Model reduction and control of complex systems

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to my parents,
Ana and Javad

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Chapter 1

Introduction

1.1 Introduction

Modeling processes in diverse disciplines such as chemistry, computer science, artificial intelligence, economics, evolutionary computation, molecular biology, neuroscience, physics, and sociology inevitably leads to system models consisting of the interconnection of a large number of subsystems. In this thesis, such system models will be referred to as *complex systems*. Studying complex systems requires investigating how the relationship between its parts gives rise to the collective behavior of the system. Two classes of complex systems considered in this thesis are *switched systems* and *networks* of dynamical agents.

Switched systems are dynamical systems composed of the interaction between a finite number of subsystems and a logical rule orchestrating the switching between these subsystems. Studying this class of systems is motivated by numerous applications in power systems, traffic control, the automotive industry, and control of mechanical systems. In case the individual subsystems of a switched system have linear dynamics, the system is called a *switched linear system* (SLS). Switched linear systems have attracted a lot of attention in the last two decades; see e.g. [7], [8], [15], [19], [48], [90]. A survey of recent results for SLS can be found in [51].

The second class of systems that we will consider in this thesis is networks of dynamical systems, which, depending on the context, are also referred to as *multi-agent systems*. These systems are composed of several agents that interact according to a communication law to fulfill a certain task. The communication law is typically given by a graph called the communication graph. The vertices of the communication graph correspond to the agents, whereas the edges (or arcs) determine the interconnection structure of the network. Consequently, the overall behavior of the network is determined by two factors, namely the agents' dynamics and the underlying communication graph.

Distributed control of networks of dynamical agents has broad applications in diverse areas including cooperative control of unmanned air vehicles (UAVs), formation control, flocking, distributed sensor networks, and many more (see e.g.

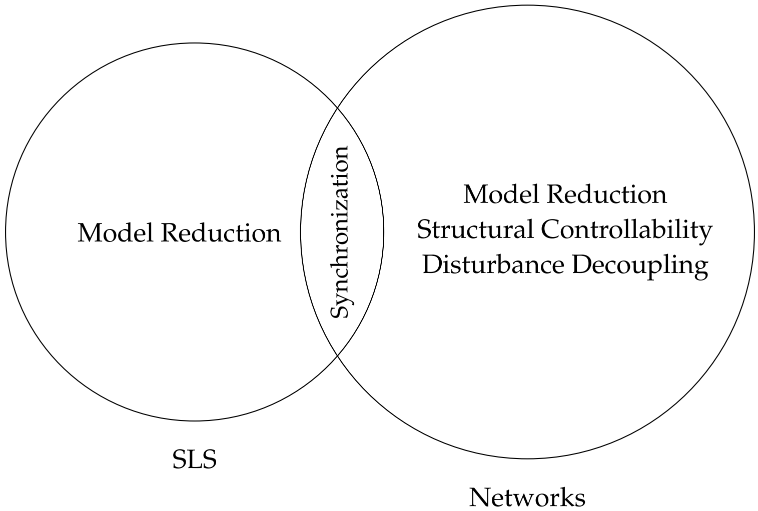

Figure 1.1: *Schematic view of the content of the thesis: topics & system classes.*

[81], [97], [27], [21]). Among numerous problems formulated and studied in the context of networks and multi-agent systems, perhaps the most well known is the *consensus* problem; see e.g. [72], [41], [80]. Consensus roughly means that the agents agree on a certain quantity of interest. A notion closely related to consensus is the so-called *synchronization* problem; see e.g. [43], [58], [88], [101], [2]. Synchronization means that the states of the agents converge to a common trajectory. Designing feedback protocols to achieve consensus/synchronziation has attracted attention of many researchers in the last decade; see e.g. [79], [47], [83], [96], [95].

After having clarified the classes of systems that we will deal with, we will now give a brief introduction to the problems we will study for these classes. An overview is provided schematically in Figure 1.1. As can be seen from this figure, *model reduction* constitutes one of the main themes of this thesis. Given a system of high order dynamics, the model reduction problem amounts to finding simpler lower order models for this system such that the reduced order models approximate relatively well the behavior of the original model. Moreover, it is of interest to preserve certain desired properties of the original model in the reduced order model.

Model reduction is among the classical problems in the field of systems and control. Reduced order models facilitate the analysis, and thus provide a better understanding of the behavior of the underlying system. Numerous model order reduction techniques have been proposed in the last decades, and can be categorized into two broad groups of balanced truncation schemes and moment matching based methods. The most well-known of these methods is Lyapunov balanced truncation first introduced in [67], where the main idea is to transform the system to an equivalent system representation in which the states which are easy (difficult) to reach are also easy (difficult) to observe. Among other types of balancing, we mention stochastic, bounded real, and positive real balancing; see [22], [74], [70]. Adopting Krylov subspace projection methods, used in numerical linear algebra (see e.g. [87]) to obtain reduced order models has led to numerous moment matching model reduction techniques; see e.g. [28], [32], [42].

In this thesis, model reduction techniques are established both for switched linear systems and multi-agent systems. For switched linear systems we establish an extended balanced truncation approach to obtain reduced order models. The idea is to seek for conditions under which there exists a single state space transformation that brings all subsystems (modes) of a given SLS in balanced coordinates. Moreover, we derive sufficient conditions under which global uniform exponential stability of the SLS is preserved in the reduced order model. Likewise, we propose conditions for preservation of positive realness or bounded realness of the SLS.

For networks and multi-agent systems two different approaches for model reduction will be developed in this thesis, namely reducing the dynamic order of the individual agents and reducing the size of the communication graph. In the first approach, bounded real balancing is adopted to reduce certain networks dynamics such that stability or synchronization of the original system is preserved in the reduced order model. In addition, we establish a priori model reduction error bounds to compare the behavior of the original network to that of the reduced order model. It should be noted that, in this technique, the interconnection structure of the network remains unchanged in the order reduction process. In contrast to this first approach, the focus of the second approach is the interconnection structure and, in particular, the size of the underlying communication graph. For this purpose, a model reduction technique is proposed in this thesis which is based on clustering the vertices (agents) of the underlying communication graph by means of suitable graph partitions. An inevitable challenge here is to preserve the spatial structure of the network. This indeed has been taken into account in the proposed model reduction method, and the reduced order models are again realized as multi-agent systems defined on a new graph with a reduced number of vertices. We provide an explicit formula for the H_2-norm of the error system obtained by comparing the input-output behaviors of the original model and the reduced order model for the case that the clusters are chosen using "almost equitable partitions" of the graph.

Synchronization in networks of dynamical agents is another property that will be studied in this thesis. Other than the synchronization preservation which has already been pointed out as an issue in model reduction of networks, we will carry out synchronization analysis in a more focused manner as an individual topic. In particular, we will deal with synchronization analysis for networks with arbitrary switching topologies. We assume that the agents have general, yet identical, linear dynamics, and the underlying communication topology may switch arbitrarily within a finite set of admissible topologies. Then, small gain and passivity types of conditions will be derived that guarantee stability or synchronization of the network. Note that this indeed is the part where the underlying classes of systems we consider in this thesis intersect as depicted in Figure 1.1.

It is true that one can adopt tools from classical control theory for the analysis and design of networks of dynamical agents. However, to facilitate the design and provide a better understanding of the network structure, studying the network properties from a topological perspective has been carried out in many recent works; see e.g. [56], [13], [106], [24]. Motivated by this, in the current dissertation we will carry out (structural) controllability analysis and address the disturbance decoupling problem for dynamical networks by establishing topological conditions. These problems are introduced next.

The notions of weak and strong structural controllability have been introduced and developed in [50] and [55] for ordinary linear systems. Recently, these properties have been investigated for networks of dynamical agents both for the case of weak (see [52]) and strong structural controllability (see [16]). One main advantage of strong structural controllability is its robustness against perturbations in the coupling strengths, which, unlike ordinary controllability analysis, leads to numerically stable methods. In this thesis, with a rather different terminology, we investigate the property of strong structural controllability for systems defined on a graph. The notion of *zero forcing sets* will be adopted here, which is closely related to the minimum rank problems of patterned matrices (see e.g. [4] and [37]). We will show that there is a one-to-one correspondence between the set of leaders rendering the network controllable and zero forcing sets. We will use special subclasses of graphs, including path, cycle, and complete graphs to illustrate the proposed results. This also provides a link between the contribution of our work to that already established in the literature; see e.g. [75], [69], [106].

Finally, the last problem we address in this thesis is disturbance decoupling of dynamical networks. Roughly speaking, a system is disturbance decoupled if its output is independent of disturbance signals affecting the states of the system. In case the system is not disturbance decoupled, one may try to design a feedback controller to render the closed loop system disturbanced decoupled. This design problem is called the disturbance decoupling problem, and is well-known in the context of linear systems and geometric control (see e.g. [94] and [102]). The notion of controlled invariance subspace plays a central role in the disturbance decoupling problem. In fact, the solution of the disturbance decoupling problem is an immediate application of the concept of controlled invariance; see e.g. [5], [103], and [91].

In this thesis, we consider the disturbance decoupling problem for networks of dynamical agents. We are interested in topological conditions that imply the disturbance decoupling of the network, and more generally guarantee the existence of a state feedback rendering the network disturbance decoupled. In particular, we will develop a class of graph partitions, which can be described as a "topological translation" of controlled invariant subspaces in the context of dynamical networks. Then, we will derive sufficient conditions in terms of graph partitions such that the network is disturbance decoupled, as well as conditions guaranteeing solvability of the disturbance decoupling problem.

1.2 Outline of the thesis

This thesis is organized as follows. In Chapter 2, an extended balanced truncation scheme is established for model reduction of switched linear systems. In Chapter 3, we propose a method to reduce the dynamic order of agents in a network, while stability or synchronization is preserved in the reduced order model. A second model reduction technique for networks of dynamical agents is established in Chapter 4, where agents are clustered by means of suitable graph partitions. Methods for stability and synchronization analysis of networks with arbitrary switching topologies are developed in Chapter 5. Structural controllability of networks is linked to zero forcing sets in Chapter 6. Finally, Chapter 7 studies disturbance decoupling problem of networks from a topological prospective.

1.3 Origins of the chapters

Chapter 2 is based on [62] which has appeared as a full paper in IEEE Transactions on Automatic Control. A brief version of this chapter has been presented at the 50^{th} IEEE Conference on Decision and Control and European Control Conference (CDC-ECC) in Orlando, USA [61]. Chapter 3 is essentially the same as [64] which has been published in Systems and Control Letters. Chapter 4 is based on the paper [59], which has been submitted for journal publication. The results provided in Chapter 5 have been partially presented at the 20^{th} International Symposium on Mathematical Theory of Networks and Systems in Melbourne, Australia [63]. In addition, the content of Chapter 5 has been submitted for journal publication [60]. Finally, Chapter 6 and Chapter 7 constitute two papers, [66] and [65], have both been submitted for journal publication.

1.4 Notation

Two types of notation will be used in this thesis, namely "local notation" and "global notation". Local notation refers to notation adopted in one or more chapters, and will be defined explicitly in the corresponding chapters. Global notation refers to notation that is used throughout the whole dissertation, and will be mostly used without an explicit definition. In this section, we recap the most important global notation used in this thesis.

Sets
The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, real numbers, and complex numbers, respectively. By $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$, we denote the spaces of all $n \times m$ matrices with real and complex elements, respectively. For a set S, we denote the cardinality of S by $|S|$.

Vectors and Matrices
For a vector v, we denote its i^{th} element by v_i. For a set of column vectors $w_1, w_2, \ldots, w_k$, we define

$$\operatorname{col}(w_1, w_2, \ldots, w_k) = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$$

For a matrix $A \in \mathbb{C}^{n\times m}$, we use $A^\top$ and A^* to denote the transpose and conjugate transpose of A, respectively. For a set of matrices $A_1, A_2, \ldots, A_N$, we define

$$\operatorname{blockdiag}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix},$$

and we use $\operatorname{diag}(A_1, A_2, \ldots, A_n)$ for the case where $n = m = 1$.

For a symmetric matrix $M \in \mathbb{R}^{n\times n}$, we write $M > 0$ $(M < 0)$ if M is positive (negative) definite. Also we write as $M \geqslant 0$ $(M \leqslant 0)$ if M is positive (negative) semi-definite.

The symbol $\mathbb{1}_n$ denotes the column vector of ones with n elements. The $n \times m$ zero matrix is denoted by $0_{n\times m}$, and I_n denotes the identity matrix with n rows and columns. We drop the indices and write $\mathbb{1}$, 0, and I whenever there is no confusion regarding the dimension.

Norms
The H_2-norm of a strictly proper stable real rational matrix G is denoted by $\|G\|_2$. For a proper stable real rational matrix G, we denote its H_∞-norm by $\|G\|_\infty$.

Mappings
Let $f : \mathcal{U} \to \mathcal{V}$ be a linear mapping from the vector space $\mathcal{U}$ to the vector space V. Then we denote the image of f by $\operatorname{im} f = \{f(u) : u \in \mathcal{U}\}$, and denote the kernel of f by $\ker f = \{u \in \mathcal{U} : f(u) = 0\}$.

Chapter 2

A simultaneous balanced truncation approach to model reduction of switched linear systems

2.1 Introduction

Seeking for simpler descriptions of highly complex or large scale systems has resulted in the development of many different model reduction methods and techniques. Models of lower complexity provide a simpler description, better understanding, and easier analysis of the system. In addition, models of lower complexity make computer simulations more tractable, and simplify controller design. Two important issues in model reduction are to obtain good error bounds, and to preserve important system properties like stability, positive realness, or bounded realness. To achieve this, extensive research, and many approaches are reported in the literature on model reduction of linear time-invariant finite-dimensional systems (for an extensive overview of the literature, see [1]). One of the most well-known techniques is model reduction by balanced truncation, first introduced in [68], and later appearing in the control system literature in [67] and [76]. In this approach, first the system is transformed into a balanced form, and next a reduced order model is obtained by truncation. There are other types of balancing methods available in the literature. Instances of those are stochastic and positive real balancing, proposed in [22], and bounded real balancing, proposed in [74]. In addition, frequency weighted balancing has been developed to approximate the system over a range of frequencies. In [25], [49], [98] and [109], different schemes are proposed for frequency weighted balancing. Another category of model reduction approaches is formed by Krylov based methods, based on moment matching. Among the pioneering work in this direction, we refer to [28], [32], and [42].

Despite the considerable research effort on model reduction for ordinary linear systems, developing methods for model reduction of systems with switching dynamics has only been studied in very few papers up to now (see e.g. [29], [85], [9]).

A switched linear system, typically, involves switching between a number of linear systems, called the *modes* of the switched linear system. Hence, to apply balanced truncation techniques to a switched linear system, we may seek for a basis of the (common) state space such that the corresponding modes are all in balanced form. A natural question that arises here is under what conditions such a basis exist. In this chapter, necessary and sufficient conditions are derived for the existence of such basis. The results obtained are not limited to a certain type of balancing, but are applicable to different types such as Lyapunov, bounded real, and positive real balancing.

It may happen that some state components of the switched linear system are difficult to reach and observe in some of the modes yet easy to reach and observe in others. In that case, deciding how to truncate the state variables and obtain a reduced order model is not trivial. A solution to this problem is proposed in this chapter. By averaging the diagonal gramians of the individual modes in the balanced coordinates, a new diagonal matrix is obtained. This *average balanced gramian* can be used to obtain a reduced order model. In the case of Lyapunov balancing, the average balanced gramian assigns an overall degree of reachability and observability to each state component. In this way, one can decide which state components should be discarded in order to obtain a reduced order switched linear system.

Another interesting issue is, in case that the original switched linear system is stable, how to ensure that the reduced order switched linear system retains this stability. It is well-known that the existence of a common quadratic Lyapunov function (CQLF) is a sufficient condition for global uniform exponential stability of the switched linear system, see [48]. In this chapter we will establish conditions under which the reduced order switched linear system inherits a CQLF from the original switched linear system, thus preserving global uniform exponential stability.

Similarly, one may be interested in preserving positive realness or bounded realness of the original switched linear system in the reduced order model after applying simultaneous positive real or bounded real balanced truncation. Conditions for preserving these properties are proposed in this chapter. The proposed conditions are similar to those obtained for stability preserving yet with a different interpretation.

It turns out that, the higher the number of modes of the original switched linear system, the more restrictive the proposed conditions are. Therefore, in addition to simultaneous balanced truncation, we develop a more general balanced truncation-like scheme as well. Independent of the number of modes of the original switched linear system, preserving desired properties like stability, positive realness or bounded realness is then guaranteed upon the satisfaction of a single condition.

This chapter is organized as follows. In Section 2.2, some preliminaries and basic material needed in this chapter are discussed. Section 2.3 is devoted to characterizing *all* balancing transformations for a single linear system. It is shown how to obtain all balancing transformations for a given pair of gramians, starting from a single transformation that diagonalizes the product of these gramians. In Section 2.4, we solve the problem of simultaneous balancing, and give necessary and sufficient conditions for the existence of a single balancing transformation for multiple pairs of gramians. In Section 2.5 we apply our results on simultaneous balancing to model reduction by balanced truncation of switched linear systems. We also address the issues of preservation of stability, positive realness, and bounded realness. Since the conditions for the existence of simultaneous balancing transformations can be rather restrictive, in Section 2.6 we propose a truncation method that is generally applicable, and, importantly, reduces to simultaneous balanced truncation if possible. Section 2.7 is devoted to a numerical example to illustrate the methods introduced in this chapter. The chapter closes with conclusions in Section 2.8.

2.2 Preliminaries

In this section we will review some basic material on balancing and on simultaneous diagonalization. Consider the finite dimensional, linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. Assume that the system is internally stable, i.e., the matrix A has all its eigenvalues in the open left half plane. Also assume (A, B) is controllable and (C, A) is observable. We shortly denote this system by $\mathcal{H} = (A, B, C, D)$. Basically, balancing the system $\mathcal{H}$ means to find a non-singular state space transformation T that diagonalizes appropriately chosen positive definite matrices P and Q in a covariant and contravariant manner, respectively. This means that P transforms to $TPT^\top$ and Q transforms to $T^{-\top}QT^{-1}$, and the transformed matrices should be diagonal, or diagonal and equal. Important special cases of balancing are classical Lyapunov balancing, bounded real (BR) balancing, and positive real (PR) balancing. In Lyapunov balancing the matrices P and Q are the reachability and observability gramian associated with the system $\mathcal{H}$, while in BR and PR balancing P and Q are the minimal real symmetric solutions of a pair of algebraic Riccati equations, see Table 2.1. Also, one could do balancing based on positive definite solutions of Lyapunov inequalities, instead of equations (see [23]), or use alternative solutions, not necessarily minimal, of the Riccati equations in Table 2.1.

In this chapter, we will study the concept of balancing without referring to a particular underlying system. Instead, we will start off with a given pair of positive definite matrices, and study how to find a suitable transformation for this pair. Thus, let P, Q be real symmetric positive definite $n \times n$ matrices. Then, the concepts of essentially-balancing and balancing transformations are defined as follows.

Type	Equations
Lyapunov	$AP + PA^\top + BB^\top = 0$ $A^\top Q + QA + C^\top C = 0$
Bounded Real	$AP + PA^\top + BB^\top + (PC^\top + BD^\top)(I - DD^\top)^{-1}(PC^\top + BD^\top)^\top = 0$ $A^\top Q + QA + C^\top C + (QB + C^\top D)(I - D^\top D)^{-1}(QB + C^\top D)^\top = 0$
Positive Real	$AP + PA^\top + (PC^\top - B)(D + D^\top)^{-1}(PC^\top - B)^\top = 0$ $A^\top Q + QA + (QB - C^\top)(D^\top + D)^{-1}(QB - C^\top)^\top = 0$

Table 2.1: *Common types of balancing*

Definition 2.2.1 Let $T \in \mathbb{R}^{n\times n}$ be nonsingular. We call T an *essentially-balancing* transformation for (P, Q) if $TPT^\top$ and $T^{-\top}QT^{-1}$ are diagonal. In this case, we say T *essentially balances* (P, Q). We call T a *balancing transformation* for (P, Q) if $TPT^\top = T^{-\top}QT^{-1} = \Sigma$, where Σ is a diagonal matrix. In this case, we say T *balances* (P, Q).

Note that essentially balancing transformations have been also referred to as *contragredient* transformations in the literature (see [45]). It is well-known, see for example [110], that for any pair of real symmetric positive definite matrices (P, Q) there exists a balancing transformation. It is clear that the diagonal elements of the matrix Σ in Definition 2.2.1 coincide with the square roots of the eigenvalues of PQ. In the case of Lyapunov balancing, the diagonal elements of the corresponding Σ are the nonzero Hankel singular values (HSV) of the system. Similarly, in the case of bounded real and positive real balancing, the diagonal elements of Σ are the nonzero bounded real and positive real characteristic values, respectively (see [74], [70], [77], [17], [36]). Next, we will address the issue of simultaneous diagonalization of matrices.

Definition 2.2.2 Let $M \in \mathbb{C}^{n\times n}$ be diagonalizable. Then the nonsingular matrix $V \in \mathbb{C}^{n\times n}$ is called a *diagonalizing transformation* for M if VMV^{-1} is diagonal. In this case, we say V *diagonalizes* M. Two diagonalizable matrices $X, Y \in \mathbb{C}^{n\times n}$ are said to be *simultaneously diagonalizable* if there exists a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that VXV^{-1} and VYV^{-1} are both diagonal.

A necessary and sufficient condition for simultaneous diagonalizability of two given matrices is well-known, and is stated in the following lemma [38]:

Lemma 2.2.3 *Let $X, Y \in \mathbb{C}^{n\times n}$ be diagonalizable matrices. Then X and Y are simultaneously diagonalizable if and only if they commute, i.e. $XY = YX$.*

The generalization of the above result to the case of three or more matrices is straightforward. In fact, a finite set of diagonalizable matrices is simultaneously diagonalizable if and only if each pair in the set commutes.

2.3 Balancing transformations for a pair of positive definite matrices

In this section we will show how we can obtain *all* balancing transformations for a given pair of real symmetric positive definite matrices (P, Q) by using a single diagonalizing transformation of the product PQ. In the first part of this section we will deal with the (generic) case that the eigenvalues of the product PQ are all distinct. In the second part, we will extend our results to the general case of repeated eigenvalues of PQ.

2.3.1 The distinct eigenvalue case

Let P and Q be real symmetric positive definite matrices and assume that the eigenvalues of PQ are all distinct. Let $\hat{T}$ be a balancing transformation for the pair (P, Q), and denote the corresponding diagonal matrix Σ by $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i \neq \sigma_j$ $(i \neq j)$. We note that the diagonal elements σ_i are not necessarily ordered in a decreasing manner. Clearly, $\hat{T}$ satisfies

$$\hat{T}PQ\hat{T}^{-1} = \Sigma^2.$$

and hence the columns of $\hat{T}^{-1}$ are eigenvectors of PQ corresponding to the distinct eigenvalues σ_i^2, $i = 1, 2, ..., n$. Since the eigenvalues of PQ are real, there exists $V \in \mathbb{R}^{n\times n}$ that diagonalizes PQ. Now, it is easy to observe that $\hat{T}$ can be written as $\hat{T} = I_P D^{-1} V$ where $D \in \mathbb{R}^{n\times n}$ is a nonsingular diagonal matrix and I_P is a permutation matrix. Clearly, the transformation T given by $T = D^{-1}V$ also balances (P, Q). Therefore, we can always obtain a balancing transformation for (P, Q) by scaling a diagonalizing transformation of PQ.

The following lemma states that there is a one-to-one correspondence between diagonalizing transformations and essentially-balancing transformations.

Lemma 2.3.1 *Let P and Q be real symmetric positive definite matrices. Assume that the eigenvalues of PQ are all distinct. Then, the matrix $V \in \mathbb{R}^{n\times n}$ is an essentially-balancing transformation for (P, Q) if and only if it is a diagonalizing transformation for PQ.*

Proof. First, assume that V is an essentially-balancing transformation for (P,Q). Then, by definition, $VPV^\top$ and $V^{-\top}QV^{-1}$ are diagonal. Consequently, the product $VPV^\top V^{-\top}QV^{-1} = VPQV^{-1}$ is diagonal. Hence, V is a diagonalizing transformation for PQ.

Conversely, suppose V is a diagonalizing transformation for PQ. Then, by using the distinct eigenvalue assumption, it follows from the introduction to this subsection that there exists a nonsingular diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $T = D^{-1}V$ balances (P,Q). Hence, by Definition 2.2.1, we have

$$D^{-1}VPV^\top D^{-1} = \Sigma = DV^{-\top}QV^{-1}D.$$

Consequently $VPV^\top$ and $V^{-\top}QV^{-1}$ are diagonal matrices, and V is, therefore, an essentially-balancing transformation for (P,Q). ■

Based on the previous discussion and Lemma 2.3.1, the following theorem characterizes balancing transformations for a given pair of positive definite matrices (P,Q).

Theorem 2.3.2 *Let P and Q be real symmetric positive definite matrices. Assume the eigenvalues of PQ are all distinct. Let $V \in \mathbb{R}^{n\times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2. Then T is a balancing transformation for (P,Q) with corresponding diagonal matrix Σ if and only if $T = D^{-1}V$ where $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix satisfying one, and, hence, all of the following equivalent equalities:*

$$D^4 = (VPV^\top)(VQ^{-1}V^\top), \tag{2.2}$$

$$D^2 = (VPV^\top)\Sigma^{-1}, \tag{2.3}$$

$$D^2 = \Sigma(VQ^{-1}V^\top). \tag{2.4}$$

Proof. First, we show that (2.2), (2.3), and (2.4) are equivalent. Clearly, we have

$$VPV^\top V^{-\top}QV^{-1} = \Sigma^2. \tag{2.5}$$

By Lemma 2.3.1, V is an essentially-balancing transformation for (P,Q); therefore, $VPV^\top$ and $V^{-\top}QV^{-1}$ are diagonal. Hence, (2.5) yields $(VPV^\top)\Sigma^{-1} = \Sigma VQ^{-1}V^\top$. Consequently, (2.3), and (2.4) are equivalent. In addition, their product results in (2.2). So, it remains to show that (2.2) implies (2.3) or (2.4). By (2.5), (2.2) can be rewritten as $D^4 = \Sigma^2(VQ^{-1}V^\top)^2$. Hence, $D^2 = \Sigma VQ^{-1}V^\top$. The last implication is due to the fact that a positive definite matrix has a unique positive definite square root.
Now, let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix satisfying (2.2). By (2.2), we have

$$D^{-2}VPV^\top = D^2V^{-\top}QV^{-1}. \tag{2.6}$$

Since $VPV^\top$ and $V^{-\top}QV^{-1}$ are diagonal, (2.6) can be written as

$$D^{-1}VPV^\top D^{-1} = DV^{-\top}QV^{-1}D. \tag{2.7}$$

Hence, $T = D^{-1}V$ is a balancing transformation for (P, Q). Clearly, the corresponding diagonal matrix is Σ.

To prove the converse, let T be a balancing transformation for (P, Q) with corresponding diagonal matrix Σ. Clearly, the columns of T^{-1} and V^{-1} are eigenvectors of PQ. Hence, T can be written as $T = D^{-1}V$ for some nonsingular diagonal matrix $D \in \mathbb{R}^{n\times n}$. By Definition 2.2.1, we have

$$D^{-1}VPV^\top D^{-1} = DV^{-\top}QV^{-1}D. \tag{2.8}$$

By Lemma 2.3.1, $VPV^\top$ and $V^{-\top}QV^{-1}$ are diagonal. Therefore, by changing the order of multiplications in (2.8) we obtain

$$D^{-2}VPV^\top = D^2V^{-\top}QV^{-1} \tag{2.9}$$

which simplifies to (2.2). ■

The above theorem provides a straightforward method for computing balancing transformations. Given real symmetric positive definite matrices P and Q, we compute, firstly, a diagonalizing transformation for PQ. Obviously, this transformation can be obtained directly from the eigenvectors of PQ. Then, as stated in the theorem, a balancing transformation can be obtained by scaling the diagonalizing transformation, and the scaling matrix can be taken as any nonsingular real diagonal matrix satisfying (2.2), (2.3), or (2.4). Note that computation of D is simple and merely requires the multiplication of two diagonal matrices.

2.3.2 The case of possibly repeated eigenvalues

We will now deal with the general case that PQ may have repeated eigenvalues. The following lemma is crucial for the proof of the subsequent results.

Lemma 2.3.3 *Let V be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2. Let T be a balancing transformation for (P, Q) with corresponding diagonal matrix Σ. Then, Σ commutes with VT^{-1}, $VPV^\top$, and $VQ^{-1}V^\top$.*

Proof. Clearly we have $TPQT^{-1} = \Sigma^2$ and $VPQV^{-1} = \Sigma^2$. Hence, $VT^{-1}\Sigma^2 = \Sigma^2VT^{-1}$. Therefore, VT^{-1} commutes with Σ^2. Thus, VT^{-1} and Σ also commute. In addition, we have

$$TPT^\top = \Sigma = T^{-\top}QT^{-1}.$$

Hence, $TV^{-1}VPV^\top V^{-\top}T^\top = \Sigma$. Thus, $VPV^\top = VT^{-1}\Sigma(VT^{-1})^\top$. Since Σ and VT^{-1} commute, we obtain $\Sigma^{-1}VPV^\top = VT^{-1}(VT^{-1})^\top$. Hence, $\Sigma^{-1}VPV^\top$ is symmetric, and Σ commutes with $VPV^\top$. In a a similar fashion, we obtain $\Sigma VQ^{-1}V^\top = VT^{-1}(VT^{-1})^\top$. Consequently, Σ and $VQ^{-1}V^\top$ commute. ■

The following theorem now provides an extension to the repeated eigenvalue case of Theorem 2.3.2:

Theorem 2.3.4 *Let P and Q be real symmetric positive definite matrices. Let $V \in \mathbb{R}^{n\times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2. Then T is a balancing transformation for (P, Q) with corresponding diagonal matrix Σ if and only if $T = \Delta^{-1}V$ where $\Delta \in \mathbb{R}^{n\times n}$ is a nonsingular matrix which commutes with Σ, and satisfies one, and, hence, all of the following equivalent equalities:*

$$(\Delta\Delta^\top)^2 = (VPV^\top)(VQ^{-1}V^\top) \tag{2.10}$$

$$\Delta\Delta^\top = (VPV^\top)\Sigma^{-1} \tag{2.11}$$

$$\Delta\Delta^\top = \Sigma(VQ^{-1}V^\top) \tag{2.12}$$

Proof. First, we show that (2.10), (2.11), and (2.12) are equivalent. Clearly, we have

$$VPV^\top V^{-\top}QV^{-1} = \Sigma^2. \tag{2.13}$$

By Lemma 2.3.3, we obtain $(VPV^\top)\Sigma^{-1} = \Sigma VQ^{-1}V^\top$. Hence, (2.11), and (2.12) are equivalent. In addition, the product of (2.11) and (2.12) results in (2.10). So, it remains to show (2.10) implies (2.11) or (2.12). By (2.13), we can rewrite (2.10) as $(\Delta\Delta^\top)^2 = \Sigma^2(VQ^{-1}V^\top)^2$. Hence, we obtain $\Delta\Delta^\top = \Sigma VQ^{-1}V^\top$. Note that $\Delta\Delta^\top$ and $\Sigma VQ^{-1}V^\top$ are positive definite matrices.

Now, assume T is a balancing transformation for (P, Q). Clearly, we can write T as $T = \Delta^{-1}V$ for a nonsingular matrix Δ. Hence,

$$\Delta^{-1}VPV^\top\Delta^{-\top} = \Sigma$$

which simplifies to (2.11) by Lemma 2.3.3. Moreover, Δ and Σ commute by Lemma 2.3.3.

To prove the converse, assume that the nonsingular matrix Δ satisfies $\Sigma\Delta = \Delta\Sigma$, (2.11), and, equivalently, (2.12). Clearly, $\Delta^\top$, Δ^{-1}, and $\Delta^{-\top}$ also commute with Σ. Hence, we have

$$\begin{aligned}\Delta\Sigma\Delta^\top &= VPV^\top\\ \Delta^{-\top}\Sigma\Delta^{-1} &= V^{-\top}QV^{-1}\end{aligned}$$

Now, it is easy to observe that the transformation T given by $T = \Delta^{-1}V$, balances (P, Q) with corresponding diagonal matrix Σ. ■

The above theorem is valid for any choice of diagonalizing transformation V, where repeated eigenvalues do not necessarily have to be grouped together on the diagonal of Σ. We will apply the statement of this theorem in this form in our subsequent results on simultaneous balancing. If, however, we would restrict ourselves to diagonalizing transformations V that do group together repreated eigenvalues on the diagonal of Σ, we obtain the following more specific result:

Corollary 2.3.5 *Let P and Q be real symmetric positive definite matrices. Let $V \in \mathbb{R}^{n\times n}$ be a diagonalizing transformation for PQ with corresponding diagonal matrix Σ^2 such that*

$$\Sigma = \begin{pmatrix} \sigma_1 I_{m_1} & 0 & \dots & 0 \\ 0 & \sigma_2 I_{m_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N I_{m_N} \end{pmatrix}, \tag{2.14}$$

with $\sigma_i \neq \sigma_j$ for $i \neq j$, and where m_i is the algebraic multiplicity of σ_i, and I_{m_i} is the identity matrix of size m_i, $i = 1, 2, ..., N$. Then T is a balancing transformation for (P, Q) with corresponding diagonal matrix Σ if and only if $T = \Delta^{-1}V$ where $\Delta \in \mathbb{R}^{n\times n}$ is a nonsingular block-diagonal matrix with N square blocks of sizes $m_1, m_2, ..., m_N$, respectively, and Δ satisfies one, and, hence, all of the equivalent equalities (2.10), (2.11), *and* (2.12).

Proof. Obviously, an $n \times n$ matrix Δ commutes with the diagonal matrix Σ in (2.14) if and only if Δ is blockdiagonal with diagonal blocks of size $m_i \times m_i$. Therefore, the result of Theorem 2.3.4 simplifies to Corollary 2.3.5. ■

2.4 Simultaneous balancing for multiple pairs of positive definite matrices

In this section, we will study the question under what conditions multiple pairs of real symmetric positive definite matrices can be simultaneously balanced by one and the same transformation. We will start off by considering the problem for two pairs of positive definite matrices (P_1, Q_1) and (P_2, Q_2).

Before stating and proving the main result of this section, we give the following instrumental lemma:

Lemma 2.4.1 *Let $M \in \mathbb{R}^{n\times n}$ be a real symmetric positive definite matrix. Let $LL^\top$ be a Cholesky decomposition of M, that is, $M = LL^\top$ where $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with positive diagonal entries. Let Σ be a real diagonal matrix. Then, M commutes with Σ if and only if L commutes with Σ.*

Proof. The proof of the 'if' part is trivial. To prove the 'only if' part, assume that M commutes with Σ. Clearly, we have $LL^\top\Sigma = \Sigma LL^\top$. Hence,

$$L^\top \Sigma L^{-\top} = L^{-1}\Sigma L. \tag{2.15}$$

Now, since the left hand side of (2.15) is upper triangular and the right hand side is lower triangular, they should be equal to a diagonal matrix. Clearly, this diagonal matrix must be equal to Σ. Hence, we have $L^{-1}\Sigma L = \Sigma$ which yields $\Sigma L = L\Sigma$. ■

The following theorem gives necessary and sufficient conditions for the existence of a transformation T that simultaneously balances (P_1, Q_1) and (P_2, Q_2).

Theorem 2.4.2 *Let (P_1, Q_1) and (P_2, Q_2) be pairs of real symmetric positive definite matrices. There exists a transformation T that balances both (P_1, Q_1) and (P_2, Q_2) if and only if the following two conditions hold:*

i) P_1Q_1 *and* P_2Q_2 *commute,*

ii) $P_1Q_2 = P_2Q_1$.

Proof. Assume T is a balancing transformation for both (P_1, Q_1) and (P_2, Q_2). Clearly, T is a diagonalizing transformation for both P_1Q_1 and P_2Q_2. Hence, by Lemma 2.2.3, P_1Q_1 and P_2Q_2 commute. In addition, by Definition 2.2.1, we have

$$\begin{aligned} TP_1T^\top &= T^{-\top}Q_1T^{-1} \\ TP_2T^\top &= T^{-\top}Q_2T^{-1}. \end{aligned} \tag{2.16}$$

Consequently, $TP_1Q_2T^{-1} = TP_2Q_1T^{-1}$ which yields $P_1Q_2 = P_2Q_1$.

