb{L} \setminus B(\gamma)$ has a unique infinite connected component, which we call the *exterior* of γ and denote by $\text{ext} \gamma$. The subgraph $\mathbb{L} \setminus B(\gamma) \setminus \text{ext} \gamma$ (which could be empty or disconnected) is called the *interior* of γ and is denoted by $\text{int} \gamma$. By $V(\gamma)$ we will mean the union of $\text{int} \gamma$ and the subgraph induced by $B(\gamma)$.[1] Let γ be a finite contour. The configuration X such that $(S(X), X)$ is connected and $\gamma = \overline{\partial} X$ (which exists by definition) is either finite or co-finite. If X is finite, we call γ a *disorder* contour, and if X is co-finite, we call γ an *order* contour. Note that, if γ is a disorder contour, all the sites appearing in γ are in the interior of γ, whereas if γ is an order contour, all the sites appearing in γ are in the exterior of γ. As a result, we can safely represent a finite contour γ by the pair $(B(\gamma), x)$ where x is a label specifying the type of the contour (disorder or order). This also means that the set of all finite contours Γ can be partitioned into two subsets: the set of disorder contours, which we denote by Γ^d, and the set of order contours, which we denote by Γ^o.

Two contours are said to be *mutually compatible* if they are disjoint (as subsets of $\mathbb{S} \times \mathbb{B}$). Let us emphasize that two mutually compatible contours are allowed to share either sites or bonds, but not pairs.

If $X \subseteq \mathbb{B}$ is an arbitrary configuration, there could be several ways to partition its boundary $\overline{\partial} X$ into mutually compatible contours. One way to construct such decomposition in an unambiguous way is as follows: first, we partition

---

[1] By the subgraph induced by a set of bonds we mean the graph obtained by those bonds and their endpoints.

(a)

(b)

(a')

(b')
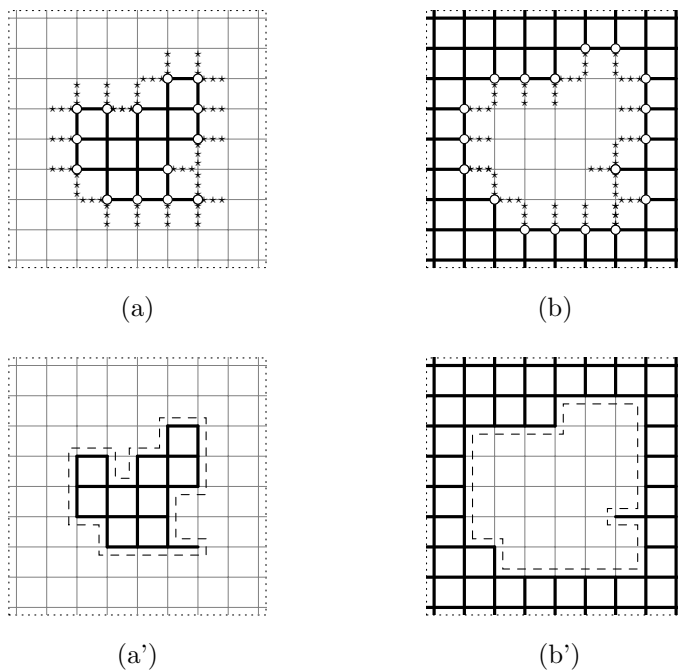
Figure 5.2:   (a) A disorder contour and its defining configuration.
(b) An order contour and its defining configuration.
(a') and (b') geometric illustrations of (a) and (b).

$(S(X), X)$ into its maximal connected components $(S(C_i), C_i)$. Then $\overline{\partial} C_i$ form a partitioning of $\overline{\partial} X$. Now, the maximal co-connected components of every C_i are contours that we identify as the *contours of* X.

   The above decomposition allows us to think of $\overline{\partial} X$ as a family of mutually compatible contours, which we call the *contour family* of X. Let us recall that the contour family of a configuration X uniquely determines X. However, note that not every family of mutually compatible contours corresponds to a configuration. In particular, in a contour family of a configuration X, between every two nested finite contours of the same type, there necessarily lies a contour of the other type. This requirement induces a long-range constraint among contours, which raises some difficulties in dealing with the contours. We will see later how to get rid of such a constraint. Let us call a family ∂ of contours *admissible* if it is the contour family of a configuration $X \subseteq \mathbb{B}$. We shall denote by Δ the set of all admissible contour families. A contour γ in a mutually compatible family ∂ of contours is said to be *external* if it is not in the interior of any other contour in ∂. Note that if ∂ is an admissible contour family with no infinite contours, all the external contours in ∂ are necessarily of the same type.

   Having formalized the notions of boundary and contours, we can now express the weight of a configuration of the r-biased random-cluster model in terms of the "energy" of its ordered and disordered regions and the correction term due to the contours separating them. The one-to-one correspondence between the configurations and the admissible families of contours allows us to write the partition functions as a sum over contour families. The ordered/disordered boundary conditions on the configurations translate into the constraints for the corresponding contour family that the outermost contours in the volume be of the order/disorder type.

   Let Λ be a *volume* in the lattice, by which, from now on, we shall mean a finite subgraph of $\mathbb{L}$ without "holes". More precisely, we assume that if we remove the subgraph Λ from $\mathbb{L}$, the remaining subgraph is connected. Let us denote by $\Delta_\Lambda^{\mathrm{disord}}$ the set of all admissible contour families whose contours are in Λ (i.e., their bonds are chosen from the bonds of Λ) and whose external contours are all of the disorder type. Similarly, let $\Delta_\Lambda^{\mathrm{ord}}$ denote the set of admissible contour families in Λ whose external contours are all of the order type. The partition function for the r-biased random-cluster model in a volume Λ with *disordered*

(resp., *ordered*) *boundary conditions* can be defined as

$$Z_{p_\beta,q,r}^{\mathsf{RC.disord}}(\Lambda) = (q+r)^{\frac{|\overline{\partial}B(\Lambda)|}{4}} \sum_{\partial \in \Delta_\Lambda^{\mathrm{disord}}} \mathrm{e}^{-|R_\Lambda^{\mathrm{o}}(\partial)|\cdot e(\mathbb{B}) - |R_\Lambda^{\mathrm{d}}(\partial)|\cdot e(\varnothing)} \prod_{\gamma \in \partial} \rho(\gamma) \,,$$

(5.27)

$$Z_{p_\beta,q,r}^{\mathsf{RC.ord}}(\Lambda) = q \sum_{\partial \in \Delta_\Lambda^{\mathrm{ord}}} \mathrm{e}^{-|R_\Lambda^{\mathrm{o}}(\partial)|\cdot e(\mathbb{B}) - |R_\Lambda^{\mathrm{d}}(\partial)|\cdot e(\varnothing)} \prod_{\gamma \in \partial} \rho(\gamma) \,, \qquad (5.28)$$

where $\rho(\gamma)$ is the *weight* of a contour γ and is given by

$$\rho(\gamma) \triangleq \begin{cases} (q+r)^{-\frac{1}{4}|\gamma|} \,, & \text{if } \gamma \text{ order,} \\ q\,(q+r)^{-\frac{1}{4}|\gamma|} \,, & \text{if } \gamma \text{ disorder,} \end{cases} \qquad (5.29)$$

and $R_\Lambda^{\mathrm{o}}(\partial)$ and $R_\Lambda^{\mathrm{d}}(\partial)$ denote, respectively, the sets of ordered and disordered bonds in Λ of the configuration corresponding to ∂.

The above definitions are consistent with the definitions given in (5.17) and (5.20) when $\Lambda = \Lambda_{n+1}$ is a square. Namely, for $\Lambda = \Lambda_{n+1}$, if X is the corresponding configuration of a family $\partial \in \Delta_{\Lambda_{n+1}}^{\mathrm{disord}}$, the restriction of X to $B(\Lambda_{n+1})$ is an element of $\mathcal{X}_{\Lambda_{n+1}}^{\mathrm{disord}}$. Conversely, every element of $\mathcal{X}_{\Lambda_{n+1}}^{\mathrm{disord}}$ has a unique infinite-volume extension whose corresponding contour family is in $\Delta_{\Lambda_{n+1}}^{\mathrm{disord}}$. A similar correspondence holds between $\mathcal{X}_{\Lambda_{n+1}}^{\mathrm{ord}}$ and $\Delta_{\Lambda_{n+1}}^{\mathrm{ord}}$. For the proof of the equivalence of the two definitions see Appendix B.

We emphasize that the factors $(q+r)^{\frac{|\overline{\partial}B(\Lambda)|}{4}}$ and q in front of the partition functions (5.27) and (5.28) do not contribute to the pressure function $f^{\mathsf{RC}}(\beta)$: in the thermodynamic limit, they are swallowed by the size of the volume. Hence, to avoid heavy notation — with all due apologies to the reader — we *re-define* the partition functions of the r-biased random-cluster model with disordered/ordered boundary conditions as

$$Z^{\mathsf{RC.disord}}(\Lambda) = \sum_{\partial \in \Delta_\Lambda^{\mathrm{disord}}} \mathrm{e}^{-|R_\Lambda^{\mathrm{o}}(\partial)|\cdot e(\mathbb{B}) - |R_\Lambda^{\mathrm{d}}(\partial)|\cdot e(\varnothing)} \prod_{\gamma \in \partial} \rho(\gamma) \,, \qquad (5.30)$$

$$Z^{\mathsf{RC.ord}}(\Lambda) = \sum_{\partial \in \Delta_\Lambda^{\mathrm{ord}}} \mathrm{e}^{-|R_\Lambda^{\mathrm{o}}(\partial)|\cdot e(\mathbb{B}) - |R_\Lambda^{\mathrm{d}}(\partial)|\cdot e(\varnothing)} \prod_{\gamma \in \partial} \rho(\gamma) \,. \qquad (5.31)$$

From now on, every time we talk about the partition function of the r-biased random-cluster model, we will be referring to the latter definitions.

As was mentioned in the introduction, we would like to express the two partition functions in terms of two (standard) contour models. The purpose

of this is to make use of the machinery available for contour models; namely, a result providing an estimate on the convergence of the corresponding free energy functions (Proposition 1), and the Peierls estimate for the probability of the appearance of a contour. The main features of the contour models that are required in the above tools are (see [47])

  i) independence, and

  ii) damping.

Unfortunately, the contours of the contour representation of the r-biased random-cluster partition functions are not independent (due to the long-range constraint). In the following section, we will see how to achieve the independence among contours, by rewriting the partition functions in terms of abstract contour models. As in the standard random-cluster model (see [55]), we need two different such contour models, one for each of the two boundary conditions.

5.4.2 Contour Models

In this section, we want to resolve the issue of long-range constraints between contours. Recall that the admissibility condition requires the contours of a family to be alternating between disorder and order contours, and this imposes a long-range constraint between contours. For example, two nested contours of the disorder type (no matter how far from each other) are "aware" of the presence of an order contour separating them. As a result, if we remove a contour from an admissible family, the admissibility could be lost.

In order to get rid of this constraint, we use two abstract contour models in which the contours are all of the same type and the admissibility condition is replaced by mere mutual compatibility. The weights of the contours in each of the abstract models will be chosen in such a way to guarantee that the ensuing partition functions are equal (up to a factor) to each of the partition functions for the r-biased random-cluster model.

A *contour model* is specified by a function $\chi : \Gamma \to \mathbb{R}$, assigning a weight $\chi(\gamma)$ to each contour $\gamma \in \Gamma$. The configurations of the model are families of mutually compatible (i.e., disjoint) contours in $\mathbb{L}$. Let us denote the set of all such families by $\mathcal{M}$, and the set of all elements of $\mathcal{M}$ whose contours are in a volume Λ by $\mathcal{M}_\Lambda$. The partition function of the model in Λ is given by

$$\mathscr{Z}(\Lambda \,|\, \chi) = \sum_{\partial \in \mathcal{M}_\Lambda} \prod_{\gamma \in \partial} \chi(\gamma) \,. \tag{5.32}$$

In the following lemma, we will see how to represent the partition functions of the r-biased random-cluster model with disordered and ordered boundary

conditions, each in terms of the partition function of a contour model, with a particular choice of the weight function. In fact, the contour model associated to the disordered boundary condition will not involve order contours. This is reflected by the fact that in this model each order contour has weight zero. Similarly, the contour model for the ordered boundary condition involves only order contours.

To set the stage for the lemma, we rewrite the partition functions $Z^{\mathsf{RC.disord}}(\Lambda)$ and $Z^{\mathsf{RC.ord}}(\Lambda)$ in a form resembling more the contour model partition function $\mathscr{Z}(\Lambda \,|\, \chi)$. That is,

$$Z^{\mathsf{RC.disord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)| \cdot e(\varnothing)} \sum_{\partial \in \Delta_\Lambda^{\mathrm{disord}}} \prod_{\gamma \in \partial} \widetilde{\rho}(\gamma) \,, \tag{5.33}$$

$$Z^{\mathsf{RC.ord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)| \cdot e(\mathbb{B})} \sum_{\partial \in \Delta_\Lambda^{\mathrm{ord}}} \prod_{\gamma \in \partial} \widetilde{\rho}(\gamma) \,, \tag{5.34}$$

where

$$\widetilde{\rho}(\gamma) = \begin{cases} \rho(\gamma) \cdot \mathrm{e}^{-|B(\mathrm{int}\,\gamma)| \cdot (e(\mathbb{B}) - e(\varnothing))} \,, & \text{if } \gamma \text{ is disorder,} \\ \rho(\gamma) \cdot \mathrm{e}^{-|B(V(\gamma))| \cdot (e(\varnothing) - e(\mathbb{B}))} \,, & \text{if } \gamma \text{ is order.} \end{cases} \tag{5.35}$$

Let us recall that the set $\Delta_\Lambda^{\mathrm{disord}}$ (resp., $\Delta_\Lambda^{\mathrm{ord}}$) does not contain only families of disorder (resp., order) contours, but all families compatible with the disordered (resp., ordered) boundary condition. To make the proof more transparent, let us define

$$Y^{\mathrm{d}}(\Lambda) = \sum_{\partial \in \Delta_\Lambda^{\mathrm{disord}}} \prod_{\gamma \in \partial} \widetilde{\rho}(\gamma) \,, \tag{5.36}$$

$$Y^{\mathrm{o}}(\Lambda) = \sum_{\partial \in \Delta_\Lambda^{\mathrm{ord}}} \prod_{\gamma \in \partial} \widetilde{\rho}(\gamma) \,, \tag{5.37}$$

so that

$$Z^{\mathsf{RC.disord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)| e(\varnothing)} \cdot Y^{\mathrm{d}}(\Lambda) \,, \tag{5.38}$$

$$Z^{\mathsf{RC.ord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)| e(\mathbb{B})} \cdot Y^{\mathrm{o}}(\Lambda) \,. \tag{5.39}$$

Notice that the above contour representation for the partition functions $Z^{\mathsf{RC.disord}}(\Lambda)$ and $Z^{\mathsf{RC.ord}}(\Lambda)$ lacks the condition of independence between compatible contours.

The following lemma is similar to Lemma 1 of [47].

