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Chahal, Jasbir S.; Top, Jaap

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TRIANGULAR NUMBERS AND ELLIPTIC CURVES

JASBIR S. CHAHAL AND JAAP TOP

Dedicated to Wolfgang Schmidt for his 60th birthday

ABSTRACT. Some arithmetic of elliptic curves and theory of elliptic surfaces is used to find all rational solutions (r, s, t) in the function field $\mathbf{Q}(m, n)$ of the pair of equations

$$r(r+1)/2 = ms(s+1)/2$$
$$r(r+1)/2 = nt(t+1)/2.$$

It turns out that infinitely many solutions exist. Several examples will be given.

1. Introduction. A triangular number Δ_r is by definition the sum

$$1 + 2 + \cdots + r = r(r+1)/2$$

of the first r natural numbers. Many properties of the triangular numbers

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

have been discovered by Legendre, Gauss, Euler and others. For example, Legendre proved that a triangular number $\Delta_r > 1$ cannot be a cube, nor a fourth power. Gauss showed that every natural number is a sum of at most three triangular numbers. Euler proved that there are infinitely many squares among the triangular numbers, and he determined all of them. Euler also showed that infinitely many pairs (Δ_r, Δ_s) exist for which $\Delta_r = 3\Delta_s$. In fact, using the well-known theory of the Pell-Fermat equation $X^2 - dY^2 = 1$, it is not hard to show that for any natural number m there are infinitely many pairs (Δ_r, Δ_s) satisfying $\Delta_r = m\Delta_s$ (cf. [**2**]). On the other hand, using a result of Mordell on integral points on curves of genus 1 [**4**] it is shown

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in [**2**] that for fixed natural numbers $m > n > 1$ the two simultaneous equations

$$(1)\qquad \begin{cases} \Delta_r = m\Delta_s \\ \Delta_r = n\Delta_t \end{cases}$$

have only finitely many solutions in natural numbers (r, s, t).

In this paper we change perspective and seek rational solutions of (1). In other words, we define $\Delta_r = r(r+1)/2$ for any rational number r and then consider (1) for fixed n and m. There exist eight obvious solutions to these equations, namely, those corresponding to $r = 0$ and to $r = -1$. The two equations in the three variables r, s, t define a curve. It turns out that, in general, this curve is of genus 1. Taking any of the eight given points on this curve as a neutral element, it is then known that all rational points on the curve constitute an abelian group. We will show that, for generic n and m, this group is isomorphic to $\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ and is generated by the eight trivial points. Adding points in this group then yields as many nontrivial solutions to (1) as desired; a few will be listed here.

2. Generalities on space quartic curves. On completing squares, the equations in (1) can be written as

$$(2)\qquad \begin{cases} (2r+1)^2 + (m-1) = m(2s+1)^2 \\ (2r+1)^2 + (n-1) = n(2t+1)^2. \end{cases}$$

Changing variables and homogenizing, this becomes

$$(3)\qquad \begin{cases} x_0^2 + (m-1)x_1^2 - mx_2^2 = 0 \\ x_0^2 + (n-1)x_1^2 - nx_3^2 = 0. \end{cases}$$

Note that the eight trivial solutions are given by $(x_0, x_1, x_2, x_3) = (1, \pm 1, \pm 1, \pm 1)$ in these coordinates. From the description (3), it is easy to deduce that these equations define a smooth irreducible quartic curve in projective 3-space $\mathbf{P}^3$, provided $n \neq m$, $n \neq 0, 1$ and $m \neq 0, 1$. Moreover, under these conditions the genus of the curve is 1: this follows, e.g., from the adjunction formula (compare [**3**], Chapter II, Proposition 8.20 and Exercise 8.4) which, for the case of a smooth intersection of two surfaces of degrees d_1 and d_2 in $\mathbf{P}^3$, says that the genus g satisfies $2g - 2 = d_1 d_2(d_1 + d_2 - 4)$. Hence, fixing a rational point (e.g., one of the points coming from a trivial solution), one has an elliptic curve. For generalities about such curves we refer to [**8**], especially Chapter III, Section 3. The points on our curve constitute an abelian group, with the chosen fixed point as zero element. Moreover, the curve can be transformed into a Weierstrass form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

in such a way that the fixed point corresponds to the point at infinity on the Weierstrass model.