Conversely, assume that the conditions *i)* and *ii)* hold. Since P_1Q_1 and P_2Q_2 commute, there exists a nonsingular matrix V that simultaneously diagonalizes P_1Q_1 and P_2Q_2 with corresponding diagonal matrices Σ_1^2 and Σ_2^2, respectively. Clearly, we have

$$VP_1Q_1P_2Q_2V^{-1} = \Sigma_1^2\Sigma_2^2. \tag{2.17}$$

By condition *ii)*, we have $Q_2P_1 = Q_1P_2$. Hence, (2.17) can be rewritten as

$$V(P_1Q_2)^2V^{-1} = \Sigma_1^2\Sigma_2^2.$$

Consequently, we have $(P_1Q_2)^2 = V^{-1}(\Sigma_1\Sigma_2)^2V$. Now, since P_1Q_2 has only positive eigenvalues, we obtain $P_1Q_2 = V^{-1}\Sigma_1\Sigma_2V$. Therefore,

$$VP_1V^\top V^{-\top}Q_2V^{-1} = \Sigma_1\Sigma_2$$

which, by Lemma 2.3.3, yields

$$VP_1V^\top\Sigma_1^{-1} = \Sigma_2VQ_2^{-1}V^\top. \tag{2.18}$$

Since (2.18) is a real symmetric positive definite matrix, it admits a Cholesky decomposition

$$VP_1V^\top\Sigma_1^{-1} = \Sigma_2VQ_2^{-1}V^\top = \Delta\Delta^\top,$$

(where Δ is a lower triangular matrix with positive diagonal entries). Now, again by Lemma 2.3.3, the identical terms in (2.18) commute with both Σ_1 and Σ_2. Hence, by Lemma 2.4.1, Δ commutes with both Σ_1 and Σ_2. Thus, by Theorem 2.3.4, $T = \Delta^{-1}V$ simultaneously balances (P_1, Q_1) and (P_2, Q_2). ■

For the sake of simplicity, the above result and conditions have been stated for two pairs of positive definite matrices (P_1, Q_1) and (P_2, Q_2). The generalization of the results to k pairs of positive definite matrices, with $k \geqslant 2$ is straightforward, and is stated in the following corollary.

Corollary 2.4.3 *Let (P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k) be k pairs of real symmetric positive definite matrices. There exists a transformation T that simultaneously balances (P_1, Q_1), (P_2, Q_2),...,(P_k, Q_k) if and only if the following conditions hold:*

i) P_iQ_i and P_jQ_j commute for all $i, j = 1, 2, ..., k$.

ii) $P_iQ_j = P_jQ_i$ for all $i, j = 1, 2, ..., k$.

It is worth to mention that in the particular case where for each i the eigenvalues of P_iQ_i are all distinct, the first condition of Corollary 2.4.3 by itself already implies the existence of a simultaneous essentially-balancing transformation for the pairs (P_i, Q_i), $i = 1, 2, ..., k$. This follows immediately from Lemma 2.3.1.

2.5 Model reduction of switched linear systems by simultaneous balanced truncation

2.5.1 Balanced truncation of switched linear systems

We will now apply our previous results to model reduction by balanced truncation of switched linear systems. As indicated before, in the context of model reduction of linear systems, the matrices P and Q can be taken to be the solutions of the Lyapunov equations, or the minimal solutions of the bounded real or positive real Riccati equations (see Table 2.1). Consequently, the results stated in the previous sections cover different types of balancing of a single linear system, and, moreover, indicate the possibility of simultaneous balancing for multiple linear systems. Thus, simultaneous balancing, if possible, provides a straightforward approach to model reduction of certain hybrid systems. Except from having the same state space dimension, no assumption is needed regarding the relation of the individual modes of the hybrid system.

In this section, we will first treat the P and Q matrices as reachability and observability gramians, thus applying our results to Lyapunov balanced truncation of switched linear systems.

A typical switched linear system (SLS) is described by (see [48]):

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x + D_\sigma u, \end{aligned} \tag{2.19}$$

where σ is a piecewise constant function of time, t, taking its value from the index set $K = \{1, 2, ..., k\}$, and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{p\times m}$ for all $i \in K$. Let $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$ denote the i^{th} mode of the given SLS. Assume that $\mathcal{H}_i$ is internally stable, controllable, and observable for every i. Let P_i and Q_i be the reachability and observability gramians of $\mathcal{H}_i$, respectively. Now, if the conditions of Corollary 2.4.3 hold for the pairs of gramians (P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k), then there exists a single state space transformation $T \in \mathbb{R}^{n\times n}$ that simultaneously balances all k modes of the given SLS. Consequently, by applying T to the individual modes of (2.19) and truncating, reduced order modes can be obtained.

As mentioned in the introduction, it may occur that some states are relatively difficult to reach and observe in some of the modes, yet easy to reach and observe in others. In order to measure the degree of reachability and observability of each of the state components of the overall SLS, we propose to take the *average over all modes* of the corresponding Hankel singular values. More precisely:

Definition 2.5.1 Assume that simultaneous balancing of the modes of the SLS (2.19) is possible, and let Σ_1, Σ_2, ..., Σ_k be the gramians of the modes of (2.19) after simultaneous balancing. Then we define the *average balanced gramian*, denoted by Σ_{av}, as

$$\Sigma_{av} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + ... + \Sigma_k). \tag{2.20}$$

Note that the average balanced gramian is unique up to permutation of its diagonal elements. Now, the diagonal matrix Σ_{av} indicates which states of the SLS are important and which are negligible, and can be used to obtain a reduced order SLS. In fact, the i^{th} diagonal element of Σ_{av} assigns an overall degree of reachability and observability to the i^{th} state component of the balanced representation. Suppose that $\bar{\sigma}_1 > \bar{\sigma}_2 > \ldots > \bar{\sigma}_N$ are the *distinct* diagonal elements of the average balanced gramian, where $\bar{\sigma}_i$ appears m_i times, $\sum_i m_i = n$. Let $1 \leqslant l \leqslant N$ be an integer. By discarding the state components corresponding to the $N-l$ smallest distinct diagonal elements of Σ_{av} we can reduce the order by $\sum_{i=l+1}^{N} m_i$ and obtain a reduced order SLS with state space dimension $r = \sum_{i=1}^{l} m_i$. This leads to a reduced order SLS

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_\sigma x + \bar{B}_\sigma u, \\ \bar{y} &= \bar{C}_\sigma x + \bar{D}_\sigma u, \end{aligned} \tag{2.21}$$

with modes $\bar{\mathcal{H}}_i = (\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i)$ ($i = 1, 2 \ldots k$), where $\bar{A}_i \in \mathbb{R}^{r\times r}$, $\bar{B}_i \in \mathbb{R}^{r\times m}$, $\bar{C}_i \in \mathbb{R}^{p\times r}$ and $\bar{D}_i = D_i$.

In the particular case that for each i the product P_iQ_i has distinct eigenvalues, equivalently mode $\mathcal{H}_i$ has distinct Hankel singular values, it is a well-known fact that the individual truncated modes $\bar{\mathcal{H}}_i$ preserve stability of the original modes $\mathcal{H}_i$ for all $i = 1, 2, ..., k$. In the case of repeated Hankel singular values, internal stability of the truncated individual modes is only guaranteed if, for each mode, all state components corresponding to a repeated, to be discarded, HSV are truncated in that mode. Preservation of internal stability of the individual modes is in fact not the kind of stability preservation that we are looking for in SLS. Instead, in Subsection 2.5.2 we will address the issue of preservation of *global uniform exponential stability*, of the SLS.

Remark 2.5.2 If the individual modes of the given SLS are not of equal importance, or if some information on the switching signal is available, the overall balanced gramian can be defined as a *weighted* average of the gramians of the individual modes in balanced coordinates. In this way one can take into account the importance of the different modes by adjusting the weighting coefficients.

Of course, our results on simultaneous balancing can also be applied to positive real and bounded real balancing, see [74], [17], [36]. The single linear system (2.1) with $p = m$ is positive real if and only if the linear matrix inequality (LMI)

$$\begin{pmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -D - D^\top \end{pmatrix} \leqslant 0 \tag{2.22}$$

has a real symmetric positive definite solution K. It is well known that in that case the LMI has extremal real symmetric solutions $K_{\min}$ and $K_{\max}$, and $0 < K_{\min} \leqslant K_{\max}$. In the context of PR balancing we take $Q = K_{\min}$ and $P = K_{\max}^{-1}$, and (2.1) is called

PR balanced if $P = Q$ is diagonal. If $D + D^\top$ is invertible, then P and Q coincide with the minimal solutions of the PR Riccati equations in Table 2.1.

For bounded real balancing the same holds. The system (2.1) is bounded real if the LMI

$$\begin{pmatrix} A^\top K + KA + C^\top C & KB + C^\top D \\ B^\top K + D^\top & -I + D^\top D \end{pmatrix} \leqslant 0 \tag{2.23}$$

has a real symmetric positive definite solution K. In that case the LMI has extremal real symmetric solutions $K_{\min}$ and $K_{\max}$ satisfying $0 < K_{\min} \leqslant K_{\max}$. Again we take $Q = K_{\min}$ and $P = K_{\max}^{-1}$, and (2.1) is called BR balanced if $P = Q$ is diagonal. If $I - D^\top D$ is invertible, then P and Q coincide with the minimal solutions of the BR Riccati equations in Table 2.1.

Now, again consider the SLS (2.19) with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$. Similar as in the case of Lyapunov balancing, both in the PR and the BR case we denote the diagonal matrices obtained after simultaneously (PR or BR) balancing the pairs (P_i, Q_i) corresponding to the modes $\mathcal{H}_i$ by Σ_i, and define the average balanced gramian by (2.20).

In subsection 2.5.3 we will review the concepts of positive realness and bounded realness of switched linear systems, and study the preservation of these properties under simultaneous PR and BR balanced truncation of the individual modes of the SLS.

2.5.2 Preservation of stability

As already observed in the previous subsection, suitable simultaneous balanced truncation of the individual modes of a switched linear system yields a reduced order switched linear system whose individual modes are internally stable. Of course, this does not mean that the SLS itself is stable. In the present subsection we will obtain conditions under which simultaneous balanced truncation preserves the stability of the SLS.

The concept of stability that we will use here is that of *global uniform exponential stability*. We call the SLS given by (2.19) globally uniformly exponentially stable if there exist positive constants α and β such that the solution $x(t)$ of $\dot{x} = A_\sigma x$ for any initial state $x(0)$ and any switching signal σ satisfies $\|x(t)\| \leqslant \beta e^{-\alpha t}\|x(0)\|$ for all $t \geqslant 0$ (see [48]). A sufficient condition for global uniform exponential stability of an SLS is that the state matrices of the individual modes share a common quadratic Lyapunov function (CQLF) (see [48]). That is, there exists a real symmetric positive definite matrix X such that $A_i^T X + XA_i < 0$ for all $i = 1, 2, ..., k$. Assuming that the state matrices of the modes of the given SLS enjoy this property, we seek for conditions under which this property is preserved in the reduced order SLS. This leads us to the following theorem.

Theorem 2.5.3 *Consider the SLS* (2.19) *with modes* $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$*,* $i = 1, 2, \ldots, k$*. Assume that there exists* $X > 0$ *such that* $A_i^T X + XA_i < 0$ *for all* $i = 1, 2, ..., k$*. Let* P_i *and* Q_i *be the reachability and observability gramians, respectively, of the* i^{th} *mode* $\mathcal{H}_i$*. Suppose that the following conditions hold:*

i) P_iQ_i *and* P_jQ_j *commute for all* $i, j = 1, 2, ..., k$*.*

ii) $P_iQ_j = P_jQ_i$ *for all* $i, j = 1, 2, ..., k$*.*

iii) $XP_iQ_i = Q_iP_iX$ *for all* $i = 1, 2, ..., k$*.*

Then there exists a state space transformation that simultaneously balances all modes $\mathcal{H}_i$ *for* $i = 1, 2, ..., k$*. Moreover, let* $\bar{\sigma}_1 > \bar{\sigma}_2 > \ldots > \bar{\sigma}_N$ *be the* distinct *diagonal elements of the average balanced gramian, where* $\bar{\sigma}_i$ *appears* m_i *times. Then, for each positive integer* $1 \leqslant l \leqslant N$*, the truncated SLS of order* $r = \sum_{i=1}^{l} m_i$ *given by* (2.21) *is globally uniformly exponentially stable.*

Proof. By Corollary 2.4.3, simultaneous balancing is possible upon satisfaction of conditions *i)* and *ii)*. By condition *iii)* we have

$$X^{\frac{1}{2}} P_i Q_i X^{-\frac{1}{2}} = X^{-\frac{1}{2}} Q_i P_i X^{\frac{1}{2}}.$$

Hence, $X^{\frac{1}{2}} P_i Q_i X^{-\frac{1}{2}}$ is symmetric. In addition, condition *i)* implies that $X^{\frac{1}{2}} P_i Q_i X^{-\frac{1}{2}}$ and $X^{\frac{1}{2}} P_j Q_j X^{-\frac{1}{2}}$ commute for all $i, j = 1, 2, ..., k$. Therefore, there exists an orthogonal matrix U which diagonalizes $X^{\frac{1}{2}} P_i Q_i X^{-\frac{1}{2}}$, for all $i = 1, 2, ..., k$ (see [38], p. 103). Hence, $UX^{\frac{1}{2}}$ is a diagonalizing transformation for P_iQ_i, $i = 1, 2, ..., k$. Consequently, based on the proof of Theorem 2.5.3, a simultaneous balancing transformation T can be obtained as $T = \Delta^{-1} U X^{\frac{1}{2}}$ for some nonsingular real matrix Δ. Without loss of generality, we assume that the corresponding diagonal matrix Σ_{av} given by (2.20) is in form of (2.14). Otherwise, we can multiply T by a permutation matrix from the left to achieve so. Applying T to the individual modes of the given SLS, the state matrices in the new coordinates are given by

$$\tilde{A}_i = \Delta^{-1} U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^\top \Delta \tag{2.24}$$

By our assumption regarding the CQLF, we have $A_i^\top X + XA_i < 0$ for all $i = 1, 2, ..., k$. Hence, for all i we have

$$\Delta^\top U X^{-\frac{1}{2}} (A_i^\top X + XA_i) X^{-\frac{1}{2}} U^\top \Delta < 0,$$

which yields

$$\Delta^\top U X^{-\frac{1}{2}} A_i^\top X^{\frac{1}{2}} U^\top \Delta + \Delta^\top U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^\top \Delta < 0.$$

This can be rewritten as

$$\Delta^\top U X^{-\frac{1}{2}} A_i^\top X^{\frac{1}{2}} U^\top \Delta^{-\top} \Delta^\top \Delta + \Delta^\top \Delta \Delta^{-1} U X^{\frac{1}{2}} A_i X^{-\frac{1}{2}} U^\top \Delta < 0,$$

which, by (2.24), simplifies to

$$\tilde{A}_i^\top \Delta^\top \Delta + \Delta^\top \Delta \tilde{A}_i < 0. \tag{2.25}$$

Since $T = \Delta^{-1} V$ is a simultaneous balancing transformation, by Lemma 2.3.3 Δ commutes with Σ_i for all $i = 1, 2, ..., k$. Hence, Δ commutes with Σ_{av}. Therefore, $\Delta^\top$ and $\Delta^\top \Delta$ commute with Σ_{av}. Since Σ_{av} is in the form of (2.14), $\Delta^\top \Delta$ will have a block-diagonal structure compatible with the multiplicities m_i of the diagonal elements of Σ_{av}. Clearly, any principal submatrix of the positive definite matrix $\Delta^\top \Delta$ is positive definite. Hence, the state matrices of the reduced order SLS (2.21) share a CQLF. Consequently, the r^{th} order reduced SLS model is globally uniformly exponentially stable. ■

Note that in the above theorem, the first two conditions in Theorem 2.5.3 implies the possibility of simultaneous balancing whereas the third condition guarantees the existence of a CQLF for the reduced order model.

2.5.3 Preservation of passivity and contractivity

It is well known that PR or BR balanced truncation of a single linear system preserves positive realness or bounded realness. As was the case with stability, this does not mean that positive realness or bounded realness of a SLS is preserved under simultaneous balanced truncation. In the present subsection we will address this issue.

We first give a definition of dissipativity for switched linear systems. Consider the SLS (2.19), and let $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ be a given function. This function is called the *supply rate*.

Definition 2.5.4 We call the SLS (2.19) dissipative with respect to supply rate $s(u, y)$ if there exists a non-negative function $V : \mathbb{R}^n \to \mathbb{R}$ such that the dissipation inequality

$$V(x(t_1)) - V(x(t_0)) \leqslant \int_{t_0}^{t_1} s(u(t), y(t))\, dt \tag{2.26}$$

holds for all $t_0 \leqslant t_1$, for all switching signals σ and trajectories (x, u, y) which satisfy the system equations. Any non-negative function V that satisfies (2.5.4) is called a *storage function*.

Then, we call the SLS (2.19) positive real if $p = m$ and if it is dissipative with respect to the supply rate $s = u^\top y$. Similarly, we call the SLS (2.19) bounded real if it is dissipative with respect to the supply rate $s = u^\top u - y^\top y$. It is easy to observe that if the individual modes of the SLS possess a common quadratic storage function (CQSF) for a given supply rate s, then the overall SLS is dissipative with respect to s. By applying this fact to the special cases of positive realness and bounded realness, we find that SLS (2.19) is positive real if there exists a real symmetric positive definite matrix K such that the following LMI's hold for all $i = 1, 2, ..., k$:

$$\begin{pmatrix} A_i^\top K + KA & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \leqslant 0 \tag{2.27}$$

Similarly, the SLS (2.19) is bounded real if there exists a real symmetric positive definite matrix K such that

$$\begin{pmatrix} A_i^\top K + KA_i + C_i^\top C_i & KB_i + C_i^\top D_i \\ B_i^\top K + D_i^\top C_i & -I + D_i^\top D_i \end{pmatrix} \leqslant 0 \tag{2.28}$$

holds for all $i = 1, 2, ..., k$. In both cases, (2.27) and (2.28), we denote by $K_{i,\min}$ and $K_{i,\max}$ the extremal real symmetric solutions to the i^{th} LMI.

Assuming the original SLS is PR or BR, it is desirable to preserve these properties in the reduced order SLS. The following theorem deals with this issue:

Theorem 2.5.5 *Consider the SLS* (2.19) *with modes $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \ldots, k$. Assume there exists $K > 0$ such that the LMI* (2.27) *(the LMI* (2.28)*) holds for all $i = 1, 2, ..., k$. Define $Q_i := K_{i,\min}$ and $P_i := K_{i,\max}^{-1}$. Suppose that the following conditions hold:*

i) P_iQ_i *and* P_jQ_j *commute for all* $i, j = 1, 2, ..., k$.

ii) $P_iQ_j = P_jQ_i$ *for all* $i, j = 1, 2, ..., k$.

iii) $KP_iQ_i = Q_iP_iK$ *for all* $i = 1, 2, ..., k$.

Then there exists a state space transformation that simultaneously PR balances (BR balances) all modes $\mathcal{H}_i$ for $i = 1, 2, ..., k$. Moreover, let $\bar{\sigma}_1 > \bar{\sigma}_2 > \ldots > \bar{\sigma}_N$ be the distinct *diagonal elements of the average balanced gramian Σ_{av}, where $\bar{\sigma}_i$ appears m_i times. Then, for each positive integer $1 \leqslant l \leqslant N$, the truncated SLS of order $r = \sum_{i=1}^{l} m_i$ given by* (2.21) *is positive real (bounded real).*

Proof. We only give the proof for the PR case, the BR case is similar. By Corollary 2.4.3, simultaneous balancing is possible if the first two conditions hold. Using the third condition, and following a similar argument as in the proof of Theorem 2.5.3, a simultaneous balancing transformation is given by $T = \Delta^{-1}UK^{\frac{1}{2}}$ for some nonsingular real matrix Δ and an orthogonal matrix U. Again, without loss of generality, we assume that the corresponding diagonal matrix Σ_{av} , given by (2.20), is in the form (2.14). Applying T to the individual modes of the given SLS, the state space representations of the modes in the new (balanced) coordinates are given by

$$\tilde{\mathcal{H}}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$$

where

$$\begin{aligned} \tilde{A}_i &= \Delta^{-1}UK^{\frac{1}{2}}A_iK^{-\frac{1}{2}}U^\top\Delta \\ \tilde{B}_i &= \Delta^{-1}UK^{\frac{1}{2}}B_i \\ \tilde{C}_i &= C_iK^{-\frac{1}{2}}U^\top\Delta,\ \tilde{D}_i = D_i \end{aligned} \tag{2.29}$$

for $i = 1, 2, ..., k$. By assumption, there exists a single positive definite matrix K such that for all $i = 1, 2, ..., k$

$$\begin{pmatrix} A_i^\top K + KA_i & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \leqslant 0. \tag{2.30}$$

Hence,

$$\begin{pmatrix} \Delta^\top UK^{-\frac{1}{2}} & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A_i^\top K + KA_i & KB_i - C_i^\top \\ B_i^\top K - C_i & -D_i - D_i^\top \end{pmatrix} \begin{pmatrix} K^{-\frac{1}{2}}U^\top\Delta & 0 \\ 0 & I_m \end{pmatrix} \leqslant 0 \tag{2.31}$$

for all $i = 1, 2, ..., k$.
Using (2.29), the inequality (2.31) simplifies to

$$\begin{pmatrix} \tilde{A}_i^\top \Delta^\top\Delta + \Delta^\top\Delta A_i & \Delta^\top\Delta\tilde{B}_i - \tilde{C}_i^\top \\ \tilde{B}_i^\top\Delta^\top\Delta - \tilde{C}_i & -\tilde{D}_i - \tilde{D}_i^\top \end{pmatrix} \leqslant 0. \tag{2.32}$$

Let $\bar{A}_i \in \mathbb{R}^{r\times r}$, $\bar{B}_i \in \mathbb{R}^{r\times m}$, $\bar{C}_i \in \mathbb{R}^{m\times r}$, $\bar{D}_i \in \mathbb{R}^{m\times m}$, and $\bar{\Delta} \in \mathbb{R}^{r\times r}$ be the matrices obtained by truncating $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$, $\tilde{D}_i$, and Δ respectively. Since this truncation is done based on Σ_{av}, $\Delta^\top\Delta$ has a block diagonal structure with respect to the multiplicities of the diagonal elements of Σ_{av} (see the proof of Theorem 2.5.3). Hence, the modes of the reduced order SLS, $\bar{\mathcal{H}}_i = (\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i)$, satisfy

$$\begin{pmatrix} \bar{A}_i^\top \bar{\Delta}^\top\bar{\Delta} + \bar{\Delta}^\top\bar{\Delta} A_i & \bar{\Delta}^\top\bar{\Delta}\bar{B}_i - \bar{C}_i^\top \\ \bar{B}_i^\top\bar{\Delta}^\top\bar{\Delta} - \bar{C}_i & -\bar{D}_i - \bar{D}_i^\top \end{pmatrix} \leqslant 0 \tag{2.33}$$

for all $i = 1, 2, ..., k$, due to the fact that (2.33) is a principal submatrix of (2.32). Therefore, the r^{th} order reduced SLS model is positive real. ■

Remark 2.5.6 For each i, let P_i and Q_i be defined as in Theorems 2.5.3 (Theorem 2.5.5). It can be proven that if there exists at least one $\ell \in \{1, 2, \dots, k\}$ such that $P_\ell Q_\ell$ has distinct eigenvalues, then the k constraints involved in the third condition of Theorem 2.5.3 (Theorem 2.5.5) can be replaced by the single constraint $XP_\ell Q_\ell = Q_\ell P_\ell X$ ($KP_\ell Q_\ell = Q_\ell P_\ell K$). In fact, this new condition guarantees that the gramian X (the gramian K) will be diagonal in the balanced coordinates, which suffices to ensure preservation of GUES (positive/bounded realness).

2.6 Model reduction of SLS by minimizing an overall cost function

As discussed in the previous section, if certain conditions are satisfied, the technique of simultaneous balanced truncation can be applied to switched linear systems (see Theorem 2.5.3 and 2.5.5). Although conceptually attractive as a model reduction method, in general the conditions are rather restrictive. In many cases, like electrical circuits with certain switching topologies, the state matrices of the different modes are not completely arbitrary, and satisfy certain conditions to comply with Kirchhoff's laws. This may decrease the restrictiveness of the obtained conditions in practice.

On the other hand, the number of conditions stated in Theorem 2.5.3 depends on the number of modes of the original SLS. Clearly, increasing the number of modes makes it harder for all conditions to be satisfied. This motivates us to propose a more general model reduction approach for the case where simultaneous balancing cannot be achieved.

It is well-known that the problem of finding a balancing transformation for a single linear system can be given a variational interpretation, and can be formulated as finding a nonsingular matrix T such that the following cost function is minimized (see [1]):

$$f(T) = \mathrm{trace}[TPT^\top + T^{-\top}QT^{-1}] \tag{2.34}$$

In a SLS with k modes, thus, we deal with minimizing the following k cost functions with respect to T.

$$f_i(T) = \mathrm{trace}[TP_iT^\top + T^{-\top}Q_iT^{-1}] \tag{2.35}$$

Clearly, if and only if the conditions of Corollary 2.4.3 hold, simultaneous balancing is possible, and there exists a transformation T which simultaneously minimizes f_i for all $i = 1, 2, ..., k$. Otherwise, simultaneous balancing is not possible, and we should seek for a single transformation that makes the above k cost functions simultaneously as small as possible. For this, we propose to introduce a single overall cost function. Since the essence of "trace" is summation, a natural choice for this overall cost function is the sum or, equivalently, the average of the cost functions of the individual modes. Hence, we define an overall cost function f_{av} as

$$f_{av}(\tilde{T}) = \frac{1}{k}\sum_{i=1}^{k} \text{trace}[\tilde{T}P_i\tilde{T}^\top + \tilde{T}^{-\top}Q_i\tilde{T}^{-1}] \tag{2.36}$$

It is interesting to note that in the case of balancing for a single linear system, we seek for a basis so that the sum of the eigenvalues of the positive definite matrices P and Q takes its minimum value (see (2.34)), while in the case of balancing for SLS we seek for a basis in which the sum of the sum of the eigenvalues of P_i and Q_i over all modes is minimal (see (2.36)). Hence, minimizing the proposed overall cost function provides a natural extension of classical balancing to the case of SLS.

The cost function (2.36) can be restated as

$$f_{av}(\tilde{T}) = \text{trace}[\tilde{T}P_{av}\tilde{T}^\top + \tilde{T}^{-\top}Q_{av}\tilde{T}^{-1}] \tag{2.37}$$

where

$$P_{av} = \frac{1}{k}\sum_{i=1}^{k} P_i \text{ and } Q_{av} = \frac{1}{k}\sum_{i=1}^{k} Q_i. \tag{2.38}$$

Therefore, the transformation $\tilde{T}$ which minimizes the proposed overall cost function is exactly the one which balances the pair (P_{av}, Q_{av}) of average gramians. Consequently, $\tilde{T}$ can be conveniently computed by the use of Corollary 2.3.5 with respect to (P_{av}, Q_{av}).

By applying $\tilde{T}$ to the individual modes and truncating, a reduced order model can be obtained. Clearly, after the transformation $\tilde{T}$, the new state space descriptions of the individual modes are not necessarily balanced, but are expected to be relatively close to being balanced. It is of course desirable that in case where simultaneous balancing *is* possible, minimizing the proposed cost function yields a transformation that simultaneously balances all modes. This important issue is addressed in the following Proposition:

Proposition 2.6.1 *Let $(P_1, Q_1), (P_2, Q_2), ..., (P_k, Q_k)$ be k pairs of positive definite matrices. Assume that there exists a simultaneous balancing transformation for these k pairs. Assume that the product $P_{av}Q_{av}$ of the average gramians* (2.38) *has all distinct eigenvalues. Then any balancing transformation $\tilde{T}$ for (P_{av}, Q_{av}), with corresponding diagonal matrix $\tilde{\Sigma}$, simultaneously balances (P_i, Q_i) for all $i = 1, 2, ..., k$. Moreover,*

$$\tilde{\Sigma} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + ... + \Sigma_k) \tag{2.39}$$

where Σ_i is the corresponding diagonal matrix obtained after applying $\tilde{T}$ to (P_i, Q_i).

Proof. Let T be a simultaneously balancing transformation for (P_i, Q_i), $i = 1, 2, ..., k$. Clearly, we have

$$TP_iT^\top = T^{-\top}Q_iT^{-1} = \Gamma_i \tag{2.40}$$

for certain diagonal matrices Γ_i, $i = 1, 2, ..., k$. Hence,

$$TP_{av}T^\top = T^{-\top}Q_{av}T^{-1} = \frac{1}{k}(\Gamma_1 + \Gamma_2 + ... + \Gamma_k).$$

Consequently, the transformation T balances (P_{av}, Q_{av}). Obviously, T is also a diagonalizing transformation for $P_{av}Q_{av}$. Thus, by Theorem 2.3.2, it is easily verified that $\tilde{T}$ can be written as $\tilde{T} = I_PST$ for some permutation matrix I_P and a sign matrix S (note that, in fact, the diagonal matrix D in equation (2.2) of Theorem 2.3.2 satisfies $D^4 = I$; hence, D is obtained as a sign matrix in this case). Now, multiplying (2.40) from the left by I_PS and from the right by $SI_P^\top$, we obtain

$$\tilde{T}P_i\tilde{T}^\top = \tilde{T}^{-\top}Q_i\tilde{T}^{-1} = \Sigma_i \tag{2.41}$$

where $\Sigma_i = I_P\Gamma_iI_P^\top$, $i = 1, 2, ..., k$. Therefore $\tilde{T}$ simultaneously balances (P_i, Q_i) for all $i = 1, 2, ..., k$. By (2.41), we obtain

$$\tilde{T}P_{av}\tilde{T}^\top = \tilde{T}^{-\top}Q_{av}\tilde{T}^{-1} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + ... + \Sigma_k).$$

Therefore, $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma} = \frac{1}{k}(\Sigma_1 + \Sigma_2 + ... + \Sigma_k).$$

■

By Proposition 2.6.1, balancing and truncation based on the average gramians contains the simultaneous balancing problem as a special case. In fact, in case that simultaneous balancing is possible, the transformation which balances (P_{av}, Q_{av}) also balances (P_i, Q_i) for all $i = 1, 2, ..., k$. Moreover in that case, the diagonal matrix $\tilde{\Sigma}$ obtained by balancing (P_{av}, Q_{av}) is equal to the average balanced gramian Σ_{av} defined in (2.20).

The state space transformation $\tilde{T}$ obtained in this way can be used for model reduction of switched linear systems. After applying the transformation $\tilde{T}$, the new state space descriptions of the individual modes are not necessarily balanced, but are, in a sense, relatively close to being balanced. Then, the truncation decision is carried out on the basis of the eigenvalues of the product $P_{av}Q_{av}$, and a reduced order SLS model is obtained. Of course, the question remains whether this method preserves the properties of the original SLS in the reduced order model. Indeed, the following theorem gives a sufficient condition for preserving the property of global uniform exponential stability:

Theorem 2.6.2 *Consider the switched linear system* (2.19) *with modes* $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \ldots, k$. *Assume that there exists* $X > 0$ *such that* $A_i^\top X + XA_i < 0$ *for all* i. *Let* $\tilde{T}$ *be a balancing transformation for* (P_{av}, Q_{av}) *with corresponding diagonal matrix* $\tilde{\Sigma}$ *of the form* (2.14). *Then for each positive integer* $1 \leqslant l \leqslant N$ *the truncated SLS of order* $r = \sum_{i=1}^{l} m_i$ *given by* (2.21) *is globally uniformly exponentially stable if*

$$XP_{av}Q_{av} = Q_{av}P_{av}X. \tag{2.42}$$

Proof. The proof is analogous to that of Theorem 2.5.3. By (2.42) we have

$$X^{\frac{1}{2}}P_{av}Q_{av}X^{-\frac{1}{2}} = X^{-\frac{1}{2}}Q_{av}P_{av}X^{\frac{1}{2}}.$$

Hence, $X^{\frac{1}{2}}P_{av}Q_{av}X^{-\frac{1}{2}}$ is a symmetric matrix. Thus, there exists an orthogonal matrix U such that $UX^{\frac{1}{2}}P_{av}Q_{av}X^{-\frac{1}{2}}U^\top = \tilde{\Sigma}^2$. Since $UX^{\frac{1}{2}}$ is a diagonalizing transformation for $P_{av}Q_{av}$, by Theorem 2.3.4 $\tilde{T}$ can be written as $\tilde{T} = \Delta^{-1}UX^{\frac{1}{2}}$ for some nonsingular real matrix Δ. Applying $\tilde{T}$ to the individual subsystems of the given SLS, the state matrices in the new coordinates are given by

$$\tilde{A}_i = \Delta^{-1}UX^{\frac{1}{2}}A_iX^{-\frac{1}{2}}U^\top\Delta. \tag{2.43}$$

By a similar argument as in the proof of Theorem 2.5.3, we obtain

$$\tilde{A}_i^\top\Delta^\top\Delta + \Delta^\top\Delta\tilde{A}_i < 0.$$

Hence the r^{th} order reduced order SLS is globally uniformly exponentially stable. ■

According to Theorem 2.6.2, regardless of the number of modes in the original SLS, the reduced order SLS model preserves globally uniform exponentially stability upon a satisfaction of the single condition (2.42).

The analogue of the result of Theorem 2.6.2 holds in the case of positive real and bounded real balancing. We state the result without proof:

Corollary 2.6.3 *Consider the SLS* (2.19) *with modes* $\mathcal{H}_i = (A_i, B_i, C_i, D_i)$, $i = 1, 2, \ldots, k$. *Assume that there exists* $K > 0$ *such that the LMI* (2.27) *(the LMI* (2.28)*) holds for all* i. *Define* $Q_i := K_{i,\min}$ *and* $P_i := K_{i,\max}^{-1}$. *Let* $\tilde{T}$ *be a balancing transformation for* (P_{av}, Q_{av}) *with corresponding diagonal matrix* $\tilde{\Sigma}$ *of the form* (2.14). *Then for each positive integer* $1 \leqslant l \leqslant N$ *the truncated SLS of order* $r = \sum_{i=1}^{l} m_i$ *given by* (2.21) *is positive real (bounded real) if*

$$KP_{av}Q_{av} = Q_{av}P_{av}K. \tag{2.44}$$

Remark 2.6.4 Instead of taking the average cost function (2.36), we can, more generally take a *weighted* average

$$f_{wav}(\tilde{T}) = \sum_{i=0}^{k} \alpha_i f_i(\tilde{T}) \tag{2.45}$$

where the α_i's are scalars satisfying $0 \leqslant \alpha_i \leqslant 1$, and $\sum_{i=1}^{k} \alpha_i = 1$. Clearly, this simplifies to

$$f_{wav}(\tilde{T}) = \text{trace}[\tilde{T}\tilde{P}\tilde{T}^{\top} + \tilde{T}^{-\top}\tilde{Q}\tilde{T}^{-1}] \tag{2.46}$$

where $P_{wav} = \displaystyle\sum_{i=1}^{k} \alpha_i P_i$ and $Q_{wav} = \displaystyle\sum_{i=1}^{k} \alpha_i Q_i$. Hence, in this case, a minimizing $\tilde{T}$ is a balancing transformation for (P_{wav}, Q_{wav}) where P_{wav} and Q_{wav} are corresponding weighted averages of the gramians. For example, in case that the individual modes are not of equal importance, or some information regarding the switching signal is available, an appropriate overall cost function can be chosen by manipulating the values of α_i's. In addition it might be possible to tune the parameters α_i in such a way that the condition (2.42) is satisfied.

2.7 Numerical example

In this section we will apply the results of this chapter to a concrete example. We will first work out an example of a switched linear system with two modes in which simultaneous balancing is possible. Next, we will modify the SLS so that it no longer allows simultaneous balancing. Then we will apply the method of Section 2.6 and transform the modes of the system using a balancing transformation for the average gramians.

Consider the bimodal SLS

$$\dot{x} = A_\sigma x + B_\sigma u, \;\; y = C_\sigma x, \tag{2.47}$$

where σ is piecewise constant, taking its values in $\{1, 2\}$, and where the state equations of the modes are given by

$$A_1 = \begin{pmatrix} -2.3333 & -3.6667 & -2.0000 \\ 3.3667 & -7.1167 & -7.8500 \\ 0.8778 & -4.1278 & -5.5500 \end{pmatrix}, \; A_2 = A_1 + \gamma I$$

$$B_1 = \begin{pmatrix} 4.1391 & -2.5590 & 1.2327 \\ 2.6502 & 3.8428 & 1.2327 \\ 1.7454 & -0.4251 & 1.2327 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 4.8423 & 1.1084 & 1.5569 \\ 2.8071 & -4.5756 & 1.5569 \\ 2.5056 & -0.7863 & 1.5569 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1.2280 & 0.6693 & -1.3123 \\ -1.0617 & 1.0617 & 1.0461 \\ -0.7121 & 0.7121 & 2.1362 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0.8747 & 0.7836 & -0.3377 \\ 1.3770 & -1.3770 & -2.2192 \\ -0.8354 & 0.8354 & 2.5062 \end{pmatrix}$$

where $\gamma \in \mathbb{R}$. We now distinguish between two cases.