Lemma 1. *The partition functions for the r-biased random-cluster model on volume Λ with the disordered and ordered boundary conditions can be written as*

$$Z^{\mathsf{RC.disord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|e(\varnothing)} \mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{d}}) \,, \tag{5.40}$$

$$Z^{\mathsf{RC.ord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|e(\mathbb{B})} \mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}}) \,, \tag{5.41}$$

where the weights ξ^{d} and ξ^{o} are defined by

$$\xi^{\mathrm{d}}(\gamma) = \begin{cases} \rho(\gamma) \dfrac{Z^{\mathsf{RC.ord}}(\operatorname{int}\gamma)}{Z^{\mathsf{RC.disord}}(\operatorname{int}\gamma)} \,, & \text{if } \gamma \text{ disorder,} \\ 0 \,, & \text{otherwise,} \end{cases} \tag{5.42}$$

and

$$\xi^{\mathrm{o}}(\gamma) = \begin{cases} \rho(\gamma)\mathrm{e}^{|B(\gamma)|e(\mathbb{B})} \dfrac{Z^{\mathsf{RC.disord}}(V(\gamma))}{Z^{\mathsf{RC.ord}}(\operatorname{int}\gamma)} \,, & \text{if } \gamma \text{ is order,} \\ 0 \,, & \text{otherwise.} \end{cases} \tag{5.43}$$

Proof. The key step to prove the lemma is to write a recursion for the above partition functions by factoring the contribution of the interior of each external contour. Let us denote by $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$ the set of mutually compatible families of disorder contours whose elements are all external. (We include the empty family in $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$.) Note that the elements of $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$ are all admissible and in $\Delta_{\Lambda}^{\mathrm{disord}}$. Moreover, for each admissible family $\partial \in \Delta_{\Lambda}^{\mathrm{disord}}$, the sub-family of ∂ consisting of its external contours is in $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$. Similarly, we denote by $\mathcal{E}_{\Lambda}^{\mathrm{ord}}$ the set of mutually compatible families of order contours whose elements are all external. The partition functions Y^{d} and Y^{o} satisfy the following recursions:

$$Y^{\mathrm{d}}(\Lambda) = \sum_{\theta \in \mathcal{E}_{\Lambda}^{\mathrm{disord}}} \prod_{\gamma \in \theta} \widetilde{\rho}(\gamma) \cdot Y^{\mathrm{o}}(\operatorname{int}\gamma) \,, \tag{5.44}$$

$$Y^{\mathrm{o}}(\Lambda) = \sum_{\theta \in \mathcal{E}_{\Lambda}^{\mathrm{ord}}} \prod_{\gamma \in \theta} \widetilde{\rho}(\gamma) \cdot Y^{\mathrm{d}}(V(\gamma)) \,. \tag{5.45}$$

Similar recursions hold for the contour model partition functions $\mathscr{Z}(\cdot \,|\, \xi^{\mathrm{d}})$ and $\mathscr{Z}(\cdot \,|\, \xi^{\mathrm{o}})$:

$$\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{d}}) = \sum_{\theta \in \mathcal{E}_{\Lambda}^{\mathrm{disord}}} \prod_{\gamma \in \theta} \xi^{\mathrm{d}}(\gamma) \cdot \mathscr{Z}(\operatorname{int}\gamma \,|\, \xi^{\mathrm{d}}) \,, \tag{5.46}$$

$$\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}}) = \sum_{\theta \in \mathcal{E}_{\Lambda}^{\mathrm{ord}}} \prod_{\gamma \in \theta} \xi^{\mathrm{o}}(\gamma) \cdot \mathscr{Z}(\operatorname{int}\gamma \,|\, \xi^{\mathrm{o}}) \,. \tag{5.47}$$

Note that, since every order contour is weighted 0 by ξ^{d}, we can ignore in $\mathscr{Z}(\cdot\,|\,\xi^{\mathrm{d}})$ the families containing order contours, and similarly the disorder contours can be ignored in $\mathscr{Z}(\cdot\,|\,\xi^{\mathrm{o}})$.

We use induction on volume Λ to prove that $Y^{\mathrm{d}}(\Lambda) = \mathscr{Z}(\Lambda\,|\,\xi^{\mathrm{d}})$. Suppose that for every sub-volume $\Lambda' \subsetneq \Lambda$ we have $Y^{\mathrm{d}}(\Lambda') = \mathscr{Z}(\Lambda'\,|\,\xi^{\mathrm{d}})$. Let $\theta \in \mathcal{E}_{\Lambda}^{\mathrm{disord}}$. We want to show that the terms corresponding to θ in the recursion formulas (5.44) and (5.46) for $Y^{\mathrm{d}}(\Lambda)$ and $\mathscr{Z}(\Lambda\,|\,\xi^{\mathrm{d}})$ are equal. If θ is empty, the equality is trivial (we consider the product over an empty set to be 1). Otherwise, for every $\gamma \in \theta$, we have $\mathrm{int}\,\gamma \subsetneq \Lambda$, which implies $\mathscr{Z}(\mathrm{int}\,\gamma\,|\,\xi^{\mathrm{d}}) = Y^{\mathrm{d}}(\mathrm{int}\,\gamma)$. Using the definitions of $\widetilde{\rho}$ and ξ^{d} we obtain that

$$\prod_{\gamma \in \theta} \widetilde{\rho}(\gamma) \cdot Y^{\mathrm{o}}(\mathrm{int}\,\gamma) = \prod_{\gamma \in \theta} \xi^{\mathrm{d}}(\gamma) \cdot Y^{\mathrm{d}}(\mathrm{int}\,\gamma)\,. \tag{5.48}$$

Therefore, $Y^{\mathrm{d}}(\Lambda) = \mathscr{Z}(\Lambda\,|\,\xi^{\mathrm{d}})$. The starting point of the induction is when the only element of $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$ is the empty family.

The argument for $Y^{\mathrm{o}}(\Lambda) = \mathscr{Z}(\Lambda\,|\,\xi^{\mathrm{o}})$ is similar. $\qquad\square$

Note that there is no complete correspondence between the configurations of the r-biased random-cluster model and the contour families of the corresponding abstract contour model. Nevertheless, the probability of appearance of a contour as an external contour is the same in both models. Let $\varphi_{\Lambda}^{\mathrm{ord}}$ denote the probability distribution associated to $Z^{\mathrm{RC.ord}}(\Lambda)$. We consider $\varphi_{\Lambda}^{\mathrm{ord}}$ as a measure on the infinite-volume bond configurations $X \subseteq \mathbb{B}$, which is concentrated on the set $\{X : \overline{\partial}X \in \Delta_{\Lambda}^{\mathrm{ord}}\}$. Likewise, $\varphi_{\Lambda}^{\mathrm{disord}}$ will denote the measure corresponding to $Z^{\mathrm{RC.disord}}(\Lambda)$, which is concentrated on the set $\{X : \overline{\partial}X \in \Delta_{\Lambda}^{\mathrm{disord}}\}$.

Corollary 1.1. *Let Λ be a finite volume and $\theta \in \mathcal{E}_{\Lambda}^{\mathrm{ord}}$ a family of external mutually compatible order contours. Then,*

$$\varphi_{\Lambda}^{\mathrm{ord}}\{X : \overline{\partial}_{\mathrm{ext}}X = \theta\} = \frac{\prod_{\gamma \in \theta} \xi^{\mathrm{o}}(\gamma)\mathscr{Z}(\mathrm{int}\,\gamma\,|\,\xi^{\mathrm{o}})}{\mathscr{Z}(\Lambda\,|\,\xi^{\mathrm{o}})}\,, \tag{5.49}$$

where $\overline{\partial}_{\mathrm{ext}}X$ is the family of external contours of X. A similar statement holds for the probability of families of external mutually compatible disorder contours under $\varphi_{\Lambda}^{\mathrm{disord}}$.

Proof.

$$\varphi_{\Lambda}^{\mathrm{ord}}\{X : \overline{\partial}_{\mathrm{ext}}X = \theta\} = \frac{\prod_{\gamma \in \theta} \widetilde{\rho}(\gamma)Y^{\mathrm{d}}(V(\gamma))}{Y^{\mathrm{o}}(\Lambda)}$$

$$= \frac{\prod_{\gamma \in \theta} \left[\rho(\gamma) \cdot \mathrm{e}^{-|B(V(\gamma))| \cdot (e(\varnothing - e(\mathbb{B})))\frac{Y^{\mathrm{d}}(V(\gamma))}{Y^{\mathrm{o}}(\mathrm{int}\,\gamma)}} \right] Y^{\mathrm{o}}(\mathrm{int}\,\gamma)}{Y^{\mathrm{o}}(\Lambda)}$$

$$= \frac{\prod_{\gamma \in \theta} \xi^{\mathrm{o}}(\gamma) \mathscr{Z}(\mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}})}{\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}})} \tag{5.50}$$

$\square$

The next corollary provides an estimate for the probability that a finite region of the lattice is surrounded by a contour (see e.g. [34]). For a finite set of sites A in the lattice, let Γ_A denote the set of all finite contours that have A in their interiors.

Corollary 1.2. *For every finite volume Λ and every finite set $A \subseteq S(\Lambda)$*

$$\varphi_\Lambda^{\mathrm{ord}}\{X : \overline{\partial} X \cap \Gamma_A \neq \varnothing\} \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) = \sum_{\substack{\gamma \in \Gamma_A \\ \gamma \text{ order}}} \xi^{\mathrm{o}}(\gamma) \,. \tag{5.51}$$

A similar bound holds in the disordered case.

Proof. Taking into account the ordered boundary condition, we have that if A is surrounded by a contour in Λ, it is also surrounded by an external order contour in Λ, that is,

$$\{X : \overline{\partial} X \cap \Gamma_A \neq \varnothing\} = \{X : \overline{\partial}_{\mathrm{ext}} X \cap \Gamma_A \neq \varnothing\} \,. \tag{5.52}$$

By the previous corollary, we can bound the probability of a contour γ appearing as an external contour by

$$\varphi_\Lambda^{\mathrm{ord}}\{X : \overline{\partial}_{\mathrm{ext}} X \ni \gamma\} = \sum_{\substack{\theta \in \mathcal{E}_\Lambda^{\mathrm{ord}} \\ \theta \ni \gamma}} \frac{\prod_{\widehat{\gamma} \in \theta} \xi^{\mathrm{o}}(\widehat{\gamma}) \mathscr{Z}(\mathrm{int}\,\widehat{\gamma} \,|\, \xi^{\mathrm{o}})}{\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}})}$$

$$= \frac{\xi^{\mathrm{o}}(\gamma) \mathscr{Z}(\mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}}) \sum_{\substack{\theta \in \mathcal{E}_\Lambda^{\mathrm{ord}} \\ \theta \ni \gamma}} \prod_{\substack{\widehat{\gamma} \in \theta \\ \widehat{\gamma} \neq \gamma}} \xi^{\mathrm{o}}(\widehat{\gamma}) \mathscr{Z}(\mathrm{int}\,\widehat{\gamma} \,|\, \xi^{\mathrm{o}})}{\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}})}$$

$$\leq \frac{\xi^{\mathrm{o}}(\gamma) \mathscr{Z}(\mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}}) \mathscr{Z}(\Lambda \setminus \mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}})}{\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}})}$$

$$\leq \xi^{\mathrm{o}}(\gamma) \,. \tag{5.53}$$

The last step follows from the fact that all the terms in the partition function $\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}})$ are non-negative, hence

$$\mathscr{Z}(\Lambda \,|\, \xi^{\mathrm{o}}) \geq \mathscr{Z}(\mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}}) \mathscr{Z}(\Lambda \setminus \mathrm{int}\,\gamma \,|\, \xi^{\mathrm{o}}) \,. \tag{5.54}$$

We obtain that

$$
\begin{aligned}
\varphi_\Lambda^{\mathrm{ord}}\{X : \overline{\partial} X \cap \Gamma_A \neq \varnothing\} &= \varphi_\Lambda^{\mathrm{ord}}\{X : \overline{\partial}_{\mathrm{ext}} X \cap \Gamma_A \neq \varnothing\} \\
&= \sum_{\gamma \in \Gamma_A} \varphi_\Lambda^{\mathrm{ord}}\{X : \overline{\partial}_{\mathrm{ext}} X \ni \gamma\} \\
&\leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) \, . \tag{5.55}
\end{aligned}
$$

$\square$

A standard argument using the positive correlation property of $\varphi_\Lambda^{\mathrm{ord}}$ (resp., $\varphi_\Lambda^{\mathrm{disord}}$) can be used to show that the thermodynamic limit of $\varphi_\Lambda^{\mathrm{ord}}$ (resp., $\varphi_\Lambda^{\mathrm{disord}}$) exists and is unique (see Appendix C). The limit measure φ^{ord} (resp., $\varphi^{\mathrm{disord}}$) satisfies the same bound as in the above corollary. If the weights ξ^{o} (resp., ξ^{d}) decay sufficiently fast, the latter bound implies that under φ^{ord} (resp., $\varphi^{\mathrm{disord}}$), the configuration of the model almost surely consists of a unique infinite sea of order (resp., disorder) with finite islands of disorder (resp., order). By a "sea" of order (resp., disorder) in a random-cluster configuration we mean a connected component of present (resp., absent) bonds.

Corollary 1.3. *For every finite set $A \subseteq \mathbb{S}$,*

$$
\varphi^{\mathrm{ord}}\{X : \overline{\partial} X \cap \Gamma_A \neq \varnothing\} \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) = \sum_{\substack{\gamma \in \Gamma_A \\ \gamma \text{ order}}} \xi^{\mathrm{o}}(\gamma) \, . \tag{5.56}
$$

Furthermore, if the sum $\sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma)$ converges,

$$
\varphi^{\mathrm{ord}} \left(\begin{array}{c} \exists \text{ unique infinite sea of order} \\ \text{with finite islands of disorder} \end{array} \right) = 1 \, , \tag{5.57}
$$

A similar statement holds in the disordered case.

Proof. As before, let Λ_n denote the $(2n+1) \times (2n+1)$ central square in the lattice. For every n let us define Γ_{A,Λ_n} as the set of all contours in Λ_n having A in their interiors. From the previous corollary, we know that for every $m > n$, the following bound holds:

$$
\varphi_{\Lambda_m}^{\mathrm{ord}}\{X : \overline{\partial} X \cap \Gamma_{A,\Lambda_n} \neq \varnothing\} \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) \, . \tag{5.58}
$$

Since the event $\{X : \overline{\partial}X \cap \Gamma_{A,\Lambda_n} \neq \varnothing\}$ is local, we obtain

$$\varphi^{\mathrm{ord}}\{X : \overline{\partial}X \cap \Gamma_{A,\Lambda_n} \neq \varnothing\} \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) . \qquad (5.59)$$

due to weak convergence of $\varphi^{\mathrm{ord}}_{\Lambda_m}$ to φ^{ord}. Letting $n \to \infty$ the first claim follows.

If $\sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma)$ converges, using a Borel-Cantelli argument, with probability 1, no infinite cascade of contours appears on the lattice. In particular, if we define

$$\mathcal{S}^{\mathrm{o}} \triangleq \left\{ X : \begin{array}{c} X \text{ has a unique infinite sea of order} \\ \text{with finite islands of disorder} \end{array} \right\} , \qquad (5.60)$$

$$\mathcal{S}^{\mathrm{d}} \triangleq \left\{ X : \begin{array}{c} X \text{ has a unique infinite sea of disorder} \\ \text{with finite islands of order} \end{array} \right\} , \qquad (5.61)$$

the latter implies that $\varphi^{\mathrm{ord}}(\mathcal{S}^{\mathrm{o}} \cup \mathcal{S}^{\mathrm{d}}) = 1$. We show that, in fact, $\varphi^{\mathrm{ord}}(\mathcal{S}^{\mathrm{d}}) = 0$.

Let A be a finite set of sites in the lattice. For every $X \in \mathcal{S}^{\mathrm{d}}$ one can find a volume Λ containing A such that $\overline{\partial}(X \cap B(\Lambda)) \in \Delta_{\Lambda}^{\mathrm{disord}}$ (i.e., the restriction of X to Λ is compatible with the disordered boundary condition). In particular, if we define

$$\mathcal{C}_{A,\Lambda_n} \triangleq$$
$$\left\{ X : \exists \text{ a finite volume } \Lambda \subseteq \Lambda_n \text{ with } S(\Lambda) \supseteq A \text{ and } \overline{\partial}(X \cap B(\Lambda)) \in \Delta_{\Lambda}^{\mathrm{disord}} \right\} , \qquad (5.62)$$

we have $\mathcal{C}_{A,\Lambda_1} \subseteq \mathcal{C}_{A,\Lambda_2} \subseteq \cdots$ and $\mathcal{S}^{\mathrm{d}} \subseteq \bigcup_n \mathcal{C}_{A,\Lambda_n}$.

If m, n are integers with $m > n$, every configuration X that is compatible with the ordered boundary condition on Λ_m (i.e., $\overline{\partial}X \in \Delta_{\Lambda_m}^{\mathrm{ord}}$) and is in $\mathcal{C}_{A,\Lambda_n}$ necessarily has an order contour surrounding A. Therefore, by the previous corollary, we have

$$\varphi^{\mathrm{ord}}_{\Lambda_m}(\mathcal{C}_{A,\Lambda_n}) \leq \varphi^{\mathrm{ord}}_{\Lambda_m}\left\{ X : \overline{\partial}X \cap \Gamma_A \neq \varnothing \right\} \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) . \qquad (5.63)$$

Since $\mathcal{C}_{A,\Lambda_n}$ is a local event, by weak convergence of $\varphi^{\mathrm{ord}}_{\Lambda_m}$ to φ^{ord} we have

$$\varphi^{\mathrm{ord}}(\mathcal{C}_{A,\Lambda_n}) \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) . \qquad (5.64)$$

Letting $n \to \infty$, we obtain

$$\varphi^{\mathrm{ord}}(\mathcal{S}^{\mathrm{d}}) \leq \lim_{n \to \infty} \varphi^{\mathrm{ord}}(\mathcal{C}_{A,\Lambda_n}) \leq \sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma) . \qquad (5.65)$$

The latter holds for every finite set $A \subseteq \mathbb{S}$, which by the convergence of the series, implies that $\varphi^{\mathrm{ord}}(\mathcal{S}^{\mathrm{d}}) = 0$. $\qquad\square$

5.5 Damping of Contour Weights

One advantage of working with contour models is that when the contour weights are sufficiently "damped" (i.e., decay exponentially in the length with a sufficiently fast rate) the free energy exists and is bounded, and moreover, the error in the finite-volume approximations of the free energy is of the order of the size of the boundary of the finite volume. This is the message of the following well-known proposition (see e.g. Section 2 of [73], or Proposition 2.3 of [68]).