A somewhat more direct explanation how such a smooth space quartic curve is related to a plane one (not necessarily given by a Weierstrass form) can be found in [**9**, pp. 135–139]. We recall this briefly, focusing on the present situation. Using the standard bilinear form in 2×4 variables $\langle x, y \rangle = \sum x_i y_i$, one can write the equations (3) as

$$\langle x, Ax \rangle = 0 = \langle x, Bx \rangle \tag{4}$$

in which A and B are diagonal 4×4 matrices with entries 1, $m-1$, $-m$, 0 and 1, $n-1$, 0, $-n$ on the diagonal, respectively. Any choice of a point on this curve now defines a birational map to the curve given by

$$\eta^2 = \det(A - \xi B) = mn\xi(\xi - 1)(m - 1 - (n-1)\xi).$$

In fact, when $(1, 1, 1, 1)$ is chosen as a fixed point, then ξ corresponds to $(x_0 + (m-1)x_1 - mx_2)/(x_0 + (n-1)x_1 - nx_3)$. If one computes the ξ-coordinates corresponding to the eight trivial points on the space quartic, one finds that four of these correspond to points (ξ, η) in which $\eta \neq 0, \infty$. Thus, by changing the sign of η, one obtains four new points, hence new solutions to the original equations (1). Although he did not phrase it in the language used here, this is basically how Fermat constructed solutions to many of his 'double equations' (compare [**9**, pp. 105–106]). To be able to translate back and forth between a Weierstrass equation and the equations (1), we will construct a very explicit isomorphism in the next section.

Yet another, more geometric, way to transform a space quartic into a plane cubic works as follows. Take any point on the quartic, and consider the projection from this point to a hyperplane $\mathbf{P}^2 \subset \mathbf{P}^3$. One readily verifies that the image of the space quartic is a cubic curve, and that actually the two curves are isomorphic (compare [**6**]). It is now a routine matter, starting with the cubic plus the image of the point we projected from, to transform it into a curve given by Weierstrass equation.

Remark 2.1. An analogous case of a space quartic is given for $0 \neq a \neq b \neq 0$ by

$$\begin{cases} x_0^2 + ax_1^2 = x_2^2 \\ x_0^2 + bx_1^2 = x_3^2 \end{cases} \tag{5}$$

If one chooses $(x_0, x_1, x_2, x_3) = (1, 0, 1, 1)$ as origin, the group law can be given in a very explicit form as a consequence of the addition formula for Jacobi's four theta functions. This is described in [**6**, Section 14]. In the special case $a = 1$, $b = -1$ these formulas can be found in [**1**, p. 36].

3. Transforming the space quartic into Weierstrass form. Translating between the equations (1) and a Weierstrass equation may be done as follows.

Proposition 3.1. *Suppose $n \neq m$ and $n \neq 0, 1$ and $m \neq 0, 1$. Then there exists a birational isomorphism between the curve given by* (1) *and the elliptic curve defined by $y^2 = x(x - mn(1-m))(x - mn(1-n))$. This correspondence is given by*

$$x = \frac{mn(nr^2t + r^2 - mns^2t - msr)}{r^2},$$
$$y = \frac{m^2n^2(mns^3t + r^3t + ms^2r + r^3 - nr^2st - mrs^2t - mr^2s - sr^2)}{r^3}.$$

The inverse transformation is

$$r = \frac{mn((m-n)x - y)(x - m^2(1-n))}{d}$$
$$s = \frac{(mx - y + m^2n(n-1))((m-n)x - y)}{d}$$

and

$$t = \frac{(x^2 + mnx(m-1))(x - m^2(1-n))}{d},$$

in which

$$\begin{aligned} d = {} & (mn^2 - m^2)x^2 - y^2 + 2mxy \\ & + (2m^3n^3 - 4m^3n^2 - 2m^3n)x \\ & + (2m^2n^2 - 2m^2n)y + m^5n^4 \\ & - 2m^5n^3 - m^4n^4 + m^5n^2 + 2m^4n^3 - m^2n^2. \end{aligned}$$

It is hardly illuminating to verify that the two maps given above are each other's inverse, and that they map one curve to the other. Hence we omit this. Instead, we will briefly explain how one finds such maps.