Case 1: $\gamma = 0.75$
Our aim is to derive a reduced order model for the given SLS. Let P_1, P_2 and Q_1, Q_2 denote the reachability and observability gramians of the first and second mode, respectively. We compute

$$P_1 = \begin{pmatrix} 4.4001 & -0.4000 & 1.9000 \\ -0.4000 & 2.0000 & -0.5000 \\ 1.9000 & -0.5000 & 1.1000 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 5.4001 & -0.0001 & 2.5000 \\ -0.0001 & 2.6999 & -0.1999 \\ 2.5000 & -0.1999 & 1.6000 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 0.5222 & 0.0777 & -0.7000 \\ 0.0777 & 0.2222 & -0.2000 \\ -0.7000 & -0.2000 & 1.1999 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0.6666 & 0.0667 & -0.9667 \\ 0.0667 & 0.3000 & -0.1333 \\ -0.9667 & -0.1333 & 1.8000 \end{pmatrix}.$$

Computation shows that the following conditions hold:

$$(P_1Q_1)(P_2Q_2) = (P_2Q_2)(P_1Q_1) \tag{2.48}$$

$$P_1Q_2 = P_2Q_1 \tag{2.49}$$

Therefore, by Theorem 2.4.2, there exists a transformation T that simultaneously balances (P_1, Q_1) and (P_2, Q_2). By computing eigenvectors, a diagonalizing transformation V for P_1Q_1 and P_2Q_2 is computed as

$$V = \begin{pmatrix} 0.5774 & -0.5774 & -1.7321 \\ -0.9428 & -0.4714 & 1.4143 \\ 0.8164 & -0.8166 & 0.0002 \end{pmatrix},$$

Consequently, a simultaneous balancing transformation T can be obtained as $T = D^{-1}V$ where $D = \mathrm{diag}(-1.7321, -1.4143, 2.4494)$ is obtained by solving the equation (2.2). The corresponding diagonal matrices obtained by applying the simultaneous balancing transformation T to (P_1, Q_1) and (P_2, Q_2) are obtained as $\Sigma_1 = \mathrm{diag}(0.3000, 0.9000, 0.8000)$ and $\Sigma_2 = \mathrm{diag}(0.7000, 1.1000, 0.8999)$, respectively. Hence, the average balanced gramian is computed as $\Sigma_{av} = \mathrm{diag}(0.5000, 1.000, 0.8500)$ by (2.20). Note that the smallest diagonal element 0.5000 corresponds to the first state component x_1 of the transformed SLS.

Computation shows that the state matrices A_1 and A_2 commute, and they, therefore, share a common quadratic Lyapunov function (see [48]). Hence, the given SLS is globally uniformly exponentially stable. Our intention is to compute a simultaneously balanced truncation of the SLS that preserves stability. For this, we should seek for a CQLF for A_1 and A_2 such that the constraints of Theorem 2.5.3 are satisfied. It can be verified computationally that the positive definite matrix X given by

$$X = \begin{pmatrix} 0.1279 & 0.0388 & -0.1800 \\ 0.0388 & 0.0446 & -0.0700 \\ -0.1800 & -0.0700 & 0.2900 \end{pmatrix}$$

satisfies the constraints:

$$\begin{aligned} A_1^\top X + XA_1 &< 0, \\ A_2^\top X + XA_2 &< 0, \\ XP_1Q_1 &= Q_1P_1X, \end{aligned}$$

Note that we do not need to check the constraint $XP_2Q_2 = Q_2P_2X$ in view of Remark 2.5.6. Therefore, by Theorem 2.5.3, the reduced SLS model obtained by applying the state space transformation T to the individual modes and discarding the first state component x_1 will be globally uniformly exponentially stable. The resulting second order SLS is computed as

$$\dot{\bar{x}} = \bar{A}_\sigma x + \bar{B}_\sigma u, \ \bar{y} = \bar{C}_\sigma x, \tag{2.50}$$

where

$$\bar{A}_1 = \begin{pmatrix} -2.0001 & -0.3334 \\ -0.7501 & -3.0000 \end{pmatrix}, \ \bar{A}_2 = \bar{A}_1 + 0.75I,$$

$$\bar{B}_1 = \begin{pmatrix} 1.8974 & 0.0002 & 0.0000 \\ 0.4961 & -2.1340 & -0.0000 \end{pmatrix}$$

$$\bar{B}_2 = \begin{pmatrix} 1.6583 & -0.0002 & -0.0000 \\ 0.6782 & 1.8947 & -0.0000 \end{pmatrix}$$

$$\bar{C}_1 = \begin{pmatrix} 1.8974 & 0.4742 \\ 0.0002 & -2.1390 \\ 0.0000 & -0.0001 \end{pmatrix}$$

$$\bar{C}_2 = \begin{pmatrix} 1.6584 & 0.6279 \\ -0.0002 & 1.9118 \\ 0.0000 & -0.0000 \end{pmatrix}.$$

It can be computed that $\Lambda = \mathrm{diag}(0.25, 0.11)$ satisfies $\bar{A}_i^\top \Lambda + \Lambda \bar{A}_i < 0$ for $i = 1, 2$. Hence, as expected, global uniform exponential stability is preserved in the reduced order SLS.

A step input is applied to the first and second input channel of the original and reduced order SLS model for two different switching signals taking values 1 or 2, randomly. For simplicity, switching between the two subsystems takes place at integer instances of time. The corresponding outputs of the original SLS and those of the reduced order SLS model are sketched in Figure 2.1.

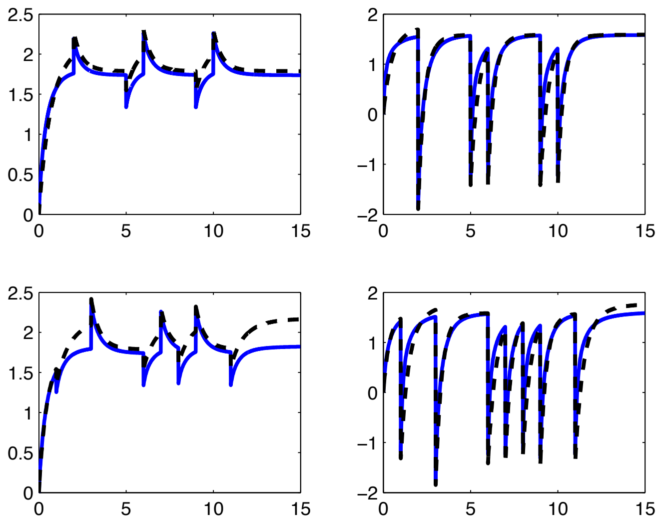

Figure 2.1: *Method of simultaneous balanced truncation: step responses of the original SLS (solid line) and reduced order SLS (dotted line) from first input to first output, and second input to second output for switching signal* $\sigma_1 = \{2, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1\}$ *(above) and switching signal* $\sigma_2 = \{1, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 2, 2, 2\}$ *(below)*

Case 2: $\gamma = -1$

After computing the reachability and observability gramians $\hat{P}_1$, $\hat{P}_2$ and $\hat{Q}_1$, $\hat{Q}_2$ of the first and second mode, it can be be checked that $\hat{P}_1\hat{Q}_1$ and $\hat{P}_2\hat{Q}_2$ do not commute and, hence, a simultaneous balancing transformation does not exist. Therefore, we apply the approach developed in Section 2.6.

The average reachability and observability gramians P_{av} and Q_{av} are computed to be

$$P_{av} = \begin{pmatrix} 3.6658 & -0.0201 & 1.5647 \\ -0.0201 & 1.8619 & -0.0691 \\ 1.5647 & -0.0691 & 0.8741 \end{pmatrix}$$

$$Q_{av} = \begin{pmatrix} 0.4263 & 0.0122 & -0.5772 \\ 0.0122 & 0.1874 & -0.0321 \\ -0.5772 & -0.0321 & 1.0200 \end{pmatrix}.$$

A balancing transformation for (P_{av}, Q_{av}), denoted by $\tilde{T}$, is computed as

$$\tilde{T} = \begin{pmatrix} 0.7175 & 0.0698 & -0.7567 \\ 0.0139 & -0.5467 & 0.1572 \\ -0.4081 & 0.1169 & 1.2491 \end{pmatrix}.$$

The corresponding diagonal matrix $\tilde{\Sigma}$ is obtained as $\tilde{\Sigma} = \mathrm{diag}(0.7029, 0.5979, 0.3863)$. Note that the state component corresponding to the smallest diagonal element is x_3.

As noted earlier, A_1 and A_2 commute and, therefore, share a CQLF, $x^\top X x$. To guarantee preservation of global uniform exponential stability in the reduced order model, we should look for a positive definite matrix X which satisfies the following constraints:

$$\begin{aligned} A_1^\top X + X A_1 &< 0, \\ A_2^\top X + X A_2 &< 0, \\ X P_{av} Q_{av} &= Q_{av} P_{av} X. \end{aligned}$$

It is easy to verify that $X = P_{av}^{-1}$ satisfies the above constraints and, hence, based on Theorem 2.6.2, the reduced order SLS model obtained by applying the state space transformation $\tilde{T}$ to the individual modes and discarding the third state component x_3 is globally uniform exponentially stable. The corresponding second order SLS model is computed as

$$\dot{\bar{x}} = \bar{A}_\sigma x + \bar{B}_\sigma u, \quad \bar{y} = \bar{C}_\sigma x, \tag{2.51}$$

where

$$\bar{A}_1 = \begin{pmatrix} -2.8846 & -0.3463 \\ -0.6022 & -5.2693 \end{pmatrix}, \quad \bar{A}_2 = \bar{A}_1 - I,$$

$$\bar{B}_1 = \begin{pmatrix} 1.8337 & -1.2462 & 0.0376 \\ -1.1172 & -2.2033 & -0.4631 \end{pmatrix}$$

$$\bar{B}_2 = \begin{pmatrix} 1.7739 & 1.0710 & 0.0475 \\ -1.0738 & 2.3934 & -0.5849 \end{pmatrix}$$

$$\bar{C}_1 = \begin{pmatrix} 1.8318 & -0.9521 \\ -1.2495 & -2.0295 \\ 0.0932 & -0.8888 \end{pmatrix}$$

and

$$\bar{C}_2 = \begin{pmatrix} 1.7657 & -1.0098 \\ 1.0607 & 2.3482 \\ 0.1095 & -1.0426 \end{pmatrix}.$$

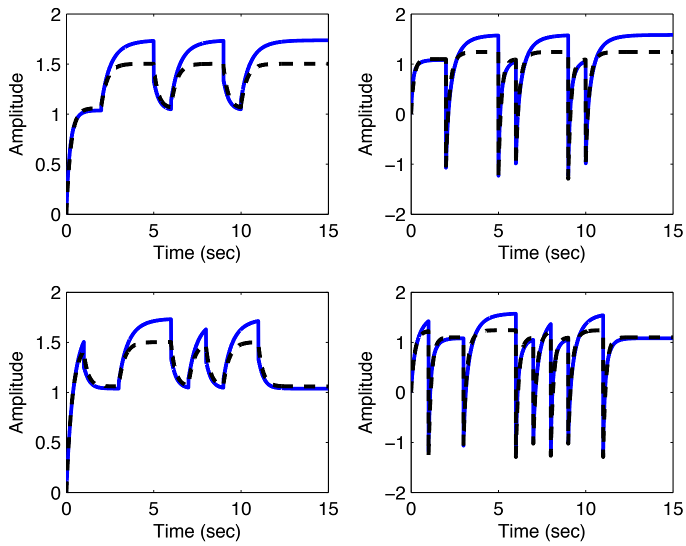

Figure 2.2: *Method of minimizing the overall cost function: step responses of the original SLS (solid line) and reduced order SLS (dotted line) from first input to first output, and second input to second output for switching signal* $\sigma_1 = \{2, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1\}$ *(above) and switching signal* $\sigma_2 = \{1, 2, 2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 2, 2, 2\}$ *(below)*

A step input is applied to the first and second input channel of the original and reduced order SLS model for two different switching signals taking values 1 or 2, randomly. For simplicity, switching between the two subsystems takes place at integer instances of time. The corresponding outputs of the original SLS and those of the reduced order SLS model are sketched in Figure 2.2.

2.8 Conclusions

In this chapter, a generalization of the balanced truncation scheme is investigated for model reduction of switched linear systems. Characterizations of all balancing transformations for a single linear system are given. Clearly, making multiple linear systems balanced in general is not possible with a single state space transformation. Hence, necessary and sufficient conditions for simultaneous balancing of multiple linear systems are derived. These conditions do not depend on the particular type of balancing, and are in terms of commutativity of products of the gramians. The results obtained are applied to balanced truncation of switched linear systems. We then address the issue of preservation of stability under simultaneous balanced truncation of switched linear systems. Starting from the assumption that the original SLS has a common quadratic Lyapunov function, we establish conditions under which global uniform exponential stability of the SLS is preserved after simultaneous balanced truncation. In a similar way, the proposed conditions, with a different interpretation, are adopted for positive real and bounded real balancing. It is shown that positive realness and bounded realness of the SLS are preserved in the reduced order SLS if these conditions are satisfied.

To overcome the restrictiveness of the derived conditions, a more general balanced truncation scheme for SLS is developed based on minimizing an overall cost function. This more general approach involves balancing the average gramians rather than simultaneously balancing all the gramians corresponding to the individual modes, and, hence, the required conditions to guarantee stability, positive realness, or bounded realness are less restrictive. In case that a simultaneous balancing transformation does exist, our more general scheme reduces to the simultaneous balanced truncation scheme studied before in this chapter. The proposed methods are illustrated by means of an extended numerical example.

Chapter 3

Stability and synchronization preserving model reduction of multi-agent systems

3.1 Introduction

The problems of consensus, coordination, and synchronization of multi-agent systems have received compelling attention in the last decade. A multi-agent system is a collection of systems (agents) that interact to fulfil a certain task. The behavior of a multi-agent system, hence, is determined by both the dynamics of the agents and the communication topology of the network which specifies admissible communication among the agents.

An important issue in the context of multi-agent system is consensus. Consensus roughly means that the agents agree on a certain quantity of interest. The focus of pioneering works in this direction has been on communication constraints concerning connectivity, time-varying topologies and time delays (see e.g. [27], [41], [73], [79]). Consequently, the dynamics of the individual agents has been somewhat ignored as the focus has been mainly on the case of simple or double integrators.

In the last few years, however, attention has shifted to analysis and design of multi-agent systems with general linear dynamics. One of the most popular frameworks that have emerged in this direction studies multi-agent systems which are composed of several copies of interacting identical linear input-output subsystems. In this rather general framework, the term "consensus" has been replaced by "synchronization" in order to put emphasis on the dynamics of the agents. In the context of synchronization, similar to consensus, the goal of communication is to achieve a common solution of the individual agents' dynamics. Among the numerous instances of available research in this direction, we refer to [47], [83], or [96]. In this chapter, similarly, we consider a network of agents with general, yet identical, linear dynamics, and we assume that partial information of the agents is transmitted via network communication. Thus, the structure we consider here also captures the case where the agents are coupled through a general static state-feedback or through an output-feedback protocol.

In the context of linear time-invariant systems, the *complexity* of a system is in general measured by its dynamic order, i.e., the number of state components in a state space representation of the system. Trying to reduce the complexity of models has led to the development of various model reduction techniques over the last decades. Perhaps the most well-known of these is Lyapunov balanced truncation (see [67], [68], [76]). In this approach, first the system is transformed into a so-called balanced form, and next a reduced order model is obtained by truncation. Other types of balancing include stochastic, bounded real, and positive real balancing (see [22], [74], [70]).

Since the multi-agent systems we consider here are composed of several linear time-invariant systems interconnected by a time-independent topology, they can be represented, as a whole, by a finite-dimensional linear time-invariant system. The dynamic order of such a representation for a network with p agents is in general p times that of the individual agents. As a result, the complexity of the network model will be reduced substantially by reducing the dynamic order of the agents, especially in large-scale networks. This motivates us to exploit available model reduction techniques to obtain a, simpler, lower order model for the network which approximates the behavior of the original one.

A critical issue in model reduction is preservation of qualitative properties of the original model in the reduced order model. For instance, stability, contractivity and passivity are preserved in the reduced order model obtained by Lyapunov balancing, bounded real and positive real balancing, respectively. In this chapter, we consider, separately, stability of the network and synchronization of the network as desired qualitative properties to be preserved in the reduced order network. Preservation of synchronization is particularly challenging as agents typically have integrators and unstable dynamics in the context of networks and multi-agent systems.

There are several factors which make this problem non-trivial and challenging, such as:

1. Although a multi-agent system as a whole can be represented as a finite-dimensional linear time-invariant system, this representation, however, has a certain structure imposed by the network communication topology. This structure, of course, must be preserved in the reduced order network, and, therefore, we deal with a kind of structure preserving model reduction problem. Besides, recall that the multi-agent systems we consider here are composed of linear agents which are identical. Thus, a direct application of available model reduction techniques may introduce heterogeneity into the network which adds to the complexity of the model, despite the fact that the dynamic order of the network might have been reduced.

2. Most of the well-established model reduction techniques rely on the assump-

tion of stability of the system. However, as mentioned, agents typically have integrators and unstable dynamics in the context of synchronization. Therefore, in this case, popular model reduction methods like Lyapunov balancing, bounded real, or positive real balancing are not directly applicable to the dynamics of the individual agents. Moreover, for linear time-invariant systems, *closeness* of the reduced order model and the original one can be estimated by the difference in $\mathcal{H}_\infty$-norm of the corresponding transfer matrices. However, this is not readily applicable to the individual agents of the network due to the presence of possible unstable dynamics, which hinders measuring how well the reduced order network approximates the original one.

The structure of the chapter is as follows. In Section 3.2, some notations and basic material needed in the sequel are provided. In Section 3.3, a stability preserving model reduction scheme for multi-agent systems is proposed. A synchronization preserving model reduction approach is established in Section 3.4, and is applied to a numerical example in Section 3.5. The chapter ends with conclusions in Section 3.6.

3.2 Preliminaries

3.2.1 Multi-agent systems

Let $G = (V, E)$ be an undirected (unweighted) graph where $V = \{1, 2, \ldots, p\}$ is the vertex set and E is the edge set. An edge is a two-element subset of V. A diffusively coupled multi-agent system consists of a collection of identical linear input/state/output systems given by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{3.1a}$$

together with the diffusive coupling rule

$$u_i(t) = -\sum_{\{i,j\}\in E} (y_i(t) - y_j(t)), \tag{3.1b}$$

where $i \in \{1, 2, \ldots, p\}$, $x_i \in \mathbb{R}^n$ is the state of agent i, and $u_i \in \mathbb{R}^m$ is the diffusive coupling term. Throughout this chapter, it is assumed that the state space representation (3.1a) is minimal. Let L denote the Laplacian matrix corresponding to the graph $G = (V, E)$. Then, the multi-agent system (3.1) can be written in compact form as

$$\dot{x}(t) = \mathcal{A}x(t) \tag{3.2}$$

where $x = \mathrm{col}(x_1, x_2, \ldots, x_p)$, $\mathcal{A} = I_p \otimes A - L \otimes BC$, and "$\otimes$" denotes the Kronecker product.

Note that the Laplacian matrix has always an eigenvalue at zero, and the multiplicity of this zero eigenvalue is associated with the connectedness of G. In particular, $G = (V, E)$ is connected if and only if the multiplicity of the zero eigenvalue of the Laplacian matrix is 1 (see e.g. [56, p.27]). In Section 3.3, we do not assume the connectedness of G, while in Section 3.4 the graph G is assumed to be connected.

Next to the Laplacian matrix, we will use another matrix associated with the graph G, the so-called *incedence matrix* of a graph. After the edges are labeled and oriented arbitrarily, the incidence matrix of G, denoted by R, is defined as (see [56, p.21]):

$$R_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of edge } j \\ -1 & \text{if vertex } i \text{ is the tail of edge } j \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, where p and q are the total number of vertices and edges, respectively. The relationship between the incidence matrix and the Laplacian matrix is captured by the following equality:

$$L = RR^\top. \tag{3.4}$$

3.2.2 Model reduction

In this subsection, we review some basic material and facts on model reduction by balanced truncation. Consider the finite dimensional, linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{3.5}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{w\times n}$, and $D \in \mathbb{R}^{w\times m}$. Assume that the matrix A is Hurwitz, and the state space representation (3.5) is minimal. We shortly denote this system by $\Gamma(A, B, C, D)$, and we use the notation $\Gamma(A, B, C)$ for the case where $D = 0$. In general, the model reduction by balanced truncation consists of two major steps, namely balancing and truncation. Balancing is, basically, finding a nonsingular state space transformation T that diagonalizes appropriately chosen positive definite matrices P and Q in a covariant and contravariant manner, respectively. This means that P transforms to $TPT^\top$ and Q transforms to $T^{-\top}QT^{-1}$, and the transformed matrices should be diagonal and equal. Notable types of balancing are Lyapunov balancing, bounded real (BR) balancing, and positive real (PR) balancing. In Lyapunov balancing the matrices P and Q are chosen to be the reachability and observability gramians, which are obtained from the following Lyapunov equations:

$$
\begin{aligned}
A^\top Q + QA + C^\top C &= 0 \\
AP + PA^\top + BB^\top &= 0.
\end{aligned}
$$

A reduced order model can be obtained by balancing the pair of positive definite matrices (P, Q), and truncating the state components which are of least importance; in other words, the states which are relatively difficult to reach and observe. Let G and G_r denote the transfer matrices of the original and the reduced order model, respectively. Then the model reduction error given by $\|G - G_r\|_\infty$ is bounded from above by twice the sum of the neglected Hankel singular values (HSV). For details, we refer to [1].

Instead of using the Lyapunov equations above, one can also work with solutions of Lyapunov inequalities, and obtain a reduced order model based on the so-called generalized gramians. More precisely, let Q_g and P_g be positive definite solutions of the inequalities

$$A^\top Q_g + Q_g A + C^\top C \leqslant 0 \tag{3.6a}$$

$$AP_g + P_g A^\top + BB^\top \leqslant 0. \tag{3.6b}$$

Then, similar to ordinary Lyapunov balancing, a reduced order model can be obtained by balancing the pair of positive definite matrices (P_g, Q_g) and truncating based on the so-called generalized Hankel singular values (GHSV) which are the square roots of the eigenvalues of the product $P_g Q_g$. Then, similarly, the corresponding model reduction error bound is twice the sum of the neglected GHSV. For details see [23, Sec. 4.7].

As mentioned before, after applying balancing transformations, the relevant gramians are equal and diagonal. In case that the state transformation only makes the gramians diagonal, but not necessarily equal, we say that the system is *essentially* balanced. It is easy to observe that, with the same truncation decision, the reduced order model obtained from balancing is equal to the one obtained from essentially balancing; that is, the transfer matrix of the reduced order model (from u to y) will be the same in both cases.

Let G denote the transfer matrix from u to y in system (3.5), i.e.

$$G(s) = C(sI - A)^{-1}B + D.$$

Then we call the linear system (3.5) *bounded real* if

$$G^\top(-j\omega)G(j\omega) \leqslant I, \qquad \forall \omega \in \mathbb{R}. \tag{3.7}$$

We call the system (3.5) *strictly bounded real* if the above inequality is strict. Under the assumption that $I - D^\top D$ is nonsingular, strict bounded realness of Γ is equivalent to the $\mathcal{H}_\infty$-norm of G being strictly less than 1. If $I - D^\top D > 0$, then (3.5) is bounded real if and only if there exists a real symmetric matrix K satisfying the Riccati equation

$$A^\top K + KA + C^\top C + (KB + C^\top D)(I - D^\top D)^{-1}(KB + C^\top D)^\top = 0. \tag{3.8}$$

In fact in that case, all real symmetric solutions of (3.8) are positive definite and any real symmetric solution K lies between two extremal solutions K_m and K_M, that is $0 < K_m \leqslant K \leqslant K_M$. Bounded real balancing involves balancing the pair of positive definite matrices (K_M^{-1}, K_m), and truncating based on the so-called bounded real characteristic values which are the square root of the eigenvalues of the product $K_M^{-1} K_m$. The error bound for Lyapunov balancing also holds for BR model reduction by considering neglected bounded real characteristic values instead of neglected HSV. Moreover, minimality, stability, and strict bounded realness are preserved in the reduced order model (see [34], [70], [74]) for more details).

3.3 Stability preserving model reduction

In this section, we assume that agents have stable internal dynamics and network (3.2) is stable. Then, we reduce the dynamic order of the agents such that stability is preserved in the reduced order network.

3.3.1 Stability of the network

First, we analyze the stability of network (3.2). As the Laplacian matrix L is symmetric, there exists an orthogonal matrix U such that $U^\top LU = \Lambda$, where Λ is a diagonal matrix having the eigenvalues of L as diagonal elements, and the columns of U are corresponding eigenvectors of L. Since L is positive semi-definite, and its row sums are zero, we can assume, without loss of generality, that

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \tag{3.9a}$$

with

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_p \tag{3.9b}$$

and the first column of U is the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I)x$ to (3.2), we obtain

$$\dot{\tilde{x}}(t) = (I \otimes A - \Lambda \otimes BC)\tilde{x}(t). \tag{3.10}$$

Observe that

$$I \otimes A - \Lambda \otimes BC = \mathrm{blockdiag}(A - \lambda_1 BC, A - \lambda_2 BC, \ldots, A - \lambda_p BC). \tag{3.11}$$

Hence, the linear system (3.2) is stable if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 1, 2, \ldots, p$.

The above necessary and sufficient stability condition cannot be directly applied to our model reduction framework. Besides, it requires information on the exact location of all eigenvalues of the Laplacian matrix which in some cases may not be available. Instead, we are interested in deducing stability of (3.2) by a small gain type of argument. Note that internal stability of the agents, i.e. A being Hurwitz, is necessary for stability of network (3.2). Assuming A to be Hurwitz and denoting the transfer matrix from u_i to y_i by G, we seek for a condition on the $\mathcal{H}_\infty$-norm of G under which the network (3.2) is stable. This brings us to the following lemma.

Lemma 3.3.1 *Assume that A is Hurwitz. Let G denote the transfer matrix from u_i to y_i in* (3.1)*, i.e. $G(s) = C(sI - A)^{-1}B$. Then, the network* (3.2) *is stable if*

$$\lambda_p \|G\|_\infty < 1 \tag{3.12}$$

Proof. Assume that (3.12) holds. Then there exists a positive definite solution $X > 0$ to the inequality (see [111])

$$A^\top X + XA + C^\top C + \lambda_p^2 XBB^\top X < 0. \tag{3.13}$$

For any $\alpha \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, we have

$$\begin{aligned}
(A - \alpha BC)^\top X + X(A - \alpha BC) &= A^\top X + XA - \alpha(C^\top B^\top X + XBC) \\
&= A^\top X + XA + C^\top C + \alpha^2 XBB^\top X \\
&\quad - (\alpha XB + C^\top)(\alpha B^\top X + C) \\
&< (\alpha^2 - \lambda_p^2)XBB^\top X - (\alpha XB + C^\top)(\alpha B^\top X + C)
\end{aligned}$$

where (3.13) is used to obtain the last inequality. Hence, $(A - \alpha BC)^\top X + X(A - \alpha BC)$ is negative definite for all $\alpha \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Consequently, $A - \lambda_i BC$ is Hurwitz for each $i = 1, 2, \ldots, p$ and the network (3.2) is stable. ■

Remark 3.3.2 The value of λ_p plays a crucial role in feasibility of the condition (3.12). Clearly, this condition tends to be more restrictive as the size of the network increases. Besides, the value of λ_p is bounded from below by the maximal degree of the vertices plus 1 (see [33]). Consequently, (3.12) is more likely to be satisfied in graphs like path or cycle graphs rather than star or complete graphs.

3.3.2 Model reduction

The result of Lemma 3.3.1 can be used to obtain a model reduction technique which preserves stability of the multi-agent system (3.1). Let $\Gamma(A, B, C)$ denote the linear system (3.1a) representing the dynamics of the individual agents. Assume that A is Hurwitz and the small gain condition (3.12) holds. Then, clearly, the linear system $\Gamma(A, \lambda_p B, C)$ is strictly bounded real. Therefore, bounded real balancing can be applied based on the Riccati equation

$$A^\top K + KA + C^\top C + \lambda_p^2 KBB^\top K = 0. \tag{3.14}$$

Let K_m and K_M denote the minimal and maximal real symmetric solutions of (3.14), respectively. Then, $0 < K_m \leqslant K_M$. In balanced coordinates we have, $K_m = K_M^{-1} = \Sigma$ where

$$\Sigma = \mathrm{diag}\{\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N}\} \tag{3.15}$$

with distinct bounded real characteristic values σ_i ordered in a decreasing manner. Consequently, for each positive integer $1 \leqslant k < N$, one can obtain a reduced order model $\Gamma_r(\bar{A}, \lambda_p \bar{B}, \bar{C})$ of order $r = \sum_{i=1}^k s_i$ by truncating the state components corresponding to the $(N-k)$ smallest distinct characteristic values. Then, obviously, the reduced agents' model $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$ can be retrieved from Γ_r. This results in the following reduced order network:

$$\dot{\bar{x}}(t) = \bar{\mathcal{A}}\bar{x}(t) \tag{3.16}$$

where $\bar{\mathcal{A}} = I_p \otimes \bar{A} - L \otimes \bar{B}\bar{C}$ and $\bar{x} = \mathrm{col}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p)$ with $\bar{x}_i \in \mathbb{R}^r$ being the reduced state component of agent i for $i = 1, 2, \ldots, p$. The reduced order network obtained in this way is stable as stated in the following theorem.

Theorem 3.3.3 *Consider the multi-agent system* (3.1), *and assume that the small gain condition* (3.12) *holds. Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation* (3.14). *Then, for each positive integer $1 \leqslant k < N$, the reduced order network* (3.16) *of order pr with $r = \sum_{i=1}^k s_i$, obtained by balancing (K_M^{-1}, K_m) and truncating according to* (3.15) *is stable.*

Proof. As observed earlier, balancing (K_M^{-1}, K_m) in (3.14) corresponds to BR balancing of $\Gamma(A, \lambda_p B, C)$. Therefore, the reduced order dynamics $\Gamma_r(\bar{A}, \lambda_p \bar{B}, \bar{C})$ is stable and strictly bounded real. Therefore, $\lambda_p \|G_r\|_\infty < 1$ where $G_r(s) = C(sI - A)^{-1}B$. Hence, the reduced network (3.16) is stable by Lemma 3.3.1. ■

3.3.3 Error bounds

Assuming that stability of the original network holds due to the small gain condition (3.12), the model reduction scheme established in the previous subsection obtains reduced order dynamics for the agents in such a way that stability of the network is preserved in the reduced model (3.16). Moreover, one can show that the reduced order agent dynamics is *close* to the original one by establishing error bounds as follows.

As observed earlier, balancing the pair (K_M^{-1}, K_m) in Riccati equation (3.14) is equivalent to bounded real balancing with respect to the linear system $\Gamma(A, \lambda_p B, C)$. Hence, a model reduction error bound for the individual agents is obtained as

$$\|G - G_r\|_\infty \leqslant \frac{2}{\lambda_p} \sum_{i=r+1}^{N} \sigma_i \tag{3.17}$$

where G and G_r are the transfer matrices corresponding to $\Gamma(A, B, C)$ and $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$, respectively. Note that to write (3.17), we have assumed that λ_p is nonzero, meaning that the graph has at least one edge. Obviously, the model reduction problem for the case where $\lambda_p = 0$ is not of current interest as it boils down to the ordinary model reduction problem of finite-dimensional linear time-invariant systems.

3.4 Synchronization preserving model reduction

3.4.1 Synchronization of the network

A synchronized network has the property that the state trajectories of the coupled agents converge to a common trajectory. More precisely, we have the following definition.

Definition 3.4.1 The multi-agent system (3.2) is *synchronized* if every solution of (3.2) satisfies $\lim_{t\to\infty} (x_j(t) - x_k(t)) = 0$ for all $j, k = 1, 2, \ldots, p$.

Thus, different from network stability, network synchronization requires that the differences of the states of the agents converge to zero as time runs off to infinity.

In contrast with the previous section, here agents are allowed to, and typically have, unstable dynamics. Therefore, throughout this section we assume that the underlying graph G is connected which is necessary for achieving synchronization in case of unstable agents' dynamics (see [47], [27]).

Let the *state disagreement vector* z be defined as

$$z(t) = (R^\top \otimes I_n)x(t) \tag{3.18}$$

where $z = \mathrm{col}(z_1, z_2, \ldots, z_{p-1})$ and R is the incidence matrix of the graph. Then it is easy to observe that the network (3.2) is synchronized if and only if $\lim_{t\to\infty} z(t) = 0$ for any solution x of (3.2). Consequently, the network (3.2) is synchronized if and only if the system (3.2) with output variable z is output stable. Necessary and sufficient conditions for synchronization have already been investigated in the literature (see e.g. [47], [84]). These conditions and the corresponding proofs are recapped in the following lemma for later use.

Lemma 3.4.2 *Assume that the graph $G = (V, E)$ is connected, and let the eigenvalues of the Laplacian matrix be given by* (3.9b)*. Then the network* (3.2) *is synchronized if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \ldots, p$.*

Proof. The Laplacian matrix L admits the spectral decomposition $U^\top LU = \Lambda$ where Λ is given by (3.9a) and U is an orthogonal matrix with its first column being the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I)x$ to system (3.2) with output equation (3.18), we obtain

$$\dot{\tilde{x}}(t) = (I_p \otimes A - \Lambda \otimes BC)\tilde{x}(t) \tag{3.19a}$$

$$z(t) = (R^\top U \otimes I_n)x(t) \tag{3.19b}$$

Observe that

$$I_p \otimes A - \Lambda \otimes BC = \mathrm{blockdiag}(A, A - \lambda_2 BC, \ldots, A - \lambda_p BC) \tag{3.20}$$

and $R^\top U \otimes I_n$ can be decomposed as

$$R^\top U \otimes I_n = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes I_n \end{bmatrix} \tag{3.21}$$

where $\tilde{U}$ is the matrix obtained by deleting the first column of U, and 0 is a zero matrix with n columns. For any initial state $\tilde{x}(0) = x_0$, the output of the system (3.19) is obtained as

$$z(t) = (R^\top U \otimes I_n)e^{(I\otimes A - \Lambda\otimes BC)t}x_0$$

which, by using (3.20) and (3.21), can be rewritten as

$$z(t) = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes I_n \end{bmatrix} \mathrm{blockdiag}(e^{At}, e^{(A-\lambda_2 BC)t}, \ldots, e^{(A-\lambda_p BC)t})x_0.$$

Since R is the incidence matrix associated with a connected graph, the matrix $R^\top \tilde{U} \otimes I_n$ has full column rank. Thus, the output variable z converges to zero for any x_0 if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \ldots, p$. ■

3.4.2 Model reduction

Starting from a synchronized network, our aim here is to derive a reduced order model for the network such that synchronization is preserved in this reduced order model. Recall that the model reduction technique proposed in the previous section uses the scaled dynamics of the agents $\Gamma(A, \lambda_p B, C)$ to obtain a lower order network (3.16). In the context of synchronization, however, the individual agents' dynamics is not necessarily stable. Therefore, usual balancing methods like Lyapunov balancing and BR balancing cannot be applied directly to the original agents' dynamics in this case. The idea, here, is to use stable dynamics present in a synchronized network; in particular, $A - \lambda_i BC$ with $i = 2, 3, \ldots, p$ (see Lemma 3.4.2). In terms of these dynamics, small gain type of conditions can also be derived to guarantee synchronization of (3.2) as stated in the following lemma.

Lemma 3.4.3 *Let the eigenvalues of the Laplacian matrix be given by* (3.9b). *Then the network* (3.2) *is synchronized if there exists an index* $\ell \in \{2, 3, \ldots, p\}$ *such that* $A - \lambda_\ell BC$ *is Hurwitz and*

$$\delta \|H_\ell\|_\infty < 1 \tag{3.22}$$

where

$$H_\ell(s) = C(sI - A + \lambda_\ell BC)^{-1} B$$

and

$$\delta = \max(\lambda_\ell - \lambda_2, \lambda_p - \lambda_\ell). \tag{3.23}$$

Proof. Suppose that (3.22) holds. Then there exists a positive definite matrix X which satisfies the Riccati inequality (see [111])

$$(A - \lambda BC)^\top X + X(A - \lambda BC) + C^\top C + \delta^2 XBB^\top X < 0 \tag{3.24}$$

where the index ℓ is dropped for notational convenience. For each $i = 2, 3, \ldots, p$, we have

$$\begin{aligned}(A - \lambda_i BC)^\top X &+ X(A - \lambda_i BC) \\ &= (A - \lambda BC)^\top X + X(A - \lambda BC) - (\lambda_i - \lambda)(C^\top B^\top X + XBC) \\ &= (A - \lambda BC)^\top X + X(A - \lambda BC) + C^\top C + \delta^2 XBB^\top X \\ &+ ((\lambda_i - \lambda)^2 - \delta^2)XBB^\top X - ((\lambda_i - \lambda)XB + C^\top)((\lambda_i - \lambda)B^\top X + C).\end{aligned} \tag{3.25}$$

Now, the right hand side is negative definite due to (3.23) and (3.24). Therefore $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \ldots, p$, and the network is synchronized by Lemma 3.4.2. ∎

Remark 3.4.4 As mentioned before, in the context of synchronization, agents typically have unstable dynamics, i.e. A is not Hurwitz. Based on the proof of Lemma 3.4.3, it can be shown that in this case, the condition (3.22) is satisfied only if $2\lambda_\ell > \lambda_p$. To see this, suppose (3.22) holds, and consider (3.25) where λ_i is replaced by zero. Then, since A is not Hurwitz, the right hand side of (3.25) cannot be negative definite. This results in $\lambda_\ell > \delta$ which yields $2\lambda_\ell > \lambda_p$. Consequently, searching for λ_ℓ in Lemma 3.4.3 can be restricted, in this generic case, to the interval $\frac{\lambda_p}{2} < \lambda_\ell \leqslant \lambda_p$.

Remark 3.4.5 The feasibility of the condition (3.22) depends both on the dynamics of the agents and the magnitude of δ. Hence, for given agent dynamics, the so-called Laplacian spread of a graph, given by $\lambda_p - \lambda_2$, plays a crucial role in feasibility of (3.22). Consequently, the condition (3.22) is expected to be more restrictive for graphs with a large Laplacian spread like star graphs, and to be less restrictive as the underlying graph tends to a complete graph. For details regarding the Laplacian spread of a graph see [26] and [105].