Proposition 1. *Let $\tau > 0$ be sufficiently large, and suppose that the weight function $\chi : \Gamma \to \mathbb{R}$ of a contour model satisfies $0 \leq \chi(\gamma) \leq \mathrm{e}^{-\tau|\gamma|}$ for every contour γ. Then, the limit*

$$g(\chi) = \lim_{n \to \infty} \frac{1}{|B(\Lambda_n)|} \log \mathscr{Z}(\Lambda_n \,|\, \chi) \tag{5.66}$$

exists and satisfies $g(\chi) \leq \sum_{\gamma : S(\gamma) \ni 0} \chi(\gamma) \leq \mathrm{e}^{-\tau/2}$. In particular $g(\chi) \to 0$ as τ tends to infinity.

Furthermore, there is a constant $C = C(\tau)$, such that $C \to 0$ as τ goes to infinity, and for each finite volume $\Lambda \in \mathbb{Z}^d$

$$\mathrm{e}^{g(\chi)|B(\Lambda)| - C(\tau)|\partial\Lambda|} \leq \mathscr{Z}(\Lambda \,|\, \chi) \leq \mathrm{e}^{g(\chi)|B(\Lambda)| + C(\tau)|\partial\Lambda|} \,, \tag{5.67}$$

where $\partial\Lambda$ denotes the boundary of the volume Λ and can be defined as the set of bonds that are not in $B(\Lambda)$ but are incident to Λ.

The main purpose of this section is to identify conditions on the parameters $(q + r)$ and β under which the weights ξ^{d} and ξ^{o} are damped (i.e., satisfy the condition of the above proposition). We will see that when $(q + r)$ is large, for any value of $\beta > 0$ at least one of ξ^{d} and ξ^{o} is damped, and moreover there exists a unique β at which both ξ^{d} and ξ^{o} are damped. Let us remark that for sufficiently damped weights, the sum appearing in Corollary 1.3 converges, implying that the corresponding phase is stable.

For $\tau > 0$ large enough, let us introduce the *truncated* (i.e., artificially damped) weights

$$\bar{\xi}^{\mathrm{o}}(\gamma) = \begin{cases} \xi^{\mathrm{o}}(\gamma) \,, & \text{if } \xi^{\mathrm{o}}(\gamma) \leq \mathrm{e}^{-\tau|\gamma|}, \\ 0 \,, & \text{otherwise,} \end{cases} \tag{5.68}$$

and similarly for $\bar{\xi}^{\mathrm{d}}(\gamma)$ (see e.g. [47]). The term truncated refers to the suppression of all contours whose weights are not damped. If we replace the original weight ξ^{o} by the artificially damped one $\bar{\xi}^{\mathrm{o}}$, we obtain the following *truncated* partition function, which can be thought of as an approximation of the partition function of the r-biased random cluster model with ordered boundary condition:

$$\bar{Z}^{\mathsf{RC.ord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|\cdot e(\mathbb{B})}\,\mathscr{Z}(\Lambda\,|\,\bar{\xi}^{\mathrm{o}})\,. \tag{5.69}$$

Similarly, replacing ξ^{d} by $\bar{\xi}^{\mathrm{d}}$ leads to the truncated partition function for the r-biased random cluster model with disordered boundary condition:

$$\bar{Z}^{\mathsf{RC.disord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|\cdot e(\varnothing)}\,\mathscr{Z}(\Lambda\,|\,\bar{\xi}^{\mathrm{d}})\,. \tag{5.70}$$

The advantage of introducing these truncated partition functions is that we can apply Proposition 1. Note that if the original weights are "damped" (that is, $\xi^{\mathrm{o}}(\gamma) \leq \mathrm{e}^{-\tau|\gamma|}$ or $\xi^{\mathrm{d}}(\gamma) \leq \mathrm{e}^{-\tau|\gamma|}$), the corresponding truncated partition functions coincide with the original ones.

From Proposition 1 we have the following bounds for the truncated partition functions:

$$\mathrm{e}^{\big(g(\bar{\xi}^{\mathrm{o}})-e(\mathbb{B})\big)|B(\Lambda)|-C(\tau)|\partial\Lambda|} \leq \ \bar{Z}^{\mathsf{RC.ord}}(\Lambda) \ \leq \mathrm{e}^{\big(g(\bar{\xi}^{\mathrm{o}})-e(\mathbb{B})\big)|B(\Lambda)|+C(\tau)|\partial\Lambda|}\,, \tag{5.71}$$

$$\mathrm{e}^{\big(g(\bar{\xi}^{\mathrm{d}})-e(\varnothing)\big)|B(\Lambda)|-C(\tau)|\partial\Lambda|} \leq \bar{Z}^{\mathsf{RC.disord}}(\Lambda) \leq \mathrm{e}^{\big(g(\bar{\xi}^{\mathrm{d}})-e(\varnothing)\big)|B(\Lambda)|+C(\tau)|\partial\Lambda|}\,. \tag{5.72}$$

The pressure functions associated to the truncated partition functions are

$$f^{\mathrm{o}}(\beta) = \lim_{n\to\infty} \frac{1}{|B(\Lambda_n)|}\log \bar{Z}^{\mathsf{RC.ord}}(\Lambda_n) = -e(\mathbb{B}) + g(\bar{\xi}^{\mathrm{o}})\,, \tag{5.73}$$

$$f^{\mathrm{d}}(\beta) = \lim_{n\to\infty} \frac{1}{|B(\Lambda_n)|}\log \bar{Z}^{\mathsf{RC.disord}}(\Lambda_n) = -e(\varnothing) + g(\bar{\xi}^{\mathrm{d}})\,. \tag{5.74}$$

The functions $f^{\mathrm{o}}(\beta)$ and $f^{\mathrm{d}}(\beta)$ are lower approximations of the pressure $f^{\mathsf{RC}}(\beta)$ of the r-biased random-cluster representation. The next lemma states that in fact when $(q+r)$ is large enough, the maximum of f^{o} and f^{d} coincides with f^{RC}. As we will see in the next section, for $(q+r)$ large enough, the functions $g(\bar{\xi}^{\mathrm{o}})$ and $g(\bar{\xi}^{\mathrm{d}})$ and their β-derivatives are small, and therefore, the dominant terms of f^{o} and f^{d} are $-e(\mathbb{B})$ and $-e(\varnothing)$. This means that f^{RC} is approximated by the maximum of the curves $-e(\mathbb{B})$ and $-e(\varnothing)$, which intersect at a unique value β, with significantly different slopes.

By the diameter of a contour γ, denoted by $\operatorname{diam}\gamma$, we shall mean the maximum lattice distance between two bonds in $B(\gamma)$. The next lemma is parallel to Lemma 2 of [47] or Theorem 3.1 of [4].

Lemma 2. *Let $(q + r)$ be sufficiently large. If $f^{\mathrm{d}} \le f^{\mathrm{o}}$, then*
  i) for every disorder contour γ with $\operatorname{diam}\gamma \le \frac{1}{f^{\mathrm{o}} - f^{\mathrm{d}}}$ we have

$$\xi^{\mathrm{d}}(\gamma) \le \mathrm{e}^{-\tau|\gamma|} , \tag{5.75}$$

  ii) for every order contour γ we have

$$\xi^{\mathrm{o}}(\gamma) \le \mathrm{e}^{-\tau|\gamma|} . \tag{5.76}$$

A similar statement holds if $f^{\mathrm{o}} \le f^{\mathrm{d}}$.

Proof. We prove the two claims simultaneously by induction on $\operatorname{diam}\gamma$. Let $K > 0$ and suppose that the claims hold for all (disorder/order) contours with diameter less than K.

Let γ be a disorder contour with diameter K that satisfies $\operatorname{diam}\gamma \le \frac{1}{f^{\mathrm{o}} - f^{\mathrm{d}}}$. Then,

$$
\begin{aligned}
\xi^{\mathrm{d}}(\gamma) &= \rho(\gamma)\frac{Z^{\mathsf{RC.ord}}(\operatorname{int}\gamma)}{Z^{\mathsf{RC.disord}}(\operatorname{int}\gamma)} \\
&= \rho(\gamma)\frac{\bar{Z}^{\mathsf{RC.ord}}(\operatorname{int}\gamma)}{\bar{Z}^{\mathsf{RC.disord}}(\operatorname{int}\gamma)} \\
&\le \rho(\gamma)\frac{\mathrm{e}^{f^{\mathrm{o}}\cdot|B(\operatorname{int}\gamma)|+C(\tau)|\partial\operatorname{int}\gamma|}}{\mathrm{e}^{f^{\mathrm{d}}\cdot|B(\operatorname{int}\gamma)|-C(\tau)|\partial\operatorname{int}\gamma|}} \\
&= \rho(\gamma)\mathrm{e}^{(f^{\mathrm{o}}-f^{\mathrm{d}})|B(\operatorname{int}\gamma)|+2C(\tau)|\partial\operatorname{int}\gamma|} ,
\end{aligned}
\tag{5.77}
$$

where in the second equality we have used the induction hypothesis. Namely, every contour in $\operatorname{int}\gamma$ has diameter less than K, allowing us to replace the original partition functions with the truncated ones. Notice that
  - $\rho(\gamma) = q(q+r)^{-\frac{1}{4}|\gamma|}$,
  - $|B(\operatorname{int}\gamma)| \le \frac{1}{2}|\gamma| \cdot \operatorname{diam}\gamma$,
  - $(f^{\mathrm{o}} - f^{\mathrm{d}}) \cdot \operatorname{diam}\gamma \le 1$, and
  - $|\partial\operatorname{int}\gamma| \le |\gamma|$.
Hence, we obtain that

$$\xi^{\mathrm{d}}(\gamma) \le q\,\mathrm{e}^{-\left(\frac{1}{4}\log(q+r)-1-2C(\tau)\right)\cdot|\gamma|} . \tag{5.78}$$

For $(q + r)$ large enough (uniformly in γ) the right-hand side is bounded by $\mathrm{e}^{-\tau|\gamma|}$, hence the claim.

Next, suppose that γ is an order contour with diameter K. We need to show

$$\xi^{\mathrm{o}}(\gamma) = \rho(\gamma)\mathrm{e}^{|B(\gamma)|e(\mathbb{B})}\frac{Z^{\mathsf{RC.disord}}(V(\gamma))}{Z^{\mathsf{RC.ord}}(\mathrm{int}\,\gamma)} \leq \mathrm{e}^{-\tau|\gamma|} \ . \tag{5.79}$$

By the induction hypothesis, the partition function $Z^{\mathsf{RC.ord}}(\mathrm{int}\,\gamma)$ is equal to the corresponding truncated partition function, which can be bounded using Proposition 1. As for $Z^{\mathsf{RC.disord}}(V(\gamma))$, if we suppress all the contours that are "big", we can get a similar bound using the induction hypothesis.

To render the argument more transparent, we work with the partition functions $Y^{\mathrm{d}}(\Lambda)$ and $Y^{\mathrm{o}}(\Lambda)$ (see (5.36) and (5.37)) for which we have

$$Z^{\mathsf{RC.disord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|\cdot e(\varnothing)}Y^{\mathrm{d}}(\Lambda) \ , \tag{5.80}$$

$$Z^{\mathsf{RC.ord}}(\Lambda) = \mathrm{e}^{-|B(\Lambda)|\cdot e(\mathbb{B})}Y^{\mathrm{o}}(\Lambda) \ . \tag{5.81}$$

Let us call a disorder contour *small* if its diameter is less than or equal to $\frac{1}{f^{\mathrm{o}}-f^{\mathrm{d}}}$. Otherwise, we call the contour *big*.

As before, let us denote by $\mathcal{E}_{\Lambda}^{\mathrm{disord}}$ the set of all mutually compatible families of disorder contours in Λ whose elements are external. Factoring the contribution of the interior of big external contours we have the recursion[2]

$$Y^{\mathrm{d}}(\Lambda) = \sum_{\substack{\theta\in\mathcal{E}_{\Lambda}^{\mathrm{disord}} \\ \theta\ \mathrm{big}}} Y_{\mathrm{small}}^{\mathrm{d}}(\Lambda \setminus \mathrm{int}\,\theta) \prod_{\gamma'\in\theta} \widetilde{\rho}(\gamma')Y^{\mathrm{o}}(\mathrm{int}\,\gamma') \tag{5.82}$$

where $\mathrm{int}\,\theta \triangleq \bigcup_{\gamma'\in\theta} \mathrm{int}\,\gamma'$, and $Y_{\mathrm{small}}^{\mathrm{d}}(\Lambda) \triangleq \mathscr{Z}(\Lambda \,|\, \xi_{\mathrm{small}}^{\mathrm{d}})$, in which the weight function $\xi_{\mathrm{small}}^{\mathrm{d}}$ is obtained from ξ^{d} by replacing the weights of all big contours with 0, that is,

$$\xi_{\mathrm{small}}^{\mathrm{d}}(\gamma') = \begin{cases} \xi^{\mathrm{d}}(\gamma') & \text{if } \gamma' \text{ small,} \\ 0 & \text{if } \gamma' \text{ big.} \end{cases} \tag{5.83}$$

---

[2]Although $\Lambda \setminus \mathrm{int}\,\theta$ does not match our requirement for being a volume (i.e., not having holes), it does not cause any problem. In fact, since the contours in $Y_{\mathrm{small}}^{\mathrm{d}}$ are small, they cannot surround the holes in $\Lambda\setminus\mathrm{int}\,\theta$, hence they do not distinguish the holes from the outside of Λ.

The expression for the weight $\xi^{\mathrm{o}}(\gamma)$ then reads

$$\xi^{\mathrm{o}}(\gamma) =$$

$$\rho(\gamma) \cdot \mathrm{e}^{|B(V(\gamma))| \cdot (e(\mathbb{B}) - e(\varnothing))} \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\mathrm{disord}} \\ \theta \text{ big}}} \frac{Y_{\mathrm{small}}^{\mathrm{d}}(V(\gamma) \setminus \mathrm{int}\,\theta) \cdot Y^{\mathrm{o}}(\mathrm{int}\,\theta)}{Y^{\mathrm{o}}(\mathrm{int}\,\gamma)} \prod_{\gamma' \in \theta} \widetilde{\rho}(\gamma') \,.$$

$$(5.84)$$

The induction hypothesis and Proposition 1 tell us:

- $Y_{\mathrm{small}}^{\mathrm{d}}(V(\gamma) \setminus \mathrm{int}\,\theta) \leq \mathrm{e}^{g(\xi_{\mathrm{small}}^{\mathrm{d}}) \cdot |B(V(\gamma) \setminus \mathrm{int}\,\theta)| + C(\tau) \cdot |\partial(V(\gamma) \setminus \mathrm{int}\,\theta)|}$,
- $Y^{\mathrm{o}}(\mathrm{int}\,\theta) \leq \mathrm{e}^{g(\bar{\xi}^{\mathrm{o}}) \cdot |B(\mathrm{int}\,\theta)| + C(\tau) \cdot |\partial\,\mathrm{int}\,\theta|}$,
- $Y^{\mathrm{o}}(\mathrm{int}\,\gamma) \geq \mathrm{e}^{g(\bar{\xi}^{\mathrm{o}}) \cdot |B(\mathrm{int}\,\gamma)| - C(\tau) \cdot |\partial\,\mathrm{int}\,\gamma|}$, and
- $g(\bar{\xi}^{\mathrm{o}}) \leq \mathrm{e}^{-\tau/2} \leq 1$.

Moreover

- $|\partial(V(\gamma) \setminus \mathrm{int}\,\theta)| \leq |\partial V(\gamma)| + |\partial\,\mathrm{int}\,\theta| \leq 3\,|\gamma| + \sum_{\gamma' \in \theta} |\gamma'|$, and
- $|B(\gamma)| \leq |\gamma|$.

Hence we have

$$\xi^{\mathrm{o}}(\gamma) \leq \rho(\gamma) \cdot \mathrm{e}^{(1 + 4C(\tau)) \cdot |\gamma|}$$

$$\times \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\mathrm{disord}} \\ \theta \text{ big}}} \mathrm{e}^{|B(V(\gamma) \setminus \theta)| \cdot \left[e(\mathbb{B}) - e(\varnothing) + g(\xi_{\mathrm{small}}^{\mathrm{d}}) - g(\bar{\xi}^{\mathrm{o}})\right]} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot \mathrm{e}^{2C(\tau) \cdot |\gamma'|}$$

$$= \rho(\gamma) \cdot \mathrm{e}^{(1 + 4C(\tau)) \cdot |\gamma|}$$

$$\times \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\mathrm{disord}} \\ \theta \text{ big}}} \mathrm{e}^{|B(V(\gamma) \setminus \theta)| \cdot \left[\left(g(\xi_{\mathrm{small}}^{\mathrm{d}}) - g(\bar{\xi}^{\mathrm{d}})\right) - \left(f^{\mathrm{o}} - f^{\mathrm{d}}\right)\right]} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot \mathrm{e}^{2C(\tau) \cdot |\gamma'|} \,.$$

$$(5.85)$$

As we shall see shortly, if $(q + r)$ is large enough, the sum appearing in the above expression can be bounded by $\mathrm{e}^{3C(\tau) \cdot |\gamma|}$, so that

$$\xi^{\mathrm{o}}(\gamma) \leq \rho(\gamma) \cdot \mathrm{e}^{(1 + 4C(\tau)) \cdot |\gamma|} \mathrm{e}^{3C(\tau) \cdot |\gamma|}$$

$$= \mathrm{e}^{-\left(\frac{1}{4} \log(q + r) - 1 - 7C(\tau)\right) \cdot |\gamma|} \tag{5.86}$$

For large $(q + r)$, the right-hand side is bounded by $\mathrm{e}^{-\tau |\gamma|}$, proving the claim.