First of all, one may start with the equations (3) instead of (1), and even with an inhomogeneous form of (3). Each of these equations defines a cylinder on a quadratic plane curve, with a rational point on it. It is easy to parametrize such a curve, e.g., by intersecting the curve with a variable line through the rational point. In this way, one can express x_0 and x_2 in terms of a new variable, say u. The other equation is then easily transformed into one of the type $v^2 =$ quartic in u. Since we have a rational point on this quartic (corresponding to any of the trivial solutions of the original equations), it is now a routine matter (as explained, e.g., in [**1**, pp. 35–36]) to exhibit an explicit transformation to a Weierstrass model.

3.2. Using Proposition 3.1 one computes that the eight trivial solutions of (1) correspond to points (x, y) on the model

(6) $$y^2 = x(x - mn(1-m))(x - mn(1-n))$$

as given in the following table.

Using the group law on the elliptic curve given by (6), the above points are related as follows. The point O is the zero element in this group. Also, $2Q_1 = 2Q_2 = 2Q_3 = O$ and $Q_1 + Q_2 + Q_3 = O$. Finally,

(r,s,t)	(x,y)
$(0,0,0)$	$Q_1=(0,0)$
$(-1,-1,0)$	$Q_2=(mn(1-m),0)$
$(-1,0,-1)$	$Q_3=(mn(1-n),0)$
$(0,-1,-1)$	$O=(\infty,\infty)$
$(-1,0,0)$	$P_1=(mn,m^2n^2)$
$(0,0,-1)$	$P_2=(n^2(1-m),n^2(m-n)(1-m))$
$(0,-1,0)$	$P_3=(m^2(1-n),m^2(n-m)(1-n))$
$(-1,-1,-1)$	$P_4=(mn(1-m)(1-n),-m^2n^2(1-m)(1-n))$

$P_1+Q_1=P_4$, $P_1+Q_2=P_2$ and $P_1+Q_3=P_3$. By taking other linear combinations of these points, one easily finds more solutions to (1). We used the computer algebra packages PARI for the group law on the curve defined by (6), and MAPLE for transforming points back to the original equation. In terms of the points P_1,P_2 in the above table, this provides, for instance, the following examples.

$-P_1$	$r=\dfrac{(mn+m-n)(mn-m+n)}{D}$, $s=\dfrac{2n(mn+m-n)}{D}$, $t=\dfrac{2m(mn-m+n)}{D}$.
$-P_2$	$r=\dfrac{-2mn(mn-m-n)}{D}$, $s=\dfrac{2n(mn+m-n)}{D}$, $t=\dfrac{-(mn-m-n)(mn+m-n)}{D}$.
$2P_1$	$r=\dfrac{-2mn(mn-m-n)}{D}$, $s=\dfrac{-(mn-m-n)(mn-m+n)}{D}$, $t=\dfrac{2mn^2-n^2-m^2n^2+m^2}{D}$.

Here $D=m^2-2m^2n-2mn^2+m^2n^2-2mn+n^2$.

Remark 3.3. On the curve described the equations (1) it is easy to find some involutions: they correspond to changing the sign of any of x_0, x_2, x_3 in (3). We write these involutions as $\sigma_r, \sigma_s, \sigma_t$, with $\sigma_r(r,s,t) = (-r-1,s,t)$ and the others defined similarly. Note that they all commute. To determine what the corresponding involutions on the model given by (6) are, recall that any isomorphism of an elliptic curve to itself that fixes the zero element necessarily gives an isomorphism for the group law. Furthermore, the only involutions that fix the zero element are ± 1. Hence, it follows that any involution is either of the type $P \mapsto P_0 - P$ (in which P_0 is an arbitrary point on the curve), or of the form $P \mapsto Q_0 + P$ (in which Q_0 is a point of order 2). In our situation, $\sigma_r(0,-1,-1) = (-1,-1,-1)$. Hence the corresponding involution on the model defined by (6) maps O to P_4. Since this is not a point of order 2, it follows that σ_r corresponds to the involution $P \mapsto P_4 - P$. The corresponding involutions for σ_s and σ_t are found analogously. Composing them one finds the following table.