The theorem above can be used for model reduction purposes. Starting from the assumption that the condition (3.22) holds for the original network, implying that the original network is synchronized, we obtain a reduced order model such that synchronization is preserved in the reduced model. This is illustrated next.

Assume that synchronization of network (3.2) is verified by condition (3.22). In addition, suppose that $\delta \neq 0$, i.e. the underlying communication topology does not corresponds to a complete graph. Model reduction for the case $\delta = 0$ is rather trivial and will be discussed later (see Remark 3.4.9). As before, let λ_ℓ be denoted shortly by λ. Choose $\gamma > 0$ such that

$$\delta \| C(sI - A + \lambda BC)^{-1} B \|_\infty < \gamma < 1 \tag{3.26}$$

Then there exists a positive definite matrix K satisfying the Riccati equation

$$(A - \lambda BC)^\top K + K(A - \lambda BC) + C^\top C + (\frac{\delta}{\gamma})^2 K B B^\top K = 0. \tag{3.27}$$

Observe that $\Gamma(A - \lambda BC, \frac{\delta}{\gamma} B, C)$ is strictly bounded real for any choice of γ satisfying (3.26). Let K_m and K_M denote the minimal and maximal real symmetric solutions of (3.27). Then BR balancing can be applied using the pair of positive definite matrices (K_M^{-1}, K_m) in order to obtain a reduced order model Γ_r from the original dynamics Γ. Obviously, the reduced order dynamics of the agents, $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$, can be then retrieved from Γ_r. Consequently, a reduced order network is obtained which can be written again as in (3.16). Moreover, synchronization is preserved in the reduced order model as stated in the following theorem.

Theorem 3.4.6 *Consider the network* (3.2)*, and assume that the condition* (3.22) *holds. Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation* (3.27)*. Then, for each positive integer $1 \leqslant k < N$, the reduced order network* (3.16) *of order pr with $r = \sum_{i=1}^{k} s_i$, obtained by balancing (K_M^{-1}, K_m) and truncating according to* (3.15) *is synchronized.*

Proof. Following the discussion preceding the theorem, if (3.22) holds then the linear system $\Gamma(A - \lambda BC, \frac{\delta}{\gamma}B, C)$ is strictly bounded real for any γ satisfying (3.26). Therefore, due to the properties of BR balancing, the reduced system $\Gamma_r(\bar{A} - \lambda\bar{B}\bar{C}, \frac{\delta}{\gamma}\bar{B}, \bar{C})$ obtained by balancing (K_M^{-1}, K_m) and truncating is stable and bounded real. Then, since $\gamma < 1$, we have

$$\|\delta C(sI - \bar{A} + \lambda\bar{B}\bar{C})^{-1}B\|_\infty < 1$$

and the reduced order network (3.16) is synchronized by Lemma 3.4.3. ■

3.4.3 Error bound

Assuming that the small gain condition (3.22) holds, the model reduction scenario proposed in the previous section obtains a reduced order network which preserves synchronization. Here, we show that the reduced order network also gives a *good* approximation of the behavior of the original network.

As mentioned earlier, in the context of synchronization agents typically have unstable dynamics, which makes it difficult to compare the output of the original dynamics of the agent to that of the reduced one. However, as observed, not the agent's state components but their difference plays a crucial role in synchronization. Therefore, to establish model reduction error bounds we look at the differences of the outputs of the agents, and we do so for each pair of agents which communicate with each other. More precisely, we define the *output disagreement vector* as

$$\xi(t) = (R^\top \otimes I_n)y(t)$$

where $y(t) = \mathrm{col}(y_1, y_2, \ldots, y_p)$ and R is the incidence matrix. This can be rewritten as

$$\xi(t) = (R^\top \otimes C)x(t) \tag{3.28}$$

Furthermore, as the network (3.2) is an autonomous system, we also need to introduce an auxiliary input in order to be able to compare the input-output behavior of the original network to that of the reduced order model. Hence, we add a disturbance term d_i in the diffusively coupled feedback law (3.1b). That is, we replace (3.1b) by

$$\tilde{u}_i(t) = -\sum_{\{i,j\}\in E} (y_i(t) - y_j(t)) + d_i(t) \tag{3.29}$$

Consequently, we obtain the compact form

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t) + (I_p \otimes B)d(t)\\ \xi(t) &= (R^\top \otimes C)x(t)\end{aligned} \tag{3.30}$$

where $d = \mathrm{col}(d_1, d_2, \dots, d_p)$ and $\mathcal{A} = I_p \otimes A - L \otimes BC$. Then, for the reduced r^{th} order dynamics we have

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{\mathcal{A}}\bar{x}(t) + (I_p \otimes \bar{B})d(t)\\ \xi(t) &= (R^\top \otimes \bar{C})\bar{x}(t)\end{aligned} \tag{3.31}$$

with $\bar{x}_i \in \mathbb{R}^r$, $\bar{x} = \mathrm{col}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)$ and $\bar{\mathcal{A}} = I_p \otimes \bar{A} - L \otimes \bar{B}\bar{C}$.

Now, let T and T_r denote the transfer matrices from d to ξ in (3.30) and (3.31), respectively. Then $\|T - T_r\|_\infty$ measures the model reduction error, and we have the following result.

Theorem 3.4.7 *Consider the network* (3.2) *and assume that there exist an ℓ and a nonzero δ such that the condition* (3.22) *holds. Let γ be a parameter satisfying* (3.26). *Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation* (3.27). *Let* (3.31) *represent the reduced order network obtained by balancing (K_M^{-1}, K_m) and truncating according to* (3.15). *Then, we have the following model reduction error bound:*

$$\|T - T_r\|_\infty \leqslant \frac{2\gamma\sqrt{\lambda_p}}{\delta(1-\gamma^2)} \sum_{i=r+1}^{N} \sigma_i, \tag{3.32}$$

where the σ_is are the diagonal elements of Σ in (3.15).

Proof. The Laplacian matrix L admits the following spectral decomposition

$$U^\top L U = \Lambda \tag{3.33}$$

where Λ is given by (3.9), and the first column of U is the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I_n)x$ to (3.30) we obtain

$$\begin{aligned}\dot{\tilde{x}}(t) &= (I_p \otimes A - \Lambda \otimes BC)\tilde{x}(t) + (U^\top \otimes B)d(t)\\ \xi(t) &= (R^\top U \otimes C)\tilde{x}(t).\end{aligned} \tag{3.34}$$

Observe that

$$I_p \otimes A - \Lambda \otimes BC = \mathrm{blockdiag}(A, A - \lambda_2 BC, \dots, A - \lambda_p BC),$$

and $R^\top U \otimes C$ is of the form

$$R^\top U \otimes C = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes C \end{bmatrix}$$

where $\tilde{U}$ is the matrix obtained by deleting the first column of U, and 0 is a zero matrix with n columns. Let $\tilde{x}$ be partitioned accordingly as $\tilde{x} = \mathrm{col}(\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_1 \in \mathbb{R}^n$. Then, the network equation (3.34) simplifies to

$$\begin{aligned} \dot{\tilde{x}}_2(t) &= (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)\tilde{x}_2(t) + (\tilde{U}^\top \otimes B)d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes C)\tilde{x}_2(t). \end{aligned} \tag{3.35}$$

where $\tilde{\Lambda} = \mathrm{diag}(\lambda_2, \ldots, \lambda_p)$. Note that $\tilde{x}_1$ does not appear in the above as it corresponds to unobservable state variables.

Analogously, for the reduced order network (3.31) we obtain the following state space representation:

$$\begin{aligned} \dot{\hat{x}}_2(t) &= (I_{p-1} \otimes \bar{A} - \tilde{\Lambda} \otimes \bar{B}\bar{C})\hat{x}_2(t) + (\tilde{U}^\top \otimes \bar{B})d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes \bar{C})\hat{x}_2(t). \end{aligned} \tag{3.36}$$

Clearly, the transfer matrix from d to ξ in (3.35) is equal to that of (3.30), i.e. to T, and the transfer matrix from d to ξ in (3.36) is equal to that of (3.31), i.e. to T_r.

Now, we write the Lyapunov inequalities (3.6) for system (3.35) as

$$(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top Q_g + Q_g(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC) + (R^\top \tilde{U} \otimes C)^\top (R^\top \tilde{U} \otimes C) \leqslant 0$$

$$(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)P_g + P_g(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top + (\tilde{U}^\top \otimes B)(\tilde{U}^\top \otimes B)^\top \leqslant 0$$

which simplifies to

$$(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top Q_g + Q_g(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC) + \tilde{\Lambda} \otimes C^\top C \leqslant 0 \tag{3.38a}$$

$$(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)P_g + P_g(I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top + I_{p-1} \otimes BB^\top \leqslant 0 \tag{3.38b}$$

Note that equality (3.4) and the fact that $\tilde{U}^\top L \tilde{U} = \tilde{\Lambda}$ are used to write (3.38a). Now, assume that there exist an ℓ and a nonzero δ such that (3.22) holds. Let again λ_ℓ be denoted shortly by λ, and recall that K_m and K_M are the minimal and maximal real symmetric solutions of (3.27), respectively. Then we claim that

$$Q_g = \frac{\lambda_p}{1 - \gamma^2}(I \otimes K_m) \tag{3.39a}$$

and

$$P_g = \frac{\gamma^2}{\delta^2(1 - \gamma^2)}(I \otimes K_M^{-1}) \tag{3.39b}$$

satisfy the Lyapunov inequalities (3.38a) and (3.38b), respectively.

It is easy to observe that Q_g given by (3.39a) satisfies (3.38a) if and only if

$$(A - \lambda_i BC)^\top K_m + K_m(A - \lambda_i BC) + \frac{\lambda_i(1 - \gamma^2)}{\lambda_p} C^\top C \leqslant 0$$

for each $i = 2, 3, \ldots, p$. We have

$$
\begin{aligned}
&(A - \lambda_i BC)^\top K_m + K_m(A - \lambda_i BC) + \frac{\lambda_i(1-\gamma^2)}{\lambda_p} C^\top C \\
&= (A - \lambda BC)^\top K_m + K_m(A - \lambda BC) - (\lambda_i - \lambda)(C^\top B^\top K_m + K_m BC) + \frac{\lambda_i(1-\gamma^2)}{\lambda_p} C^\top C \\
&= -\frac{\delta^2}{\gamma^2} K_m BB^\top K_m - (1 - \frac{\lambda_i(1-\gamma^2)}{\lambda_p}) C^\top C - (\lambda_i - \lambda)(C^\top B^\top K_m + K_m BC) \\
&= -\begin{bmatrix} K_m B & C \end{bmatrix} \begin{bmatrix} \frac{\delta^2}{\gamma^2} I & (\lambda_i - \lambda) I \\ (\lambda_i - \lambda) I & (1 - \frac{\lambda_i(1-\gamma^2)}{\lambda_p}) I \end{bmatrix} \begin{bmatrix} B^\top K_m \\ C \end{bmatrix}
\end{aligned}
$$

where (3.27) is used to derive the second equality. Therefore, Q_g given by (3.39a) satisfies (3.38a) if

$$
\begin{bmatrix} \frac{\delta^2}{\gamma^2} & \lambda_i - \lambda \\ \lambda_i - \lambda & 1 - \frac{\lambda_i(1-\gamma^2)}{\lambda_p} \end{bmatrix} \geqslant 0 \tag{3.40}
$$

for $i = 2, 3, \ldots, p$. This holds if and only if

$$
\frac{\lambda_i(1-\gamma^2)}{\lambda_p} + \frac{\gamma^2}{\delta^2}(\lambda_i - \lambda)^2 \leqslant 1. \tag{3.41}
$$

Recall that, by definition, $\delta = \max(\lambda_p - \lambda, \lambda - \lambda_2)$. Hence, the first term on the left hand side of (3.41) is not greater than $(1 - \gamma^2)$, and the second term is not greater than γ^2. Therefore, Q_g given by (3.39a) satisfies Lyapunov inequalities (3.38a).

Now, we will show that P_g given by (3.39b) satisfies (3.38b). Clearly this holds if and only if

$$
(A - \lambda_i BC) K_M^{-1} + K_M^{-1}(A - \lambda_i BC)^\top + \frac{\delta^2(1-\gamma^2)}{\gamma^2} BB^\top \leqslant 0 \tag{3.42}
$$

for each $i = 2, 3, \ldots, p$. By multiplying (3.42) from the left and right by K_M, we obtain

$$
(A - \lambda_i BC)^\top K_M + K_M(A - \lambda_i BC) + \frac{\delta^2(1-\gamma^2)}{\gamma^2} K_M BB^\top K_M \leqslant 0. \tag{3.43}
$$

We have

$$
\begin{aligned}
&(A - \lambda_i BC)^\top K_M + K_M(A - \lambda_i BC) + \frac{\delta^2(1-\gamma^2)}{\gamma^2} K_M BB^\top K_M \\
&= (A - \lambda BC)^\top K_M + K_M(A - \lambda BC) - (\lambda_i - \lambda)(C^\top B^\top K_M + K_M BC) \\
&\qquad + \frac{\delta^2(1-\gamma^2)}{\gamma^2} K_M BB^\top K_M \\
&= -C^\top C - \delta^2 K_M BB^\top K_M - (\lambda_i - \lambda)(C^\top B^\top K_M + K_M BC) \\
&= -\begin{bmatrix} K_M B & C^\top \end{bmatrix} \begin{bmatrix} \delta^2 I & (\lambda_i - \lambda) I \\ (\lambda_i - \lambda) I & I \end{bmatrix} \begin{bmatrix} B^\top K_M \\ C \end{bmatrix}
\end{aligned}
$$

where (3.27) is used to derive the second equality. Therefore, P_g given by (3.39b) satisfies (3.38b) if

$$\begin{bmatrix} \delta^2 & \lambda_i - \lambda \\ \lambda_i - \lambda & 1 \end{bmatrix} \geqslant 0 \tag{3.44}$$

for each $i = 2, 3, \ldots, p$. This holds if and only if

$$\delta \geqslant |\lambda_i - \lambda|$$

which is true by the definition of δ. Hence, P_g given by (3.39b) satisfies Lyapunov inequalities (3.38b). Consequently, P_g and Q_g are generalized gramians for system (3.35).

Now, in the balanced coordinates, we have $K_m = K_M^{-1} = \Sigma$ where Σ is given by (3.15). Hence, after balancing, P_g and Q_g are obtained as

$$P_g = \frac{\gamma^2}{\delta^2(1-\gamma^2)}(I \otimes \Sigma)$$

and

$$Q_g = \frac{\lambda_p}{1-\gamma^2}(I \otimes \Sigma).$$

Thus, both P_g and Q_g will become diagonal after balancing (K_M^{-1}, K_m). Therefore, balancing (K_M^{-1}, K_m) yields essentially balancing of the generalized gramians (P_g, Q_g) of the network, see Subsection 3.2.2. Note that the generalized Hankel singular values are the square roots of the eigenvalues of the product $P_g Q_g$ which in this case are the diagonal elements of the matrix $\frac{\gamma\sqrt{\lambda_p}}{\delta(1-\gamma^2)}\Sigma$. This establishes the model reduction error bound (3.32). ■

Remark 3.4.8 Recall that the parameter γ is chosen such that (3.26) holds. Obviously, different choices of γ lead to different reduced order models. Although the error bound proposed in Theorem 3.4.7 is not optimal in any norm, heuristically, one can choose γ such that the guaranteed error bound in (3.32) is as small as possible. Note that the singular values σ_i also depend on γ.

Remark 3.4.9 In case where the communication topology corresponds to a complete graph, reducing dynamics from d to ξ in (3.30) boils down to an ordinary model reduction problem of a linear system (see e.g. [56, p.28] for the Laplacian spectrum of complete graphs). In particular, (3.35) can be written as

$$\begin{aligned} \dot{\tilde{x}}_2(t) &= (I_{p-1} \otimes (A - pBC))\tilde{x}_2 + (\tilde{U} \otimes B)d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes C)\tilde{x}_2(t). \end{aligned}$$

Note that for a complete graph we have $\lambda_2 = \lambda_3 = \ldots = \lambda_p = p$. Then, one can write the corresponding Lyapunov equations as

$$
\begin{aligned}
(I_{p-1} \otimes (A - pBC))^\top Q + Q(I_{p-1} \otimes (A - pBC)) + pI_{p-1} \otimes C^\top C = 0 \\
(I_{p-1} \otimes (A - pBC))P + P(I_{p-1} \otimes (A - pBC))^\top + I_{p-1} \otimes BB^\top = 0,
\end{aligned}
$$

which is simplified to

$$
\begin{aligned}
(A - pBC)^\top Q + Q(A - pBC) + pC^\top C = 0 \\
(A - pBC)P + P(A - pBC) + BB^\top = 0.
\end{aligned}
$$

Consequently, one can apply Lyapunov balanced truncation to the stable linear system $\Gamma(A - pBC, B, \sqrt{p}C)$, and obtain a reduced order network.

3.5 Numerical example

Here, we apply the proposed synchronization preserving model reduction approach established in the previous section to a numerical example. Consider the spacecraft formation problem studied in [47]. The dynamics of the individual agents is given by

$$
\begin{bmatrix} \dot{r}_i \\ \ddot{r}_i \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r_i \\ \dot{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_i \tag{3.45}
$$

with

$$
A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 \times 10^{-6} & 0 \\ 0 & 0 & -10^{-6} \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 2 \times 10^{-3} & 0 \\ -2 \times 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

where $r_i \in \mathbb{R}^3$ is the position vector of the i^{th} agent (satellite). Let A denote the overall state matrix in (3.45). Suppose that state information of the agents is transmitted over the network through the static state feedback C given by [47, Ex. 4]

$$
C = \begin{bmatrix} 0.6596 & -0.0013 & 0 & 1.9789 & 0 & 0 \\ 0.0013 & 0.6596 & 0 & 0 & 1.9789 & 0 \\ 0 & 0 & 0.6596 & 0 & 0 & 1.9789 \end{bmatrix},
$$

and the agents communicate according to the communication topology given by Figure 3.1. Then, the network equations can be written in compact as

$$
\dot{x} = (I_4 \otimes A - L \otimes BC)x \tag{3.46}
$$

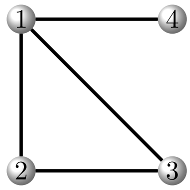

Figure 3.1: *Communication topology*

where $x = \text{col}(r_1, \dot{r}_1, r_2, \dot{r}_2, r_3, \dot{r}_3, r_4, \dot{r}_4)$, $B = \text{col}(0, I_3)$, and the Laplacian matrix is given by

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \tag{3.47}$$

The eigenvalues of the Laplacian matrix (3.47) are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$, and $\lambda_4 = 4$. Observe that

$$(\lambda_4 - \lambda_2)\|C(sI - A + \lambda_4 BC)^{-1}B\|_\infty = 0.7746 < 1,$$

and, hence, (3.46) is synchronized by Lemma 3.4.3.

Now let $\gamma = 0.8$, which clearly satisfies (3.26). Let again K_m and K_M be the minimal and maximal real symmetric solutions of the corresponding Riccati (3.27). Then a balancing transformation for (K_M^{-1}, K_m) is computed as

$$T = \begin{bmatrix} 0.2194 & -0.0006 & 0.0000 & 0.7300 & -0.0002 & 0.0000 \\ -0.0006 & -0.2194 & 0.0000 & 0.0002 & -0.7300 & 0.0000 \\ 0.0000 & 0.0000 & 0.2194 & 0.0000 & 0.0000 & 0.7300 \\ -0.2194 & -0.0006 & 0.0000 & 0.0718 & 0.0003 & 0.0000 \\ -0.0006 & 0.2194 & 0.0000 & 0.0003 & -0.0718 & 0.0000 \\ 0.0000 & 0.0000 & -0.2194 & 0.0000 & 0.0000 & 0.0718 \end{bmatrix}, \tag{3.48}$$

and the corresponding diagonal matrix Σ is obtained as

$$\Sigma = \begin{bmatrix} 0.7827 I_3 & 0 \\ 0 & 0.1927 I_3 \end{bmatrix}$$

Note that the only admissible truncation is to discard three state components. Now, by truncating the three state components corresponding to the smallest diagonal elements of Σ, the reduced order dynamics of the agents $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$ is obtained as

$$\bar{A} = \begin{bmatrix} 0.2736 & -0.0012 & 0.0000 \\ 0.0012 & 0.2736 & 0.0000 \\ 0.0000 & 0.0000 & 0.2736 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.7300 & -0.0002 & 0.0000 \\ -0.0002 & -0.7300 & 0.0000 \\ 0.0000 & 0.0000 & 0.7300 \end{bmatrix},$$

and

$$\bar{C} = \begin{bmatrix} 2.7374 & -0.0010 & 0.0000 \\ -0.0010 & -2.7374 & 0.0000 \\ 0.0000 & 0.0000 & 2.7374 \end{bmatrix}.$$

Consequently, the corresponding reduced order network can be represented as

$$\dot{\bar{x}}(t) = (I_4 \otimes \bar{A} - L \otimes \bar{B}\bar{C})\bar{x}(t). \tag{3.49}$$

It is easy to verify that $\bar{A} - \lambda_i \bar{B}\bar{C}$ is Hurwitz for each $i \in \{2, 3, 4\}$, and, hence, the reduced order network is synchronized by Lemma 3.4.2. Alternatively, one can verify that

$$(\lambda_4 - \lambda_2)\|\bar{C}(sI - \bar{A} + \lambda_4 \bar{B}\bar{C})^{-1}\bar{B}\|_\infty = 0.7766 < 1,$$

which implies that (3.49) is synchronized by Lemma 3.4.3. Note that, in fact, while the static feedback C synchronizes the original agents' dynamics, the truncated matrix $\bar{C}$ synchronizes the reduced order dynamics of the agents.

Now, to compare the behavior of the reduced order network to the original one, as in Subsection 3.4.3, we introduce auxiliary inputs and outputs to obtain the forms (3.30) and (3.31) for the original and the reduced order network, respectively. Note that in this example we have $p = 4$, $n = 6$, $r = 3$, $\xi \in \mathbb{R}^{12}$, and the incidence matrix of the graph can be given as

$$R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Now, we apply step disturbances to some, randomly chosen, channels of (3.30) and (3.31). In particular, for each $i = 1, 2, \ldots, 12$, we consider a disturbance $d_i = \alpha_i \mathbb{1}(t)$ where $\mathbb{1}(t)$ is the unit step function and α_i is some nonnegative integer indicating the amplitude of the step disturbance d_i. Clearly, the value of α_i is set to zero whenever the i^{th} channel is not affected by the disturbance. For two different choices of α_is, the output responses of (3.30) and (3.31), i.e. ξ variables, are compared in Figures 3.2 and 3.3. It is also worth to compute the actual model reduction error and the proposed error bound which corresponds to the left and right hand side of (3.32), respectively. In this case, the actual model reduction error is obtained as 0.1587 and the error bound is computed as 0.5674.

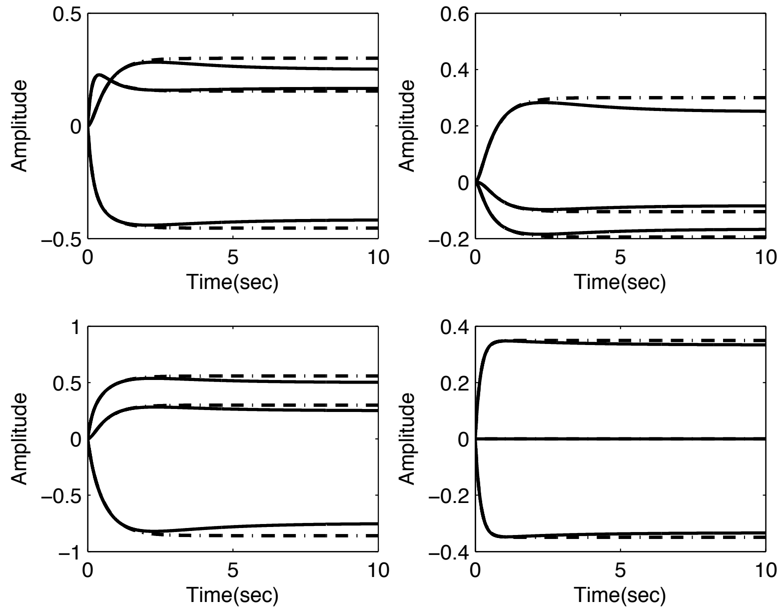

Figure 3.2: *Network model order reduction of Example 1: responses to the step disturbances corresponding to* $\alpha_i = [1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0]_i$*; original network (solid line) and reduced order network (dashed dotted line);* ξ_1, ξ_2, ξ_3 *(top left),* ξ_4, ξ_5, ξ_6 *(top right),* ξ_7, ξ_8, ξ_9*(down left),* $\xi_{10}, \xi_{11}, \xi_{12}$ *(down right).*

3.6 Conclusions

In this chapter, we have studied the problem of model order reduction for multi-agent systems. The identical agents are assumed to be general finite dimensional linear time-invariant systems. Small gain types of condition were derived to guarantee stability and synchronization of networks. It was observed that these conditions depend both on the communication topology and dynamics of the agents. Two different scenarios for model reduction of multi-agent systems were considered. First, assuming that the agents have stable internal dynamics, and the overall network is stable, a stability preserving model reduction approach was established. Consequently, model reduction error bounds on the dynamics of the individual agents were obtained. In the second scheme, we started off by the assumption that the original network is synchronized and a certain small gain condition holds. Then, a synchronization preserving model reduction technique was proposed by using bounded real balancing of some network dynamics. After adding appropriate auxiliary inputs

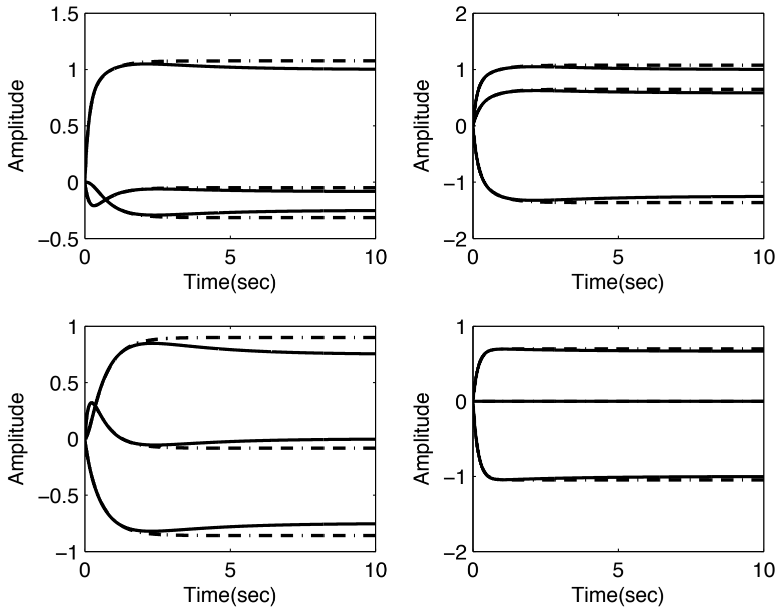

Figure 3.3: *Network model order reduction of Example 1: responses to the step disturbances corresponding to* $\alpha_i = [2\ 0\ 3\ 3\ 0\ 0\ 1\ 3\ 0\ 3\ 0\ 1]_i$*; original network (solid line) and reduced order network (dashed dotted line);* ξ_1, ξ_2, ξ_3 *(top left),* ξ_4, ξ_5, ξ_6 *(top right),* ξ_7, ξ_8, ξ_9*(down left),* $\xi_{10}, \xi_{11}, \xi_{12}$ *(down right).*

and outputs to the initial network representation, the behavior of the original and the reduced order network were compared by establishing model reduction error bounds. The proposed model reduction scheme was applied to a numerical example. The simulation results shows that the reduced order network gives a good approximation of the original one, providing that the neglected bounded real characteristic values are relatively small.

Chapter 4

A projection based approximation of multi-agent systems by using graph partitions

4.1 Introduction

Multi-agent systems and distributed control of networks of dynamic agents have received compelling attention in the last decade. In particular, reaching an agreement among agents in a network has been widely studied in terms of consensus and synchronization (see e.g. [41], [73], [79], [47], [83], [95]). Among numerous research directions in this area we mention formation control, flocking, placement of mobile sensors, and controllability analysis of networks; see e.g. [27], [71], [21], [20], [78], [107].

In order to analyze or control a large-scale system, model reduction techniques are highly advantageous. Clearly, lower order models admit easier analysis and provide a better understanding of the system behavior. Various model reduction techniques, such as balanced truncation, Hankel-norm approximation and Krylov projection are available in the literature; see e.g. [1]. Naively, one may think of exploiting these model reduction techniques to deal with analysis or control of large-scale networks. However, a major drawback here is that the spatial structure of the network may collapse by direct application of classical model reduction tools. Of course, related to this issue, structure preserving model reduction techniques have been established in the literature. In particular, preservation of the Langrangian structure, the second-order structure, and the interconnection structure of several subsystems have been studied in [44], [46], and [82]. Nevertheless, multi-agent systems and dynamical networks have their own structural characteristics and this motivates us to study the model reduction problem for this class of systems in a more focused manner. Obviously, the key structure needed to be preserved in the reduced order model is the network topology. Some recent work in this direction is [40] and [39], where a clustering based algorithm was proposed for asymptotically stable networks.

In the present chapter we consider multi-agent systems defined on weighted undirected graphs, and we propose a projection based technique to obtain reduced order models for these systems. The projection used is formulated in terms of the characteristic matrix of a graph partition. The reduction procedure preserves the spatial structure of the network, meaning that the reduced order model is realized as a new multi-agent system. The communication graph of the reduced order multi-agent system is weighted, symmetric and directed, and has a reduced number of nodes.

In the previous chapter, a model reduction scheme was established in which the dynamic order of the agents is reduced, but the communication graph remains unchanged. As a counterpart of Chapter 3, in the present chapter we consider single integrator dynamics with a consensus type of protocol, and we aim at reducing the size of the underlying communication graph. It should be noted that the Laplacian matrix of the communication graph serves as the state matrix in the model of a typical multi-agent system with a consensus based feedback protocol. Thus, inevitably, the system is not asymptotically stable, and hence most of the aforementioned existing result do not directly apply to this case. Another issue which is relevant here is the preservation of consensus in the reduced order model. As we will observe, clustering the agents does not jeopardize the consensus property of the original multi-agent system.

An important challenge is to compare the behavior of the reduced and original network. In this chapter, we will establish an explicit expression for the model reduction error in the sense of the $\mathcal{H}_2$-norm and with respect to an appropriate choice of inputs and outputs of the network. In particular, we work with a leader-follower set up, meaning that some agents, often called *leaders*, may receive an external command, a disturbance, or a reference signal. Moreover, as outputs we consider the differences among the states of the communicating agents, as these differences play a crucial role in the context of distributed control. The expression provided for the associated model reduction error is simple, easy to compute, and can be derived directly from the graph partition involved in obtaining the reduced order model.

The structure of the chapter is as follows. In Section 4.2, we review some basic notions and preliminaries that are needed in the sequel. The proposed model reduction scheme is discussed in Section 4.3. The input-output behaviors of the reduced order and the original multi-agent system are compared in Section 4.4. Finally, Section 4.5 concludes the chapter.

4.2 Preliminaries

In this section, we will provide some preliminaries and basic material needed in the sequel. In particular, we will discuss some basic notions from graph theory, describe the model used in this chapter for multi-agent systems, and finally recap the notion of Petrov-Galerkin projection.

4.2.1 Graph theory

In this chapter we consider both weighted undirected graphs and weighted directed graphs. A weighted *undirected* graph is a triple $G = (V, E, A)$ where $V = \{1, 2, \ldots, n\}$ is the vertex set, E is the edge set, and $A = [a_{ij}]$ is the adjacency matrix, with nonnegative elements a_{ij} called the weights. The edge set of G is a set of unordered pairs $\{i, j\}$ of distinct vertices of G. Similarly, a weighted *directed* graph is a triple $G = (V, E, A)$ where $V = \{1, 2, \ldots, n\}$ is the vertex set, E is the arc set, and $A = [a_{ij}]$ is the adjacency matrix with nonnegative elements a_{ij}, again called the weights. The arc set of G is a set of ordered pairs (i, j) of distinct vertices of G. For an arc $(i, j) \in E$, we say i is the tail, and j is the head of the arc. In this chapter we consider *simple* graphs meaning that self-loops and multiple edges (multiple arcs in the same direction) between one particular pair of vertices are not permitted. We have $a_{ij} > 0$ whenever there is an edge between i and j (an arc from j to i). Clearly, $a_{ij} = a_{ji}$ for undirected graphs. A directed graph is called *symmetric* if whenever (i, j) is an arc also (j, i) is. We note that for symmetric directed graphs the weights a_{ij} and a_{ji} can be distinct. Clearly any weighted undirected graph can be identified with a symmetric directed graph in which the weights satisfy $a_{ij} = a_{ji}$.

Both for undirected and directed graphs the degree matrix of G is the diagonal matrix, denoted by $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, with

$$d_i = \sum_{j=1}^{n} a_{ij}.$$

The Laplacian matrix of G is defined as $L = D - A$. For directed graphs, the incidence matrix of G, denoted by $R = [r_{ij}]$, is defined as

$$r_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of arc } j \\ -1 & \text{if vertex } i \text{ is the tail of arc } j \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, where k is the total number of arcs. In order to obtain an incidence matrix for a given undirected graph, we first assign an arbitrary orientation to each of the edges and next take the incidence matrix of the corresponding directed graph (see [56, p.21]). Corresponding to the incidence matrix R, let

$$W = \mathrm{diag}(w_1, w_2, \ldots, w_k) \tag{4.2}$$

be a $k \times k$ matrix such that w_j indicates the weight associated to the edge (arc) j, for each $j = 1, 2, \ldots, k$. For undirected graphs, the relationship between the incidence matrix and the Laplacian matrix is then captured by the following equality:

$$L = RWR^\top. \tag{4.3}$$

4.2.2 Multi-agent systems

Let $G = (V, E, A)$ be a weighted undirected graph where $V = \{1, 2, \ldots, n\}$. Let $V_L = \{v_1, v_2, \ldots, v_m\}$ be a subset of V, and let $V_F = V \setminus V_L$. By a leader-follower multi-agent system, we mean the following dynamical system:

$$\dot{x}_i = \begin{cases} z_i & \text{if } i \in V_F \\ z_i + u_\ell & \text{if } i \in V_L \end{cases} \tag{4.4}$$

where $x_i \in \mathbb{R}$ denotes the state of agent i, $u_\ell \in \mathbb{R}$ is the external input applied to agent $i = v_\ell$, and $z_i \in \mathbb{R}$ is the coupling variable for the agent i which is given by

$$z_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{4.5}$$

Let $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$, $u = \operatorname{col}(u_1, u_2, \ldots, u_m)$, and the matrix $M \in \mathbb{R}^{n \times m}$ be defined as

$$M_{i\ell} = \begin{cases} 1 & \text{if } i = v_\ell \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Then we can write the above leader-follower linearly diffusively coupled multi-agent system associated with the graph G in a compact form as

$$\dot{x} = -Lx + Mu, \tag{4.7}$$

where L is the Laplacian matrix of G, and M is given by (4.6).

4.2.3 Petrov-Galerkin projections

Consider the input/state/output system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{4.8}$$

where $x = \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y = \mathbb{R}^p$ is the output of the system. Let $\mathbf{W}, \mathbf{V} \in \mathbb{R}^{n \times r}$ such that $\mathbf{W}^\top \mathbf{V} = I$. By using the projection $\Gamma = \mathbf{V}\mathbf{W}^\top$, a reduced order model (projected model) is obtained as

$$\begin{aligned}\dot{\hat{x}} &= \mathbf{W}^\top A\mathbf{V}\hat{x} + \mathbf{W}^\top Bu \\ y &= C\mathbf{V}\hat{x}\end{aligned} \tag{4.9}$$

where $\hat{x} \in \mathbb{R}^r$ denotes the state of the reduced model. This projection is called a Petrov-Galerkin projection. Note that Γ defines a projection onto the image of V and along the kernel of $W^\top$. In case that $\mathbf{W}$ is equal to $\mathbf{V}$, the projection Γ is orthogonal and is called a Galerkin projection. The Petrov-Galerkin projection is a rather general reduction framework meaning that many of the model reduction techniques including Krylov based and truncation methods essentially use this projection with appropriate choice of matrices $\mathbf{V}$ and $\mathbf{W}$. In particular, depending on the application, one can choose the matrix $\mathbf{V}$, and consequently $\mathbf{W}$, to preserve stability, passivity, or to match certain moments and Markov parameters (see [1] for more details).

4.3 Projection by graph partitions

It is not hard to see that a direct application of Petrov-Galerkin projection will, in general, destroy the spatial structure of the network. In particular, the relationship between the reduced order network and the original one is not transparent, the structure of the Laplacian matrix may be lost, and the reduced order model may not be in the form of a leader-follower multi-agent system as given by (4.7). Therefore, we propose to use graph partitions in order to preserve the structure of the network in the reduced order model. First, we need to recap the notions of cells and graph partitions.