It remains to show that for $(q + r)$ sufficiently large,

$$\sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\text{disord}} \\ \theta \text{ big}}} e^{|B(V(\gamma) \backslash \text{int } \theta)| \cdot \left[\left(g(\xi_{\text{small}}^{\text{d}}) - g(\bar{\xi}^{\text{d}})\right) - \left(f^{\text{o}} - f^{\text{d}}\right)\right]} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot e^{2C(\tau) \cdot |\gamma'|} \leq e^{3C(\tau) \cdot |\gamma|} \,.$$

$$(5.87)$$

To show this, let us consider a contour model with weight function

$$\widehat{\rho}(\gamma') = \begin{cases} \rho(\gamma') \cdot e^{3C(\tau) \cdot |\gamma'|} \,, & \text{if } \gamma' \text{ big and disorder,} \\ 0 \,, & \text{otherwise.} \end{cases} \qquad (5.88)$$
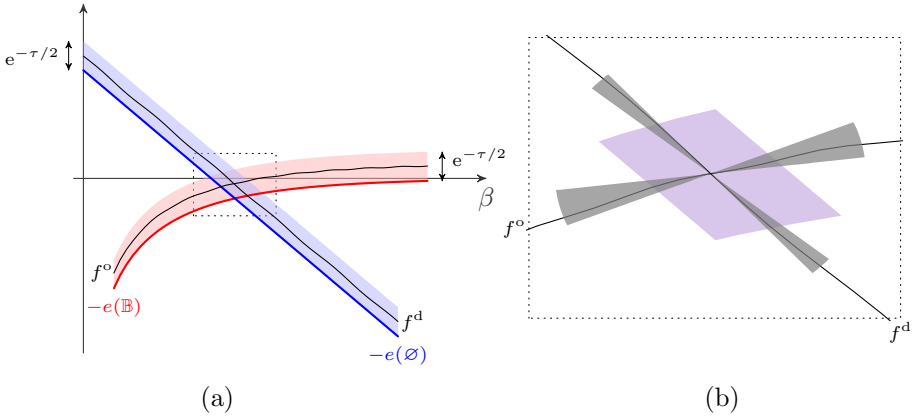
Assuming that $g(\widehat{\rho}) \leq f^{\text{o}} - f^{\text{d}} + g(\bar{\xi}^{\text{d}}) - g(\xi_{\text{small}}^{\text{d}})$, and $(q + r)$ in such a way that $\widehat{\rho}(\gamma') \leq e^{-\tau|\gamma'|}$, we can use Proposition 1 to obtain

$$\sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\text{disord}} \\ \theta \text{ big}}} e^{|B(V(\gamma) \backslash \text{int } \theta)| \cdot \left[\left(g(\xi_{\text{small}}^{\text{d}}) - g(\bar{\xi}^{\text{d}})\right) - \left(f^{\text{o}} - f^{\text{d}}\right)\right]} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot e^{2C(\tau) \cdot |\gamma'|}$$

$$\leq \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\text{disord}} \\ \theta \text{ big}}} e^{-|B(V(\gamma) \backslash \text{int } \theta)| \cdot g(\widehat{\rho})} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot e^{2C(\tau) \cdot |\gamma'|} \qquad (5.89)$$

$$= e^{-|B(V(\gamma))| \cdot g(\widehat{\rho})} \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\text{disord}} \\ \theta \text{ big}}} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot e^{2C(\tau) \cdot |\gamma'|} \cdot e^{|B(\text{int } \gamma')| \cdot g(\widehat{\rho})}$$

$$\leq e^{-|B(V(\gamma))| \cdot g(\widehat{\rho})} \sum_{\substack{\theta \in \mathcal{E}_{V(\gamma)}^{\text{disord}} \\ \theta \text{ big}}} \prod_{\gamma' \in \theta} \rho(\gamma') \cdot e^{2C(\tau) \cdot |\gamma'|} \cdot \mathscr{Z}(\text{int } \gamma' \,|\, \widehat{\rho}) \cdot e^{C(\tau) \cdot |\gamma'|}$$

$$= e^{-|B(V(\gamma))| \cdot g(\widehat{\rho})} \cdot \mathscr{Z}(V(\gamma) \,|\, \widehat{\rho})$$

$$\leq e^{3C(\tau) \cdot |\gamma|} \,.$$

Finally, to see that $g(\widehat{\rho}) \leq f^{\text{o}} - f^{\text{d}} + g(\bar{\xi}^{\text{d}}) - g(\xi_{\text{small}}^{\text{d}})$, note that $g(\bar{\xi}^{\text{d}}) - g(\xi_{\text{small}}^{\text{d}}) \geq 0$ (due to the fact that $\xi_{\text{small}}^{\text{d}} \leq \bar{\xi}^{\text{d}}$) and by Proposition 1

$$g(\widehat{\rho}) \leq \sum_{\substack{\gamma' : S(\gamma') \ni 0 \\ \gamma' \text{ big}}} \widehat{\rho}(\gamma') \leq e^{-\tau/(f^{\text{o}} - f^{\text{d}})} \,, \qquad (5.90)$$

using the fact that γ' is big only if $|\gamma'| \geq \frac{2}{f^{\text{o}} - f^{\text{d}}}$. For τ not too small we have $e^{-\tau/(f^{\text{o}} - f^{\text{d}})} \leq f^{\text{o}} - f^{\text{d}}$. $\qquad \square$

Figure 5.3: (a) The curves of $f^{\mathrm{o}}(\beta)$ and $f^{\mathrm{d}}(\beta)$ are within a narrow margin above $-e(\mathbb{B})$ and $-e(\varnothing)$. (b) The slopes of $f^{\mathrm{o}}(\beta)$ and $f^{\mathrm{d}}(\beta)$ are close to the slopes of $-e(\mathbb{B})$ and $-e(\varnothing)$.

From the above lemma, we know that the pressure $f^{\mathsf{RC}}(\beta)$ of the r-biased random-cluster representation coincides with the maximum of the functions $f^{\mathrm{o}}(\beta)$ and $f^{\mathrm{d}}(\beta)$, that is,

$$f^{\mathsf{RC}}(\beta) = \max\left\{ f^{\mathrm{o}}(\beta), f^{\mathrm{d}}(\beta) \right\} . \tag{5.91}$$

Recall that

$$f^{\mathrm{o}}(\beta) = -e(\mathbb{B}) + g(\bar{\xi}^{\mathrm{o}}) , \tag{5.92}$$

$$f^{\mathrm{d}}(\beta) = -e(\varnothing) + g(\bar{\xi}^{\mathrm{d}}) . \tag{5.93}$$

If τ is large, Proposition 1 says that $g(\bar{\xi}^{\mathrm{o}})$ and $g(\bar{\xi}^{\mathrm{d}})$ are small, so that $f^{\mathsf{RC}}(\beta)$ can be nearly expressed in terms of the "energy" per bond of the fully ordered and fully disordered configurations. More precisely, if we define

$$F(\beta) \triangleq \max\left\{ -e(\mathbb{B}), -e(\varnothing) \right\} . \tag{5.94}$$

we have

$$0 \le f^{\mathsf{RC}}(\beta) - F(\beta) \le \mathrm{e}^{-\tau/2} . \tag{5.95}$$

The two curves $-e(\mathbb{B})$ and $-e(\varnothing)$ (as functions of β) intersect at a single point

$$\bar{\beta}_{\mathrm{c}} = \log\left(1 + \sqrt{q+r} \right) , \tag{5.96}$$

above which $F(\beta) = -e(\mathbb{B})$ and below which $F(\beta) = -e(\varnothing)$. Furthermore, these two curves have significantly different slopes, implying that $F(\beta)$ is not differentiable at $\bar{\beta}_{\mathrm{c}}$. What we are after is to infer that $f^{\mathsf{RC}}(\beta)$ has a similar behaviour. In other words, we would like to show that there exists a unique solution β_{c} for the equation $f^{\mathrm{o}}(\beta) = f^{\mathrm{d}}(\beta)$, at which $f^{\mathsf{RC}}(\beta)$ is not differentiable, above which $f^{\mathsf{RC}}(\beta) = f^{\mathrm{o}}(\beta)$ and below which $f^{\mathsf{RC}}(\beta) = f^{\mathrm{d}}(\beta)$. Note that condition (5.95) guarantees that $f^{\mathsf{RC}}(\beta) = f^{\mathrm{o}}(\beta) > f^{\mathrm{d}}(\beta)$ for $\beta \gg \bar{\beta}_{\mathrm{c}}$ and $f^{\mathsf{RC}}(\beta) = f^{\mathrm{d}}(\beta) > f^{\mathrm{o}}(\beta)$ for $\beta \ll \bar{\beta}_{\mathrm{c}}$. In fact, it states that $f^{\mathsf{RC}}(\beta)$ lives in a margin of width $\mathrm{e}^{-\tau/2}$ above $F(\beta)$ (see Figure 5.3(a)). To infer such a sharp transition we further need to give bounds for the derivatives of $g(\bar{\xi}^{\mathrm{o}})$ and $g(\bar{\xi}^{\mathrm{d}})$ (see Figure 5.3(b)). This is addressed in the following lemma, which is analogous to Theorem 3.3 of [4].

Lemma 3. *We have*

$$\frac{\partial}{\partial \beta} g(\bar{\xi}^{\mathrm{o}}) \leq -2 \frac{\partial e(\mathbb{B})}{\partial \beta} \sum_{\gamma : S(V(\gamma)) \ni 0} \bar{\xi}^{\mathrm{o}}(\gamma) \leq \frac{2}{\mathrm{e}^{\beta} - 1} \, \mathrm{e}^{-\tau/2} \,, \tag{5.97}$$

$$\frac{\partial}{\partial \beta} g(\bar{\xi}^{\mathrm{d}}) \leq -2 \frac{\partial e(\mathbb{B})}{\partial \beta} \sum_{\gamma : S(\mathrm{int}\,\gamma) \ni 0} \bar{\xi}^{\mathrm{d}}(\gamma) \leq \frac{2}{\mathrm{e}^{\beta} - 1} \, \mathrm{e}^{-\tau/2} \,. \tag{5.98}$$

Proof. For a finite volume Λ, if we denote by $\Gamma(\Lambda)$ the set of all contours in Λ, we have

$$\frac{\partial}{\partial \beta} \log \mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\mathrm{o}}) = \frac{1}{\mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\mathrm{o}})} \sum_{\partial \in \mathcal{M}_{\Lambda}} \frac{\partial}{\partial \beta} \prod_{\gamma \in \partial} \bar{\xi}^{\mathrm{o}}(\gamma)$$

$$= \frac{1}{\mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\mathrm{o}})} \sum_{\partial \in \mathcal{M}_{\Lambda}} \sum_{\gamma \in \partial} \frac{\partial \bar{\xi}^{\mathrm{o}}(\gamma)}{\partial \beta} \prod_{\substack{\widehat{\gamma} \in \partial \\ \widehat{\gamma} \neq \gamma}} \bar{\xi}^{\mathrm{o}}(\widehat{\gamma})$$

$$= \sum_{\gamma \in \Gamma(\Lambda)} \frac{\partial \bar{\xi}^{\mathrm{o}}(\gamma)}{\partial \beta} \frac{\mathscr{Z}(\mathrm{int}\,\gamma \,|\, \bar{\xi}^{\mathrm{o}}) \cdot \mathscr{Z}(\Lambda \cap \mathrm{ext}\,\gamma \,|\, \bar{\xi}^{\mathrm{o}})}{\mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\mathrm{o}})}$$

$$\leq \sum_{\gamma \in \Gamma(\Lambda)} \frac{\partial \bar{\xi}^{\mathrm{o}}(\gamma)}{\partial \beta} \,. \tag{5.99}$$

Recall that $\bar{\xi}^{\mathrm{o}}(\gamma)$ is equal to either 0 or $\xi^{\mathrm{o}}(\gamma)$, hence the derivative of $\bar{\xi}^{\mathrm{o}}(\gamma)$ is bounded by the derivative of $\xi^{\mathrm{o}}(\gamma)$. Using the definition of $\xi^{\mathrm{o}}(\gamma)$ (Eq. (5.43)) we have

$$\frac{\partial \xi^{\mathrm{o}}(\gamma)}{\partial \beta} = |B(\gamma)| \cdot \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \xi^{\mathrm{o}}(\gamma) \; + \; \rho(\gamma) \cdot \mathrm{e}^{|B(\gamma)| \cdot e(\mathbb{B})} \cdot \frac{\partial}{\partial \beta} \frac{Z^{\mathsf{RC.disord}}(V(\gamma))}{Z^{\mathsf{RC.ord}}(\mathrm{int}\,\gamma)} \,. \tag{5.100}$$

The derivative of the partition functions appearing on the right-hand side can be bounded directly from the definitions (Eq. (5.30) and (5.31)) by

$$0 \le \frac{\partial}{\partial \beta} Z^{\text{RC.disord}}(V(\gamma)) \le - |B(V(\gamma))| \cdot \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot Z^{\text{RC.disord}}(V(\gamma)) \,, \quad (5.101)$$

$$0 \le \frac{\partial}{\partial \beta} Z^{\text{RC.ord}}(\text{int}\,\gamma) \le - |B(\text{int}\,\gamma)| \cdot \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot Z^{\text{RC.ord}}(\text{int}\,\gamma) \,, \quad (5.102)$$

leading to

$$\frac{\partial}{\partial \beta} \frac{Z^{\text{RC.disord}}(V(\gamma))}{Z^{\text{RC.ord}}(\text{int}\,\gamma)} \le - |B(V(\gamma))| \cdot \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{Z^{\text{RC.disord}}(V(\gamma))}{Z^{\text{RC.ord}}(\text{int}\,\gamma)} \,. \quad (5.103)$$

Therefore,

$$\frac{\partial \bar{\xi}^{\text{o}}(\gamma)}{\partial \beta} \le - |B(V(\gamma))| \cdot \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \bar{\xi}^{\text{o}}(\gamma) \,. \quad (5.104)$$

We can now write

$$
\begin{aligned}
\frac{\partial}{\partial \beta} \frac{1}{|S(\Lambda)|} \log \mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\text{o}}) &\le - \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{1}{|S(\Lambda)|} \sum_{\gamma \in \Gamma(\Lambda)} |B(V(\gamma))| \cdot \bar{\xi}^{\text{o}}(\gamma) \\
&= - \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{1}{|S(\Lambda)|} \sum_{\gamma \in \Gamma(\Lambda)} \sum_{b \in B(V(\gamma))} \bar{\xi}^{\text{o}}(\gamma) \\
&= - \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{1}{|S(\Lambda)|} \sum_{b \in B(\Lambda)} \sum_{\substack{\gamma \in \Gamma(\Lambda) \\ B(V(\gamma)) \ni b}} \bar{\xi}^{\text{o}}(\gamma) \\
&\le - \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{1}{|S(\Lambda)|} \sum_{b \in B(\Lambda)} \sum_{\substack{\gamma \in \Gamma \\ B(V(\gamma)) \ni b}} \bar{\xi}^{\text{o}}(\gamma) \\
&\le - \frac{\partial e(\mathbb{B})}{\partial \beta} \cdot \frac{|B(\Lambda)|}{|S(\Lambda)|} \sum_{\substack{\gamma \in \Gamma \\ S(V(\gamma)) \ni 0}} \bar{\xi}^{\text{o}}(\gamma) \\
&\le - 2 \frac{\partial e(\mathbb{B})}{\partial \beta} \sum_{\substack{\gamma \in \Gamma \\ S(V(\gamma)) \ni 0}} \bar{\xi}^{\text{o}}(\gamma) \,. \quad (5.105)
\end{aligned}
$$

Since τ is large, the above sum converges and leads to a uniform bound (with respect to Λ) for $\frac{\partial}{\partial \beta} \frac{1}{|S(\Lambda)|} \log \mathscr{Z}(\Lambda \,|\, \bar{\xi}^{\text{o}})$. Using the dominated convergence theorem the first claim follows. The proof of the other claim is similar. □

5.6 The Main Result

Theorem. *Let $\varepsilon > 0$ and $(q + r)$ large enough. The two-dimensional (q, r)-Potts model undergoes a first-order transition in temperature with breaking of permutation symmetry:*

  i) *Above the transition temperature, the model has a unique Gibbs state μ^{free}, which is "disordered".*

 ii) *Below the transition temperature, there exist at least q different "ordered" Gibbs states $\mu^1, \mu^2, \ldots, \mu^q$.*

iii) *At the transition temperature, q "ordered" Gibbs states $\mu^1, \mu^2, \ldots, \mu^q$ co-exist with a "disordered" Gibbs state μ^{free}.*

The "ordered" and "disordered" states can be distinguished by

$$\mu^k(\{\sigma : \sigma_i = k\}) > 1 - \varepsilon , \qquad \textit{for every visible } k, \tag{5.106}$$

$$\mu^{\mathrm{free}}(\{\sigma : \sigma_i = k\}) < \varepsilon , \qquad \textit{for every } k, \tag{5.107}$$

for every site i in the lattice.