involution on (1)	corresponding involution on (6)
σ_r	$P \mapsto P_4 - P$
σ_s	$P \mapsto P_3 - P$
σ_t	$P \mapsto P_2 - P$
$\sigma_r\sigma_s$	$P \mapsto Q_2 + P$
$\sigma_r\sigma_t$	$P \mapsto Q_3 + P$
$\sigma_s\sigma_t$	$P \mapsto Q_1 + P$
$\sigma_r\sigma_s\sigma_t$	$P \mapsto P_1 - P.$

This explains why the 'new' solutions given above appear so similar: $2P_1$ is the image of $-P_1$ under the involution corresponding to $\sigma_r\sigma_s\sigma_t$. Similarly, $-P_2$ is the image of $-P_1$ under the involution corresponding to $\sigma_r\sigma_s$. Hence, it is easy to relate the corresponding solutions of (1).

4. The main result. We will now describe all the solutions to (6) for generic m and n. By this we mean that one considers m and n as algebraically independent variables over $\mathbf{Q}$. By definition, $\mathbf{Q}(n,m)$ denotes the field of all rational functions in the variables n and m which have coefficients in $\mathbf{Q}$. The object of interest is the set (group, in fact, if the point at infinity is included) of solutions to (6) in functions $x, y \in \mathbf{Q}(n, m)$. This group can be described as follows.

Theorem 4.1. *The group* $E(\mathbf{Q}(n, m))$ *of all solutions to* (6) (*including the one at infinity*) *is isomorphic to* $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

It is generated by the elements $P_1 = (mn, m^2n^2)$ *of infinite order, and* $Q_1 = (0, 0)$ *and* $Q_2 = (mn(1-m), 0)$ *which each have order* 2.

Proof. The proof to be presented here relies on some theory of elliptic surfaces. A useful general reference for this is [**7**]. First of all, fix an inclusion of fields $\mathbf{Q}(m) \subset \mathbf{C}$. Using this, one may regard $\mathbf{Q}(m, n)$ as a subfield of the field $K = \mathbf{C}(n)$ of rational functions in the variable n with complex coefficients. The equation (6) then defines an elliptic curve E over $\mathbf{Q}(m, n)$, which for the moment will be regarded as an elliptic curve over K. The group $E(K)$ of K-rational points on E is a finitely generated abelian group (compare [**7**, Theorem 1.1]). To compute the rank and a set of generators of this group, one starts by regarding (6) as an equation which defines a surface over $\mathbf{C}$. The surface comes equipped with the map $(x, y, n) \mapsto n$, and for all but finitely many $n \in \mathbf{C}$ the fiber of this map over n is an elliptic curve over $\mathbf{C}$. The theory of elliptic surfaces now allows one to slightly modify this surface, giving a surface S which is projective, smooth and minimal, and which comes with a surjective morphism $f : S \to \mathbf{P}^1$, with the property that all but finitely many fibers of f are elliptic curves (in fact, they are the curves we already had in our original surface).

The fibers of f which are not elliptic curves can be found using an algorithm due to Tate, which can be found in [**5**, pp. 46–52]. Using the notation of loc. cit., one finds a fiber of type I_0^* at $n = 0$, and fibers of type I_2 at both $n = 1$ and $n = m$. To find the fiber over $n = \infty \in \mathbf{P}^1$, one changes coordinates by writing $\xi = x/n^2$, $\eta = y/n^3$ and $t = 1/n$. This transforms the equation (6) into

$$\eta^2 = \xi(\xi - mt(1-m))(\xi - m(t-1)). \tag{7}$$

From this, it is easy to deduce that over $n = \infty$ (which corresponds to $t = 0$), one has another fiber of type I_2.

The next important property of the surface S is that it is a rational surface. This is immediate from the criterion given in [**7**]. Since the surface is rational, a formula for the rank of the free part of $E(K)$ is given by (cf. [**7**, Theorem 10.3])

$$\operatorname{rank} E(K) = 8 - \sum (m_v - 1),$$

where the sum is taken over all singular fibers of $f : S \to \mathbf{P}^1$ and where m_v denotes the number of irreducible components of such a fiber. A fiber of type I_2 has two components, and one of type I_0^* has five components, so one concludes $\operatorname{rank} E(K) = 8 - 4 - 1 - 1 - 1 = 1$ in our case.