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of a graph G. We call any nonempty subset of V a *cell* of V. We call a collection of cells, given by $\pi = \{C_1, C_2, \ldots, C_r\}$, a *partition* of V if $\cup_i C_i = V$ and $C_i \cap C_j = \varnothing$ whenever $i \neq j$. With a little abuse of notation, we say π is a partition of $G = (V, E)$, or shortly G, meaning that π is a partition of V. We say i is a *cellmate* of j in π if i and j belong to the same cell of π. For a cell $C \subseteq V$, the *characteristic vector* of C is defined as the n-dimensional column vector $p(C)$ with

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

For a partition $\pi = \{C_1, C_2, \ldots, C_r\}$, we define the *characteristic matrix* of π as

$$P(\pi) = \begin{bmatrix} p(C_1) & p(C_2) & \cdots & p(C_r) \end{bmatrix}. \tag{4.11}$$

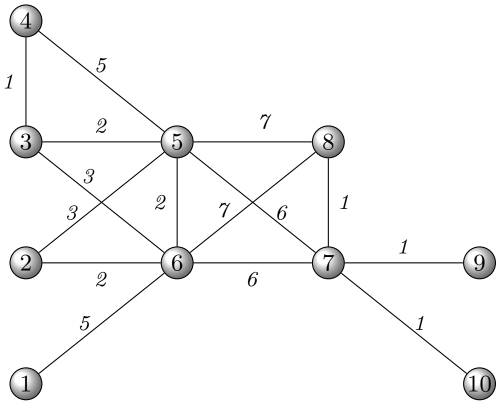

Figure 4.1: $G = (V, E, A)$

Example 4.3.1 As an example, consider the graph G depicted in Figure 4.1.
Then,

$$\pi = \{\{1,2,3,4\},\{5,6\},\{7\},\{8\},\{9,10\}\} \tag{4.12}$$

is a partition of $V = \{1, 2, \ldots, 10\}$, and its characteristic matrix is given by

$$P(\pi) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\top . \tag{4.13}$$

□

Now, consider again in general the multi-agent system (4.7) with associated graph $G = (V, E, A)$. Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be a partition of V, and $P(\pi)$ be the characteristic matrix of π. Recall the Petrov-Galerkin projection discussed in Section 4.2.3. We propose the following choices for the matrices $\mathbf{V}$ and $\mathbf{W}$:

$$\mathbf{W} = P(\pi)(P^\top(\pi)P(\pi))^{-1} \tag{4.14a}$$

$$\mathbf{V} = P(\pi). \tag{4.14b}$$

Note that the columns of $P(\pi)$ are orthogonal, thus the matrix $P^\top(\pi)P(\pi)$ is a diagonal matrix. Moreover, its i^{th} diagonal element is equal to the number of vertices in cell C_i of π. Hence, $P^\top(\pi)P(\pi)$ is invertible. Also note that $\mathbf{W}^\top\mathbf{V} = I$, and the projection $\Pi = \mathbf{V}\mathbf{W}^\top$ is orthogonal since $\operatorname{im}\mathbf{V} = (\ker\mathbf{W}^\top)^\perp$. Then, by applying the

Petrov-Galerkin projection to (4.7), with the choices of $\mathbf{V}$ and $\mathbf{W}$ given by (4.14), we obtain the reduced order system

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{4.15}$$

where $\hat{x} \in \mathbb{R}^r$ is the state of the reduced order model, and the matrices $\hat{L}$ and $\hat{M}$ are given by

$$\hat{L} = (P^\top P)^{-1} P^\top L P, \tag{4.16}$$

$$\hat{M} = (P^\top P)^{-1} P^\top M, \tag{4.17}$$

where $P(\pi)$ is denoted shortly by P.

Next, we show that the reduced model (4.15) is associated with a leader-follower multi-agent system defined on a graph, in a similar form as (4.7). First, observe that $\hat{M}$ has a similar structure as M. More precisely, each column of $\hat{M}$ contains exactly one nonzero-element, indicating a leader. The only difference is that the non-zero elements do not need to be 1 anymore. This can be interpreted by saying that input signals are now weighted.

It is easy to observe that the matrix $\hat{L}$ is equal to the Laplacian matrix of a weighted *directed* graph, say $\hat{G} = (\hat{V}, \hat{E}, \hat{A})$. In fact, as a consequence of the aforementioned projection, the underlying graph G is mapped to the graph $\hat{G}$. In particular, each cell of π in G is mapped to a vertex in $\hat{G}$. Hence, the number of vertices in $\hat{G}$ is equal to the cardinality of π, i.e. the number of cells in π. Moreover, there is an arc from vertex p to vertex q in $\hat{G}$ if and only if there exist $i \in C_p$ and $j \in C_q$ with $p \neq q$ such that $\{i, j\} \in E$. Therefore, $\hat{G}$ is a *symmetric directed* graph, i.e. $(i, j) \in \hat{E} \Leftrightarrow (j, i) \in \hat{E}$. For the relationship between the matrices A and $\hat{A} = [\hat{a}_{pq}]$, we have

$$\hat{a}_{pq} = \frac{1}{|C_p|} \sum_{i \in C_p, j \in C_q} a_{ij}, \tag{4.18}$$

for $p \neq q$, where $|.|$ denotes the cardinality of a set. Observe that the row sums of $\hat{L}$ are indeed zero as $P(\pi)\mathbb{1} = \mathbb{1}$ and $L\mathbb{1} = 0$, where $\mathbb{1}$ denotes the vector of ones of appropriate dimension. Note that the matrix $\hat{L}$ is not necessarily symmetric, as the number of vertices may differ from cell to cell in π. However, $\hat{L}$ is similar to the symmetric matrix $(P^\top P)^{-\frac{1}{2}} P^\top L P (P^\top P)^{\frac{1}{2}}$, thus $\hat{L}$ inherits nice properties of L, like diagonalizability and having real eigenvalues.

As observed, the reduced order model (4.15) is associated with a new multi-agent system where the diffusive coupling rule is defined based on the graph $\hat{G}$. The idea behind the proposed projection is that the partition π clusters some vertices (agents) together, and these vertices are mapped to a single vertex in the reduced order (projected) model. In addition, note that the components of the reduced state $\hat{x}$ approximate the averages of the states of the agents that are cellmates in π. In case the agents that are cellmates in π have a "similar" interconnection to the rest of the network, then this approximation tends to be exact. We will clarify what we mean by "similar" in the next section.

Example 4.3.2 We will now return to our example in Figure 4.1. Suppose that agents (vertices) 1 and 7 are leaders. Then the multi-agent system associated with the graph G is given by:

$$\dot{x} = -Lx + Mu \tag{4.19}$$

where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $P(\pi)$, given by (4.13), be denoted in short by P. Then, the reduced order model obtained by clustering the agents according to π, given by (4.12), is given by

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{4.20}$$

where $\hat{L}$ and $\hat{M}$ are computed as

$$\hat{L} = (P^\top P)^{-1}P^\top LP = \begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \hat{M} = (P^\top P)^{-1}P^\top M = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The graph $\hat{G}$ associated with the reduced order system (4.20) is shown in Figure 4.2.

Observe that $\hat{G}$ has 5 vertices, each of which corresponds to a cell in π. For instance, vertex 1 in $\hat{G}$ corresponds to the cell $C_1 = \{1, 2, 3, 4\}$, and vertex 2 corresponds to the cell $C_2 = \{5, 6\}$. Then the arcs $(1, 2)$ and $(2, 1)$ of $\hat{G}$ account for the coupling between C_1 and C_2 in the graph G given by Figure 4.1. In particular, the weight associated to the arc $(2, 1) \in \hat{E}$ is indeed equal to the average of the weights of the edges $\{i, j\} \in E$ with $i \in C_1$ and $j \in C_2$, as given by (4.18). Observe that the input weights indicated by $\hat{M}$ depend on the cardinality of the cells in π. For instance, $\hat{x}_1$ receives one fourth of u_1 in the reduced order model (4.20). This value indeed indicates the average of the input signals received by the agents in C_1. $\square$

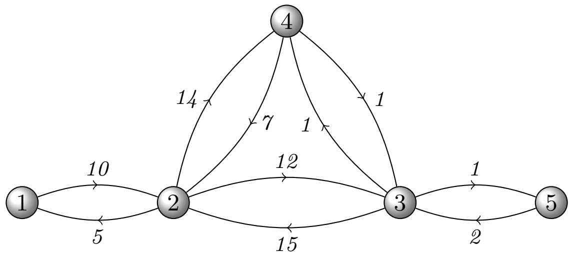

Figure 4.2: $\hat{G} = (\hat{V}, \hat{E}, \hat{A})$

Before proceeding, we point out another useful relationship between the Laplacian matrices L and $\hat{L}$. We need to recap the notion of interlacing first. Let X be a real symmetric $n \times n$ matrix, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of X in an increasing order. Also let Y be a real symmetric $m \times m$ matrix, where $m \leqslant n$. Moreover, let $\mu_1, \mu_2, \ldots, \mu_m$ denote the eigenvalues of Y in an increasing order. Then we say that the eigenvalues of Y *interlace* the eigenvalues of X if

$$\lambda_i \leqslant \mu_i \leqslant \lambda_{n-m+i} \tag{4.21}$$

for each $i = 1, 2, \ldots, m$.

The eigenvalues of $\hat{L}$, given by (4.16), interlace the eigenvalues of L as stated in the following lemma.

Lemma 4.3.3 *Let L be a symmetric matrix, and let $\hat{L}$ be given by* (4.16) *for a given partition π. Then the eigenvalues of $\hat{L}$ interlace the eigenvalues of L.*

Proof. Recall that $P^\top P$ is a diagonal matrix with strictly positive diagonal elements. Clearly, the matrix $\hat{L}$ is similar to the matrix $F^\top LF$ for $F = P(P^\top P)^{-\frac{1}{2}}$. Now, noting that $F^\top F = I$, the result immediately follows by [30, Thm. 9.5.1]. ■

Next, we discuss consensus and convergence rate preservation in the reduced order model. Roughly speaking, consensus means that the agents agree on a certain quantity of interest. Consensus is defined in the absence of the external input, thus we deal with the following multi-agent system:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{4.22}$$

We say that the multi-agent system (4.22) reaches consensus if for any arbitrary initial condition we have:

$$\lim_{t \to \infty} x_i(t) - x_j(t) = 0 \text{ for all } i, j \in V \tag{4.23}$$

Now, suppose that the original multi-agent system reaches consensus, and hence (4.23) holds. Then, consensus is preserved in the reduced order system:

Theorem 4.3.4 *Consider the multi-agent system* (4.7) *with* $u = 0$, *i.e.*

$$\dot{x} = -Lx, \tag{4.24}$$

and suppose that consensus is reached for this system. For any given partition π, *the reduced order multi-agent system*

$$\dot{\hat{x}} = -\hat{L}\hat{x} \tag{4.25}$$

also reaches consensus, where $\hat{L}$ *is given by* (4.16).

Proof. Clearly, as the multi-agent system (4.24) reaches consensus, zero is a simple eigenvalue of the Laplacian matrix L. In addition, note that $\hat{L}\mathbb{1} = 0$. By Lemma 4.3.3, we conclude that zero is also a simple eigenvalue of $\hat{L}$, and the rest of the eigenvalues are real and strictly positive. This implies that the reduced order model (4.25) reaches consensus. ∎

Note that, because of the interlacing property provided in Lemma 4.3.3, the rate of convergence in the reduced order model is at least as fast as that of the original model.

4.4 Input-output approximation of multi-agent systems

We have observed that by applying an appropriate projection to the original multi-agent system defined on G, we obtain a reduced-order model that represents a multi-agent system defined on a new graph $\hat{G}$. Moreover, consensus and the convergence rate are preserved by this model reduction. In this section, we discuss appropriate choices of partitions such that the behavior, in particular the input-output behavior, of the reduced and the original multi-agent system are "close" in a certain sense. Without loss of generality, assume that graph G is connected. Obviously, in case G is not connected, one can apply the proposed model reduction technique on disconnected components of G, individually.

We first include some output variables in system (4.7). Note that in the context of distributed control, differences of the states of the agents play a crucial role. In fact, these differences reflect the disagreement among the agents, and the network reaches consensus if this disagreement vanishes as time evolves. Observe that the differences of the states of communicating agents are embedded in the incidence matrix. Therefore, we choose the output variables as

$$y = W^{\frac{1}{2}} R^{\top} x, \tag{4.26}$$

where W is given by (4.2). Hence, the disagreement in the states of a pair of agents is reflected in the output variables (4.26) in accordance with the weight of the edge connecting those agents (vertices). Furthermore note that, as G is connected, the multi-agent system (4.7) reaches consensus if and only if $\lim_{t\to\infty} y(t) = 0$ for all initial states $x(0)$. It is also worth mentioning that, by (4.3), we have $\|y\|^2 = x^\top Lx = \frac{1}{2}\sum_{i,j} a_{ij}(x_i - x_j)^2$ which is a measure of group disagreement (see e.g. [73]).

Consequently, we obtain the following input/state/output model for the original multi-agent system defined on the graph G:

$$\dot{x} = -Lx + Mu, \tag{4.27a}$$

$$y = W^{\frac{1}{2}} R^\top x, \tag{4.27b}$$

where $x \in \mathbb{R}^n$, L is the Laplacian, and R is the incidence matrix of G. Now, let again π be a partition of G. Then, the input/state/output model for the reduced order (projected) model is obtained as

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{4.28a}$$

$$y = W^{\frac{1}{2}} \hat{R}^\top \hat{x}, \tag{4.28b}$$

where $\hat{x} \in \mathbb{R}^r$ with $r \leqslant n$, $\hat{L}$ is given by (4.16), $\hat{M}$ is given by (4.17), and $\hat{R} = P^\top R$.

Recall that $\hat{L}$ is the Laplacian matrix of the weighted symmetric directed graph $\hat{G}$. It is worth mentioning that the matrix $\hat{R}$ is related to the incidence matrix of the graph $\hat{G}$, which we denote by R'. Indeed, it can be shown that each column of $\hat{R}$ is either equal to zero or is equal to a column of R'. Note that the zero columns of $\hat{R}$ indeed correspond to the difference of the states of cellmate agents, which are approximated to be identical in deriving the reduced order model.

Clearly, different choices of graph partitions lead to different reduced order models, and one may think of choosing an appropriate partition to approximate the behavior of the original multi-agent system relatively well. Note that we have two trivial partitions here, one is taking each vertex (agent) as a singleton and the other one is $\pi = \{V\}$. In the first case, no order reduction occurs and the corresponding model reduction error is zero. In the latter case, the network topology is neglected, and the reduced model is a single agent with a zero transfer matrix from u to y. Thus, these two trivial partitions indicate the finest and the coarsest approximation by graph partitions. Clearly, similar to model reduction in ordinary linear systems, this leads to a compromise between the order of the reduced model and the accuracy of the approximation.

Recall that the dynamics of the individual nodes are the same. So, an appropriate partitioning (clustering) decision solely depends on the graph topology. Hence, in order to achieve a better approximation, it is expected that the agents (vertices) that are connected to the rest of the network in a "similar" fashion should be clustered in one cell. In order to formalize this heuristic idea, in what follows we distinguish a class of partitions, namely *almost equitable partitions*, from other partitions. An easily computable model reduction error in the sense of the $\mathcal{H}_2$-norm will be provided for this class of partitions. The notion of almost equitability is recapped next.

Let $G = (V, E)$ be an unweighted undirected graph. For a given cell $C \subseteq V$, we write $N(i, C) = \{j \in C \mid \{i, j\} \in E\}$. Note that $N(i, C)$ indicates the number of neighbors vertex i has in cell C. We call a partition $\pi = \{C_1, C_2, \ldots, C_r\}$ an *almost equitable partition* (AEP) of G if for each $p, q \in \{1, 2, \cdots, r\}$ with $p \neq q$ there exists an integer d_{pq} such that $|N(i, C_q)| = d_{pq}$ for all $i \in C_p$.

An almost equitable partition, say π, has the key property that $\operatorname{im} P(\pi)$ is L-invariant (see e.g. [107, Lem. 2]). Note that we call a subspace $\mathcal{X} \subseteq \mathbb{R}^n$ $\mathbf{A}$-*invariant* if $\mathbf{A}\mathcal{X} \subseteq \mathcal{X}$ where $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$. To incorporate the case of weighted graphs, the notion of almost equitability can be extended as follows.

Let $G = (V, E, A)$ be a weighted undirected graph. Recall that a_{ij} indicates the weight associated to the edge $\{i, j\}$. We call a partition $\pi = \{C_1, C_2, \ldots, C_r\}$ an *almost equitable partition* (AEP) of G if for each $p, q \in \{1, 2, \cdots, r\}$ with $p \neq q$ there exists an integer d_{pq} such that $\sum_{j \in N(i, C_q)} a_{ij} = d_{pq}$ for all $i \in C_p$.

As an example consider the graph G in Figure 4.1. It is easy to verify that π given by (4.12) is an AEP of G.

Note that the definition of almost equitability for weighted graphs includes the case of unweighted graphs as a special case. In fact, for unweighted graphs, $a_{ij} = 1$ whenever $\{i, j\} \in E$. Hence, the quantity $\sum_{j \in N(i, C_q)} a_{ij}$ coincides with the cardinality of $N(i, C_q)$. Moreover, the L-invariance property remains valid for weighted graphs, as stated in the following lemma.

Lemma 4.4.1 *Let π be a partition of a weighted undirected graph G, and let L denote the Laplacian matrix of G. Then π is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant, i.e.*

$$L \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi) \tag{4.29}$$

Proof. The proof is analogous to that of [14, Prop. 1]. ■

Now, assume that $\pi = \{C_1, C_2, \ldots, C_r\}$ is an AEP of a weighted undirected graph G, and suppose that the reduced order model (4.28) is obtained from (4.27) by choosing the partition π. Moreover, recall that $V_L = \{v_1, v_2, \ldots, v_m\}$, and for each $i = \{1, 2, \ldots, m\}$, let k_i be an integer such that $v_i \in C_{k_i}$. Then, the (normalized) model reduction error involved in obtaining the reduced order model (4.28) is provided in the following theorem.

Theorem 4.4.2 *Let G be a weighted undirected graph, and assume G is connected. Let $\pi = \{C_1, C_2, \dots, C_r\}$ be an almost equitable partition of G. Suppose that the reduced order multi-agent system* (4.28) *is obtained from* (4.27) *by choosing the partition π. Also, let S and $\hat{S}$ denote the transfer matrices from u to y in* (4.27) *and* (4.28)*, respectively. Then, we have*

$$\frac{\|S - \hat{S}\|_2^2}{\|S\|_2^2} = \frac{\sum_{i=1}^{m}(1 - \frac{1}{|C_{k_i}|})}{m(1 - \frac{1}{n})}, \tag{4.30}$$

where n is the total number of vertices (agents) in G and the k_is are defined as before.

Proof. Note that the columns of $P(\pi)$ form an orthogonal set of vectors in $\mathbb{R}^n$. We complete this set to an orthogonal basis for $\mathbb{R}^n$. In particular, we construct a matrix T as $T = \begin{bmatrix} P & Q \end{bmatrix}$, where $P(\pi)$ is denoted in short by P, and Q is an $n \times (n-k)$ matrix such that the columns of T are orthogonal. Observe that we have

$$P^\top Q = 0. \tag{4.31}$$

Now, we apply the state space transformation $x = T\tilde{x}$ to system (4.27). Consequently, we obtain the following input/state/output system:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = -\begin{bmatrix} (P^\top P)^{-1}P^\top LP & (P^\top P)^{-1}P^\top LQ \\ (Q^\top Q)^{-1}Q^\top LP & (Q^\top Q)^{-1}Q^\top LQ \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} (P^\top P)^{-1}P^\top M \\ (Q^\top Q)^{-1}Q^\top M \end{bmatrix} u \tag{4.32a}$$

$$y = \begin{bmatrix} W^{\frac{1}{2}}R^\top P & W^{\frac{1}{2}}R^\top Q \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{4.32b}$$

Clearly, the transfer matrices from u to y in (4.27) and (4.32) are identical. Moreover, observe that the reduced order model (4.28) is the system obtained by truncating the state components $\tilde{x}_2$ in (4.32). Since π is an AEP of G, $\operatorname{im} P$ is L-invariant by Lemma 4.4.1. Thus, there exists a matrix X such that $LP = PX$. Hence, we obtain

$$Q^\top LP = 0. \tag{4.33}$$

Therefore, the transfer matrices S and $\hat{S}$ of the original system (4.27) and its reduced order model (4.28), respectively, are related by:

$$S(s) = \hat{S}(s) + \Delta(s), \tag{4.34}$$

where

$$\Delta(s) = W^{\frac{1}{2}}R^\top Q(sI + (Q^\top Q)^{-1}Q^\top LQ)^{-1}(Q^\top Q)^{-1}Q^\top M. \tag{4.35}$$

By using (4.3) and (4.33), we have $\hat{S}^\top(-s)\Delta(s) = 0$. Hence, we have

$$\|S\|_2^2 = \|\hat{S}\|_2^2 + \|\Delta\|_2^2. \tag{4.36}$$

Now, let the matrices $X_1 \in \mathbb{R}^{n\times n}$ and $Y_1 \in \mathbb{R}^{r\times r}$ be defined as:

$$X_1 = \int_0^\infty e^{-Lt} L e^{-Lt} dt, \tag{4.37}$$

$$Y_1 = \int_0^\infty e^{-\hat{L}^\top t} P^\top L P e^{-\hat{L}t} dt, \tag{4.38}$$

where

$$\hat{L} = (P^\top P)^{-1} P^\top L P.$$

Of course, one should address the issue of convergence of these improper integrals. Since the original model reaches consensus we have $e^{-Lt}Le^{-Lt} \to 0$ as $t \to \infty$. Hence, since the components of $e^{-Lt}Le^{-Lt}$ are products of polynomials and exponentials, the integral defining X_1 exists. Similarly, by Theorem 4.3.4 the reduced order model (4.25) also reaches consensus and therefore the integral defining Y_1 exists as well.

Let $L = U\Lambda U^\top$ be a spectral decomposition of the Laplacian, where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

with $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n$ are the eigenvalues of L and U is an orthogonal matrix. Note that $\lambda_2 > 0$, due to the connectedness of G. Moreover, the first column of the matrix U is equal to the normalized vector of ones, i.e. $\frac{1}{\sqrt{n}}\mathbb{1}$. Thus X_1 is computed as

$$\begin{aligned} X_1 = \int_0^\infty e^{-Lt} L e^{-Lt} dt &= \int_0^\infty L e^{-2Lt} dt \\ &= -\frac{1}{2} e^{-2Lt} \mid_0^\infty dt = -\frac{1}{2} U e^{-2\Lambda t} U^\top \mid_0^\infty = \frac{1}{2} I_n - \frac{1}{2n} \mathbb{1}\mathbb{1}^\top. \end{aligned} \tag{4.39}$$

In addition, with T given as above, we have

$$\begin{aligned} T^\top X_1 T &= \int_0^\infty T^\top e^{-Lt} L e^{-Lt} T dt \\ &= \int_0^\infty e^{-(T^{-1}LT)^\top t} T^\top L T e^{-T^{-1}LTt} dt \\ &= \int_0^\infty e^{-\begin{bmatrix} \hat{L}^\top & 0 \\ 0 & * \end{bmatrix} t} \begin{bmatrix} P^\top L P & 0 \\ 0 & * \end{bmatrix} e^{-\begin{bmatrix} \hat{L} & 0 \\ 0 & * \end{bmatrix} t} dt, \end{aligned}$$

where we have used (4.33) to derive the last equality, and where " $*$ " denotes values that are not of interest. Hence, we obtain that

$$T^\top X_1 T = \begin{bmatrix} Y_1 & 0 \\ 0 & * \end{bmatrix}.$$

This yields, $Y_1 = P^\top X_1 P$. Therefore, by (4.39), Y_1 is computed as

$$Y_1 = \frac{1}{2} P^\top P - \frac{1}{2n} P^\top \mathbb{1}\mathbb{1}^\top P.$$

Next we compute the values $\|S\|_2^2$ and $\|\hat{S}\|_2^2$. By (4.3), it is easily seen that

$$\|S\|_2^2 = \operatorname{trace} M^\top X_1 M.$$

Hence, by (4.39), $\|S\|_2^2$ is computed as

$$\begin{aligned} \|S\|_2^2 &= \operatorname{trace} M^\top X_1 M = \frac{1}{2} \operatorname{trace} M^\top (I_n - \frac{1}{n}\mathbb{1}\mathbb{1}^\top) M \\ &= \frac{1}{2} \operatorname{trace} M M^\top (I_n - \frac{1}{n}\mathbb{1}\mathbb{1}^\top). \end{aligned}$$

Note that $MM^\top$ is a diagonal matrix, where the diagonal elements are either zero or 1. In particular, the i^{th} diagonal element is equal to 1 if $i \in V_L$, and is equal to zero otherwise. Thus, we conclude that

$$\|S\|_2^2 = \frac{m}{2}(1 - \frac{1}{n}), \tag{4.40}$$

where m is the cardinality of V_L as before. Moreover, we have

$$\begin{aligned} \|\hat{S}\|_2^2 &= \operatorname{trace} M^\top P (P^\top P)^{-1} Y_1 (P^\top P)^{-1} P^\top M \\ &= \frac{1}{2} \operatorname{trace} M^\top P (P^\top P)^{-1} (P^\top P - \frac{1}{n} P^\top \mathbb{1}\mathbb{1}^\top P)(P^\top P)^{-1} P^\top M \\ &= \frac{1}{2} \operatorname{trace} M M^\top (P(P^\top P)^{-1} P^\top - \frac{1}{n} P (P^\top P)^{-1} P^\top \mathbb{1}\mathbb{1}^\top P (P^\top P)^{-1} P^\top). \end{aligned}$$

It is easy to verify that $P(P^\top P)^{-1} P^\top \mathbb{1} = \mathbb{1}$. Hence, we obtain that

$$\|\hat{S}\|_2^2 = \frac{1}{2} \operatorname{trace} M M^\top (P(P^\top P)^{-1} P^\top - \frac{1}{n}\mathbb{1}\mathbb{1}^\top) \tag{4.41}$$

Recall the diagonal structure of $MM^\top$. Also recall that $v_i \in C_{k_i}$ for each $v_i \in V_L$. Then it is straightforward to check that (4.41) yields

$$\|\hat{S}\|_2^2 = \frac{1}{2} \sum_{i=1}^{m} \frac{1}{|C_{k_i}|} - \frac{m}{2n}. \tag{4.42}$$

Therefore, by (4.36) and (4.40) we obtain that

$$\|\Delta\|_2^2 = \frac{m}{2}(1 - \frac{1}{n}) - \frac{1}{2} \sum_{i=1}^{m} \frac{1}{|C_{k_i}|} + \frac{m}{2n} = \frac{1}{2} \sum_{i=1}^{m} (1 - \frac{1}{|C_{k_i}|}). \tag{4.43}$$

This together with (4.40) completes the proof. ■

Theorem 4.4.2 provides a simple and easily computable expression for the model reduction error in case π is an AEP of G. Depending on the structure of π, the normalized model reduction error (4.30) takes a value between 0 and 1. Moreover, for a given multi-agent system, this value is determined by the *population*, i.e. cardinality, of those cells of π containing the leaders. Clearly, the less populated these cells are, the less model reduction error we have. An interesting case is when all the leaders appear as singleton in π. Then, the corresponding model reduction error is zero by (4.30). Indeed, this case is a special almost equitable partition, which is studied in the context of controllability of multi-agent systems (see e.g. [54], [107]). In particular, if π^* is an AEP with each leader appearing as a singleton, then $\operatorname{im} P(\pi^*)$ is an upper bound for the controllable subspace of (4.7) (see [107, Thm. 3]). Therefore, the proposed model reduction technique in the case of $\pi = \pi^*$, indeed corresponds to removing uncontrollable modes. Thus, the input-output behavior remains unchanged, which is in accordance with the model reduction error being zero in (4.30).

Example 4.4.3 As an example, consider again the multi-agent system (4.19) corresponding to the graph G given in Figure 4.1, and the reduced order model (4.20) obtained from the partition π given by (4.12). For system (4.19), include output variables as in (4.27b). The output equations for the reduced order system (4.20) are given in (4.28b). Recall that π is an AEP of G. Also recall that the leader set is $\{1, 7\}$ in this case. Clearly, we have $1 \in C_1 = \{1, 2, 3, 4\}$ and $7 \in C_3 = \{7\}$. For an arbitrary partition π, let $\Xi(\pi)$ denote the corresponding normalized model reduction error given by the left hand side of (4.30). Then, by Theorem 4.4.2, the normalized model reduction error $\Xi(\pi)$ in our example is computed as:

$$\Xi(\pi) = \frac{(1 - \frac{1}{|C_1|}) + (1 - \frac{1}{|C_3|})}{10(1 - \frac{1}{2})} = 0.15.$$

Remark 4.4.4 As mentioned in Subsection 4.3, the reduced order model (4.15) approximates the dynamics of the average of the states of the agents that are cellmates in π. By (4.32a) and (4.33), it can be observed that this approximation is exact in case π is an almost equitable partition of G. That is, (4.15) indeed describes the dynamics of the average of the states of cellmates agents.

Previously, we have established an explicit formula for the model reduction error in case clustering is performed with respect to an almost equitable partition. Of course we are also interested in computing or estimating the errors associated with *arbitrary* partitions of the graph.

In order to attack this issue, we will first compare the model reduction error corresponding to an almost equitable partition, say π_0, to that of an arbitrary, not necessarily almost equitable, partition, say π. We restrict our attention to the case in which the partitions π and π_0 are *comparable* in the sense that one is *finer* than the other.

Given two partitions π_1 and π_2 of the graph G, we call π_1 *finer* than π_2 if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leqslant \pi_2$. Alternatively, π_2 is called *coarser* than π_1. It is immediate that

$$\pi_1 \leqslant \pi_2 \quad \iff \quad \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1). \tag{4.44}$$

Now, we have the following result.

Theorem 4.4.5 *Let π_0 be an almost equitable partition of G. Then for every partition π that is coarser than π_0 we have*

$$\Xi(\pi_0) \leqslant \Xi(\pi).$$

Proof. Suppose that π is a partition of G and $\pi \geqslant \pi_0$. Let S_0 and $\tilde{S}$ denote the transfer matrices from u to y in the reduced order model (4.28) corresponding to the partitions π_0 and π, respectively. Also let S denote the transfer matrix from u to y in (4.27), as before. Then, clearly, it suffices to show that

$$\|S - \tilde{S}\|_2^2 \geqslant \|S - S_0\|_2^2. \tag{4.45}$$

We have

$$\begin{aligned}
\|S - \tilde{S}\|_2^2 &= \|S - S_0 + S_0 - \tilde{S}\|_2^2 \\
&= \|S - S_0\|_2^2 + \|S_0 - \tilde{S}\|_2^2 + 2\langle S - S_0, S_0 - \tilde{S}\rangle \\
&\geqslant \|S - S_0\|_2^2 + 2\langle S - S_0, S_0\rangle - 2\langle S - S_0, \tilde{S}\rangle,
\end{aligned}$$

where $\langle S_1, S_2\rangle = \int_{-\infty}^{\infty} \operatorname{trace} S_1^\top(-i\omega)S_2(i\omega)d\omega$ is the inner product in $\mathcal{H}_2$.

Since π_0 is an AEP of G, by Lemma 4.4.1 we have

$$L \operatorname{im} P(\pi_0) \subseteq \operatorname{im} P(\pi_0). \tag{4.46}$$

In addition, as $\pi \geqslant \pi_0$, the subspace inclusion $\operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi_0)$ holds. Hence, by (4.46), we obtain that $L \operatorname{im} P(\pi) \subseteq L \operatorname{im} P(\pi_0) \subseteq \operatorname{im} P(\pi_0)$. Therefore, there exist matrices X and Y such that

$$LP(\pi_0) = P(\pi_0)X \tag{4.47}$$

and

$$LP(\pi) = P(\pi_0)Y. \tag{4.48}$$

Now, recall that $S(s) - S_0(s) = \Delta(s)$ where Δ is given by (4.35). Then, by using (4.3), (4.31), (4.47), and (4.48), it is easy to verify that $\langle S - S_0, S_0 \rangle = 0$ and $\langle S - S_0, \tilde{S} \rangle = 0$. Hence, (4.45) holds which completes the proof. ■

As a consequence of the above, if, starting from a given AEP, we choose an arbitrary partition that is coarser than this AEP and perform reduction based upon the latter, then the error will be at least as big as the error associated with the AEP.

Using the previous result, we are now able to estimate a lower bound for the error associated with an arbitrary, not necessarily AEP, partition. let π be an arbitrary partition of G. Consider the set

$$\Pi_{\mathrm{AEP}}(\pi) = \{\pi_0 \mid \pi_0 \text{ is an AEP of } G \text{ and } \pi_0 \leqslant \pi \}$$

of all almost equitable partitions of G that are finer than π. This set can be shown to contain a unique maximal element (see [107]), which we will call *the maximal almost equitable partition finer than* π, and that will be denoted by $\pi^*_{\mathrm{AEP}}(\pi)$. It follows from Theorem 4.4.5 that a lower bound on the error associated with the given partition π is given by the error associated with the maximal almost equitable partition finer than π:

$$\Xi(\pi^*_{\mathrm{AEP}}(\pi)) \leqslant \Xi(\pi)$$

An algorithm to actually compute $\pi^*_{\mathrm{AEP}}(\pi)$ for a given partition π was given in [107]. Note that $\Xi(\pi^*_{\mathrm{AEP}}(\pi))$ can then be computed using Theorem 4.4.2.

4.5 Conclusions

In this chapter, by means of graph partitions we have established a projection based model reduction method for multi-agent systems defined on a graph. Reduced order models are obtained by clustering the vertices (agents) of the underlying communication graph in accordance with suitable graph partitions. In particular, the states of the vertices that are clustered together are approximated to be identical in deriving the reduced order models. As observed, the spatial structure of the network is preserved in this reduction process, and the reduced order models are realized as multi-agent systems defined on a new graph of smaller size. We have shown that if the original multi-agent system reaches consensus, then so does the reduced order model. As observed the underlying intuitive idea is to cluster together the vertices (agents) which are connected to the rest of the network in a similar fashion. This heuristic idea is formally formulated in terms of almost equitable partitions. Corresponding to an almost equitable partition, an explicit formula for the $\mathcal{H}_2$-norm of the error system has been provided. The proposed formula is simple, easy to compute, and can be derived directly from the graph partition involved in the reduction procedure. We also have shown that the error obtained by taking an arbitrary partition of the graph is at least as big as the one obtained by using the maximal almost equitable partition finer than the given partition. We have adopted a running example for illustration of the proposed results.

Chapter 5

Stability and synchronization analysis of multi-agent systems with arbitrary switching topologies

5.1 Introduction

The distributed control of multi-agent systems has gained a lot of attention during the last decade. In particular, the consensus problem has been widely investigated for networks of agents. Consensus roughly means that the agents of a network reach an agreement on the state components' values. The pioneering work in this direction has been carried out in [73], [27, 41] and [79] for the case where the agents have simple dynamics, like single or double integrators. . An excellent review can be found in [72]. Results on consensus for the case where the agents have general, yet identical, linear dynamics with time-independent communication topology are reported in some recent papers (see e.g. [47], [96], [53]).

Despite the extensive amount of research available in the context of consensus and synchronization of multi-agent systems, relatively few works have considered network of agents with general linear dynamics together with a time-dependent communication topology. This situation is considered in [83] and [84] where consensus protocols are proposed for possibly time-varying communication structures. However, in these papers, the agents are not allowed to have exponentially unstable dynamics. Assuming nonzero dwell time for the switching among possible communication graphs, the consensus problem for multi-agent systems with general linear dynamics is studied in [99]. Sufficient conditions for achieving synchronization via fast switching are established in [89].

In this chapter, we consider a network of agents with general, but identical, linear dynamics, and the communication topology may switch within a finite set of admissible topologies. We allow switching to be arbitrary, meaning that the switching may occur at arbitrary time instances. Our aim is to analyze the stability and synchronization of such networks for the case where the output information of the agents is transmitted via network communication. For stability analysis, assuming the dynamics of the agents to be stable, we derive small gain type of conditions guaranteeing overall stability of the network as the agents communicate with respect to a time-dependent topology.

Conditions verifying the synchronization of networks with time-independent topologies are obtained in the literature mostly by applying a certain state space transformation to obtain an appropriate decomposition of the overall network. These conditions depend on the agent dynamics together with the nontrivial Laplacian eigenvalues (see e.g [47, 84]). However, the same technique cannot be adopted directly for the case of networks with switching topologies unless the admissible topologies satisfy certain constraints like commutativity or simultaneous triangularizability of the corresponding Laplacian matrices (see [108]).

Summarizing, we want to highlight the following points which together make the contribution of the present chapter novel compared to earlier references.

- We consider general linear dynamics for the agents, thus the agents are allowed to have high order or unstable dynamics.

- The underlying communication topology is time-dependent and may switch within a finite set of undirected graphs.

- No a priori relationship among the underlying communication graphs is assumed.

- The switching is arbitrary, meaning that no restriction on the rate of the switching signal is imposed.