Proof. In the previous sections we have introduced the main ingredients to prove the occurrence of a first-order transition. Below, we first put these ingredients together so as to obtain a recipe for the proof. Afterwards, we shall see how these ingredients along with basic properties of the biased random-cluster model (see Appendix C) can be used to prove the symmetry breaking at the transition temperature.

The first step was to reduce the partition function of the (q, r)-Potts model to the partition function of the r-biased random-cluster model. This was done for the free and homogeneous visible boundary conditions, which led to the disordered and ordered boundary conditions for the r-biased random-cluster model (see Eq. (5.16) and (5.19)). By means of this we could rewrite the pressure $f(\beta)$ for the (q, r)-Potts model as

$$f(\beta) = 2\beta + 2f^{\mathsf{RC}}(\beta) , \tag{5.108}$$

where

$$f^{\mathsf{RC}}(\beta) = \lim_{n \to \infty} \frac{\log Z^{\mathsf{RC.ord}}_{p_\beta,q,r}(\Lambda_n)}{|B(\Lambda_n)|} = \lim_{n \to \infty} \frac{\log Z^{\mathsf{RC.disord}}_{p_\beta,q,r}(\Lambda_n)}{|B(\Lambda_n)|} . \tag{5.109}$$

The second step consisted in re-expressing the partition functions $Z^{\mathsf{RC.ord}}$ and $Z^{\mathsf{RC.disord}}$ in terms of the partition functions of two abstract contour models $\mathscr{Z}(\cdot \,|\, \xi^{\mathrm{o}})$ and $\mathscr{Z}(\cdot \,|\, \xi^{\mathrm{d}})$ (Lemma 1) so that we could write

$$f^{\mathsf{RC}}(\beta) = -e(\mathbb{B}) + g(\xi^{\mathrm{o}}) = -e(\varnothing) + g(\xi^{\mathrm{d}}) , \tag{5.110}$$

where $g(\xi^{\mathrm{o}})$ and $g(\xi^{\mathrm{d}})$ are the pressure functions for the two contour models (Eq. (5.66)).

If the weight function χ of a contour model is sufficiently "damped" (i.e., $\chi(\gamma) \le \mathrm{e}^{-\tau|\gamma|}$ for τ large enough), the corresponding pressure $g(\chi)$ can be made arbitrarily small (Proposition 1). In order to exploit this result, in the third step, we truncat**ed** the weight functions ξ^{o} and ξ^{d} so as to render them artificially damped (Eq. (5.68)). We could then define two functions

$$f^{\mathrm{o}}(\beta) = -e(\mathbb{B}) + g(\bar{\xi}^{\mathrm{o}}) \,, \qquad (5.111)$$

$$f^{\mathrm{d}}(\beta) = -e(\varnothing) + g(\bar{\xi}^{\mathrm{d}}) \,, \qquad (5.112)$$

which approximate $f^{\mathsf{RC}}(\beta)$ from below, and which can be thought of (for sufficiently large τ) as perturbations of the functions $-e(\mathbb{B})$ and $-e(\varnothing)$, respectively (see Figure 5.3(a)).

In the fourth step, we proved that, for $(q+r)$ large (relative to τ), the pressure $f^{\mathsf{RC}}(\beta)$ is the maximum of these two approximations. This was achieved by proving that whenever $f^{\mathrm{o}} \ge f^{\mathrm{d}}$, the weight function ξ^{o} is "naturally" damped (i.e., $\xi^{\mathrm{o}} = \bar{\xi}^{\mathrm{o}}$) and vice versa (Lemma 2). Therefore, $f^{\mathsf{RC}}(\beta)$ can be closely approximated by the maximum between $-e(\mathbb{B})$ and $-e(\varnothing)$. Due to the continuity of the pressure functions, the latter implies that the curves $f^{\mathrm{o}}(\beta)$ and $f^{\mathrm{d}}(\beta)$ intersect.

In the last step, we showed that for τ sufficiently large, $\frac{\partial}{\partial\beta}f^{\mathrm{o}}(\beta) > \frac{\partial}{\partial\beta}f^{\mathrm{d}}(\beta)$ (Lemma 3). Therefore, the two functions meet at a unique point β_{c} at which $f^{\mathsf{RC}}(\beta)$ is non-differentiable (Figure 5.3(b)). Hence, the (q,r)-Potts model undergoes a first-order phase transition at β_{c}.

We now prove the breaking of permutation symmetry at the transition temperature. Note that if $\beta \ge \beta_{\mathrm{c}}$, we have $f^{\mathrm{o}}(\beta) \ge f^{\mathrm{d}}(\beta)$ and therefore, in view of Lemma 2, the weights ξ^{o} satisfy $\xi^{\mathrm{o}}(\gamma) \le \mathrm{e}^{-\tau|\gamma|}$. Since τ was chosen large, for every finite set of sites A, the sum $\sum_{\gamma \in \Gamma_A} \xi^{\mathrm{o}}(\gamma)$ converges. Hence, it follows from Corollary 1.3 that

$$\varphi_{p_\beta}^{\mathrm{ord}} \left(\begin{array}{c} \exists \text{ unique infinite sea of order} \\ \text{with finite islands of disorder} \end{array} \right) = 1 \,. \qquad (5.113)$$

(The subscript p_β is added to emphasize the dependence on β.) Similarly, if $\beta \le \beta_{\mathrm{c}}$, the sum $\sum_{\gamma \in \Gamma_A} \xi^{\mathrm{d}}(\gamma)$ converges and thus

$$\varphi_{p_\beta}^{\mathrm{disord}} \left(\begin{array}{c} \exists \text{ unique infinite sea of disorder} \\ \text{with finite islands of order} \end{array} \right) = 1 \,. \qquad (5.114)$$

For a visible colour k, let μ_β^k be, as introduced in Section 5.2, a weak limit of Boltzmann distributions with homogeneous boundary conditions ω^k. Likewise, let μ_β^{free} be a weak limit of free-boundary Boltzmann distributions.

Similar to the finite-volume couplings, there exists a coupling of μ_β^k and $\varphi_{p_\beta}^{\text{ord}}$ with the property that with probability 1, every site incident to an infinite connected component of present bonds has colour k (see Appendix C). If $\beta \geq \beta_c$, we know that almost surely the bond configuration consists of a unique sea of order with finite islands of disorder. In particular, the probability that a given site i takes a colour other than k is bounded by the probability that site i is surrounded by an order contour; that is,

$$\mu_\beta^k(\{\sigma : \sigma_i \neq k\}) \leq \varphi_{p_\beta}^{\text{ord}}\{X : \overline{\partial}X \cap \Gamma_i \neq \varnothing\} . \qquad (5.115)$$

In this region of β, Corollary 1.3 and Lemma 2 ensure that $\varphi_{p_\beta}^{\text{ord}}\{X : \overline{\partial}X \cap \Gamma_i \neq \varnothing\}$ can be made arbitrarily small by tuning τ. Hence, for every $\varepsilon > 0$, choosing $q + r$ large enough, we have $\mu_\beta^k(\{\sigma : \sigma_i = k\}) > 1 - \varepsilon$.

The measures μ_β^{free} and $\varphi_{p_\beta}^{\text{disord}}$ can also be coupled, in such a way that, given a configuration of bonds, the colour of the isolated sites are chosen independently and uniformly among the $q + r$ possibilities. Using this coupling, and conditioning on whether a given site i is isolated or not, we obtain

$$\mu_\beta^{\text{free}}(\{\sigma : \sigma_i = k\}) \leq \frac{1}{q + r} + \varphi_{p_\beta}^{\text{disord}}\{X : i \text{ not isolated in } (\mathbb{S}, X)\} . \quad (5.116)$$

If $\beta \leq \beta_c$, the bond configuration almost surely consists of a unique sea of disorder with finite islands of order. Hence, the probability that site i is not isolated is bounded by the probability that site i is surrounded by a disorder contour; that is,

$$\varphi_{p_\beta}^{\text{disord}}\{X : i \text{ not isolated in } (\mathbb{S}, X)\} \leq \varphi_{p_\beta}^{\text{disord}}\{X : \overline{\partial}X \cap \Gamma_i \neq \varnothing\} . \quad (5.117)$$

As in the previous case, Corollary 1.3 and Lemma 2 guarantee that for every $\varepsilon > 0$, choosing $q + r$ large enough, $\mu_\beta^{\text{free}}(\{\sigma : \sigma_i = k\}) < \varepsilon$.

It remains to show that for $\beta < \beta_c$, the (q, r)-Potts Gibbs measure is unique.

As in the standard random-cluster model, there exists a critical value $0 < p_c < 1$ such that

- for $p < p_c$, almost surely with respect to φ_p^{ord} and $\varphi_p^{\text{disord}}$, there is no infinite connected component of bonds (order does not "percolate"), whereas
- for $p > p_c$, the event that a given site is in an infinite connected component happens with positive probability under both φ_p^{ord} and $\varphi_p^{\text{disord}}$.

(See Appendix C). It follows that $p_c = p_{\beta_c}$. Namely, if $p_\beta < p_c$, then order does not percolate under $\varphi_{p_\beta}^{\mathrm{ord}}$. Therefore, equation (5.113) does not hold, implying that $\beta < \beta_c$. Conversely, if $p_\beta > p_c$, then order percolates with positive probability under $\varphi_{p_\beta}^{\mathrm{disord}}$, refuting (5.114). Hence, we must have $\beta > \beta_c$.

On the other hand, for every β at which

$$\varphi_{p_\beta}^{\mathrm{ord}}(\exists \text{ an infinite connected component of bonds}) = 0 , \qquad (5.118)$$

the measure $\mu_\beta^{\mathrm{free}}$ is the only Gibbs measure for the (q,r)-Potts model (see Appendix C). The latter condition is guaranteed whenever $p_\beta < p_c$, which is equivalent to $\beta < \beta_c$. Thus the uniqueness of Gibbs measure for $\beta < \beta_c$ follows. $\qquad\qquad\square$

5.7 Conclusion

In this last chapter, we presented a proof that the two-dimensional Potts model with q visible colours and r invisible colours undergoes a first-order phase transition in temperature accompanied by a q-fold symmetry breaking, provided the number of invisible colours is large enough. On the other hand, for $r = 0$ (no invisible colours), the model reduces to the standard q-colour Potts model, for which it is known that if $q = 2, 3, 4$, the transition in two dimensions is second-order. Tamura, Tanaka and Kawashima [70, 71, 72] introduced the Potts model with r invisible colours as a simple two-dimensional example with short-range interactions in which, tuning the parameter r, the same symmetry breaking could accompany phase transitions of different orders. The impossibility to infer the order of the phase transition from the broken symmetry was already noticed in other examples, such as the two-dimensional 3-colour Kac-Potts model [31]. For this model, Gobron and Merola proved that a 3-fold symmetry breaking might be accompanied with either a first-order or a second-order phase transition, by changing the finite range of the interactions.

The first-order phase transition in the (q,r)-Potts model occurs as long as $q+r$ is large enough. In particular, even for small values of q (say, $q = 1, 2, 3, 4$), the presence of many invisible colours assures a first-order transition. The argument is very similar to the one for the standard q-Potts model, in which q is required to be large [47, 55]. The transition point is asymptotically (in $q+r$) given by $\beta_c \approx \frac{1}{2}\log(q+r)$. For $q+r$ large, the latent heat is approximately given by $2\left(-\frac{\partial e(\varnothing)}{\partial\beta} + \frac{\partial e(\mathbb{B})}{\partial\beta}\right) = 2 + \frac{2}{\sqrt{q+r}}$, which tends to 2 as $q + r \to \infty$.

The proof relies on a formulation of the Potts model with invisible colours in terms of a variant of the random-cluster model, which we named the biased random-cluster model. The difference between this new model and the original random-cluster model is that it weights singleton connected components differently from non-singleton connected components. Such a disparity allows one to increase the entropy by increasing the number of invisible colours, while keeping the number of ground states (i.e., the number of visible colours) unchanged. The random-cluster representation allows for a clear formulation of order and disorder: order is associated with the presence of bonds while disorder with the absence of bonds. This leads to a simple notion of contours describing the interface between order and disorder. Hence, the random-cluster representation lends itself to a Pirogov-Sinai analysis, which is used to prove the existence of a first-order phase transition.

We remark that the above analysis extends to higher dimensions. In fact in the performed analysis the dimension entered mainly in the counting arguments, which however can be refined in higher dimensions. The 2 dimensional case however, has the asset of simplifying many of those counting and furthermore it allows for a much easier geometric visualization of the contours.

Appendices

Appendix A

Boundary Laws, Beyond Homogeneity

It is the purpose of this appendix to explain the relation between the notion of a boundary law as it is used in the book by Georgii [29] and the one-sided simple recursions which are used in Chapter 3. The notion of a boundary law is necessary to describe all the extremal phases (or more generally, all Markov chains on trees).

To follow the notation used in Georgii, let us denote, for $i \sim j$, by $Q_{ij}(\sigma_i, \sigma_j) = e^{\beta \sigma_i \sigma_j + g_i \sigma_i + g_j \sigma_j}$ the transition matrix of the random field Ising model on the tree with Hamiltonian $-\beta \sum_{\{i,j\} \in E} \sigma_i \sigma_j - \sum_i h_i \sigma_i$, where $g_i = h_i/(d+1)$, so the local field at each site has been symmetrically distributed among the edges to its neighbours.

Every extremal Gibbs measure μ for the random field Ising model on the Cayley tree is a Markov chain on the tree (Theorem 12.12 of Georgii). To define what it means to be a Markov chain on the tree, consider an oriented bond ij, draw this bond horizontally such that i lies to the left of j, and draw the tree embedded into the plane in such a way that there is no intersection between the tree and the axis crossing the oriented bond ij in a perpendicular way. A measure μ is a Markov chain on the tree if conditioning on the semi-infinite spin configurations extending from i to the left (the past) is the same as conditioning on the spin configuration at the site i alone, and this holds for all oriented bonds ij. Not all Markov chains are extremal Gibbs measures however, as the example of the free boundary condition Gibbs measure of the Ising model in zero field at sufficiently low temperatures shows. The meaning and importance of a boundary law lies in the following fact. A Markov chain on the tree always has a representation in terms of a boundary law $l_{ij}(a)$, $a = \pm$, that is for the finite-volume marginals it holds

$$\mu[h](\sigma_{\Lambda \cup \partial_+ \Lambda}) = \frac{1}{Z_\Lambda(\beta, h)} \prod_{k \in \partial_+ \Lambda} l_{kk_\Lambda}(\sigma_k) \prod_{\{ij\} \cap \Lambda \neq \emptyset} Q_{ij}(\sigma_i, \sigma_j) \qquad (A.1)$$

where $\partial_+ \Lambda$ denotes the outer boundary of Λ and k_Λ is the unique nearest neighbour of k in Λ. A boundary law is a function on oriented edges ij which

depends on the possible spin values. From its appearance in the last formula
we see that, at any ij, it is defined only up to a multiplicative constant, not
depending on the spin configuration a. Define therefore $q_{ij} = \frac{1}{2} \log \frac{l_{ij}(+)}{l_{ij}(-)}$ in
the Ising case. This quantity has the character of a local field at the site i and
contains the full information about the boundary law in the Ising case. More
precisely q_{kk_Λ} has the meaning of a local field acting on the spin σ_k which has
to be added to the Hamiltonian with free boundary conditions in the volume
$\Lambda \cup \partial_+ \Lambda$ if the site k is attached at the site k_Λ.

Assuming the validity of the last formula for the finite-volume marginals one
arrives at a Q-dependent consistency (or recursion) relation that a boundary
law has to satisfy. This recursion is formulated as (12.10) in Georgii; in the
case of the Ising model with site-dependent fields it translates equivalently into
the recursion

$$q_{ij} = \sum_{k \in \partial_+ i \backslash j} \frac{1}{2} \log \frac{e^{2q_{ki}+\beta+g_k+g_i} + e^{-\beta-g_k+g_i}}{e^{2q_{ki}-\beta+g_k-g_i} + e^{\beta-g_k-g_i}} \tag{A.2}$$

Conversely, a function q_{ij} on all oriented bonds which is consistent in the sense
of (A.2) defines a Markov chain by formula (A.1) with the corresponding bound-
ary law l_{ij}.