There are several methods for finding the torsion subgroup of $E(K)$. We follow a method in the spirit of the remainder of this proof. Write k for the number of elements of the torsion subgroup of $E(K)$. Using again [**7**, Theorem 10.3], it follows that $k^2 | \prod m'_v$, where the product is taken over all singular fibers of f, and where now m'_v denotes the number of *simple* irreducible components of such a fiber. This number is 2 for a fiber of type I_2 and 4 for a fiber of type I_0^*. Hence, $k^2 | 4 \cdot 2 \cdot 2 \cdot 2 = 32$, hence $k | 4$. Since we already know that the torsion subgroup consists of at least four elements, the conclusion is that we have found all the torsion points.

What remains to be done is to find a generator of $E(K)$ modulo torsion. For this, a canonical height on $E(K)$ modulo torsion will be used. Recall that we already proved that $E(K)$ has rank 1. By a canonical height in this case one simply means a function

$$h : E(K) \longrightarrow \mathbf{R}$$

with the properties $h(P) = 0$ whenever P is a torsion point, $h(P+Q) = h(P)$ whenever Q is a torsion point, and $h(lP) = l^2 h(P) > 0$ whenever P is a point of infinite order. Such a function $P \mapsto h(P)$ indeed exists; using [**7**, Lemma 10.1] a formula for it can be found in [**7**, Theorem 8.6]. It has the form

$$\begin{aligned} h(P) = 2 + 2(PO) &- \mathrm{contr}_0(P) \\ &- \mathrm{contr}_1(P) - \mathrm{contr}_m(P) - \mathrm{contr}_\infty(P). \end{aligned} \tag{8}$$

Here (PO) denotes an intersection number (of the sections in S defined by the points P and O); this is an integer which for $P \neq O$ is nonnegative. If the surface S has a singular fiber over $n = v$, then one may also have a contribution $\mathrm{contr}_v(P)$ to the height of P. In our case, this contribution is easy to compute using [**7**, p. 229]:

1. For $n = 0$, if by substituting $n = 0$ into the coordinates of P one obtains the singular point $(0, 0)$ of the curve with defining equation $y^2 = x^3$ (obtained by substituting $n = 0$ in (6)), then $\mathrm{contr}_0(P) = 1$ and otherwise $\mathrm{contr}_0(P) = 0$.

2. Similarly, if by substituting $n = 1$ into the coordinates of P one obtains the singular point $(0, 0)$ of the curve given by $y^2 = x^2(x - m(1 - m))$, then $\mathrm{contr}_1(P) = 1/2$ and otherwise $\mathrm{contr}_1(P) = 0$.

3. If by substituting $n = m$ into the coordinates of P one obtains the singular point $(m^2(1 - m), 0)$ of the curve given by $y^2 = x(x - m^2(1 - m))^2$, then $\mathrm{contr}_m(P) = 1/2$ and otherwise $\mathrm{contr}_m(P) = 0$.

4. Finally, for $n = \infty$ one first rewrites P and the equation (6) in terms of the coordinates $\xi = x/n^2$, $\eta = y/n^3$ and $t = 1/n$. This leads to the equation (7). If substituting $t = 0$ into the new coordinates of P leads to the singular point $(0, 0)$ of the curve given by $\eta^2 = \xi^2(\xi + m)$, then $\mathrm{contr}_\infty(P) = 1/2$, and otherwise $\mathrm{contr}_\infty(P) = 0$.

From this and (8) it is clear that any point $P \in E(K)$ has $h(P) \in (1/2)\mathbf{Z}$. Furthermore, the point $P_1 = (mn, m^2n^2)$, which is written as (mt, m^2t) in (ξ, η, t)-coordinates, does not coincide with the point (∞, ∞) for any value of n (or t). Hence the intersection number (OP_1) equals 0. One concludes that the point P_1 has height $2 + 0 - 1 - 0 - 0 - 1/2 = 1/2$. Since this is the minimal positive value a function to $(1/2)\mathbf{Z}$ can possibly attain, it follows that P_1 generates $E(K)$ modulo torsion. Hence $E(K)$ is generated by P_1, Q_1 and Q_2. Moreover, since these points are all in the subgroup $E(\mathbf{Q}(m, n)) \subset E(K)$, it follows that $E(\mathbf{Q}(m, n)) = E(K)$. This proves the theorem. □