In the present chapter, the synchronization problem of the network is, firstly, viewed as an *output stability* problem (see [102, sec. 4.4.] or [94, exc. 4.10]). Then, by performing a suitable decomposition, the synchronization problem boils down to asymptotic stability of certain subsystems of the network. Consequently, given the agents' dynamics, a coupling rule, and a set of admissible topologies, we derive conditions under which synchronization is guaranteed. We show that synchronization under arbitrary switching is achieved if a certain pair of linear matrix inequalities (LMI) admits a solution. The LMI conditions obtained are easy to check and, similar to the case of time-independent topologies, they depend only on the agents' dynamics and the nontrivial Laplacian eigenvalues. Next, solvability of the proposed LMI is discussed. It is observed that necessary and sufficient conditions for solvability of the proposed LMI are available for the special case of single-input single-output dynamics. In addition, for the case of multi-input multi-output, conditions in terms of small gain and passivity of certain subsystems of the networks are derived which guarantee the existence of a solution for the proposed LMI, and, thus synchronization of the network.

In case the overall information on the relative states of the neighboring agents are not available, observer- based protocols achieving synchronization are proposed in the literature (see e.g. [47], [83], [84]), which use only the relative output information of the adjacent agents. We show that the proposed conditions for synchronization analysis carry over to the case of these observer-based protocols.

The chapter is organized as follows. First, in Section 5.2, we introduce some notation and review some basic definitions. Stability analysis of networks with fixed and switching topologies is carried out in Section 5.3. In Section 5.4, first synchronization of networks with a fixed topology is revisited. The results are then extended to the case of switching topologies, and small gain and passivity types of conditions are proposed. Synchronization analysis for the case of observer-based protocols are also provided at the end of this section. Finally, Section 5.5 is dedicated to conclusions.

5.2 Preliminaries

For $i = 1, 2, \dots, N$, let $G_i = (V, E_i)$ be a simple undirected (unweighted) graph with vertex set $V = \{1, 2, \dots, p\}$ and edge set E_i where an edge is a two-element subset of V. Corresponding to this set of graphs, a diffusively coupled multi-agent system

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t) \tag{5.1a}$$

$$y_j(t) = Cx_j(t) \tag{5.1b}$$

together with the diffusive coupling rule

$$u_j(t) = \sum_{\{i,j\} \in E_{\sigma(t)}} (y_i(t) - y_j(t)), \tag{5.1c}$$

where $j \in V$, $x_j \in \mathbb{R}^n$ is the state of agent j, $u_j \in \mathbb{R}^m$ is the diffusive coupling term, and $\sigma : \mathbb{R}^+ \to \{1, 2, \dots, N\}$ is a right-continuous piecewise constant function.

Throughout this chapter it is assumed that (A, B) is stabilizable. This is a necessary condition for stability or synchronization of the multi-agent system (5.1), even in the special case of a time-independent topology, i.e. $N = 1$. We introduce the following nomenclature for later use. For each $i = 1, 2, \dots, N$, let L_i denote the Laplacian matrix corresponding to the graph $G_i = (V, E_i)$. The eigenvalues of L_i are denoted by $0 = \lambda_1^i \leqslant \lambda_2^i \leqslant \cdots \leqslant \lambda_p^i$. We define

$$\underline{\lambda} = \min\{\lambda_j^i \mid i \in \{1, 2, \dots, N\}, j \in \{2, 3, \dots, p\}\} \tag{5.2a}$$

and

$$\overline{\lambda} = \max\{\lambda_j^i \mid i \in \{1, 2, \dots, N\}, j \in \{2, 3, \dots, p\}\}. \tag{5.2b}$$

Note that $\lambda_1^i = 0$ for $i = 1, 2, \dots, N$ are excluded in the above definitions.

The multi-agent system (5.1) can be written in a compact form as

$$\dot{x}(t) = \mathcal{A}_{\sigma(t)} x(t), \tag{5.3}$$

where $x = \mathrm{col}(x_1, \dots, x_p)$, and $\mathcal{A}_{\sigma(t)} = I_p \otimes A - L_{\sigma(t)} \otimes BC$, where "$\otimes$" denotes the Kronecker product. The following basic properties of the Kronecker product are frequently used in the sequel: $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A \otimes B)^\top = A^\top \otimes B^\top$, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

We call an absolutely continuous function x a solution of (5.3) if there exists a switching signal σ such that (5.3) is satisfied for almost all $t \geqslant 0$. We call the switched linear system (5.3) globally uniformly exponentially stable (GUES) if there exist positive constants α and β such that for any solution x of (5.3) we have $\|x(t)\| \leqslant \beta e^{-\alpha t} \|x(0)\|$ for all $t \geqslant 0$.

Remark 5.2.1 A sufficient condition for GUES is the existence of a common quadratic Lyapunov function (CQLF), i.e. the existence of a positive definite matrix X such that $\mathcal{A}_i^\top X + X\mathcal{A}_i < 0$ for all $i = 1, 2, ..., N$. (see e.g. [48]).

5.3 Stability analysis of networks

5.3.1 Fixed topology

First, consider the special case of the network (5.1) where $N = 1$. In this case, the edge set, namely E, is time independent, and the network state space representation is given by

$$\dot{x}(t) = (I \otimes A - L \otimes BC)x(t), \tag{5.4}$$

where $x = \mathrm{col}(x_1, \dots, x_p)$ and L is the Laplacian matrix corresponding to $G = (V, E)$. Since the Laplacian matrix L is symmetric, it can be diagonalized as

$$U^\top L U = \Lambda, \tag{5.5}$$

where Λ is a diagonal matrix given by

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_p) \tag{5.6}$$

with $0 = \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_p$, and U is an orthogonal matrix with the first column being the normalized vector of ones.

Applying the state space transformation $\tilde{x} = (U^\top \otimes I)x$, we obtain

$$\dot{\tilde{x}}(t) = (I \otimes A - \Lambda \otimes BC)\tilde{x}(t). \tag{5.7}$$

Note that

$$I \otimes A - \Lambda \otimes BC = \mathrm{blockdiag}(A - \lambda_1 BC, A - \lambda_2 BC, \dots, A - \lambda_p BC). \tag{5.8}$$

Hence, the linear system (5.4) is asymptotically stable if and only if $A - \lambda_j BC$ is Hurwitz for all $j = 1, 2, \dots, p$. Obviously, since $\lambda_1 = 0$, a necessary condition is that A is Hurwitz.

5.3.2 Switching topology

Here, we are interested in deducing the GUES of (5.3) by a small gain type of argument. Clearly, a necessary condition for GUES of (5.3) is that each agent in (5.1) has a stable internal dynamics, i.e. A is Hurwitz. Let H denote the transfer matrix from u_j to y_j. We seek for a condition on the $\mathcal{H}_\infty$-norm of H under which the overall network given by (5.3) is GUES as the agents interact with each other based on (5.1c). This leads us to the following theorem.

Theorem 5.3.1 *Consider the network* (5.3). *Assume that the matrix A is Hurwitz. Let H denote the transfer matrix from u_j to y_j in* (5.1), *i.e. $H(s) = C(sI - A)^{-1}B$. Then, the network* (5.3) *is GUES if*

$$\overline{\lambda}\|H\|_\infty < 1 \tag{5.9}$$

where $\overline{\lambda}$ is given by (5.2b).

Proof. Suppose that (5.9) holds. Then there exists a positive definite solution $K > 0$ to the Riccati inequality (see [111])

$$A^\top K + KA + C^\top C + \overline{\lambda}^2 KBB^\top K < 0. \tag{5.10}$$

We claim that $I_p \otimes K$ generates a common quadratic Lyapunov function for the matrices $(I_p \otimes A - L_i \otimes BC)$, that is

$$(I_p \otimes A - L_i \otimes BC)^\top (I_p \otimes K) + (I_p \otimes K)(I_p \otimes A - L_i \otimes BC) < 0 \tag{5.11}$$

for all $i = 1, 2, \dots, N$. To see this, let L_i be diagonalized as $U_i^\top L_i U_i = \Lambda_i$, where U_i is an orthogonal matrix, and $\Lambda_i = \mathrm{diag}(\lambda_1^i, \lambda_2^i, \dots, \lambda_p^i)$ with $0 = \lambda_1^i \leqslant \lambda_2^i \leqslant \cdots \leqslant \lambda_p^i$. By using $I_p \otimes K = (U_i \otimes I_n)(I_p \otimes K)(U_i^\top \otimes I_n)$, multiplying (5.11) from the left and right by $U_i^\top \otimes I_n$ and $U_i \otimes I_n$, respectively, we see that (5.11) holds if and only if

$$(I_p \otimes A - \Lambda_i \otimes BC)^\top (I_p \otimes K) + (I_p \otimes K)(I_p \otimes A - \Lambda_i \otimes BC) < 0 \tag{5.12}$$

for all $i = 1, 2, \dots, N$. Therefore, it suffices to show that

$$(A - \lambda_j^i BC)^\top K + K(A - \lambda_j^i BC) < 0 \tag{5.13}$$

for all $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, p$. Clearly, the LMI (5.13) is satisfied for all i and j if

$$(A - \alpha BC)^\top K + K(A - \alpha BC) < 0$$

holds for all $\alpha \in [0, \overline{\lambda}]$. Note that

$$\begin{aligned}(A - \alpha BC)^\top K + K(A - \alpha BC) &= A^\top K + KA - \alpha(C^\top B^\top K + KBC) \\ &= A^\top K + KA + C^\top C + \alpha^2 KBB^\top K - (\alpha KB + C^\top)(\alpha B^\top K + C).\end{aligned} \tag{5.14}$$

It follows from (5.10) that the right hand side of (5.14) is negative definite for all $\alpha \in [0, \overline{\lambda}]$. Consequently, $I_p \otimes K$ generates a common quadratic Lyapunov function for the matrices $I_p \otimes A - L_i \otimes BC$, $i = 1, 2, \ldots, N$, hence the network (5.3) is GUES. ■

Remark 5.3.2 The result of Theorem 5.3.1 can also be deduced from a result available in the context of switched linear systems (see [48, p.48]) by reformulating the stability problem of network (5.3). However, here, we prefer providing the proof above as it is insightful for the results presented later in this chapter.

5.4 Network synchronization

Here, we discuss the so-called synchronization problem for networks which is the main focus of this chapter. A synchronized network has the property that the state trajectories of the coupled agents converge to a common trajectory. More precisely, we have the following definition.

Definition 5.4.1 The multi-agent system (5.3) is *synchronized* if every solution of (5.3) satisfies $\lim_{t\to\infty}(x_j(t) - x_k(t)) = 0$ for all $j, k = 1, 2, \ldots, p$.

Note that sometimes we say the network is *synchronized under an arbitrary switching topology* meaning that the network is synchronized in the sense of Definition 5.4.1.

Observe that network synchronization requires that the differences of the agents' state components converge to zero, which can be viewed as *output stability* (see [102, sec. 4.4.] or [94, exc. 4.10]). Let $z_j = x_j - x_{j+1}$ for each $j = 1, 2, \ldots, p-1$. Then, we have the compact form

$$z(t) = (Q \otimes I_n)x(t) \tag{5.15}$$

where $z = \mathrm{col}(z_1, z_2, \ldots, z_{p-1})$ and Q is the $(p-1) \times p$ matrix given by

$$Q = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{pmatrix}. \tag{5.16}$$

Clearly, the network (5.3) is synchronized if and only if $\lim_{t\to\infty} z(t) = 0$ for any solution x of (5.3). Consequently, the network (5.3) is synchronized if and only if the system (5.3) with output variable z is output stable. Unfortunately, the output stability problem for switched linear systems is far more complicated than that for linear time-invariant systems. Therefore, to facilitate the derivation of the results for the case of networks with switching topologies, we suggest to use the structure of the network and the properties of the Laplacian matrix to convert the output stability problem into an internal stability problem. First, we discuss the synchronization analysis for the case where the topology of the network is fixed. Then, we extend the results to the case of networks with switching topologies.

5.4.1 Synchronization with respect to a fixed topology

Consider the network (5.4) together with the output given by (5.15). We will apply a state space transformation such that the state x is transformed to $\tilde{z} = \mathrm{col}(z, x_p)$. In order to do this, define $\tilde{Q}$ to be the $p \times p$ matrix given by

$$\tilde{Q} = \begin{pmatrix} Q \\ e_p^\top \end{pmatrix} \tag{5.17}$$

with e_p the p^{th} standard basis vector for $\mathbb{R}^p$, i.e. $e_p = (0, 0, \ldots, 1)^\top$, and where Q is given by (5.16). Note that $\tilde{Q}$ is nonsingular. Then, on the one hand we indeed have $\tilde{z} = \mathrm{col}(z, x_p) = (\tilde{Q} \otimes I_n)x(t)$. On the other hand, the transformed dynamics of (5.4) is given by

$$\dot{\tilde{z}}(t) = (I_p \otimes A - \tilde{Q}L\tilde{Q}^{-1} \otimes BC)\tilde{z}(t).$$

It is easy to observe that $\tilde{Q}^{-1}$ is an upper triangular matrix where the entries on and above its diagonal are all equal to 1. As the row sums of L are zero, the matrix $\tilde{Q}L\tilde{Q}^{-1}$ can be partitioned as

$$\tilde{Q}L\tilde{Q}^{-1} = \begin{pmatrix} \tilde{L} & 0 \\ * & 0 \end{pmatrix} \tag{5.18}$$

where $\tilde{L}$ is a $(p-1) \times (p-1)$ matrix, and "*" denotes the values which are not of interest to us. It follows from the structure of (5.18) that the dynamics of the state components $z_1, z_2, \ldots, z_{p-1}$ are not affected by that of x_p, and we have

$$\dot{z}(t) = (I_{p-1} \otimes A - \tilde{L} \otimes BC)z(t). \tag{5.19}$$

Therefore, synchronization of (5.4) is equivalent to asymptotic stability of the linear time-invariant system (5.19).

Remark 5.4.2 In fact, the proposed state space transformation brings the linear system (5.4) together with the output equation (5.15) into Kalman canonical form (see [35, p. 149]). Consequently, the synchronization analysis can be carried out by verifying the internal stability of the observable part of the system, which is given by (5.19).

The structure of (5.18) reveals the intimate relationship between the matrices L and $\tilde{L}$ as stated in the following lemma.

Lemma 5.4.3 *Let L, U, Λ, Q, $\tilde{Q}$, and $\tilde{L}$ be as defined before. Let the product QU be partitioned as*

$$QU = \begin{pmatrix} 0_{(p-1)\times 1} & \tilde{U} \end{pmatrix}. \tag{5.20}$$

Then the matrix $\tilde{L}$ admits a spectral decomposition $\tilde{L} = \tilde{U}\tilde{\Lambda}\tilde{U}^{-1}$ where $\tilde{\Lambda}$ is the diagonal matrix obtained by deleting the first row and column of Λ. Moreover, $\tilde{U}\tilde{U}^\top = QQ^\top$.

Proof. Since the first column of U is the normalized vector of ones, we have

$$\tilde{Q}U = \begin{pmatrix} 0_{(p-1)\times 1} & \tilde{U} \\ \frac{1}{\sqrt{p}} & * \end{pmatrix} \tag{5.21}$$

Therefore, we obtain

$$\tilde{Q}L\tilde{Q}^{-1} = \tilde{Q}U\Lambda U^\top \tilde{Q}^{-1} = \begin{pmatrix} 0 & \tilde{U} \\ \frac{1}{\sqrt{p}} & * \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} \begin{pmatrix} 0 & \tilde{U} \\ \frac{1}{\sqrt{p}} & * \end{pmatrix}^{-1}$$

where Λ is partitioned as $\mathrm{blockdiag}(0, \tilde{\Lambda})$. Hence, we have

$$\tilde{Q}L\tilde{Q}^{-1} = \begin{pmatrix} \tilde{U}\tilde{\Lambda}\tilde{U}^{-1} & 0 \\ * & 0 \end{pmatrix}. \tag{5.22}$$

Comparing the right hand side of (5.22) to that of (5.18), we obtain $\tilde{L} = \tilde{U}\tilde{\Lambda}\tilde{U}^{-1}$ which corresponds to a spectral decomposition of $\tilde{L}$. The rest follows from

$$QQ^\top = QUU^\top Q^\top = \begin{pmatrix} 0 & \tilde{U} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{U}^\top \end{pmatrix} = \tilde{U}\tilde{U}^\top.$$

■

According to (5.19), the network (5.4) is synchronized if and only if $(I_{p-1} \otimes A - \tilde{L} \otimes BC)$ is Hurwitz. By applying the similarity transformation $\tilde{U}^{-1} \otimes I_n$ to the matrix $(I_{p-1} \otimes A - \tilde{L} \otimes BC)$, we obtain

$$(\tilde{U}^{-1} \otimes I_n)(I_{p-1} \otimes A - \tilde{L} \otimes BC)(\tilde{U} \otimes I_n) = I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC. \tag{5.23}$$

Therefore, the network (5.4) is synchronized if and only if $A - \lambda_j BC$ is Hurwitz for all $j = 2, 3, \ldots, p$. This is in accordance with the result presented in [47] and [84], in case the underlying graph is connected. Obviously, in case the graph is not connected, λ_2 is zero and the matrix A has to be Hurwitz for synchronization.

5.4.2 Switching topology

In this subsection, we carry out the synchronization analysis for the switching topology case (5.3). To further motivate the problem, we provide an example showing that maintaining connectivity and/or synchronization of the individual modes does not imply synchronization of the network (5.3).

Example 5.4.4 Consider the agents' dynamics in (5.1) is given by

$$A = \begin{pmatrix} 0 & 0.6619 & -0.2194 & 0.4494 & 0.2791 & 0.4553 & 0 \\ 0 & -0.1762 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4775 & 0.7807 & 0.4849 & 0.7909 & 0 \\ 0 & 0 & 0 & -0.6957 & 0.3125 & 0.5097 & 0 \\ 0 & 0 & 0 & 0 & -1.52 & 0.3166 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.5 & 1.936 \\ 0 & 0 & 0 & 0 & 0 & -1.936 & -3.5 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 9.155 \end{pmatrix}^\top,$$
$$C = \begin{pmatrix} 0.2018 & 0.3328 & -0.1103 & 0.2259 & 0.1403 & 0.2289 & 0 \end{pmatrix}.$$

Suppose that $N = 2$ in (5.1) and the corresponding communication graphs G_1 and G_2 are as depicted in Figure 5.1 and Figure 5.2, respectively.

Let L_1 and L_2 denote the Laplacian matrices of G_1 and G_2 respectively. The eigenvalues of L_1 are 0, 1, 3, and the eigenvalues of L_2 are 0, 3, 3. It is easy to verify that the matrices $A - BC$ and $A - 3BC$ are both Hurwitz. Therefore, synchronization is achieved with respect to the individual modes, i.e. the network (5.3) is synchronized for $\sigma(t) = 1, t \geqslant 0$, and also for the case where $\sigma(t) = 2, t \geqslant 0$. Now, suppose that the switching signal is chosen as the $2T$-period signal:

$$\sigma(t) = \begin{cases} 1 & \lfloor \frac{t}{T} \rfloor \text{ is odd} \\ 2 & \lfloor \frac{t}{T} \rfloor \text{ is even} \end{cases} \tag{5.24}$$

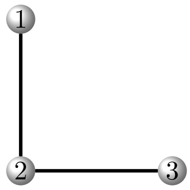

Figure 5.1: *Communication graph G_1*

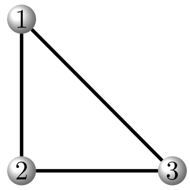

Figure 5.2: *Communication graph G_2*

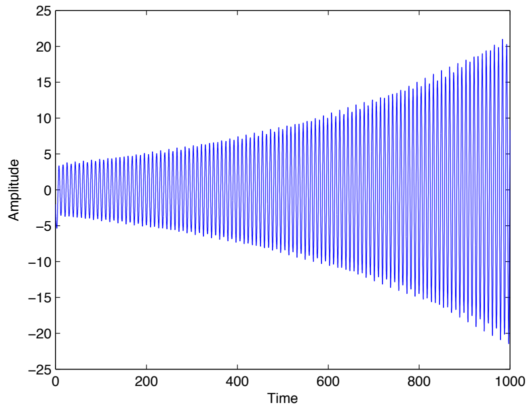

Figure 5.3: *The value of $(x_2)_1 - (x_3)_1$ in Example 5.4.4 depicted over time for the switching signal σ given by* (5.24) *with $T = 1$, and initial states $x_1(0) = [2\ 2\ 2\ 2\ 2\ 2\ 2]^\top$, $x_2(0) = [-1\ -1\ -1\ -1\ -1\ -1\ -1]^\top$, $x_3(0) = [1\ 1\ 1\ 1\ 1\ 1\ 1]^\top$.*

where $\lfloor x \rfloor$ denotes the biggest integer k such that $k \leqslant x$. Then, one can verify that the network (5.3) may not be synchronized for sufficiently small T. In Figure 5.3, the value of $(x_2)_1 - (x_3)_1$ is depicted over time for $T = 1$ and a given initial state x_0. Here, $(x_i)_j$ denotes the j^{th} component of x_i. As can be seen from this figure, the value of $(x_2)_1 - (x_3)_1$ does not converge to zero, thus the network is not synchronized.

Note that the communication graphs G_1 and G_2 are both connected. Moreover, the network is synchronized for constant switching signals, which corresponds to the fixed topology case. However, as observed above the network is not synchronized for the particular choice of the switching signal σ given by (5.24). Therefore, synchronization analysis for the switching topology case (5.3) requires additional effort, and cannot be deduced solely based on the connectivity of the underlying communication graphs or synchronization of the individual modes.

As observed in the previous subsection, similar to [84] and [47], the synchronization analysis in the case where the network communication is time-independent, is essentially based on a suitable state space transformation of the system (5.4). As also pointed out in [108], the main challenge in extending the available results to the arbitrary switching topology case is that the aforementioned state space transformation depends directly on the Laplacian, in particular on the Laplacian eigenvectors; hence, in the time-dependent case where the Laplacian matrix switches as time evolves, the spectral decomposition (5.5) and transformed equations (like (5.7) or (5.23)) are not valid in the same coordinate frame. An idea to overcome this difficulty is to impose constraints like commutativity or simultaneous triangularizability on the Laplacian matrices of underlying communication graphs. The latter one is assumed in [108] and synchronization analysis for a more general coupling configuration, with possibly directed communication graphs, is carried out. Here, we do not make any assumptions on the relationship among the corresponding Laplacian matrices of the communication graphs. To overcome the technical challenge discussed above, we use the decomposition proposed in the previous subsection, along with some properties of the Laplacian L and the matrix $\tilde{L}$ which are partially summarized in Lemma 5.4.3. This is discussed in detail next.

Clearly, one can mimic the approach illustrated in the previous subsection and obtain

$$\tilde{Q}L_i\tilde{Q}^{-1} = \begin{pmatrix} \tilde{L}_i & 0 \\ * & 0 \end{pmatrix} \tag{5.25}$$

where $\tilde{L}_i$ is a $(p-1)\times(p-1)$ matrix for each $i = 1, 2, \dots, N$. Then, (5.19) turns into

$$\dot{z}(t) = (I_{p-1} \otimes A - \tilde{L}_{\sigma(t)} \otimes BC)z(t). \tag{5.26}$$

Therefore, following the discussion in the previous subsection, it is easy to observe that synchronization analysis of the network (5.3) boils down to stability analysis of the switched linear system (5.26). Consequently, synchronization analysis of (5.3) can be carried out by dwell time, average dwell time, or Lyapunov based arguments available in the context of switched linear systems. In this chapter, however, we will investigate the existence of a CQLF as we are interested in synchronization under an arbitrary switching topology. Observe that existence of a CQLF for the SLS (5.26) implies GUES of (5.26) which results in synchronization of the network (5.3).

From (5.26) it readily follows that (5.3) is synchronized if there exists a CQLF for the state matrices $I_{p-1} \otimes A - \tilde{L}_i \otimes BC$ where $i = 1, 2, \dots, N$. Similar to the fixed topology case, it would be desirable to relate synchronization of (5.3) to the dynamics of the agents and the Laplacian eigenvalues, more specifically to the matrices $A - \lambda_j^i BC$ where $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, p$. This brings us to the following theorem.

Theorem 5.4.5 *Let $\underline{\lambda}$ and $\overline{\lambda}$ be defined as in* (5.2)*. Then, the network* (5.3) *is synchronized if there exists a positive definite matrix X satisfying both of the following linear matrix inequalities:*

$$(A-\underline{\lambda}BC)^\top X + X(A-\underline{\lambda}BC) < 0 \tag{5.27a}$$

$$(A-\overline{\lambda}BC)^\top X + X(A-\overline{\lambda}BC) < 0. \tag{5.27b}$$

Proof. Assume that there exists $X > 0$ such that (5.27) holds. Then we have

$$(A-\lambda_j^i BC)^\top X + X(A-\lambda_j^i BC) < 0 \tag{5.28}$$

for all $j = 2, 3, \dots, p$ and $i = 1, 2, \dots, N$. Here, we have used the fact that $A - \lambda_j^i BC$ can be written as a convex combination of the matrices $A - \underline{\lambda}BC$ and $A - \overline{\lambda}BC$ for any $j = 2, 3, \dots, p$ and $i = 1, 2, \dots, N$. The LMIs (5.28) can be rewritten as

$$(I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC)^\top (I_{p-1} \otimes X) + (I_{p-1} \otimes X)(I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC) < 0, \tag{5.29}$$

where $\tilde{\Lambda}_i = \mathrm{diag}\{\lambda_2^i, \lambda_3^i, \dots, \lambda_p^i\}$ for each $i = 1, 2, \dots, N$. By Lemma 5.4.3, there exist diagonalizing transformations $\tilde{U}_i$ such that $\tilde{L}_i = \tilde{U}_i \tilde{\Lambda}_i \tilde{U}_i^{-1}$ for $i = 1, 2, \dots, N$. Clearly, the LMIs (5.29) can be restated as

$$\begin{aligned}(I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC)^\top (\tilde{U}_i^\top \otimes I_n)(\tilde{U}_i^{-\top} \otimes I_n)(I_{p-1} \otimes X) + \\ (I_{p-1} \otimes X)(\tilde{U}_i^{-1} \otimes I_n)(\tilde{U}_i \otimes I_n)(I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC) < 0\end{aligned} \tag{5.30}$$

for $i = 1, 2, \dots, N$. Multiplying (5.30) from the left and right by $(\tilde{U}_i^{-\top} \otimes I_n)$ and $(\tilde{U}_i^{-1} \otimes I_n)$, respectively, we obtain

$$(I_{p-1} \otimes A - \tilde{L}_i \otimes BC)^\top (\tilde{U}_i^{-\top} \tilde{U}_i^{-1} \otimes X) + (\tilde{U}_i^{-\top} \tilde{U}_i^{-1} \otimes X)(I_{p-1} \otimes A - \tilde{L}_i \otimes BC) < 0. \tag{5.31}$$

Now, by Lemma 5.4.3, the product $\tilde{U}_i \tilde{U}_i^\top$ is independent of i for $i = 1, 2, \dots, N$, and is equal to $QQ^\top$, where Q is given by (5.16). Therefore, the quadratic function W defined by $W(z) = z^\top((QQ^\top)^{-1} \otimes X)z$ serves as a CQLF for the switched linear system (5.26). Hence, (5.26) is GUES, and the network (5.3) is synchronized. ■

Clearly, solvability of the LMIs (5.27) is equivalent to the existence of a CQLF for the pair of matrices $A - \underline{\lambda}BC$ and $A - \overline{\lambda}BC$. In general, there are conditions to guarantee the existence of a CQLF based on commutativity, simultaneous triangularizability, and solvable Lie algebras (see [48], [51]). However, these conservative conditions involve direct constraints on the matrices A and BC, namely A and BC must commute or at least be simultaneously triangularizable. A subtle point is that these conditions are basically for an SLS with arbitrary given modes whereas in our case certain structures and system properties are present in the network dynamics.

Thus, we can take advantage of these available structures and properties to derive sensible conditions guaranteeing network synchronization.

First, consider the special case of single-input single output (SISO), i.e. $m = 1$ in (5.1). In this case, the difference of the matrices $A - \overline{\lambda}BC$ and $A - \underline{\lambda}BC$ is equal to $(\overline{\lambda} - \underline{\lambda})BC$ which is of rank 1. Hence, the result of [86, Thm. 4] gives a necessary and sufficient condition for solvability of the LMIs (5.27), in the SISO case. This result leads to the following corollary.

Corollary 5.4.6 *Consider the multi-agent system* (5.1), *and assume that* $m = 1$. *Let* $\underline{\lambda}$ *and* $\overline{\lambda}$ *be defined as in* (5.2). *Let* $A_1 = A - \underline{\lambda}BC$ *and* $A_2 = A - \overline{\lambda}BC$. *Then, the network* (5.3) *is synchronized if the matrices* A_1 *and* A_2 *are both Hurwitz, and* $A_1 + \eta A_2^{-1}$ *is nonsingular for all* $\eta \in [0, +\infty)$.

Example 5.4.7 Consider the multi-agent system composed of a group of N harmonic oscillators described by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{5.32}$$

Suppose that the communication graph switches, arbitrarily, among a given set of graphs $G_i = (V, E_i)$ with $i = 1, 2, \ldots, N$. Obviously, according to the Subsection 5.4.1, synchronization is achieved only if the matrices $A - \underline{\lambda}BC$ and $A - \overline{\lambda}BC$ are both Hurwitz, which results in the condition $\underline{\lambda} > 0$. This means that G_i must be connected for each $i = 1, 2, \ldots, N$. Now, to ensure synchronization by Corollary 5.4.6, it remains to check whether the matrix $(A - \underline{\lambda}BC) + \eta(A - \overline{\lambda}BC)^{-1}$ is nonsingular for all $\eta \in [0, +\infty)$. This matrix is computed as

$$\begin{pmatrix} \eta\overline{\lambda} & 1 - \eta \\ -1 + \eta & \underline{\lambda} \end{pmatrix},$$

which is nonsingular for any nonnegative η if $\underline{\lambda} > 0$. Therefore, as long as connectivity is preserved switching does not jeopardize the synchronization of the network with harmonic oscillators given by (5.32).

Next, we return to the multi-input multi-output case. As observed earlier, the network dynamics associated with $A - \lambda_j^i BC$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, p$ plays a crucial role in synchronization of the network. We will now show that the solvability of the LMIs (5.27) is guaranteed by imposing certain structural properties, namely bounded realness and passivity, to this dynamics.

First, small gain conditions are provided to guarantee the solvability of (5.27), thus synchronization of the network (5.3).

Theorem 5.4.8 *Let $\underline{\lambda}$ and $\overline{\lambda}$ be defined as in* (5.2). *The network* (5.3) *is synchronized if there exists at least one pair of indices (k, ℓ) with $k \in \{1, 2, \ldots, N\}$ and $\ell \in \{2, 3, \ldots, p\}$ such that $A - \lambda_\ell^k BC$ is Hurwitz and*

$$\max\{\lambda_\ell^k - \underline{\lambda}, \overline{\lambda} - \lambda_\ell^k\} \|H_\ell^k\|_\infty < 1 \tag{5.33}$$

where

$$H_\ell^k(s) = C(sI - A + \lambda_\ell^k BC)^{-1} B.$$

Proof. Suppose that (5.33) holds. For simplicity, denote $\max\{\lambda_\ell^k - \underline{\lambda}, \overline{\lambda} - \lambda_\ell^k\}$ by d and λ_ℓ^k by λ. Then there exists a positive definite matrix K which satisfies the Riccati inequality (see [111])

$$(A - \lambda BC)^\top K + K(A - \lambda BC) + C^\top C + d^2 KBB^\top K < 0. \tag{5.34}$$

We claim that the matrix $K > 0$ in (5.34) satisfies

$$(A - \alpha BC)^\top K + K(A - \alpha BC) < 0 \tag{5.35}$$

both for $\alpha = \underline{\lambda}$ and $\alpha = \overline{\lambda}$, and, therefore, the network (5.3) is synchronized based on Theorem 5.4.5. Indeed, we have

$$\begin{aligned}
&(A - \alpha BC)^\top K + K(A - \alpha BC) = \\
&(A - \lambda BC)^\top K + K(A - \lambda BC) - (\alpha - \lambda)(C^\top B^\top K + KBC) = \\
&(A - \lambda BC)^\top K + K(A - \lambda BC) + C^\top C + d^2 KBB^\top K + \\
&((\alpha - \lambda)^2 - d^2) KBB^\top K - ((\alpha - \lambda)KB + C^\top)((\alpha - \lambda)B^\top K + C).
\end{aligned} \tag{5.36}$$

Now, by (5.34), the right hand side of (5.36) is negative definite. Clearly this holds both for $\alpha = \underline{\lambda}$ and $\alpha = \overline{\lambda}$ since $d = \max\{\lambda - \underline{\lambda}, \overline{\lambda} - \lambda\}$. ■

Remark 5.4.9 According to the results in Subsection 5.4.1 and considering the special case of constant switching signals, the network (5.3) is synchronized only if the matrix $A - \lambda_j^i BC$ is Hurwitz for each $i = 1, 2, \ldots, N$ and $j = 2, 3, \ldots, p$. Therefore, there is no conservatism involved in the Hurwitzness condition provided in Theorem 5.4.8.

For the analysis and design of protocols that achieve synchronization, assumptions on Lyapunov stability or passivity of the agents' dynamic have been made in the literature (see e.g. [83], [84]). Next, we show that synchronization can be guaranteed by imposing passivity conditions on certain network dynamics, namely $(A - \underline{\lambda} BC, B, C)$. Note that, in this case, the matrix A is allowed to have eigenvalues in the right half plane, whereas $A - \underline{\lambda} BC$ is required to be Hurwitz which is indeed a necessary condition for synchronization of (5.3). The notion we use here is *strict passivity*. Consider the finite-dimensional linear time-invariant system

$$\dot{w} = Aw + Bv \tag{5.37a}$$

$$z = Cw + Dv. \tag{5.37b}$$

We will denote this system by $\Sigma(A, B, C, D)$. We call the system (5.37) strictly passive if there exist $\epsilon > 0$ and $X > 0$ such that

$$\begin{pmatrix} A^\top X + XA + \epsilon X & XB - C^\top \\ B^\top X - C & -(D + D^\top) \end{pmatrix} \leqslant 0. \tag{5.38}$$

For relations of the above definition to other types of passivity, positive realness, and strict positive realness we refer to [100]. Now, we have the following result.

Theorem 5.4.10 *Assume that* $\underline{\lambda} \neq \overline{\lambda}$*. Then, the network* (5.3) *is synchronized if the linear system*

$$\Sigma(A - \underline{\lambda}BC, B, C, \frac{1}{\overline{\lambda} - \underline{\lambda}} I_m)$$

is strictly passive.

Proof. Suppose $\Sigma(A - \underline{\lambda}BC, B, C, \frac{1}{\overline{\lambda}-\underline{\lambda}} I_m)$ is strictly passive. Then, using an appropriate Schur complement, by (5.38) we have

$$(A - \underline{\lambda}BC)^\top X + X(A - \underline{\lambda}BC) + \frac{1}{2}(\overline{\lambda} - \underline{\lambda})(XB - C^\top)(XB - C^\top)^\top < 0 \tag{5.39}$$

Therefore, we obtain

$$\begin{aligned} (A - \overline{\lambda}BC)^\top X + X(A - \overline{\lambda}BC) + (\overline{\lambda} - \underline{\lambda})(C^\top B^\top X + XBC) \\ + \frac{1}{2}(\overline{\lambda} - \underline{\lambda})(XB - C^\top)(XB - C^\top)^\top < 0 \end{aligned}$$

which can be rewritten as

$$(A - \overline{\lambda}BC)^\top X + X(A - \overline{\lambda}BC) + \frac{1}{2}(\overline{\lambda} - \underline{\lambda})(XB + C^\top)(XB + C^\top)^\top < 0. \tag{5.40}$$

Consequently, since both (5.39) and (5.40) hold, the network (5.3) is synchronized by Theorem 5.4.5. ■

Remark 5.4.11 Note that the case $\underline{\lambda} = \overline{\lambda}$ is not of our interest since it corresponds to a fixed topology network associated with a complete graph. Obviously, in this case, synchronization is achieved if and only if $A - \underline{\lambda}BC$ is Hurwitz.

Remark 5.4.12 Let $\Sigma_\alpha = \Sigma(A - \underline{\lambda} BC, B, C, \alpha I_m)$. It can be verified from (5.38) that if Σ_α is strictly passive, then so is Σ_β for any $\beta \geqslant \alpha$. Hence, by Theorem 5.4.10, the network (5.3) is synchronized if Σ_α is strictly passive for some $\alpha \in [0, \frac{1}{\overline{\lambda} - \underline{\lambda}}]$.

Remark 5.4.13 The condition proposed in Theorem 5.4.10 can be useful also for the special case of network (5.4) where the topology is time-independent. While the available necessary and sufficient conditions (see Subsection 5.4.1) need information on all eigenvalues of the Laplacian matrix, the proposed passivity condition merely requires knowledge of the (nontrivial) extremal eigenvalues. Moreover, one can also use lower/upper bounds for these extremal Laplacian eigenvalues (see e.g. [30]) to conclude synchronization of network (5.3) from Theorem 5.4.10. In fact, it is easy to observe that, both in Theorem 5.4.8 and Theorem 5.4.10, the values of $\underline{\lambda}$ and $\overline{\lambda}$ can be replaced by lower and upper bounds, say $\underline{\mu}$ and $\overline{\mu}$, respectively; i.e. $\underline{\mu} \leqslant \underline{\lambda}$ and $\overline{\lambda} \leqslant \overline{\mu}$. Besides, as it is clear from Remark 5.4.12, the information on $\overline{\lambda}$ is included to obtain less restrictive condition in Theorem 5.4.10. Indeed, strict passivity of Σ_0 in Remark 5.4.12 does not depend on $\overline{\lambda}$.