Note that (A.2) is a one-sided recursion which has no beginning and no end.
It is interesting in the first step to look at homogeneous solutions, i.e. solutions
not depending on the bond ij, but there may be also many other solutions,
even in the case when the local magnetic field in the initial Hamiltonian is
site-independent. In that case there can be non-homogeneous solutions when
there are more than one fixed points for the homogeneous recursion. Indeed,
to construct a non-homogeneous solution one picks a site j and looks to all
oriented bonds ij pointing to it, and picks values of q_{ij} not at the fixed point.
Then one defines a boundary law by preimages for q_b's for the oriented bonds
b going up to ij. In order to make sure that there are such preimages under
all orders of iterations, the value has to be chosen such that it lies between a
stable and an unstable fixed point.

To see the meaning of the boundary law in a more intuitive or physical way
let us make explicit the difference to the field which is already present in the
original Hamiltonian. We look at the asymmetric quantity which is centred at
the local field for the first spin, namely $f_{ij} = q_{ij} - g_i d$ and note that it satisfies
the equation

$$f_{ij} = \sum_{k \in \partial_+ i \backslash j} \varphi_\beta(f_{ki} + h_k) \tag{A.3}$$

with $\varphi_\beta(t) = \frac{1}{2} \log \frac{\cosh(t+\beta)}{\cosh(t-\beta)}$. With this variable we have

$$\mu[h](\sigma_{\Lambda\cup\partial_+\Lambda}) = \frac{e^{\sum_{\{ij\}\cap\Lambda\neq\emptyset} \beta\sigma_i\sigma_j + \sum_{i\in\Lambda\cup\partial_+\Lambda} h_i\sigma_i + \sum_{k\in\partial_+\Lambda} f_{kk_\Lambda}\sigma_k}}{Z_\Lambda(\beta, h)} \qquad (A.4)$$

So the f_{ij} has the meaning of an additional boundary field at the site i acting on top of the local fields which are present already in the Hamiltonian, when one computes the finite-volume marginals in a volume with a boundary site i when i is attached via the site j to the inside of the volume.

Now, let us enter in more detail the discussion on the dependence of boundary laws on a variation of local fields entering in the Hamiltonian. Suppose that a boundary law $l[h]$, not necessarily homogeneous, is given for the (not necessarily but possibly homogeneous) Hamiltonian with a field h. Recall that, as we just explained, homogeneous fields h may have very well inhomogeneous boundary laws. Let us consider the system now in the presence of a local perturbation of the field $h + \Delta h$, possibly site-dependent, but bounded, i.e. $\sup_k |\Delta h_k| < \infty$. Any Gibbs measure $\mu[h]$ gives rise to a Gibbs measure $\mu[h + \Delta h]$ which is related by the formula involving the local perturbation of the Hamiltonian of the form

$$\mu[h + \Delta h](\varphi(\tilde{\sigma})) = \frac{\mu[h](\varphi(\tilde{\sigma})e^{\sum_i \Delta h_i \tilde{\sigma}_i})}{\mu[h](e^{\sum_i \Delta h_i \tilde{\sigma}_i})} \qquad (A.5)$$

where it is understood that integration is over $\tilde{\sigma}$. If the original Gibbs measure is actually a Markov chain described by the boundary law $l_{ij} \equiv l_{ij}[h]$, the perturbed measure is described by the boundary law $l_{ij}[h + \Delta h]$ which is obtained by putting $l_{ij}[h + \Delta h] := l_{ij}[h]$ for oriented bonds ij in the outside of the region of the perturbation of the fields which are pointing towards the perturbation region. When passing with the recursion through the perturbation region of the local fields the l_{ij}'s obtain a dependence on the size of the perturbations. Then the forward iteration is used to obtain an assignment of l's to all oriented bonds.

Summarizing we have the following lemma.

Lemma 1. *Suppose that h is an arbitrary external-field configuration, Δh is an arbitrary finite-volume perturbation of the external fields, and $\mu[h + \Delta h]$ is the measure which results from a local perturbation of a Markov chain $\mu[h]$ which is described by a boundary law $l[h]$.*

Then $\mu[h + \Delta h]$ behaves in a quasilocal way (i.e. all expected values $\mu[h + \Delta h](\varphi)$ on local spin functions φ are quasilocal functions of Δh) if and only if the boundary laws $\Delta h \mapsto l_{ij}[h + \Delta h]$, depending on field perturbations Δh_k's

for k in the past of the oriented bond ij, behave in a quasilocal way, and this holds for all oriented bonds ij.

Here a vertex k is said to be in the past of ij if the path from k to j passes through i. Quasilocality is meant in the same way as it has been introduced in the context of finite-volume variations of spins, i.e. we say that l depends quasilocally on a variation of fields iff

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{\Lambda': \Lambda' \supset \Lambda} \sup_{\Delta h|_\Lambda = \Delta h'|_\Lambda} |l[\Delta h|_{\Lambda'}] - l[\Delta h'|_{\Lambda'}]| = 0 \tag{A.6}$$

where the supremum is taken over perturbations $\Delta h|_{\Lambda'}, \Delta h'|_{\Lambda'}$ in the finite volume Λ' which look the same on Λ.

Proof. The proof follows from the representation of the finite-volume Gibbs measures $\mu[h + \Delta h]$ in terms of the boundary laws $l_{ij}[h + \Delta h]$. □

We note again that there is a one-to-one correspondence between simple directed field recursions with d neighbours, as used in Chapter 3, and boundary laws. So we obtain the following corollary, which is used extensively in Chapter 3.

Corollary 1.1. *Suppose that h is a homogeneous external field, Δh is an arbitrary finite-volume perturbation of external fields, and $\mu[h + \Delta h]$ is the measure which results from a local perturbation of either one of the homogeneous measures $\mu[h]$, corresponding to the plus, the minus or the unstable fixed points. Then the measures $\mu[h + \Delta h]$ behave in a non-quasilocal way on the field perturbations Δh if and only if, the corresponding solutions of the one-sided simple recursions for the effective fields behave in a non-quasilocal way.*

Some non-homogeneous Gibbs measures. The discussion just given has consequences also for those Gibbs measures $\mu = \mu_{(l^b),\Lambda}$ which are obtained by pasting boundary laws l^b for oriented bonds b of the form kk_Λ for some fixed subtree Λ, so that (A.1) is true for the particular volume Λ. Then extend the boundary laws to have a prescription in the whole volume. Then the parameter region for non-quasilocal behavior of the resulting measure will be the union of the parameter regions of non-Gibbsianness of the original measures taken over the b's.

Connection to Gibbs vs. non-Gibbs under time evolution. Since the Gibbs properties of time-evolved Ising measures in infinite-temperature evolution can be expressed via quasilocality properties of $\Delta h \mapsto \mu[h + \Delta h]$, for *finite-volume* Δh, we are left with the investigation of the locality properties

of the boundary law iteration. A local variation of the image spins amounts to a local perturbation Δh of the local fields. Indeed, denoting the time-evolved measure by $\hat{\mu}_t(d\eta)$, starting from the measure $\mu(d\sigma)$, we have for finite $\Lambda \ni 0$ the formula

$$
\begin{aligned}
\hat{\mu}_t(\eta_0|\eta_{\Lambda\backslash 0}) &= \frac{\int \mu(d\sigma) P_t(\sigma_0, \eta_0) e^{h_t \sum_{i\in\Lambda\backslash 0} \eta_i\sigma_i}}{\int \mu(d\sigma) e^{h_t \sum_{i\in\Lambda\backslash 0} \eta_i\sigma_i}} \\
&=: \int \mu[\eta_{\Lambda\backslash 0}](d\sigma_0) P_t(\sigma_0, \eta_0)
\end{aligned}
\tag{A.7}
$$

with a measure $\mu[\eta_{\Lambda\backslash 0}](d\sigma)$ of the form $\mu[h+\Delta h]$ with a perturbation in the finite volume $\Lambda\backslash 0$. Finite-volume marginals of this measure have a representation, according to Lemma 1, of the form (A.1) with an η-dependent transition matrix

$$
Q_{ij}[\eta](\sigma_i, \sigma_j) = e^{\frac{h_t \eta_i \mathbb{1}_{i\in\Lambda\backslash 0}}{d+1} + \frac{h_t \eta_j \mathbb{1}_{j\in\Lambda\backslash 0}}{d+1}} Q_{ij}(\sigma_i, \sigma_j) \,,
$$

where $Q_{ij}(\sigma_i, \sigma_j)$ is the transition matrix for the initial measure μ, and an $\eta_{\Lambda\backslash 0}$-dependent boundary law $l_{ij}[\eta_{\Lambda\backslash 0}]$ which obeys the locally modified iterations for the boundary law described below (A.5). Hence, non-Gibbsianness of time-evolved measures is detected by non-quasilocality of the perturbed boundary laws $l_{ij}[\eta_{\Lambda\backslash 0}]$.

A consequence of these remarks is that a time-evolved measure resulting from an initial Gibbs measure which is constructed by pasting finitely many boundary laws l^b as described above, will be non-Gibbsian at a parameter regime which is the union of the non-Gibbsian parameter regimes of the time-evolved Markov chains corresponding to l^b, over b.

Appendix B

Derivation of the Biased Random-Cluster Representation

To derive the relation (5.16), we start from (5.15) and write

$$
\begin{aligned}
Z_\beta(\Lambda_n) &= \mathrm{e}^{\beta|B(\Lambda_n)|} \cdot Z^{\mathsf{RC}}_{p_\beta,q,r}(\Lambda_n) \\
&= \sum_{X \subseteq B(\Lambda_n)} (\mathrm{e}^\beta - 1)^{|X|}(q+r)^{\kappa_0(S(\Lambda_n),X)} q^{\kappa_1(S(\Lambda_n),X)} \\
&= (q+r)^{-|S(\Lambda_{n+1}\backslash\Lambda_n)|} \sum_{\substack{Y \subseteq B(\Lambda_{n+1}) \\ Y \cap (B(\Lambda_{n+1})\backslash B(\Lambda_n))=\varnothing}} (\mathrm{e}^\beta - 1)^{|Y|}(q+r)^{\kappa_0(S(\Lambda_{n+1}),Y)} q^{\kappa_1(S(\Lambda_{n+1}),Y)} \\
&= (q+r)^{-|S(\Lambda_{n+1}\backslash\Lambda_n)|} \cdot \mathrm{e}^{\beta|B(\Lambda_{n+1})|} \cdot Z^{\mathsf{RC.disord}}_{p_\beta,q,r}(\Lambda_{n+1}) \ .
\end{aligned}
\tag{B.1}
$$

To obtain the relation (5.19), we need to take the homogeneous boundary condition for the (q,r)-Potts model into account. Denoting the set of (q,r)-Potts configurations on Λ_n by Ω_{Λ_n}, we start from the definition (5.5) and write

$$
Z^{\omega^k}_\beta(\Lambda_n) = \sum_{\sigma \in \Omega_{\Lambda_n}} \exp\left\{ \beta \sum_{\{i,j\} \in B(\Lambda_n)} \delta(\sigma_i = \sigma_j \leq q) \ + \ \beta \sum_{\substack{\{i,j\} \in B(\Lambda_{n+1}) \\ i \in S(\Lambda_n),\, j \notin S(\Lambda_n)}} \delta(\sigma_i = \omega^k_j = k) \right\}
\tag{B.2}
$$

$$
= \sum_{\sigma \in \Omega_{\Lambda_n}} \prod_{\{i,j\} \in B(\Lambda_n)} \left(1 + \delta(\sigma_i = \sigma_j \leq q)(\mathrm{e}^\beta - 1)\right) \times
$$

$$
\prod_{\substack{\{i,j\} \in B(\Lambda_{n+1}) \\ i \in S(\Lambda_n),\, j \notin S(\Lambda_n)}} \left(1 + \delta(\sigma_i = k)(\mathrm{e}^\beta - 1)\right) \ .
\tag{B.3}
$$

Denoting

$$
\partial\Lambda_n \triangleq \{\{i,j\} \in B(\Lambda_{n+1}) : i \in S(\Lambda_n) \text{ and } j \notin S(\Lambda_n)\} \ ,
\tag{B.4}
$$

we can expand the products to obtain

$$Z_\beta^{\omega^k}(\Lambda_n) = \sum_{\sigma \in \Omega_{\Lambda_n}} \sum_{X_1 \subseteq B(\Lambda_n)} \sum_{X_2 \subseteq \partial \Lambda_n} (e^\beta - 1)^{|X_1| + |X_2|} \, \delta(\sigma \in \Xi_k(X_1, X_2)) \,,$$

(B.5)

where

$$\Xi_k(X_1, X_2) \triangleq \{\sigma \in \Omega_{\Lambda_n} : \sigma_i = \sigma_j \leq q \text{ for all } \{i, j\} \in X_1 \text{ and}$$
$$\sigma_i = k \text{ for all } i \in S(\Lambda_n) \cap S(X_2)\} \,.$$

(B.6)

To impose the ordered boundary condition, we multiply and divide by $(e^\beta - 1)^{|B(\Lambda_{n+1} \setminus \Lambda_n)|}$ to emulate the presence of the bonds in $B(\Lambda_{n+1} \setminus \Lambda_n)$. This gives

$$Z_\beta^{\omega^k}(\Lambda_n)$$
$$= (e^\beta - 1)^{-|B(\Lambda_{n+1} \setminus \Lambda_n)|} \sum_{\tilde\sigma \in \Omega_{\Lambda_{n+1}}} \sum_{\substack{X \subseteq B(\Lambda_{n+1}) \\ X \supseteq B(\Lambda_{n+1} \setminus \Lambda_n)}} (e^\beta - 1)^{|X|} \, \delta(\tilde\sigma \in \Theta_k(X)) \,,$$

(B.7)

where $\tilde\sigma$ is the extension of σ to a configuration in $\Omega_{\Lambda_{n+1}}$ with $\tilde\sigma_i = k$ for $i \in S(\Lambda_{n+1} \setminus \Lambda_n)$ and

$$\Theta_k(X) \triangleq \{\tilde\sigma \in \Omega_{\Lambda_{n+1}} : \tilde\sigma_i = \tilde\sigma_j \leq q \text{ for all } \{i, j\} \in X \text{ and}$$
$$\tilde\sigma_i = k \text{ for all } i \in S(\Lambda_{n+1} \setminus \Lambda_n)\} \,.$$

(B.8)

Changing the order of the sums gives

$$Z_\beta^{\omega^k}(\Lambda_n) = (e^\beta - 1)^{-|B(\Lambda_{n+1} \setminus \Lambda_n)|} \sum_{\substack{X \subseteq B(\Lambda_{n+1}) \\ X \supseteq B(\Lambda_{n+1} \setminus \Lambda_n)}} (e^\beta - 1)^{|X|} \, |\Theta_k(X)| \,.$$

(B.9)

Note that for X satisfying $B(\Lambda_{n+1} \setminus \Lambda_n) \subseteq X \subseteq B(\Lambda_n)$, the size of $\Theta_k(X)$ is

$$q^{-1} \cdot (q + r)^{\kappa_0(S(\Lambda_{n+1}), X)} q^{\kappa_1(S(\Lambda_{n+1}), X)}$$

(B.10)

we obtain

$$Z_\beta^{\omega^k}(\Lambda_n) = q^{-1}(e^\beta - 1)^{-|B(\Lambda_{n+1} \setminus \Lambda_n)|}$$
$$\times \sum_{\substack{X \subseteq B(\Lambda_{n+1}) \\ X \supseteq B(\Lambda_{n+1} \setminus \Lambda_n)}} (e^\beta - 1)^{|X|} \cdot (q + r)^{\kappa_0(S(\Lambda_{n+1}), X)} q^{\kappa_1(S(\Lambda_{n+1}), X)}$$

(B.11)

$$= q^{-1} \cdot (e^\beta - 1)^{-|B(\Lambda_{n+1} \setminus \Lambda_n)|} \cdot e^{\beta|B(\Lambda_{n+1})|} \cdot Z_{p_\beta, q, r}^{\mathsf{RC.ord}}(\Lambda_{n+1}) \,.$$

Derivation of the Contour Representation

We show that for $\Lambda = \Lambda_{n+1}$, the definitions (5.17) and (5.27) (resp., (5.20) and (5.28)) agree.

The weight of a configuration $X \subseteq B(\Lambda_{n+1})$ is

$$p_\beta^{|X|} (1 - p_\beta)^{|B(\Lambda_{n+1}) \setminus X|} (q + r)^{\kappa_0(S(\Lambda_{n+1}), X)} q^{\kappa_1(S(\Lambda_{n+1}), X)} \ . \tag{B.12}$$

For a configuration X in $\mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}$ or $\mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}$, let ∂_X be the corresponding contour family in $\Delta_{\Lambda_{n+1}}^{\text{disord}}$ or $\Delta_{\Lambda_{n+1}}^{\text{ord}}$. More precisely, $\partial_X \triangleq \overline{\partial} X$ if $\mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}$ and $\partial_X \triangleq \partial \left(X \cup B(\Lambda_{n+1})^{\complement} \right)$ if $\mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}$.