5. Examples. Although Theorem 4.1 shows that the rank of the curve defined by (6) is 1 when m and n are treated as independent variables, it may be larger for special values of m and n. Using Ian Connell's MAPLE-program 'apecs,' we computed this rank for all natural numbers m and n with $1 < m < n \leq 12$. Of these 55 pairs, 41 give rank 1 and 14 give rank 2. One finds rank 2 for the following values of (m,n):

$$\begin{array}{lllllll} (3,8) & (4,10) & (4,11) & (5,7) & (5,8) & (5,12) & (6,11) \\ (6,12) & (7,9) & (7,11) & (7,12) & (8,12) & (10,12) & (11,12). \end{array}$$

By transforming rational points on these curves back to the original equations (1), the following examples of solutions are found.

1. In the case $(m,n) = (3,8)$, generators of the rational points on the curve given by (6) modulo torsion are provided by $(-84, 504)$ and $(-56, 224)$. These points add up to $(3\cdot 8, 3^2\cdot 8^2)$, so by Remark 3.3 they basically lead to the same solution of (1), which is $(r,s,t) = (2,1,1/2)$.

2. For $(m,n) = (4,10)$ one finds generators $(-200, 1600)$ and $(-144, 864)$. Plus and minus the second one of these leads to the trivial solution with $r = 0$ of (1). Hence, from Remark 3.3, these points must be related to $(40, 1600)$ via involutions. The point $(-200, 1600)$ corresponds up to such involutions to $(r,s,t) = (35/13, 15/13, 8/13)$. Its negative $(-200, -1600)$ yields $(5/3, 2/3, 1/3)$.

3. If one takes $(m,n) = (5,7)$, a basis modulo torsion for the corresponding group of rational points is given by $(-150, 300)$ and $(-160, 400)$. While the first of these only leads to a trivial solution of (1), the second one yields $(r,s,t) = (14, 6, -6)$. With our involutions, this is transformed into the solution in positive integers $(14, 6, 5)$. This expresses the easily verifiable fact that $\Delta_{14} = 5\Delta_6 = 7\Delta_5$.

4. When $(m,n) = (7,9)$ one finds especially many integral solutions to (6), for instance when $x = -486$, $x = -441$, $x = -432$, $x = 63$, $x = 72$, $x = 1800$, $x = 2646$ and $x = 84672$. However, the only 'small' values for (r,s,t) that correspond to such points are $(7/3, 2/3, 5/9)$ and $(35/9, 11/9, 28/27)$. In particular, we found no nontrivial triples of integers (r,s,t) in this way.

The referee of this paper pointed out to us that even a very modest attempt to extend our list of examples reveals cases such as $(m,n) = (13,17)$ and $(m,n) = (17,19)$ where the rank turns out to equal 3. We have not tried to find cases with even higher rank by a more extensive search.

Remark 5.1. Although the above list of examples gave us only one solution in positive integers, it is in fact not hard at all using elementary considerations to find many more. Namely, what one looks for are numbers $2\Delta_r$, that is, products of two consecutive natural numbers, which have at least two different factors which are themselves products of two consecutive natural numbers. So, for instance, when r itself contains a factor $s(s+1)\cdot t(t+1)$ for arbitrary natural numbers $s < t$, then $\Delta_r = m\Delta_s = n\Delta_t$ for natural numbers $m \neq n$ depending on r, s and t. So, in particular, there exist infinitely many triples of natural numbers (r, s, t) satisfying (1), when one allows m and n to vary.

Using similar considerations, it is not hard to find solutions to related sets of equations. For instance, one has

$$\Delta_{15} = 120\Delta_1 = 40\Delta_2 = 20\Delta_3 = 12\Delta_4 = 8\Delta_5.$$

Using the symmetries $r \leftrightarrow -r-1$, etc., as well as the trivial solutions, this gives $2^6 + 2^6 = 128$ integral points on a (smooth) curve given as the intersection of five quadrics (such a curve has genus 49).

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Department of Mathematics, Brigham Young University, Provo, UT 84602-6539
E-mail address: jasbir@math.byu.edu

Vakgroep Wiskunde, RuG, P.O. Box 800, 9700 AV Groningen, The Netherlands
E-mail address: top@math.rug.nl