As an example to show how the result of Theorem 5.4.10 can be employed, next we design a protocol to achieve synchronization of (5.3). Consider again the diffusively coupled multi-agent system

$$\dot{x}_j = Ax_j + Bu_j,$$

$j = 1, 2, \dots, p$, together with the static protocol

$$u_j = K \sum_{\{i,j\} \in E_{\sigma(t)}} (x_i - x_j), \tag{5.41}$$

for some matrix K which will be determined later. For each $i = 1, 2, \dots, N$, assume that $G_i = (V, E_i)$ is connected. Assume that (A, B) is stabilizable . Then, obviously, there exists a positive definite matrix P satisfying the Riccati inequality

$$A^\top P + PA - 2\underline{\lambda} PBB^\top P < 0. \tag{5.42}$$

Observe that

$$\begin{pmatrix} (A - \underline{\lambda} BK)^\top X + X(A - \underline{\lambda} BK) + \epsilon X & XB - K^\top \\ B^\top X - K & 0 \end{pmatrix} \leqslant 0 \tag{5.43}$$

for the choices $K = B^\top P$, $X = P$, and a sufficiency small ϵ. In other words, the system $\Sigma(A - \underline{\lambda} BK, B, K, 0)$ is strictly passive. Thus, it follows from Theorem 5.4.10 and Remark 5.4.12 that the protocol (5.41) with $K = B^\top P$ achieves synchronization.

Note that two assumptions are made to derive the proposed protocol. First, the pair (A, B) is assumed to be stabilizable. This assumption is obviously necessary for synchronization. In addition, the underlying graphs corresponding to the network (5.3) are assumed to be connected. This is also necessary for the case where the agents have exponentially unstable dynamics as we allow arbitrary switching.

5.4.3 Synchronization with respect to observer-based protocols

Recall that in the multi-agent system (5.1), the output information of the agents is transmitted through the network. Hence, the result obtained in the previous section, inherently, can be used to analyze the synchronization of the agents for a given static state feedback protocol in the form of (5.41). However, as the overall information on relative states of the neighboring agents is not always available, observer-based protocols are proposed in the literature (see e.g. [47], [83], [84]), which use only the relative output information of the adjacent agents. Here, we employ the observer based protocol proposed in [47], which is originally for multi-agent systems with a fixed topology, to multi-agent systems with switching topologies, and we show that, interestingly, the result of the previous section carries over to this case.

Consider the observer based protocol:

$$\begin{aligned}
\dot{v}_j(t) &= (A+BK)v_j(t) + GC \sum_{\{i,j\}\in E_{\sigma(t)}} (v_i(t)-v_j(t)) - (x_i(t)-x_j(t)) \\
u_j(t) &= Kv_j(t)
\end{aligned}$$

Attaching this protocol to the agents (5.1) results in

$$\dot{\xi}_j(t) = \mathbf{A}\xi_j(t) - \sum_{i=1}^{N}(L_{\sigma(t)})_{ij}\mathbf{H}\xi_j \tag{5.44}$$

where

$$\xi_j = \begin{pmatrix} x_j \\ v_j \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A & BK \\ 0 & A+BK \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 & 0 \\ -GC & GC \end{pmatrix},$$

$j \in V$, v_j is the state of the observer protocol, K is the feedback gain, and G is the observer gain. Note that the matrix $\mathbf{H}$ can be written as $\mathbf{H} = \mathbf{BC}$, where

$$\mathbf{B} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -C & C \end{pmatrix}.$$

Hence, the multi-agent system (5.44) can be written in compact form as

$$\dot{\xi}(t) = (I_p \otimes \mathbf{A} - L_{\sigma(t)} \otimes \mathbf{BC})\xi(t) \tag{5.45}$$

where $\xi = \mathrm{col}(\xi_1, \xi_2, \ldots, \xi_p)$. This coincides with the network representation (5.3), where the matrices A, B, and C are replaced by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively. Therefore, the proposed results in the previous subsections can also be applied to the case of observer-based protocols. In particular, regarding the key result of Theorem 5.4.5, we obtain that the network (5.45) achieves synchronization under an arbitrary switching topology if there exists a CQLF for the pair of matrices $\mathbf{A} - \underline{\lambda}\mathbf{BC}$ and $\mathbf{A} - \overline{\lambda}\mathbf{BC}$. These matrices are computed as follows:

$$\mathbf{A} - \underline{\lambda}\mathbf{BC} = \begin{pmatrix} A & BK \\ \underline{\lambda}GC & A + BK - \underline{\lambda}GC \end{pmatrix}, \quad \mathbf{A} - \overline{\lambda}\mathbf{BC} = \begin{pmatrix} A & BK \\ \overline{\lambda}GC & A + BK - \overline{\lambda}GC \end{pmatrix}.$$

Now, we apply the similarity the transformation $T = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}$ to obtain

$$T(\mathbf{A} - \underline{\lambda}\mathbf{BC})T^{-1} = \begin{pmatrix} A - \underline{\lambda}GC & 0 \\ \underline{\lambda}GC & A + BK \end{pmatrix}$$

and

$$T(\mathbf{A} - \overline{\lambda}\mathbf{BC})T^{-1} = \begin{pmatrix} A - \overline{\lambda}GC & 0 \\ \overline{\lambda}GC & A + BK \end{pmatrix}.$$

As the existence of a CQLF is invariant under similarity transformation, (5.45) is synchronized if there exists a CQLF for the pair of matrices $T(\mathbf{A} - \underline{\lambda}\mathbf{BC})T^{-1}$ and $T(\mathbf{A} - \overline{\lambda}\mathbf{BC})T^{-1}$. Observe that these matrices are in a block triangular form. Hence, according to [18, Thm. 5.1], they share a CQLF if and only if $A + BK$ is Hurwitz and the pair of matrices $A - \underline{\lambda}GC$ and $A - \overline{\lambda}GC$ share a CQLF. This brings us to the following proposition.

Proposition 5.4.14 *Let $\underline{\lambda}$ and $\overline{\lambda}$ be defined as in* (5.2)*. Then, the network* (5.45) *is synchronized if $A + BK$ is Hurwitz and there exists a positive definite matrix X satisfying both of the following linear matrix inequalities:*

$$\begin{aligned}(A - \underline{\lambda}GC)^\top X + X(A - \underline{\lambda}GC) < 0 \\ (A - \overline{\lambda}GC)^\top X + X(A - \overline{\lambda}GC) < 0.\end{aligned}$$

The proposition above can be exploited to design a feedback gain K and an observer gain G such that the network (5.45) is synchronized under an arbitrary switching topology. The assumptions required are stabilizability of (A, B), detectability of (C, A), and connectedness of L_i for $i = 1, 2, \ldots, N$. As mentioned before, the connectedness assumption is required as we consider arbitrary switching and the agents are allowed to have unstable dynamics. Now, choose K such that $A + BK$ is Hurwitz. Also choose $G = QC^\top$ where Q is a positive definite matrix satisfying the Riccati inequality $AQ + QA^\top - 2\underline{\lambda}QC^\top CQ < 0$. Clearly, these choices of K and G satisfy the conditions of Proposition 5.4.14 with $X = Q^{-1}$. This, together with

Proposition 5.4.14, extend the result of [47] to the the case of networks with arbitrary (undirected) switching topologies. Finally note that, similarly, one can modify the results of Theorems 5.4.8 and 5.4.10 to incorporate the case of observer-based protocols.

5.5 Conclusions

In this chapter, we have studied stability and synchronization of networks with arbitrary switching topologies. The agents of the network are assumed to have identical, but general linear dynamics, and the underlying communication topology may switch arbitrarily within a finite set of admissible topologies. For the case that the individual dynamics of the agents is stable, a small gain condition to guarantee the stability of the overall network is proposed. By viewing synchronization as an output stability problem, network synchronization is revisited for the case where the topology is fixed. By using a particular choice of decomposition of the overall network dynamics, the output stability problem is translated to an internal stability problem. Consequently, conditions in terms of existence of a common quadratic Lyapunov function guaranteeing synchronization of networks under arbitrary switching topologies are derived. These conditions require that a certain pair of LMI should have a solution. The solvability of this pair of LMI is discussed for both SISO and MIMO cases. Sufficient conditions for synchronization are established in terms of bounded realness and passivity of certain network dynamics. The proposed conditions depend on the agents dynamics and the (nontrivial) extremal eigenvalues of the Laplacian matrices. The results established are extended to also incorporate the case of observer-based protocols. It is explained how static feedback protocols as well as observer-based protocols guaranteeing synchronization can be designed based on the proposed conditions.

Chapter 6

Zero forcing sets and controllability of systems defined on graphs

The study of networks of dynamical systems became one of the most popular themes within systems and control theory in the last two decades. Roughly speaking, networks of dynamical systems can be seen as dynamical systems that inherit certain structural properties from the topology of a graph that captures the network structure. Across many scientific disciplines, one encounters such systems in a variety of applications. Typical examples include biological, chemical, social, power grid, and robotic networks (see e.g. [57, Ch. 1]). The research on numerous aspects of these kind of systems have already resulted in a vast literature that still keeps growing.

One line of research in this fast growing literature is devoted to the controllability analysis of linear input/state systems of the form

$$\dot{x} = Xx + Uu$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input with the distinguishing feature that the matrix X is associated with a given graph and the matrix U encodes the vertices (often called leaders) through which external inputs are applied.

Up to our knowledge, [92] is the first paper which addressed controllability problem within this framework when X is the Laplacian matrix of an undirected graph. This early paper was followed by a number of papers dealing with different aspects of controllability when X is the Laplacian matrix (see e.g.[78], [24], [106]) and when X is the adjacency matrix (see e.g. [31]). On the one hand, controllability was investigated from a graph topology perspective in [78], [24], [54], [12], [106], [104], [31] which established necessary/sufficient conditions for controllability as well as lower and/or upper bounds on the controllable subspace. These conditions are based on graph theoretical tools such as graph symmetry [78], (almost) equitable partitions [78], [24], [54], [106], walks of a graph [31], distance partitions [106], or pseudo monotonically increasing sequences [104]. On the other hand, the minimum number of leaders that render the system controllable, with X being the Laplacian matrix of a simple undirected graph, was explored for several classes of graphs such as path graphs [78], [75], cycle graphs [75], [106], complete graphs [78], [106], and circulant graphs [69] which all provide also a leader selection procedure.

Another thread in the study of controllability of systems defined by a graph was centered around structural controllability. Structural controllability deals with a family of pairs (X, U) rather than a particular instance and asks whether the family contains a controllable pair (weak structural controllability [52]) or all members of the family are controllable (strong structural controllability [16]). In the latter case, the authors of [16] have established necessary and sufficient conditions for strong structural controllability in terms of constrained matchings over the bipartite graph representation of the network.

In this chapter, we deal with a family of X matrices carrying the structure of a directed graph G. This family is called the qualitative class of G, and we investigate the controllability of the network with respect to this qualitative class, under a fixed set of vertices (leaders). Note that essentially this is the same as studying strong structural controllability, but we carry out controllability analysis through the notion of zero forcing sets, similar to [10], rather than through the constrained matching which has been treated in [16].

The notions of zero forcing sets and zero forcing number have an intimate relationship with minimum rank problems of patterned matrices, and have been well studied in the literature (see e.g. [4] and [37]). Moreover, in these papers and the references therein, lower/upper bounds for the zero forcing number has been provided, and also the exact value has been obtained for some special classes of graphs, either directly or in terms of some graph parameters such as path cover number. Note that computing the zero forcing number as well as finding a minimal zero forcing set for a general loop directed graph is an NP-hard problem (see [93, Thm. 2.6]). Utilizing the notion of zero forcing sets for controllability analysis of networks has been carried out recently in [10], where controllability of networks with quantum and linear dynamics is studied. As a side result, for the case where the underlying graph is undirected and all off-diagonal elements of X have the same sign, a sufficient condition for controllability of a network with linear dynamics is provided in terms of zero forcing sets.

In this chapter, for the case where the underlying graph is directed, we establish a one-to-one correspondence between the set of leaders rendering the network controllable and zero forcing sets. Consequently, we obtain that the minimum number of leaders required to render the network controllable, with respect to the whole qualitative class, is indeed equal to the zero forcing number of the underlying graph. Note that in some applications extra assumptions and constraints such as symmetry may be present on the entries of the matrix X. Hence, in these cases, one may be interested in some subsets of the qualitative class of G rather than the whole class. This will be addressed through the notion of sufficiently rich subclasses, and we explore how the results established in this chapter boils down or can be applied to certain qualitative subclasses. Then, we study the controllability problem for some special classes of graphs, namely path, cycle, and complete graphs. In addition, we establish a connection between the existing results on the minimum number of leaders in these cases where the matrix X is the Laplacian matrix, and the results proposed in this manuscript.

An advantage of the proposed results of this chapter is that one can deduce conclusions on the minimum number of leaders for controllability as well as how to choose such leaders in particular cases, by utilizing the existing results in graph theory regarding the zero forcing sets of graphs. For instance, in case where the underlying graph has a structure of a (directed) tree, we conclude that the minimum number of leaders rendering the network controllable, for all matrices in the qualitative class, is equal to the corresponding path cover number of the graph. Moreover, initial vertices in a minimal path cover can be selected as the choice of leaders in this case. Likewise, one can draw similar conclusions for other classes of graphs for which the zero forcing sets has been already studied in the literature. Finally, thanks to the result of the present chapter, the problem of verifying whether a given set of leaders render the network controllable, for all matrices in the qualitative class, boils down to checking whether this leader set constitutes a zero forcing set or not.

The organization of the chapter is as follows. In Section 6.1, the problem at hand is mathematically formulated, and is motivated by establishing connection to the existing results in the literature. In Section 6.2, zero forcing sets, zero forcing number, and the involved notions are recapped. The main result of this chapter is reported in Section 6.3, where a necessary and sufficient condition for controllability of networks is established in terms of zero forcing sets. In addition, controllability of the network with respect to qualitative subclasses is studied in this section, and finally some special cases are provided for further illustration of the proposed results. The chapter ends with concluding remarks in Section 6.4.

6.1 Problem formulation and motivation

For a given simple directed graph G, the vertex set of G is a nonempty set and is denoted by $V(G)$. The arc set of G, denoted by $E(G)$, is a subset of $V \times V$, and $(i,i) \notin E$ for all $i \in V(G)$. The cardinality of a given set V is denoted by $|V|$. Also we use $|G|$ to denote in short the cardinality of $V(G)$. We say vertex j is an out-neighbor of vertex i if $(i,j) \in E$. The family of matrices described by G is called *qualitative class* of G, and is given by

$$\mathcal{Q}(G) = \{X \in \mathbb{R}^{|G|\times|G|} : \text{ for } i \neq j,\ X_{ij} \neq 0 \Leftrightarrow (j,i) \in E(G)\}. \tag{6.1}$$

For $V = \{1, 2, \ldots, n\}$ and $V_L = \{v_1, v_2, \ldots, v_m\} \subseteq V$, we define the $n \times m$ matrix $U(V; V_L) = [U_{ij}]$ by:

$$U_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

By a leader/follower system defined on a graph G, we mean a finite-dimensional linear input/state system of the form

$$\dot{x}(t) = Xx(t) + Uu(t) \tag{6.3}$$

in continuous-time and

$$x(t+1) = Xx(t) + Uu(t) \tag{6.4}$$

in discrete-time where $x \in \mathbb{R}^{|G|}$ is the state, $u \in \mathbb{R}^m$ is the input, $X \in \mathcal{Q}(G)$, and $U = U(V(G); V_L)$ for some given leader set $V_L \subseteq V(G)$.

Systems of the form (6.3) or (6.4) where $X \in \mathcal{Q}(G)$ for a given graph G are encountered in various contexts. Examples include the cases where X is adjacency [31], (in-degree or out-degree) Laplacian [57], normalized Laplacian [3], etc. matrices associated to a graph.

In this chapter, we deal with the controllability of the systems of the form (6.3) or (6.4). With a slight abuse of notation, we sometimes write $(X; V_L)$ is controllable meaning that (X, U) is controllable. For a given graph G and a leader set V_L we say $(G; V_L)$ is controllable if the pair $(X; V_L)$ is controllable for all $X \in \mathcal{Q}(G)$.

In particular, we are interested in determining the set of leaders rendering systems of the form (6.3) controllable. For a given graph G and a matrix $X \in \mathcal{Q}(G)$, we denote the minimum number of leaders rendering the system (6.3) controllable by $\ell_{\min}(X)$, that is

$$\ell_{\min}(X) = \min_{V_L \subseteq V(G)} \{|V_L| : (X; V_L) \text{ is controllable}\}.$$

For a given graph G, we denote the minimum number of leaders rendering all systems of the form (6.3) controllable by $\ell_{\min}(G)$, that is

$$\ell_{\min}(G) = \min_{V_L \subseteq V(G)} \{|V_L| : (G; V_L) \text{ is controllable}\}. \tag{6.5}$$

Controllability of systems of the form (6.3) has been studied in the literature from different angles. In what follows, we give an account of the existing results/approaches in the literature.

One particular line of research within the context of controllability has been devoted to systems of the form

$$\dot{x}(t) = -Lx(t) + Uu(t) \tag{6.6}$$

where L is the Laplacian matrix of an undirected graph. This line of research has been initiated by [92] and further developed by [78]. Within this framework, the two main themes were graph theoretical characterization of controllability properties in terms of certain graph partitions [54], [24], [106] and (minimum) leader selection for rendering a system of the form (6.6) controllable for particular classes of undirected graphs [106], [75], [69].

The work on the leader selection led to a number of interesting results by exploiting the structure of the Laplacian matrices for several graph classes. It has been shown in [78] that $\ell_{\min}(L) = 1$ for path graphs. In this case, one can choose one of the two terminal vertices as the leader. By [106], $\ell_{\min}(L) = 2$ for undirected cycle graphs and any two neighbours can be chosen as leaders. The paper [75] further studied cycle graphs and has proved that any two leaders would render the system controllable in case the number of all vertices is a prime number. For an undirected complete graph with n vertices, we know from [106], [78] that $\ell_{\min}(L) = n - 1$ and any choice of $n - 1$ leaders would render the system controllable. Another rather specific class of undirected graphs that has been studied within the same context is distance regular graphs. In [106], it was shown that $\ell_{\min}(L) \leqslant n - d$ where n is the number of vertices and d is the diameter of the graph. The paper [106] provided also a recipe to select $n - d$ leaders that render the system controllable. In case the underlying graph is a circulant graph, the authors of [69] proved that $\ell_{\min}(L)$ is equal to the maximum algebraic multiplicity of Laplacian eigenvalues.

Another particular class of systems that has been studied in the context of the controllability is given by

$$\dot{x}(t) = Ax(t) + Uu(t) \tag{6.7}$$

where A is the adjacency matrix of an undirected graph, see e.g. [31]. The same class of systems was studied in [52] from the weak structural controllability viewpoint.

In this chapter, we will mainly deal with the controllability of families of systems given by (6.3) where $X \in \mathcal{Q}(G)$ for a graph G and provide results concerning $\ell_{\min}(G)$ rather than $\ell_{\min}(X)$ for a specific choice of $X \in \mathcal{Q}(G)$. However, our treatment, as a side result, will reveal that the aforementioned existing results on the number of minimum leaders are not intrinsic to the Laplacian but hold for any matrix within the corresponding qualitative class given by the underlying graph.

6.2 Zero forcing sets

First, we review the notion of zero forcing sets together with the involved notations and terminology which will be used in the sequel. For more details see e.g. [37].

Let G be a given graph, where each vertex is colored either white or black. Consider the following coloring rule:

◐ : If u is a black vertex and exactly one out-neighbor v of u is white, then change the color of v to black.

Following terminology will be used when we apply the color-change rule above to a graph G:

– When the color-change rule is applied to $u \in V(G)$ to change the color of $v \in V(G)$, we say u *forces* or *infects* v, and write $u \to v$.

– Given a coloring set $C \subseteq V(G)$, i.e. C indexes the initially black vertices of G, the *derived* set of C is denoted by $\mathcal{D}(C)$, and is the set of black vertices obtained by applying the color-change rule until no more changes are possible.

– The set $Z \subseteq V(G)$ is a *zero forcing set* (ZFS) for G if $\mathcal{D}(Z) = V(G)$.

– The *zero forcing number* $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$. A set Z is called a *minimal zero forcing set* if $|Z| = Z(G)$.

Figures 6.1 and 6.2 illustrate the zero forcing set and the notions defined above. First, consider the graph depicted in Figure 6.1 where the vertex 1 is initially colored black. Then, by the color-change rule it is clear that $1 \to 2$. Consequently, $2 \to 3$, and $3 \to 4$. Therefore, the derived set of $\{1\}$ is equal to $\{1, 2, 3, 4\}$, and thus $\{1\}$ is not a zero forcing set. Now, suppose that we choose $\{1, 5\}$ to be the initially colored black vertices as shown in Figure 6.2. Then by applying the color-change rule, we conclude that this set is a zero forcing set. Moreover, note that no singleton set constitutes a zero forcing set in this case, thus the zero forcing number is indeed equal to 2.

6.3 Zero forcing sets and controllability

In this section, we characterize a set of leaders which renders $(G; V_L)$ controllable for a given graph G. Clearly, a pair (X, U) is controllable if and only if the matrix $\begin{bmatrix} X - \lambda I & U \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$. Here, we deal with a family of matrices based on a given graph G, and thus we should consider whether the matrix $\begin{bmatrix} X - \lambda I & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$ and $\lambda \in \mathbb{C}$. It turns out that this property does not depend on the parameter λ due to the structure of the matrix family $\mathcal{Q}(G)$.

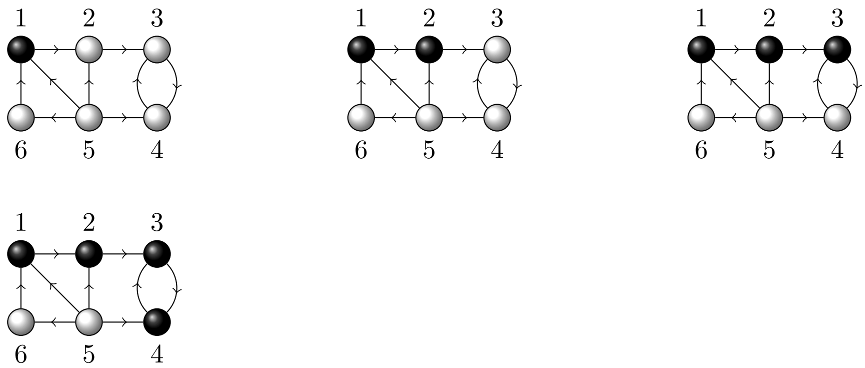

Figure 6.1: *An example for the coloring rule*

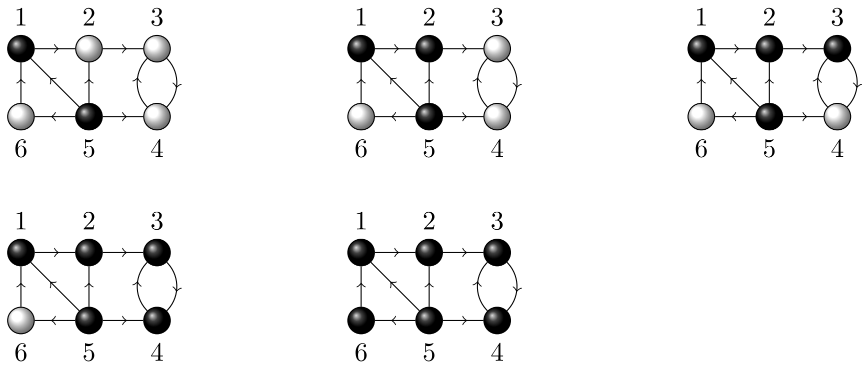

Figure 6.2: *An example for the zero forcing set*

Lemma 6.3.1 *Let G be a graph and $V_L \subseteq V(G)$. Then, $(G; V_L)$ is controllable if and only if the matrix $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$ where $U = U(V; V_L)$ given by* (6.2).

Proof. Clearly, $(G; V_L)$ is controllable if and only if the matrix $\begin{bmatrix} X - \lambda I & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$ and all $\lambda \in \mathbb{C}$. Hence, the "only if" part follows trivially. Now, suppose that $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$. Let $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^{|G|}$ be such that $z^* \begin{bmatrix} X - \lambda I & U \end{bmatrix} = 0$ for some $X \in \mathcal{Q}(G)$. Let $z = p + jq$ for real vectors p and q where j is the imaginary number. Define $x \in \mathbb{R}^{|G|}$ as $x = p + \alpha q$ where α is a real number. Choose α such that

$$\alpha \notin \{-\frac{p_i}{q_i} : q_i \neq 0; i = 1, 2, \ldots, |G|\} \tag{6.8}$$

where p_i and q_i denote the i^{th} element of p and q, respectively. Then one can show that $x_i = 0$ if and only if $z_i = 0$. In fact, if $z_i = 0$ then obviously $x_i = 0$. In addition, if $x_i = 0$ then we obtain $p_i + \alpha q_i = 0$, which yields $q_i = 0$ by (6.8). Hence, we have $p_i = 0$, and thus $z_i = 0$.

Next, we claim that the following implication holds:

$$x_i = 0 \Rightarrow (x^\top X)_i = 0. \tag{6.9}$$

To prove this claim, suppose that $x_i = 0$. Then, we have $z_i = 0$. Since $z^* X = \lambda z^*$, we obtain $(z^* X)_i = 0$. Hence, $(p^\top X)_i = 0 = (q^\top X)_i$. Consequently, $((p^\top + \alpha q^\top)X)_i = (x^\top X)_i = 0$.

Now, we define the diagonal matrix $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$ with

$$d_i = \begin{cases} 0 & \text{if } x_i = 0 \\ \frac{(x^\top X)_i}{x_i} & \text{otherwise.} \end{cases} \tag{6.10}$$

By using (6.9), it holds that $x^\top X = x^\top D$. Besides, $z^* U = 0$ results in $p^\top U = 0 = q^\top U$ which yields $x^\top U = 0$. Now, choose $\hat{X} = X - D$. Clearly, $\hat{X} \in \mathcal{Q}(G)$ and $x^\top \hat{X} = 0$. Then due to full row rank assumption of $\begin{bmatrix} \hat{X} & U \end{bmatrix}$ we obtain $x = 0$, thus $z = 0$. Therefore, $\begin{bmatrix} X - \lambda I & U \end{bmatrix}$ has full row rank, and the result follows. ■

Next, we explore the relationship between zero forcing sets and controllability of $(G; V_L)$. First we show that the process of coloring/infecting vertices, according to the change-color rule, does not affect the controllability. This issue is addressed in the following lemma.

Lemma 6.3.2 *Let G be a graph and C be a (coloring) set. Suppose that $v \to w$ where $v \in C$ and $w \notin C$. Then $(G; C)$ is controllable if and only if $(G; C \cup \{w\})$ is controllable.*

Proof. The "only if" part is trivial. Now, let $C' := C \cup \{w\}$ and suppose that $(G; C')$ is controllable. Hence, (X, U) is controllable for all $X \in \mathcal{Q}(G)$ where $U = U(V(G); C')$ is given by (6.2). Without loss of generality, we can assume that

$$(X, U) = \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \right) \tag{6.11}$$

is controllable for all $X \in \mathcal{Q}(G)$, where the first row corresponds to the vertex w, the second corresponds to v, the third row block corresponds to the vertices indexed by $C \setminus \{v\}$, and the last row block corresponds to remaining white vertices, i.e. $V(G) \setminus C'$.

By Lemma 6.3.1, we know that $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank, which implies that the last row block of X in (6.11) has full row rank. Since $v \to w$, we have $x_{12} \neq 0$ and $X_{42} = 0$. Therefore, the submatrix

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \tag{6.12}$$

has full row rank. Consequently, the pair

$$\left(\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \right) \tag{6.13}$$

is controllable, and hence $(G; C)$ is controllable. ■

Roughly speaking, this lemma states that controllability is invariant under infection. As such, we can obtain the following corollary by repeated application of Lemma 6.3.2.

Corollary 6.3.3 *Let G be a graph and a C be a coloring set. Then, $(G; C)$ is controllable if and only if $(G; \mathcal{D}(C))$ is controllable.*

Next, we state one of the main results of this chapter based on the above auxiliary lemmas.

Theorem 6.3.4 *Let G be a graph and $V_L \subseteq V(G)$. Then, $(G; V_L)$ is controllable if and only if V_L is a zero forcing set.*

Proof. If V_L is a zero forcing set, then $\mathcal{D}(V_L) = V(G)$ by definition. Hence, it follows from Corollary 6.3.3 that controllability of $(G; V_L)$ is equivalent to that of $(G; V(G))$. Since $(G; V(G))$ is trivially controllable, so is $(G; V_L)$. To prove the converse, suppose that $(G; V_L)$ is controllable, but V_L is *not* a zero forcing set. Then, we have $\mathcal{D}(V_L) \neq V(G)$. We also know that $(G; \mathcal{D}(V_L))$ is controllable by Corollary 6.3.3. Without loss of generality, we can assume that $V_L = \{1, 2, ..., m\}$ and $\mathcal{D}(V_L) = V_L \cup \{m + 1, m + 2, \ldots, m + r\}$ where $m + r < |G|$. Since $(G; \mathcal{D}(V_L))$ is controllable, it follows from Lemma 6.3.1 that the matrix $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$ where $U = U(V; \mathcal{D}(V_L)) = \operatorname{col}(I_{m+r}, 0)$. Hence, the matrix

$$\begin{bmatrix} X_{11} & X_{12} & I_{m+r} \\ X_{21} & X_{22} & 0 \end{bmatrix} \tag{6.14}$$

has full row rank for all $X \in \mathcal{Q}(G)$ where $X_{11} \in \mathbb{R}^{(m+r)\times(m+r)}$, $X_{12} \in \mathbb{R}^{(m+r)\times k}$, $X_{21} \in \mathbb{R}^{k\times(m+r)}$, and $X_{22} \in \mathbb{R}^{k\times k}$ with $k = |G| - (m+r)$ constitute the corresponding partitioning of the matrix X.

Now, we distinguish two cases. First, suppose that there exists a column of X_{21} with exactly one nonzero element. This implies that there is a vertex, say $v \in \mathcal{D}(V_L)$, which has exactly one (white) out-neighbor, say $w \notin \mathcal{D}(V_L)$. Consequently, v can infect w, and we reach a contradiction. On the other hand, suppose that there does not exist a column of X_{21} with exactly one nonzero element. Then, clearly the nonzero elements of X_{21} can be chosen such that we have $\mathbb{1}^\top X_{21} = 0$, where $\mathbb{1}$ denotes the vector of ones with an appropriate dimension. In addition, note that the diagonal elements of X can be chosen arbitrarily due to the the definition of $\mathcal{Q}(G)$, and thus can be assigned such that $\mathbb{1}^\top X_{22} = 0$. Therefore, we obtain that

$$\begin{bmatrix} 0_{m+r}^\top & \mathbb{1}^\top \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & I_{m+r} \\ X_{21} & X_{22} & 0 \end{bmatrix} = 0,$$

for some X in $Q(G)$, and again we reach a contradiction. ■

Theorem 6.3.4 establishes a one-to-one correspondence between leader sets rendering systems of the form (6.3) controllable and zero forcing sets of the corresponding graphs. An immediate consequence of this result yields the following result on the minimum number of leaders required for controllability.

Corollary 6.3.5 *Let G be a given graph. Then, $\ell_{\min}(G) = Z(G)$.*

6.3.1 Sufficiently rich qualitative subclasses

So far, we have investigated controllability of systems given by (6.3) where the matrices X belongs to the family $\mathcal{Q}(G)$ which is described by the graph G. In many examples, one encounters matrices of X carrying more structure than that imposed by $\mathcal{Q}(G)$. For instance, consider a graph G_1 for which $E(G_1)$ is symmetric, i.e. $(v, w) \in E(G_1)$ if and only if $(w, v) \in E(G_1)$ and the matrices X belonging to

$$\mathcal{Q}_s(G_1) = \{X \in \mathcal{Q}(G_1) : X = X^\top\} \subseteq \mathcal{Q}(G_1). \tag{6.15}$$

Note that undirected graphs can be identified with directed graphs having symmetric arc sets. As such, the class $\mathcal{Q}_s(G_1)$ naturally appears whenever the underlying graph structure is induced by an undirected graph as in the systems of the form (6.6) and (6.7)

In what follows, we focus on controllability with respect to subclasses of $\mathcal{Q}(G)$. For a graph G, (leader) set $V_L \subseteq V(G)$, and a qualitative subclass $\mathcal{Q}'(G) \subseteq \mathcal{Q}(G)$, we say V_L controls $\mathcal{Q}'(G)$ if $(X; V_L)$ is controllable for all $X \in \mathcal{Q}'(G)$.

If V_L is a zero a forcing set for the graph G, then V_L controls $\mathcal{Q}(G)$ by Theorem 6.3.4. Consequently, such a V_L controls $\mathcal{Q}'(G)$ for any $\mathcal{Q}'(G) \subseteq \mathcal{Q}(G)$. However, the converse is not true in general. For instance, consider $G_1 = (V_1, E_1)$ where $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}$. Let $V_L = \{2\}$ and take the Laplacian matrix of G_1, denoted by L_1, as the qualitative subclass in this case. Then, by [75], $(L_1; V_L)$ is controllable whereas obviously V_L is not a zero forcing set.

Therefore, we conclude that V_L is not necessarily a zero forcing set for G even though it controls $\mathcal{Q}'(G)$ for some nonempty $\mathcal{Q}'(G) \subseteq Q(G)$. Next, we investigate under what conditions, controlling $\mathcal{Q}'(G) \subseteq \mathcal{Q}(G)$ by V_L, implies that V_L also controls $\mathcal{Q}(G)$, or equivalently, V_L is a zero forcing set. For this purpose, the following notion is needed.

Definition 6.3.6 Let $\mathcal{Q}'(G)$ be a non-empty subset of $\mathcal{Q}(G)$. We say that $\mathcal{Q}'(G)$ is a *sufficiently rich* subclass of $\mathcal{Q}(G)$ if the following implication holds:

$$z \in \mathbb{R}^{|G|},\ X \in \mathcal{Q}(G),\ z^\top X = 0 \implies \exists\, X' \in \mathcal{Q}'(G) \text{ such that } z^\top X' = 0. \tag{6.16}$$

Now, we have the following result.

Theorem 6.3.7 *Let G be a graph and $V_L \subseteq V(G)$ be a (leader) set. Suppose that $\mathcal{Q}' \subseteq \mathcal{Q}(G)$ is a sufficiently rich subclass of $\mathcal{Q}(G)$. Then the following statements are equivalent:*

1. *The set V_L is a zero forcing set.*
2. *The set V_L controls $\mathcal{Q}(G)$.*
3. *The set V_L controls $\mathcal{Q}'(G)$.*

Proof. The first two statements are equivalent by Theorem 6.3.4. Besides, the second statement trivially implies the third one. Hence, it suffices to show that statement 3 implies 2. Suppose that statement 3 holds. In view of Lemma 6.3.1, it suffices to show that the matrix $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$, where U is given by (6.2). Now suppose that $x^\top \begin{bmatrix} X & U \end{bmatrix} = 0$ for some $x \in \mathbb{R}^{|G|}$ and $X \in \mathcal{Q}(G)$. Since $\mathcal{Q}'(G)$ is a sufficiently rich subclass of $\mathcal{Q}(G)$, there exists $X' \in \mathcal{Q}'(G)$ such that $x^\top \begin{bmatrix} X' & U \end{bmatrix} = 0$. This results in $x = 0$ due to the assumption that V_L controls $\mathcal{Q}'(G)$. Consequently, the matrix $\begin{bmatrix} X & U \end{bmatrix}$ has full row rank for all $X \in \mathcal{Q}(G)$. Thus, V_L also controls $\mathcal{Q}(G)$. ■

By Theorem 6.3.7, controlling sufficiently rich subclasses is equivalent to controlling the corresponding qualitative classes, which can be further characterized by zero forcing sets. Next, we focus on two notable subclasses of $\mathcal{Q}(G_1)$. Bare in mind that $E(G_1)$ is symmetric. The first subclass we consider here is $\mathcal{Q}_s(G_1)$ given by (6.15). This subclass is a sufficiently rich subclass of $\mathcal{Q}(G_1)$ as stated next.