Claim. *For $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}$ we have the relation*

$$2 \left| B(\Lambda_{n+1}) \setminus X \right| = 4 \kappa_0(S(\Lambda_{n+1}), X) + \sum_{\gamma \in \partial_X} |\gamma| - \left| \overline{\partial} B(\Lambda_{n+1}) \right| \ , \tag{B.13}$$

and for $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}$ we have

$$2 \left| B(\Lambda_{n+1}) \setminus X \right| = 4 \kappa_0(S(\Lambda_{n+1}), X) + \sum_{\gamma \in \partial_X} |\gamma| \ . \tag{B.14}$$

Proof. We first decompose the set $\{(i, b) : b \in B(\Lambda_{n+1}) \setminus X \text{ and } i \sim b\}$ as

$$\{(i, b) : b \in B(\Lambda_{n+1}) \setminus X \text{ and } i \sim b \text{ and } i \in S(X)\} \tag{B.15}$$
$$\cup$$
$$\{(i, b) : b \in B(\Lambda_{n+1}) \setminus X \text{ and } i \sim b \text{ and } i \notin S(X)\} \ , \tag{B.16}$$

and furthermore note that the latter set can be expressed as

$$\{(i, b) : i \in S(\Lambda_{n+1}) \setminus S(X) \text{ and } i \sim b\} \tag{B.17}$$
$$\setminus$$
$$\{(i, b) : i \in S(\Lambda_{n+1}) \setminus S(X) \text{ and } i \sim b \text{ and } b \notin B(\Lambda_{n+1})\} \ . \tag{B.18}$$

We have

$$\left| \{(i, b) : b \in B(\Lambda_{n+1}) \setminus X \text{ and } i \sim b\} \right| = 2 \left| B(\Lambda_{n+1}) \setminus X \right| \ , \tag{B.19}$$

$$\left| \{(i, b) : b \in B(\Lambda_{n+1}) \setminus X \text{ and } i \sim b \text{ and } i \in S(X)\} \right| = \sum_{\gamma \in \partial_X} |\gamma| \ , \tag{B.20}$$

$$|\{(i,b) : i \in S(\Lambda_{n+1}) \setminus S(X) \text{ and } i \sim b\}| = 4\kappa_0(S(\Lambda_{n+1}), X) , \qquad \text{(B.21)}$$

and the cardinality of $\{(i,b) : i \in S(\Lambda_{n+1}) \setminus S(X) \text{ and } i \sim b \text{ and } b \notin B(\Lambda_{n+1})\}$ equals

$$\begin{cases} |\overline{\partial}B(\Lambda_{n+1})| , & \text{if } X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}, \\ 0 , & \text{if } X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}. \end{cases} \qquad \text{(B.22)}$$

$$\square$$

Using the relations (B.13) and (B.14) the weight of X takes the form

$$(q+r)^{\frac{1}{4}|\overline{\partial}B(\Lambda_{n+1})|}\mathrm{e}^{-e(\mathbb{B})\cdot|X|}\mathrm{e}^{-e(\varnothing)\cdot|B(\Lambda_{n+1})\setminus X|}(q+r)^{-\frac{1}{4}\sum_{\gamma\in\partial_X}|\gamma|}q^{\kappa_1(S(\Lambda_{n+1}),X)} \tag{B.23}$$

if $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}$ and

$$\mathrm{e}^{-e(\mathbb{B})\cdot|X|}\mathrm{e}^{-e(\varnothing)\cdot|B(\Lambda_{n+1})\setminus X|}(q+r)^{-\frac{1}{4}\sum_{\gamma\in\partial_X}|\gamma|}q^{\kappa_1(S(\Lambda_{n+1}),X)} \tag{B.24}$$

if $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}$. If $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}$, every non-singleton connected component in $(S(\Lambda_{n+1}), X)$ contains all the sites of a unique disorder contour in ∂_X, so that the number of disorder contours in ∂_X is the same as $\kappa_1(S(\Lambda_{n+1}), X)$, and we have

$$(q+r)^{-\frac{1}{4}\sum_{\gamma\in\partial_X}|\gamma|}q^{\kappa_1(S(\Lambda_{n+1}),X)} = \prod_{\gamma\in\partial_X} \rho(\gamma) . \qquad \text{(B.25)}$$

On the other hand, if $X \in \mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}$, the outermost non-singleton connected component of $(S(\Lambda_{n+1}), X)$ has no associated disorder contour in ∂_X, thus

$$(q+r)^{-\frac{1}{4}\sum_{\gamma\in\partial_X}|\gamma|}q^{\kappa_1(S(\Lambda_{n+1}),X)} = q\prod_{\gamma\in\partial_X} \rho(\gamma) . \qquad \text{(B.26)}$$

In conclusion, summing over all configurations, we obtain

$$Z^{\mathsf{RC.disord}}(\Lambda_{n+1}) = (q+r)^{\frac{|\overline{\partial}B(\Lambda_{n+1})|}{4}} \sum_{X\in\mathcal{X}_{\Lambda_{n+1}}^{\text{disord}}} \mathrm{e}^{-|X|\cdot e(\mathbb{B})-|B(\Lambda_{n+1})\setminus X|\cdot e(\varnothing)} \prod_{\gamma\in\partial} \rho(\gamma) , \tag{B.27}$$

$$Z^{\mathsf{RC.ord}}(\Lambda_{n+1}) = q \sum_{X\in\mathcal{X}_{\Lambda_{n+1}}^{\text{ord}}} \mathrm{e}^{-|X|\cdot e(\mathbb{B})-|B(\Lambda_{n+1})\setminus X|\cdot e(\varnothing)} \prod_{\gamma\in\partial} \rho(\gamma) . \qquad \text{(B.28)}$$

We remark that the definitions (5.17) and (5.20) may also be extended to general finite volumes of the lattice in a compatible fashion. Namely, if for a volume Λ, we define

$$\mathcal{X}_{\Lambda}^{\text{disord}} \triangleq \left\{ X \subseteq B(\Lambda) : \overline{\partial} X \in \Delta_{\Lambda}^{\text{disord}} \right\} , \tag{B.29}$$

$$\mathcal{X}_{\Lambda}^{\text{ord}} \triangleq \left\{ X \subseteq B(\Lambda) : \overline{\partial} \left(X \cup B(\Lambda)^{\complement} \right) \in \Delta_{\Lambda}^{\text{ord}} \right\} , \tag{B.30}$$

the compatibility of the definitions can be verified similarly.

Appendix C

A Few Properties of the Biased Random-Cluster Model

Much information about the standard Potts model can be detected by studying the corresponding random-cluster model (see [30, 34, 35]). Many of the properties of the standard random-cluster model can be extended to the biased random-cluster model. These properties, in turn, can be used, in a similar fashion, to obtain information about the Potts model with invisible colours. In this appendix, we briefly sketch some of these properties that we exploit in the proof of the main theorem. The proofs are straightforward modifications of the standard case, which can be found in [30, 34, 35].

Let $\mathbb{G} = (S, B)$ be a finite graph. The configurations of the biased random-cluster model on $\mathbb{G}$ can be ordered according to the inclusion ordering. A configuration $X \subseteq B$ is considered to be *smaller than or equal to* a configuration $Y \subseteq B$, if and only if every bond present in X is also present in Y. An event $\mathcal{E} \subseteq 2^B$ is *increasing* if for every two configurations X and Y such that $X \in \mathcal{E}$ and $Y \supseteq X$, we have $Y \in \mathcal{E}$. We say that a probability distribution ν is *positively correlated*, if $\nu(\mathcal{E}_1 \cap \mathcal{E}_2) \geq \nu(\mathcal{E}_1)\nu(\mathcal{E}_2)$. The inclusion ordering on the configuration space 2^B induces an ordering on the space of probability distributions on 2^B. If ν_1 and ν_2 are probability distributions on 2^B, we write $\nu_1 \preceq \nu_2$ if $\nu_1(\mathcal{E}) \leq \nu_2(\mathcal{E})$ for every increasing event $\mathcal{E} \subseteq 2^B$. In this case, we say that ν_1 is *stochastically dominated* by ν_2.

For every $0 < p < 1$, $q \geq 1$ and $r \geq 0$, the r-biased random-cluster distribution $\varphi_{p,q,r}$ on $\mathbb{G}$ is positively correlated. This follows from the Fortuin-Kasteleyn-Ginibre theorem (Theorem 4.11 of [30]; see Corollary 6.7). It follows that, if $\mathcal{E}$ is an increasing (resp., decreasing) event with $\varphi_{p,q,r}(\mathcal{E}) > 0$, then the conditional distribution $\varphi_{p,q,r}(\cdot \,|\, \mathcal{E})$ stochastically dominates (resp., is dominated by) $\varphi_{p,q,r}$. Furthermore, if $0 < p_1 \leq p_2 < 1$, it follows from Holley's theorem (Theorem 4.8 of [30]) that $\varphi_{p_1,q,r} \preceq \varphi_{p_2,q,r}$ (see Corollary 6.7 in [30]).

Let Λ be a finite volume in the lattice and φ_Λ the biased random-cluster distribution on Λ (as a graph, without boundary condition). Let us denote by $\varphi_\Lambda^{\mathrm{ord}}$ and $\varphi_\Lambda^{\mathrm{disord}}$ the biased random-cluster distributions on Λ with ordered and disordered boundary conditions, respectively. By an application of the positive

correlation property of φ_Λ we have

$$\varphi_\Lambda^{\mathrm{disord}} \preceq \varphi_\Lambda \preceq \varphi_\Lambda^{\mathrm{ord}} . \tag{C.1}$$

Moreover, by a further application of the Fortuin-Kasteleyn-Ginibre theorem, the distributions $\varphi_\Lambda^{\mathrm{disord}}$ and $\varphi_\Lambda^{\mathrm{ord}}$ are also positively correlated. This implies that if Λ_1 is a sub-volume of Λ_2, we have

$$\varphi_{\Lambda_1}^{\mathrm{disord}} \preceq \varphi_{\Lambda_2}^{\mathrm{disord}} \qquad \text{and} \qquad \varphi_{\Lambda_1}^{\mathrm{ord}} \succeq \varphi_{\Lambda_2}^{\mathrm{ord}} . \tag{C.2}$$

As in Lemma 6.8 of [30], this implies that the weak limits

$$\varphi^{\mathrm{disord}} \triangleq \lim_{\Lambda\uparrow\mathbb{L}} \varphi_\Lambda^{\mathrm{disord}} \qquad \text{and} \qquad \varphi^{\mathrm{ord}} \triangleq \lim_{\Lambda\uparrow\mathbb{L}} \varphi_\Lambda^{\mathrm{ord}} \tag{C.3}$$

exist, where the limit $\Lambda \uparrow \mathbb{L}$ can be taken along the net of all finite volumes in $\mathbb{L}$ with the inclusion ordering.

To emphasize the dependence on the parameter p, let us write $\varphi_{\Lambda,p}^{\mathrm{ord}}$ and $\varphi_{\Lambda,p}^{\mathrm{disord}}$ for the biased random-cluster distributions with parameter p. Then, by an application of Holley's theorem, if $0 < p_1 \leq p_2 < 1$, we have

$$\varphi_{\Lambda,p_1}^{\mathrm{disord}} \preceq \varphi_{\Lambda,p_2}^{\mathrm{disord}} \qquad \text{and} \qquad \varphi_{\Lambda,p_1}^{\mathrm{ord}} \preceq \varphi_{\Lambda,p_2}^{\mathrm{ord}} . \tag{C.4}$$

Let $i \overset{\mathrm{o}}{\leftrightarrow} \infty$ denote the event that there exists an infinite path of bonds passing through site i ("order" *percolates* from site i to infinity). The latter stochastic inequalities imply that the probabilities $\varphi_p^{\mathrm{ord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty)$ and $\varphi_p^{\mathrm{disord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty)$ are increasing in p. This monotonicity assures the existence of *critical probabilities* $0 \leq p_{\mathrm{c}}^{\mathrm{ord}}, p_{\mathrm{c}}^{\mathrm{disord}} \leq 1$ such that for every $p < p_{\mathrm{c}}^{\mathrm{ord}}$ we have $\varphi_p^{\mathrm{ord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty) = 0$ while for every $p > p_{\mathrm{c}}^{\mathrm{ord}}$ we have $\varphi_p^{\mathrm{ord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty) > 0$, and similarly for $p_{\mathrm{c}}^{\mathrm{disord}}$. The critical probabilities are given by

$$p_{\mathrm{c}}^{\mathrm{ord}} \triangleq \sup\left\{ p : \varphi_p^{\mathrm{ord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty) = 0 \right\} , \tag{C.5}$$

$$p_{\mathrm{c}}^{\mathrm{disord}} \triangleq \sup\left\{ p : \varphi_p^{\mathrm{disord}}(i \overset{\mathrm{o}}{\leftrightarrow} \infty) = 0 \right\} . \tag{C.6}$$

It turns out that the two critical probabilities are actually the same, hence we define $p_{\mathrm{c}} \triangleq p_{\mathrm{c}}^{\mathrm{ord}} = p_{\mathrm{c}}^{\mathrm{disord}}$. This follows from the fact that the probability measures φ_p^{ord} and $\varphi_p^{\mathrm{disord}}$ may differ for at most countably many values of p. The latter can be proved in a very similar manner as done in Theorem 8.17 of [35] for the standard random-cluster measures.

By means of the coupling, many properties of the (q,r)-Potts measures can be derived from the corresponding r-biased random-cluster measures. For instance, one can show that the thermodynamic limits μ^k and μ^{free} do not depend

on the sequence $\{\Lambda_n\}_n$ of volumes along which the limits are taken. In fact, the limits

$$\mu^k = \lim_{\Lambda \uparrow \mathbb{L}} \mu^k_\Lambda \qquad \text{and} \qquad \mu^{\text{free}} = \lim_{\Lambda \uparrow \mathbb{L}} \mu^{\text{free}}_\Lambda \tag{C.7}$$

can be taken along the net of all finite volumes in $\mathbb{L}$. In particular, this implies the translation-invariance of μ^k and μ^{free}. The proofs are similar to those of the standard case (Proposition 6.9 of [30]).

Uniqueness and multiplicity of the (q, r)-Potts measures are related to the percolation of "order" in the r-biased random-cluster model. More specifically, if φ^{ord}(order percolate) $= 0$, then the (q, r)-Potts model admits a unique Gibbs measure (as in Theorem 6.10 in [30]). On the other hand, if

$$\varphi^{\text{ord}} \left(\begin{array}{c} \exists \text{ unique infinite sea of order} \\ \text{with finite islands of disorder} \end{array} \right) = 1 \,, \tag{C.8}$$

then the measures $\mu^1, \mu^2, \dots, \mu^q$ are distinct and satisfy

$$\mu^k \left(\begin{array}{c} \exists \text{ unique infinite uni-colour sea,} \\ \text{which has colour } k \end{array} \right) = 1 \,. \tag{C.9}$$

(Recall that a "sea" of order in a random-cluster configuration is simply a connected component of bonds. A "uni-colour sea" in a Potts configuration refers to a maximal connected subgraph of the lattice induced by sites having the same colour.)

The latter claim is a consequence of the existence of a coupling between φ^{ord} and μ^k (for visible k), which can be constructed as follows:

i) We first sample a bond configuration X according to φ^{ord}.

ii) For every site i that $i \overset{\text{o}}{\leftrightarrow} \infty$ in $(\mathbb{S}, X)$, we colour i with colour k.

iii) For every finite non-singleton connected component of $(\mathbb{S}, X)$, we choose a random visible colour uniformly among the q possibilities and colour all the sites in the component with this colour.

iv) For every isolated site i in $(\mathbb{S}, X)$, we choose a random colour uniformly among the $q + r$ possible colours.

The fact that the marginal of this construction on spin configurations is μ^k is parallel to Theorem 4.91 in [34] and has a similar proof. An analogous coupling exists between φ^{disord} and μ^{free}.

Post Scriptum

A cursory glance at Cayley's formula

I would like to start this post scriptum apologizing to the brave and stubborn readers who had the perseverance to go so far in reading this thesis. These very few—I reckon, might be wondering why an apology. The reason has to do with the fact that what follows is not very tight to the main core of this work. However, I want to strike a blow to myself, and referring to the literal meaning of the two words "post scriptum", I need not owe any apology.

Here, we provide a short combinatorial proof of Cayley's formula, which states that the number of (different!) trees with n labelled vertices is n^{n-2}. The proof is obtained by means of a bijective map to an outcome space of an urn-drawing problem. In addition, we introduce an algebraic structure on the set of labelled trees, which provides a more standard approach to Cayley's formula. More precisely, we introduce an equivalence relation on the set of labelled trees which decomposes the set into classes. Each class is uniquely determined by the degrees of the labelled vertices. By means of our bijection, it is straightforward to count the number of trees in each class which is similar to the original approach of Cayley. Summing over classes turns out to be "easy". For those who were not satisfied with — I have to admit — the rather weak semantic argument I gave above to justify the content of this section, and find the presence of this post scriptum a bit contrived, perhaps they would be pleased to know that Cayley's formula plays a role also in statistical mechanics. More precisely, it is used in cluster expansion, which is a technique to study the convergence of the log of the partition function, which we exploited in Chapter 5.

The Cayley formula is well-known in mathematics, especially in graph theory. Many proofs have been provided since the formula first appeared in the year 1860 in a paper of Carl Wilhelm Borchardt. The formula became notorious when in the year 1889, in a short note, Cayley proved the formula by taking into account the degrees of the vertices. Among the different proofs, see [40] for a good overview, a very famous one is due to Prüfer in the year 1918. It establishes a bijective map between the number of labelled trees with n

vertices and sequences of length $n - 2$. We provide here a different bijective proof which—I think— offers a more intuitive combinatorial insight of Cayley's formula. Indeed, it is not trivial to count the number of labelled trees for a given number n of vertices, though it is much more immediate to count the number of ways to allocate $n - 2$ labelled balls to n labelled urns.

The first time I came across the Cayley's formula I was astonished to realize how easily it was possible, by means of this, to count the number of (labelled)[3] trees for a given number of vertices. In fact, although trees (in the mathematical sense!) are very easy to sketch, they are not really easy to imagine. I cannot figure neatly in my mind a tree with more than four or five (labelled) vertices. Cayley was able to count them though. My sudden reaction was: How? The first proof of the Cayley's formula I encountered, used an inductive argument. It was very beautiful of course, but it did not really answer my question, or at least it was not what I was after. Since high school, counting for me has always been a one-to-one correspondence between entities and natural numbers. Often this correspondence might not be so evident and it might well elude even a watchful eye. This happens mostly in case we attack the problem brutally, that is we look for a "direct" correspondence. It is well known that composition of one-to-one correspondences is still one-to-one. This enables us to lift the problem of counting the elements of a set, to the one of counting the elements of a different set, under the condition that there exists a bijection between one set and the other. Along the infinitely many sets of objects to which a particular set may be bijectively related to, there are some for which the correspondence with the natural numbers is more evident, i.e., it is easier to count their elements. If we ask somebody, who is not aware of Cayley's formula, yet possesses a certain knowledge of combinatorics, to count the number of labelled trees for a given number n of vertices, perhaps we will not get an instant reply, though he or she might be able to count the number of ways to allocate $n - 2$ labelled balls in n labelled urns.

Drawing balls and Cayley's formula

Let n be an integer and A be an ordered set of n elements, where each element has a label. Now the set A can be identified with the ordered set $\{1, \ldots, n\}$. By a (finite) *tree* we shall mean any finite, connected graph, without cycles. By a *vertex* we shall mean an end point of a line segment occurring in the tree [40]. By *stub* we shall mean a line segment having a vertex at only one of its end

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[3]To be honest, I did not know at that time what the appellative "labelled" was standing for. Trees were for me simply these weird patterns one could easily sketch. However, I could perceive the importance of the vertices' *name* in order to identify the different patterns.

points. One should think of a stub as a half-edge. By *number of incidence* of a vertex we shall mean the number of stubs having that vertex in common. By T_n we shall mean the set of unrooted labelled trees having $\{1, \ldots, n\}$ as a set of vertices. We will provide a simple combinatorial argument, which seems to be new, to prove the well-known Cayley formula for labelled trees, which states $|T_n| = n^{n-2}$, Chapter 11 [40]. Observe that in a tree $\tau \in T_n$ the numbers of incidence $d_1^\tau, \ldots, d_n^\tau$, respectively at vertices $1, \ldots, n$, satisfy [39]

$$\sum_{j=1}^{n} d_j^\tau = 2n - 2 \qquad \text{(PS.1)}$$

Moreover, every vertex has a number of incidence in $\{1, \ldots, n-1\}$.

The problem of counting the elements of T_n can be mapped to a simple combinatorial problem, namely counting the number of ways to allocate $n - 2$ labelled balls into n labelled urns. This formulation is a result of the following argument. Let n be the number of vertices. Associate to any vertex j, $j = 1, \ldots, n$ the urn U_j, and to any of the $2n - 2$ stubs a ball. The number of balls in the urn U_k will correspond to the number of incidence of the vertex k. We must take into account the fact that the number of incidence of any vertex j in a tree has to satisfy $d_j \geq 1$, therefore we put initially one ball into each urn leaving us with $n - 2$ balls. This concludes the setup of the problem, that is to distribute $n - 2$ balls into n urns.

By a *drawing* we shall mean the allocation of a ball in a urn. In order to allocate all the $n - 2$ balls we need to perform $n - 2$ drawings. We can think that the drawings are executed one at a time so that the sequence of their times of execution introduces a natural order on the set of balls.

Remark. *The first ball we put in each of the urns does not get labelled. We remind the reader that these balls were allocated in order to satisfy the constraint that the number of incidence at every vertex has to be bigger than one. This represents a common property of any possible tree, that is why we do not take it into account as a distinguishable feature. In other words, the trees cannot yet be distinguished just knowing that every vertex has number of incidence at least one.*

Define an *outcome* of $n - 2$ balls' drawings as a finite sequence of vectors, each of them having n components; we denote it by $(O^j)_{j=0,\ldots,n-2}$, where $O^j = (O_1^j, \ldots, O_n^j)$. The k-th vector of the sequence will correspond to the configuration of balls in the urns after the k-th ball has been drawn. The

component i of the k-th vector, namely O_i^k, correspond to the number of balls in the urn U_i after the k-th ball has been drawn. The vector $O^{j=0}$ is nothing but the unit vector. Notice that, by the way an outcome has been defined, two vectors O^j and O^{j+1} will differ only in one component; this component will correspond to the urn where the j-th ball has been allocated.

What we need to show is the existence of a bijective map from the space of outcomes to T_n. The map is as follows: let $(O^j)_{j=0,\dots,n-2}$ be an outcome then

1. Sketch the vertex i for which $O_i^1 \neq O_i^0$ (which is unique!) with two stubs labelling them (0) and (1)

2. Suppose the k-th ball was drawn and it ended in the m-th urn, that is, m is such that $O_m^k \neq O_m^{k-1}$, then
   - if $O_m^{k-1} = O_m^0 = 1$, attach the vertex m to the stub carrying the smallest label, then sketch an extra stub at the vertex m, and label it (k)
   - if $O_m^{k-1} \neq O_m^0 = 1$, then sketch an extra stub to the vertex m and label it (k)

3. Once we have performed the $n-2$ drawings, the vector O^{n-2} will contain a certain amount of 1's. Any i for which $O_i^{n-2} = 1$ corresponds to a *not-used* urn. We attach then the vertices corresponding to not-used urns (and therefore labelled by indexes of not-used urns) to the remaining stubs, in such a way that the vertex with the smallest label is attached to the stub with the smallest label.

We shall refer to this map as *"way of drawing"* map.

Note that the number of not-used urns corresponds to the number of leaves in the tree. Another warning, to better understand the second step, is that the stubs get labelled only after having been drawn.

It can be proved by contradiction that the above described map is injective and also surjective. We have just proved the following

Lemma. *The "way of drawing" map is a bijective map between ordered drawings of $n-2$ balls into n urns and labelled trees with n vertices.*

An equivalence class on trees

By *incidence function* we shall mean a function from T_n to $\{1,\dots,n-1\}^n$, which we denote $\mathcal{N}_\mathcal{I}$, defined as follows: $\mathcal{N}_\mathcal{I} : \tau \mapsto (d_1^\tau,\dots,d_n^\tau)$. Let $\tau_1,\tau_2 \in T_n$ and let ρ be the equivalence relation on T_n defined by: $\tau_1 \rho \tau_2$ iff $\mathcal{N}_\mathcal{I}(\tau_1) = \mathcal{N}_\mathcal{I}(\tau_2)$. In other words we shall call τ_1,τ_2 ρ-equivalent iff they have the same number of incidence at every vertex. This equivalence relation partitions the set T_n

in classes of the form $[\tau_1]_\rho := \{\tau \in T_n | \tau \rho \tau_1\}$. We might see the relation ρ as being a way to zoom out the set T_n and get the quotient set T_n/ρ. The previous observation represents a nice trick to determine the cardinality of T_n. The general idea is that in order to determine the number of elements in a set one can, after having partitioned the set, sum the element in every part[4]. The point is that, although we are not able to determine the cardinality of T_n straight away, we can do it for every class and then sum over classes. Indeed it turns out to be quite easy to compute the cardinality of any of the ρ-classes and not a big deal either to sum them. Thanks to the bijectivity of the "way of drawing " map we can easily compute how many trees are in every ρ-class. First of all notice that every ρ-class is uniquely determined by the last element of an outcome. This is the vector $O^{n-2} = (O_1^{n-2}, \ldots, O_n^{n-2})$ which corresponds to the distribution of the balls in the urns after the $(n-2)$-th drawing has been performed. Then, by simple combinatorics the number of way to draw such a configuration is given by $\frac{(n-2)!}{\prod_{j=1}^n (O_j^{n-2}-1)!}$. What is left then is to sum these factors over all the possible ρ-classes. This sum can be written as

$$\sum_{\substack{O_1^{n-2},\ldots,O_n^{n-2}:1\le O_j^{n-2}\le n-1 \\ O_1^{n-2}+\cdots+O_n^{n-2}=2n-2}} \frac{(n-2)!}{\prod_{j=1}^n (O_j^{n-2}-1)!} = \sum_{\substack{s_1,\ldots,s_n:0\le s_j\le n-2 \\ s_1+\cdots+s_n=n-2}} \frac{(n-2)!}{\prod_{j=1}^n s_j!}$$

where $s_j = O_j^{n-2} - 1$. The usage of the multinomial formula gives the value n^{n-2}.

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[4] " Divide et Impera"

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Summary

As the (sub)title of this thesis enunciates, the present work is but a very short stroll through the realm of statistical mechanics. The common link between the different works discussed here is the will to understand the cooperative effects arising from a very large system, whose (microscopic) components interact with each other. A turning point, in pursuing this understanding, is due to the introduction of the concept of a Gibbs measure.

To deal with the microscopic complexity (which is due to the huge number of components), the system is cast in a probabilistic framework by treating its constituents as random variables. Furthermore, for any finite part (volume) of the system, local laws are prescribed via the Boltzmann-Gibbs distributions, which enclose the information of how the components inside the volume interact with each other and with their exterior. These prescribed laws are conditional probabilities on a finite part of the system, conditioned on what happens outside this volume. A Gibbs measure is defined to be a probability measure on the whole (infinite) system, which is compatible with the prescribed local laws. The climax of the Gibbsian description of an infinite system is the possibility that several Gibbs measures exist for the same prescribed family of local laws. This mathematical dichotomy accounts for the physical phenomenon of phase transitions, which describes the possibility that a physical system with a certain interaction between constituents can admit different equilibria. Another important feature of the Gibbsian description, which has a physical counterpart as well, is the continuity of the conditional probabilities as a function of the conditioning. This continuity requirement relates to the physical concept of locality which can also be summarized as :"no action at distance". In fact, in the Gibbsian description, we describe what happens locally in terms of a distribution on a finite volume which, however, depends on what happens outside this volume. The continuity requirement then says, that the dependence on components which are far outside this volume ought to be small. In this thesis, we have encountered two examples which are strictly related to the continuity requirement and to the concept of phase transition.

As far as the continuity of the conditional probabilities is concerned, we have provided in Chapter 4 an example of how such a continuity can be lost if the

system is subjected to a dynamics. More precisely, we have shown that under an infinite-temperature Glauber dynamics the homogeneous Ising Gibbs measures on a Cayley tree lose the continuity if we let the system evolve for sufficiently long time. Moreover, if the time is sufficiently large, a very peculiar behaviour displays itself. For one of these evolved homogeneous Gibbs measures (three in total) the non-Gibbsianness (non-continuity) will get worse as the time goes by, in the sense that the number of discontinuity points increases. For the other two evolved homogeneous Gibbs measures, instead, the big times "cure" the non-continuity illness, i.e., the discontinuity points are cleared as the time goes by. The above difference in behaviour arises from the fact that one of the phases is unstable while the other two are stable phases. In our context, instability for a phase means that this phase converges to one of the stable ones when a boundary field is either increased or decreased.

In Chapter 5, we studied phase transition for the Potts model with invisible colours on $\mathbb{Z}^2$. This model reduces to the standard Potts model when the number of invisible colours is set to 0. However, the model with invisible colours displays a remarkable phenomenon which is alien to the standard Potts model. In fact, when the number of visible colours is fixed to q, depending on the number of invisible colours, the model undergoes either a second-order or a first-order phase transition in temperature, both accompanied by a q-fold symmetry breaking. The proof that a first-order phase transition occurs for the Potts model with invisible colours if the number of invisible colours is large enough, is based on a random-cluster representation of the model, which we baptised biased random-cluster. The biased random-cluster model differs from the original Fortuin-Kasteleyn representation because it weights singleton connected components differently than non-singleton connected components.

In Chapter 3 a simple example of disordered systems is analysed, i.e., disordered mean-field models where both spin variables and disorder variables take finitely many values. Disordered models are difficult in the sense that they show the undesired and unfamiliar phenomenon that the Boltzmann-Gibbs distributions on an increasing sequence of volumes may fail to converge to a thermodynamic limit. Many different limits might exists along different subsequences of volumes which depend on the disorder (chaotic size-dependence). Weaker types of convergence might be sought, such as convergence in distribution. For these reasons the notion of metastate has been introduced, being a probability measure on the set of Gibbs measures. In Chapter 3, we provide the construction of the metastate and we compute the probability weights which give us the appearance of the candidate states. The weights are obtained by studying the fluctuations of the free energy with respect to the disorder variables.

Samenvatting

Het verband tussen de verschillende onderwerpen die hier worden behandeld, is de wil om de collectieve, emergente eigenschappen te begrijpen die in een bijzonder groot systeem optreden waarvan de microscopische componenten met elkaar wisselwerken. Een doorbraak bij de pogingen om zo'n begrip te verkrijgen, is de introductie van het concept van een Gibbs-maat. Om met de microscopische complexiteit als gevolg van het enorme aantal componenten te kunnen rekenen, is het systeem in een probabilistisch frame gegoten door de componenten te behandelen als toevalsvariabelen. Bovendien hebben we voor elk eindig deel (volume) van het systeem te maken met lokale voorschriften, namelijk de Boltzmann-Gibbs-kansverdelingen. Een Gibbs-maat, die dient om een evenwichtstoestand te beschrijven, is een kansmaat voor het gehele (oneindige) systeem die compatibel is met de al genoemde voorwaardelijke Boltzmann-Gibbs-kansverdelingen. Een van de voornaamste rechtvaardigingen van de Gibbsiaanse beschrijving van een oneindig systeem is de mogelijkheid dat er meerdere Gibbs-maten bestaan met dezelfde voorwaardelijke Boltzmann-Gibbs-kansen. Dit beschrijft het natuurkundige verschijnsel van een faseovergang. Een volgend belangrijk kenmerk van de Gibbsiaanse beschrijving is de continuïteit van de voorwaardelijke kansverdelingen als functie van de randvoorwaarden. Deze continuïteit is gerelateerd aan het natuurkundig concept van lokaliteit. In dit proefschrift zijn we twee voorbeelden tegengekomen die te maken hebben met de noodzaak van continuïteit en met het concept faseovergang.

In hoofdstuk 4 geven we een voorbeeld van de wijze waarop de genoemde continuïteit verloren kan gaan als het systeem wordt onderworpen aan een bepaalde toevalsdynamica. Meer in het bijzonder laten we zien dat, als we de homogene Ising-Gibbs-maten voor een Cayley-boom lang genoeg aan een 'oneindige-temperatuur Glauber dynamica' onderwerpen, deze hun continuïteit kunnen verliezen, op een toestandsafhankelijke manier.

In hoofdstuk 5 bestuderen we de faseovergang voor het Potts-model met onzichtbare kleuren op $\mathbb{Z}^2$. Dit model met de onzichtbare kleuren kont overeen met het standaard Potts-model als we het aantal onzichtbare kleuren op 0 stellen. We bewijzen dat er bij dit model een eerste-orde-faseovergang optreedt, als het aantal onzichtbare kleuren groot genoeg is. Het bewijs is gebaseerd op

een random-cluster-representatie van dit model dat we 'biased random-cluster model' noemen.

In hoofdstuk 3 analyseren we een eenvoudig voorbeeld van wanordelijke systemen, namelijk wanordelijke mean-field-modellen waar zowel spinvariabelen als 'wanorde'-variabelen een groot maar een eindig aantal waarden kunnen aannemen. We geven een constructie van de 'metastate' en we berekenen de gewichten van de kansen waarmee de mogelijke limiettoestanden optreden. We verkrijgen deze gewichten door de fluctuaties van de vrije energie te bestuderen als functie van de 'wanorde'-variabelen.

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158

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Giulio Iacobelli, Groningen, 2012

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