Proposition 6.3.8 *The set $\mathcal{Q}_s(G_1)$ is a sufficiently rich subclass of $\mathcal{Q}(G_1)$.*

Proof. Assume that there exists $z \in \mathbb{R}^{|G_1|}$ such that $z^\top X = 0$ for some $X \in \mathcal{Q}(G_1)$. We distinguish two cases. First, suppose that $z_i \neq 0$ for each $i = 1, 2, \ldots, |G_1|$. Define the matrix X' as $X' = \hat{X} + D$ where $\hat{X} \in \mathcal{Q}_s(G_1)$ and D is a real diagonal matrix. Obviously, we have $X' \in \mathcal{Q}_s(G_1)$ for any choices of X' and D. Then, since $z_i \neq 0$ for each i, one can choose D such that $z^\top X' = 0$. Next, consider the case where $z_i = 0$ for some i. Without loss of generality, the vector z can be then partitioned as $z = [\hat{z} \quad 0]^\top$ such that the vector $\hat{z}$ does not contain any zero element. Correspondingly, let the matrix X be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Hence, we have $\hat{z}^\top X_{11} = 0$ and $\hat{z}^\top X_{12} = 0$ by the assumption. Now, choose a matrix $\hat{X} \in \mathcal{Q}_s(G_1)$ and let

$$\hat{X} = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{12}^\top & \hat{X}_{22} \end{bmatrix}.$$

Let D be a real diagonal matrix such that $\hat{z}^\top(\hat{X}_{11} + D) = 0$. Note that such D exists as $\hat{z}_i \neq 0$ for each i. Then, we construct a matrix X' as

$$X' = \begin{bmatrix} \hat{X}_{11} + D & X_{12} \\ X_{12}^\top & \hat{X}_{22} \end{bmatrix}.$$

Clearly, we have $X' \in \mathcal{Q}_s(G_1)$. Moreover, it holds that $z^\top X' = 0$, and thus $\mathcal{Q}_s(G_1)$ is a sufficiently rich subclass of $\mathcal{Q}(G_1)$. ∎

Now, we consider a second subclass of $\mathcal{Q}(G_1)$ by imposing an additional constraint to $\mathcal{Q}_s(G_1)$. More precisely, let $\mathcal{Q}_s^-(G_1) \subseteq \mathcal{Q}_s(G_1) \subseteq \mathcal{Q}(G_1)$ be defined as

$$\mathcal{Q}_s^-(G_1) = \{X \in \mathcal{Q}_s(G_1) : \text{ for } i \neq j,\ X_{ij} < 0 \text{ whenever } X_{ij} \neq 0\} \tag{6.17}$$

Note that ordinary Laplacian matrices and adjacency matrices are among the special cases of this subclass. Structural controllability with respect to this subclass has been studied in [10]. In particular, it has been shown that the set V_L controls $\mathcal{Q}_s^-(G_1)$ if V_L is a zero forcing set. However, the converse does not hold as pointed out by [10, Ex. 4.3]. The following proposition shows that indeed $\mathcal{Q}_s^-(G_1)$ is not a sufficiently rich subclass, except for some pathological cases.

Proposition 6.3.9 *Assume that the graph G_1 has a vertex with at least two (out) neighbors. Then, the set $\mathcal{Q}_s^-(G_1)$ is not a sufficiently rich subclass of $\mathcal{Q}(G_1)$.*

Proof. Let k be a vertex of G_1 with at least two (out) neighbors. Define $z \in \mathbb{R}^{|G_1|}$ as

$$z_i = \begin{cases} 1 & \text{if } i \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(z^\top X')_k$ is a negative number for any $X' \in \mathcal{Q}_s^-(G_1)$. Hence, $z^\top X' \neq 0$ for any $X' \in \mathcal{Q}_s^-(G_1)$. Therefore, to conclude that $\mathcal{Q}_s^-(G_1)$ is not sufficiently rich, it suffices to show that $z^\top X = 0$ for some $X \in \mathcal{Q}(G_1)$. It is easy to see that one can choose a matrix $X \in \mathcal{Q}(G_1)$ such that $(z^\top X)_i = 0$ for each $i \neq k$. Also note that, by the assumption, the matrix X has at least two nonzero off-diagonal elements in its k^{th} column. Hence, these (two or more) nonzero elements can be further chosen such that we have $(z^\top X)_k = 0$, and thus $z^\top X = 0$. This completes the proof. ■

6.3.2 Special cases

Next, we study some special cases to demonstrate how the proposed results can be used in particular applications.

As we mentioned earlier, controllability of systems of the form (6.6) has been extensively studied in the literature. In particular, minimum number of leaders that render the system (6.6) controllable was investigated for some special classes of undirected graphs. To apply our results to the special case of undirected graphs, we identify an undirected graph H by a corresponding directed graph G whose arc set is symmetric. As an example, three undirected graphs together with the corresponding directed graphs are depicted in Figure 6.3. For an undirected graph H, we denote the corresponding directed graph by $\theta(H)$. Note that, clearly, the Laplacian matrix L of H belongs to the qualitative class $\mathcal{Q}(\theta(H))$.

In case of an undirected path graph P_n with n vertices, it has been shown in [78] that $\ell_{\min}(L) = 1$. For an undirected cycle graph C_n, it has been shown in [106, Thm. 3] that $\ell_{\min}(L) = 2$, and any two neighbors can be chosen as leaders. For an undirected complete graph K_n with n vertices, we have $\ell_{\min}(L) = n - 1$, and any $n - 1$ vertices can be chosen as leaders (see [106, Thm. 4]). By looking at Figure 6.3, it is easy to verify that $\ell_{\min}(L)$ coincides with the zero forcing number in these three cases, i.e. path, cycle, and complete graphs. Note that the set $\{1\}$ or $\{3\}$ is a minimal ZFS for the path graph in Figure 6.3. Moreover, any two neighboring vertices constitutes a minimal zero forcing set for the cycle graph, and any three out of the four vertices forms a minimal ZFS for the complete graph in Figure 6.3. Obviously, this is not limited to the depicted examples, and holds true for any undirected path, cycle, or complete graphs. Therefore, we obtain that $Z(\theta(P_n)) = 1$, $Z(\theta(C_n)) = 2$, and $Z(\theta(K_n)) = n - 1$. Then, by Corollary 6.3.5, we conclude that the existing results

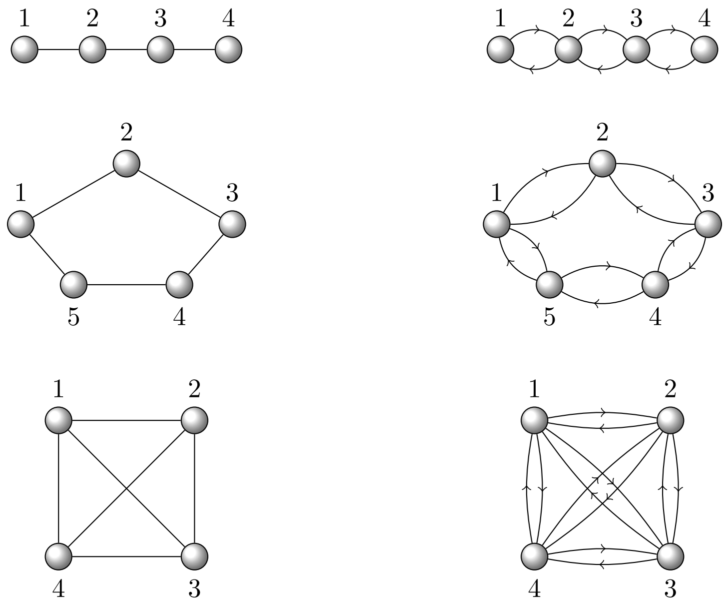

Figure 6.3: *Undirected graphs and the associated symmetric directed graphs: path, cycle and complete graphs*

for the minimum number of leaders rendering the system (6.6) controllable, carries over unchanged to the class of systems whose dynamics is given by (6.3).That is, we have $\ell_{\min}(X) = 1$ for any $X \in \mathcal{Q}(\theta(P_n))$, $\ell_{\min}(X) = 2$ for any $X \in \mathcal{Q}(\theta(C_n))$, and $\ell_{\min}(X) = n - 1$ for any $X \in \mathcal{Q}(\theta(K_n))$.

It is worth mentioning that one should not conjecture based on the aforementioned special cases that $\ell_{\min}(L)$ is equal to the zero forcing number for any graph. As a counter example, consider a 6-regular circulant graph with 10 vertices. It follows directly from [69, Thm. III.1] that $\ell_{\min}(L) = 2$, whereas it is easy to observe that no pair of vertices results in a zero forcing set.

After the discussion of undirected graph classes for which $\ell_{\min}(L)$ has been characterized in the literature, we turn our attention to a class of directed graphs, namely directed trees (ditrees). We use the symbol T to denote a ditree to avoid possible confusion with the general case. The notions of a path, the path cover number, and a minimal path cover are required before stating the result for this case (see e.g. [37]) for more details on these notions).

Definition 6.3.10 A *path* P in G is an ordered set of distinct vertices $(v_1, v_2, \ldots, v_k)$ of G such that $(v_i, v_{i+1}) \in E(G)$ for each $i = 1, 2, \ldots, k-1$. The vertex v_1 is called *the initial point* of P and v_k is the *final point* of P. The *path cover number* of G, denoted by $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G; such a set of paths realizing $P(G)$ is called a *minimal path cover*.

Now, we have the following result in case of tree structures.

Proposition 6.3.11 *Let T be a ditree. Then, we have $\ell_{\min}(T) = P(T)$. Moreover, the initial points of the vertex disjoint paths realizing a minimal path cover form a minimal zero forcing set.*

Proof. The result follows directly by applying Theorem 6.3.4 and Corollary 6.3.5, together with [37, Thm. 3.5] and the proof provided therein. ■

6.4 Conclusion

Controllability of systems defined on graphs has been discussed in this chapter. We have considered the problem of controllability of the network for a family of matrices carrying the structure of an underlying directed graph. This family of matrices is called the qualitative class, and as observed, there is a one-to-one correspondence between the set of leaders rendering the network controllable for all matrices in the qualitative class and zero forcing sets. We have also dealt with the case where one is interested in some subset of this qualitative class, through the notion of sufficiently rich subclasses. To further illustrate the proposed results, special cases including path, cycle, and complete graphs are discussed. In addition, we have shown how the proposed results of the present chapter together with the existing results on the zero forcing sets lead to a minimal leader selection scheme in particular cases, such as graphs with a tree structure. Based on the results of the present chapter, our knowledge about (minimal) leader selection for controllability of a network is intimately related to the knowledge we have for zero forcing sets (number). Indeed, for each class of graphs whose zero forcing number has been known or will be established later on, we immediately obtain the minimum number of leaders for controllability, and, in principle, a minimal leader selection scheme.

Chapter 7

Disturbance decoupling problem for multi-agent systems: A graph topological approach

Analysis and design of multi-agent systems and networks of dynamical agents have turned to an extremely popular research target in the last decade. Studying consensus/synchronization and designing feedback protocols to achieve consensus/synchronization have perhaps been the most popular framework in this direction; see e.g. [72], [41], [79], [47], [83], [95].

An important issue in studying networks of dynamical agents is to deduce certain network properties from the "network topology" which is typically given in terms of the so-called "communication graph" of the network. For instance, it is well-known that connectivity of the communication graph plays a crucial role in the consensus problem (see e.g. [72]). Recently, studying other network properties from a topological perspective has attracted the attention of many researchers see e.g. [56], [13], [78], [24]. A notable instance is controllability analysis; see e.g. [107], [24], [52], [78], [75], [69]. In this framework, agents are labeled as *leaders* and *followers*. Leaders are agents through which external input signals are injected to the network, and the rest of the agents are called followers. Then, controllability analysis amounts to investigate the possibility of deriving the states of the agents to arbitrary values by appropriate input signals applied to the leaders. Graph partitions, and in particular "almost equitable partitions", has been proven to be a useful tool in controllability analysis; see e.g. [107]. These partitions can be considered as a topological translation of $L-$invariant subspaces, with L denoting the Laplacian matrix of the network communication graph; see e.g. [14], [107].

As a step forward to investigate network properties from a topological perspective, in this chapter, we study the "disturbance decoupling problem" of diffusively coupled leader-follower networks, where each vertex has single integrator dynamics and some agents are affected by disturbance signals. Roughly speaking, for a classical linear system with inputs and outputs, the disturbance decoupling problem (DDP) amounts to find a feedback (typically, state feedback) such that the output of the closed-loop system is not affected by disturbance signals acting on the states of the system, see e.g. [94]. If such a feedback exists, then we say the DDP for the system is solvable.

Our contribution comes from but goes beyond the disturbance decoupling solution in the *geometric approach* for linear systems. The geometric approach to linear system synthesis was inaugurated by the recognition of controlled invariant subspaces, due independently to Basile and Marro [6] and to Wohnam and Morse [103]. Disturbance decoupling problem is in fact an immediate application of the controlled invariance property, see e.g.[5, 91]. To the authors' best knowledge, the current manuscript is the first attempt to study the disturbance decoupling problem for networks of dynamical agents from a topological perspective. Studying network properties such as controllability and DDP from a topological perspective provides valuable insights into the structure/behaviour of the network, and it will facilitate the design.

In this chapter, we introduce a new class of partitions almost equitability with respect to a cell in order to provide an appropriate topological translation for controlled invariant subspaces in the context of dynamical networks. Then, by using this extended notion of almost equitability, we derive sufficient (topological) conditions for the network to be disturbance decoupled. More precisely, we consider both open loop and closed loop disturbance decoupling problem. In the first case, we investigate if the network is already disturbance decoupled without applying input signals to the leaders. In the latter case, we consider the solvability of the DDP for the network that amounts to find (if possible) a state feedback controller rendering the network disturbance decoupled. In particular, we establish sufficient topological conditions guaranteeing the network to be disturbance decoupled (open loop) as well as conditions guaranteeing the solvability of DDP (closed loop). Both in the closed loop as well as the open loop case, algorithms to verify the proposed conditions are provided. A crucial point in the context of distributed control is to exploit the relative (local) information of the states of the agents rather than absolute (global) information of the states. As desired, it will be observed that in case the DDP for the network is solvable then the controller rendering the network disturbance decoupled is indeed using relative information of the states of the agents.

The structure of this chapter is as follows. In Section 7.1, some preliminary materials are provided, and the open loop and closed loop disturbance decoupling problems for multi-agent systems are formulated. In Section 7.2, the notion of almost equitability with respect to a cell is proposed and characterized in terms of controlled invariant subspaces. In Section 7.3, we establish sufficient conditions guaranteeing the network to be disturbance decoupled as well as conditions guaranteeing the solvability of DDP. Algorithms to verify the proposed topological conditions are proposed in Section 7.4. To illustrate the proposed results, a numerical example is provided in Section 7.5. Finally, the chapter ends with a summary in Section 7.6.

7.1 Diffusively coupled multi-agent systems and disturbance decoupling

7.1.1 Leader-follower diffusively coupled multi-agent systems with disturbance

In this chapter, we consider a multi-agent system consisting of $n > 1$ agents labeled by the set $V = \{1, 2, \ldots, n\}$. We assign three subsets of V as follows: $V_{\mathrm{L}} = \{\ell_1, \ell_2, \ldots, \ell_m\}$ where $m \leqslant n$, $V_{\mathrm{F}} = V \setminus V_{\mathrm{L}}$ and $V_{\mathrm{D}} = \{w_1, w_2, \ldots, w_r\}$ where $r \leqslant n$.

We associate the dynamics

$$\dot{x}_i(t) = \begin{cases} z_i(t) + u_k(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) + u_k(t) & \text{otherwise} \end{cases} \tag{7.1}$$

to each agent $i = \ell_k \in V_{\mathrm{L}}$, and

$$\dot{x}_i(t) = \begin{cases} z_i(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) & \text{otherwise} \end{cases} \tag{7.2}$$

to each agent $i \in V_{\mathrm{F}}$, where $x_i \in \mathbb{R}$ represents the state of agent $i \in V$, z_i indicates the coupling variable of agent $i \in V$, $u_k \in \mathbb{R}$ is an external control input signal received by agent $i = \ell_k \in V_{\mathrm{L}}$, and $d_l \in \mathbb{R}$ is taken as an external disturbance signal influencing agent $i = w_l \in V_{\mathrm{D}}$.

Considering the roles of the defined subsets of V, we refer to V_{L} as the *leader set*, V_F as the *follower set*, and V_D as the *disturbance set*. Correspondingly, we say i is a *leader* if $i \in V_L$, and i is a *follower* if $i \in V_F$.

We consider a simple (unweighted) directed graph $G = (V, E)$, where V is the vertex set and $E \subseteq V \times V$ is the arc set of G. For two distinct vertices $i, j \in V$, we have $(i, j) \in E$ if there is an arc from i to j with i being the *tail* and j being the *head* of the arc. Then i is said to be a *neighbor* of j. The coupling variable z_i admits the following *diffusive coupling* rule:

$$z_i(t) = -\sum_{(j,i) \in E} (x_i(t) - x_j(t)). \tag{7.3}$$

By defining $x(t) = \mathrm{col}(x_1(t), x_2(t), \cdots, x_n(t))$, $u(t) = \mathrm{col}(u_1(t), u_2(t), \cdots, u_m(t))$ and $d(t) = \mathrm{col}(d_1(t), d_2(t), \cdots, d_r(t))$, we write the above leader-follower diffusively coupled multi-agent system (7.1), (7.2) and (7.3) into a compact form as follows:

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \tag{7.4}$$

where L is the in-degree Laplacian of the simple directed graph G (see e.g. [56, p.26]), the matrix $M \in \mathbb{R}^{n\times m}$ is defined by

$$M_{ik} = \begin{cases} 1 & \text{if } i = \ell_k \\ 0 & \text{otherwise} \end{cases} \tag{7.5}$$

and the matrix $S \in \mathbb{R}^{n\times l}$ is defined by

$$S_{il} = \begin{cases} 1 & \text{if } i = w_l \\ 0 & \text{otherwise.} \end{cases} \tag{7.6}$$

Next we consider another simple directed graph $\tilde{G} = (V, \tilde{E})$ and define the output $y(t)$ of the system (7.4) as follows:

$$y(t) = R^\top x(t) \tag{7.7}$$

where R is the incidence matrix of $\tilde{G}$ (see e.g. [56, p.23]). The output variables (7.7) capture the differences between the state components of certain pairs of agents determined by the arc set $\tilde{E}$ of $\tilde{G}$. In particular, an arc from i to j in $\tilde{G}$ corresponds to the output variable $x_i - x_j$ in (7.7).

In this chapter, we study the so-called *disturbance decoupling problem* for multi-agent system (7.4) by establishing graph topological conditions. Roughly speaking, our aim is to investigate the effect of the disturbance signal d on the output y, given by (7.7). For a formal description of the problem and discussing the proposed results, we first review the disturbance decoupling problem and its solution for ordinary linear systems.

7.1.2 Review: disturbance decoupling problem of linear systems

Consider the linear system

$$\dot{x}(t) = Ax(t) + Ed(t) \tag{7.8a}$$

$$y(t) = Cx(t) \tag{7.8b}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}^q$ is the output, and all involved matrices are of appropriate sizes. We denote the state trajectory of the system (7.8) for the initial state $x(0) = x_0$ and the disturbance d by $x^{x_0,d}$ and the corresponding output trajectory by $y^{x_0,d}$. We quote the following definition for later use.

Definition 7.1.1 The linear system (7.8) is said to be *disturbance decoupled* if $y^{x_0,d_1}(t) = y^{x_0,d_2}(t)$ for all $x_0 \in \mathbb{R}^n$, all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$. Due to linearity, this is equivalent to the condition $y^{0,d_1}(t) = y^{0,d_2}(t)$ for all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$.

In what follows, we quickly review the geometric approach for DDP. For more details, we refer to [94] and [102].

Let $\langle A \mid \operatorname{im} E\rangle$ denote the controllable subspace corresponding to the matrix pair (A, E), that is $\langle A \mid \operatorname{im} E\rangle = \operatorname{im} E + A \operatorname{im} E + \cdots + A^{n-1} \operatorname{im} E$. As it is well-known, the subspace $\langle A \mid \operatorname{im} E\rangle$ is the smallest A-invariant subspace that contains $\operatorname{im} E$. Note that we call a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$ where $A : \mathbb{R}^n \to \mathbb{R}^n$. For the matrix pair, (A, C), the unobservable subspace is denoted by $\langle \ker C \mid A\rangle$, that is $\langle \ker C \mid A\rangle = \ker C \cap A^{-1} \ker C \cap \cdots \cap A^{-n+1} \ker C$. Here, for a given subspace $\mathcal{X}$, $A^{-1}\mathcal{X}$ denotes the subspace $\{x : Ax \in \mathcal{X}\}$. It is well-known that the unobservable subspace $\langle \ker C \mid A\rangle$ is the largest A-invariant subspace that is contained in $\ker C$.

Necessary and sufficient conditions for the system (7.8) to be disturbance decoupled is well-known and are recapped in the following lemma.

Lemma 7.1.2 *The following conditions are equivalent.*

1. *The system* (7.8) *is disturbance decoupled.*
2. *There exists an A-invariant subspace $\mathcal{V}$ such that* $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker C$.
3. *The inclusion* $\operatorname{im} E \subseteq \langle \ker C \mid A\rangle$ *holds.*
4. *The inclusion* $\langle A \mid \operatorname{im} E\rangle \subseteq \ker C$ *holds*

Note that the equivalence between the first three statements is quite standard and can be, for instance, found in [94, Ch. 4]. The forth statement immediately follows from the first two and will be employed in the context of multi-agent systems later.

Now, suppose that the linear system (7.8) is not disturbance decoupled. Then, one may think of applying control inputs to manipulate the system dynamics such that the closed loop system will be disturbance decoupled. This is discussed next.

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{7.9a}$$

$$y(t) = Cx(t) \tag{7.9b}$$

where $u \in \mathbb{R}^m$ is the input and $B \in \mathbb{R}^{n\times m}$. The disturbance decoupling problem by state feedback is defined as follows.

Definition 7.1.3 The *disturbance decoupling problem* by state feedback for the system (7.9) amounts to finding a state feedback of the form $u = Kx$ such that the resulting closed loop system

$$\dot{x}(t) = (A + BK)x(t) + Ed(t) \tag{7.10a}$$

$$y(t) = Cx(t) \tag{7.10b}$$

is disturbance decoupled. Moreover, if such a state feedback exists, then we say the disturbance decoupling problem for system (7.9) is *solvable*.

Necessary and sufficient conditions for solvability of disturbance decoupling problem are among the classical results of the geometric approach. In order to state these classical results, we need to review a few more notions of geometric approach. We say a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is *controlled invariant* for the pair (A, B) if there exists K such that $(A + BK)\mathcal{V} \subseteq \mathcal{V}$. Moreover, we have

$$\mathcal{V} \text{ is controlled invariant for } (A, B) \Leftrightarrow A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B \tag{7.11}$$

For the pair (A, B), we denote the set of all controlled invariant subspaces which are contained in $\ker C$ by $\mathscr{V}(A, B, C)$. Let $\mathcal{V}^*(A, B, C)$ denote the maximal element of the set $\mathscr{V}(A, B, C)$ with respect to the partial order induced by the subspace inclusion, that is $\mathcal{V} \subseteq \mathcal{V}^*(A, B, C)$ for all $\mathcal{V} \in \mathscr{V}(A, B, C)$. The existence and uniqueness of such an element immediately follow from finite-dimensionality. It is well-known that $\mathcal{V}^*(A, B, C) \in \mathscr{V}(A, B, C)$. Now, the following lemma states the necessary and sufficient condition for the solvability of the disturbance decoupling problem for system (7.10).

Lemma 7.1.4 *Considering the system* (7.10)*, the following statements are equivalent:*

1. *The disturbance decoupling problem for system* (7.10) *is solvable.*
2. *There exists a controlled invariant subspace $\mathcal{V}$ for the pair (A, B) such that $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker C$.*
3. *The inclusion $\operatorname{im} E \subseteq \mathcal{V}^*(A, B, C)$ holds.*

7.1.3 Disturbance decoupling problem of diffusively coupled multi-agent system

In this subsection, we formally state the disturbance decoupling problem for multi-agent systems, which we will study in this chapter. Now, recall the multi-agent system (7.4) together with the output (7.7). Similar to Subsection 7.1.2, we first consider the open loop case where no external control input is applied to the agents. Consequently, we propose the following problem.

Problem 7.1.5 Consider the input/state/output system given by

$$\dot{x}(t) = -Lx(t) + Sd(t) \tag{7.12a}$$

$$y(t) = R^\top x(t) \tag{7.12b}$$

where the matrices L, S, and R are defined as before. Under what (topological) conditions the system (7.12) is disturbanced decoupled?

This problem together with the proposed solutions will be discussed in details in Subsections 7.3.1 and 7.4.1. In case the system (7.12) is not disturbanced decoupled, similar to the idea in Subsection 7.1.2, we investigate the possibility of rendering the system disturbance decoupled by choosing some agents as leaders and apply appropriate inputs to these agents. This leads us to the following problem.

Problem 7.1.6 Consider the input/state/output system given by

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \tag{7.13a}$$

$$y(t) = R^\top x(t) \tag{7.13b}$$

where the matrices L,M, S, and R are defined as before. Under what (topological) conditions the disturbance decoupling problem for system (7.13) is solvable in the sense of Definition 7.1.3?

The corresponding results and discussions for Problem 7.1.6 are provided in Subsections 7.3.2 and 7.4.2.

7.2 Graph partitions and almost equitability with respect to a cell

Before discussing solutions for the aforementioned problems, in this section we review some notions from graph theory, including graph partitions and, in particular, almost equitable partitions. Moreover, for our purpose, we define an extended version of almost equitability, and that is almost equitability with respect to a cell. This notion together with the results established in this section, provides the main foundation for the subsequent results which will be developed in Section 7.3.

Let $G = (V, E)$ be a simple (unweighted) directed graph where $V = \{1, 2, \ldots, n\}$, $E \subseteq V \times V$, and $(i, i) \notin E$. By $L(G)$, we denote the in-degree Laplacian of G (see [56, p. 26]). We simply use L to denote the Laplacian matrix when the underlying graph is clear from the context.

We call any subset of V a *cell* of V. We call a collection of cells, given by $\rho = \{C_1, C_2, \ldots, C_k\}$, a *partial partition* of V if $C_i \cap C_j = \varnothing$ whenever $i \neq j$. In addition, we call ρ a *partition* of V if it is a partial partition and $\cup_i C_i = V$. In some occasions, to clarify the underlying graph we say ρ is a (partial) partition of $G = (V, E)$, or shortly G, meaning that ρ is a (partial) partition of V.

For a cell $C \subseteq V$, we define the *characteristic vector* of C as

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases} \tag{7.14}$$

For a partial partition $\rho = \{C_1, C_2, \ldots, C_k\}$, we define the *characteristic matrix* of π as

$$P(\rho) = \begin{bmatrix} p(C_1) & p(C_2) & \cdots & p(C_k) \end{bmatrix}. \tag{7.15}$$

Finally, the notion of partial ordering for partitions is defined as follows. We say that a partition π_1 is *finer* than another partition π_2, or alternatively π_2 is *coarser* than π_1, if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leqslant \pi_2$. Also we write as $\pi_1 \nleq \pi_2$ meaning that π_1 is not finer than π_2. It is a direct consequence of the definition that

$$\pi_1 \leqslant \pi_2 \iff \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1). \tag{7.16}$$

7.2.1 Almost equitable partitions

Here, we adopt the notion of almost equitability (see e.g. [14]) for directed graphs. For a given cell $C \subseteq V$, we write

$$N(j, C) = \{i \in C : (i, j) \in E\}.$$

We call a partition $\pi = \{C_1, C_2, \ldots, C_k\}$ an *almost equitable partition* (AEP) of G if for each $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$ there exists an integer d_{ij} such that $|N(v, C_j)| = d_{ij}$ for all $v \in C_i$.

Example 7.2.1 Consider the graph H depicted in Figure 7.1. It is easy to verify that the partition π given by

$$\pi = \{\{1, 2\}, \{7, 8\}, \{4, 6\}, \{3\}, \{5\}\} \tag{7.17}$$

is an almost equitable partition of H.

For a given matrix A, we denote its $(i, j)^{th}$ element by A_{ij}. Then, associated to an almost equitable partition $\pi = \{C_1, C_2, \ldots, C_k\}$, we define the matrix $\mathcal{L}_\pi$ as:

$$(\mathcal{L}_\pi)_{ij} = \begin{cases} -d_{ij} & \text{if } i \neq j \\ s_i & \text{otherwise} \end{cases} \tag{7.18}$$

where $s_i = \sum_{j \neq i} d_{ij}$.

For undirected graphs, characterization of almost equitable partitions in terms of invariant subspaces has been provided in [106]. In particular, it is shown that a

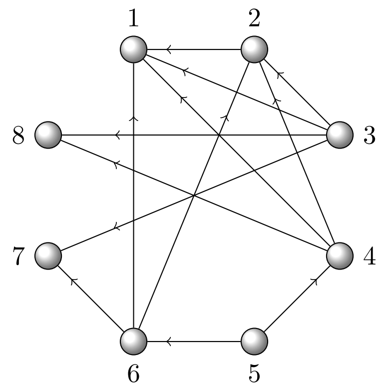

Figure 7.1: *A simple directed graph H*

partition is almost equitable if and only if the image of its characteristic matrix is L-invariant. This result can be extended to the case of directed graphs as stated in the following lemma.

Lemma 7.2.2 *A partition $\pi = \{C_1, C_2, \ldots, C_k\}$ is an AEP of G if and only if $\operatorname{im} P(\pi)$ is L-invariant.*

Proof. First, we prove the "only if" part. Assume that π is an AEP of G, and let $\mathcal{L}_\pi$ be defined as in (7.18). We claim that

$$LP(\pi) = P(\pi)\mathcal{L}_\pi, \tag{7.19}$$

and, hence $\operatorname{im} P(\pi)$ is L-invariant. First, we show that

$$(LP(\pi))_{rj} = (P(\pi)\mathcal{L}_\pi)_{rj} \tag{7.20}$$

for $r = \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, k\}$, and $r \notin C_j$. Clearly, the left hand side is equal to $-|N(r, C_j)|$. Now, since π is an AEP, we have $-|N(r, C_j)| = -d_{ij}$ where i is such that $r \in C_i$. The right hand side of (7.20) is equal to $(\mathcal{L}_\pi)_{ij}$ which is again equal to $-d_{ij}$ by definition. Hence, it remains to show that the equality (7.20) also holds for the remaining n elements indicated by $r \in C_j$. To show this, obviously, it suffices to prove that the row sums of the matrix $LP(\pi)$ is equal to that of $P(\pi)\mathcal{L}_\pi$. Let $\mathbb{1}_k$ denote the vector of ones with the length k. Then by multiplying the left hand side of (7.19) by $\mathbb{1}_k$, we obtain that $LP(\pi)\mathbb{1}_k = L\mathbb{1}_n = 0$. Similarly, for the right hand side we have $P(\pi)\mathcal{L}_\pi\mathbb{1}_k = 0$ as the row sums of $\mathcal{L}_\pi$ is zero. Therefore, (7.19) holds, and thus $\operatorname{im} P(\pi)$ is L-invariant.

Conversely, assume that $\operatorname{im} P(\pi)$ is L-invariant. Then, for each $j = 1, 2, \ldots, k$, we have

$$Lp(C_j) \in \operatorname{im} P(\pi) \tag{7.21}$$

as $p(C_j)$ is contained in the image of $P(\pi)$. Observe that the q^{th} element of $Lp(C_j)$ is equal to $|N(q, C_j)|$ for each $q \notin C_j$. Hence, based on (7.21), for any $q_1, q_2 \in C_i$, we have $|N(q_1, C_j)| = |N(q_2, C_j)|$ for all $j \neq i$. Consequently, π is an AEP of G. ■

Note that, if π is an AEP, then based on the proof of Lemma 7.2.2 we have

$$LP(\pi) = P(\pi)X \tag{7.22}$$

for $X = \mathcal{L}_\pi$ where $\mathcal{L}_\pi$ is given by (7.18). Moreover, $\mathcal{L}_\pi$ is the unique solution of (7.22) as $P(\pi)$ has full column rank.

7.2.2 Almost equitability with respect to a cell

Next, we define almost equitability with respect to a given cell as follows. Given a cell C and a partition $\pi = \{C_1, C_2, \ldots, C_k\}$, we call π an AEP with respect to C if for each $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$ there exists an integer d_{ij} such that $|N(v, c_j)| = d_{ij}$ for all $v \in C_i \backslash C$.

Observe that if π is an AEP, then the number of neighbors that a vertex in C_i has in C_j is independent of the choice of the vertex in C_i, for all $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$. The notion of almost equatability with respect to a cell C is obtained by exempting the nodes in C from satisfying neighborhood constraints of the ordinary almost equitability. Clearly, π is an AEP of G if and only if it is an AEP with respect to the empty cell. Moreover, if π is an AEP of G, then it is an AEP with respect to any arbitrary cell of V.

Example 7.2.3 Consider the graph G given by Figure 7.2, and let the partition π be given by (7.17). Then it is easy to observe that π is an almost equitable partition of G with respect to $C = \{2\}$.

Note that if π is an almost equitable partition with respect to C, then, similar to ordinary almost equitable partitions, we can define the matrix $\mathcal{L}_\pi$ as in (7.18). In this case we use the notation $\mathcal{L}_\pi^C$ to distinguish with the case of ordinary almost equitability.

Our aim, here, is to characterize the property of almost equitability with respect to a cell. To state this characterization, we need some additional notation and auxiliary results.

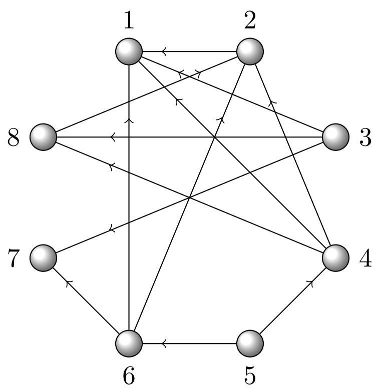

Figure 7.2: *A simple directed graph G*

For a given matrix $M \in \mathbb{R}^{m \times m}$, let M^{α} with $\alpha \subseteq \{1, 2, \cdots, m\}$ denote the submatrix of M obtained by collecting the rows of M indexed by α. Then, the following result holds.

Lemma 7.2.4 *A partition π is an AEP with respect to cell C if and only if*

$$L^{\alpha} \operatorname{im} P(\pi) \subseteq \operatorname{im} P^{\alpha}(\pi) \tag{7.23}$$

where $\alpha = V \backslash C$.

Proof. The proof is analogous to that of Lemma 7.2.2 by restricting the rows of L and P to those which are indexed by α. ■

Note that, if π is an AEP with respect to C, we have

$$L^{\alpha} P(\pi) = P^{\alpha}(\pi) \mathcal{L}_{\pi}^{C} \tag{7.24}$$

where $\alpha = V \backslash C$, and $\mathcal{L}_{\pi}^{C}$ is given by (7.18) with d_is obtained from the definition of almost equitability with respect to C. Now, we have the following characterization for almost equitability with respect to a cell.

Theorem 7.2.5 *Let $C = \{\ell_1, \ell_2, \ldots, \ell_m\}$ be a cell of V and $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of G. Then the following statements are equivalent:*

1. *The partition π is an AEP of G with respect to C.*

2. *$L \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi) + \operatorname{im} P(\rho)$ where $\rho = \{\{\ell_1\}, \{\ell_2\}, \dots, \{\ell_m\}\}$.*

3. *There exists a simple (unweighted) directed graph $H = (V, F)$ obtained from $G = (V, E)$ by adding some non-existing or removing some existing arcs from a vertex in V to a vertex in C such that π is an almost equitable partition of H.*

Proof.

First we show that the first two statements are equivalent. It is easy to observe that the second statement is equivalent to:

$$\begin{bmatrix} L_{\bar{\alpha}} \\ L^{\alpha} \end{bmatrix} \operatorname{im} P(\pi) \subseteq \operatorname{im} \begin{bmatrix} P^{\bar{\alpha}}(\pi) \\ P^{\alpha}(\pi) \end{bmatrix} + \operatorname{im} \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{7.25}$$

where $\alpha = V \backslash C$ and $\bar{\alpha} = C$. This holds if and only if

$$L^{\alpha} \operatorname{im} P(\pi) \subseteq \operatorname{im} P^{\alpha}(\pi)$$

which is equivalent to almost equitability of π with respect to C by Lemma 7.2.4.

Now, by assuming that the first two statements hold, we prove the third statement as follows. Since π is an AEP with respect to C, the equality (7.24) holds. Let the matrices X and Y be defined as $X = \mathcal{L}_{\pi}^{C}$ and $Y = L^{\bar{\alpha}} P(\pi) - P^{\bar{\alpha}}(\pi) \mathcal{L}_{\pi}^{C}$. Then, clearly, we have

$$\begin{bmatrix} L^{\bar{\alpha}} \\ L^{\alpha} \end{bmatrix} P(\pi) = \begin{bmatrix} P^{\bar{\alpha}}(\pi) \\ P^{\alpha}(\pi) \end{bmatrix} X + \begin{bmatrix} I_m \\ 0 \end{bmatrix} Y.$$

Now, for each $i = \{1, 2, \cdots, m\}$, let r_i be an integer such that $\ell_i \in C_{r_i}$. Then, it is easy to observe that the matrix Y is obtained as

$$Y_{ij} = -|N(\ell_i, C_j)| - (\mathcal{L}_{\pi}^{C})_{r_i j} \tag{7.26}$$

for each $i \in \{1, 2, \dots, m\}$, $j = \{1, 2, \dots, k\}$, and $j \neq r_i$. The remaining m elements of Y are such that $Y\mathbb{1} = 0$. By (7.18), the equality (7.26) can be rewritten as

$$Y_{ij} = -|N(\ell_i, C_j)| + d_{r_i j} \tag{7.27}$$

where $d_{r_{ij}}$ are obtained from the definition of almost equitability with respect to C.

Now, we construct the graph $H = (V, F)$ by adding some non-existing arcs or removing some existing arcs of G as follows. For each $i \in \{1, 2, \cdots, m\}$ and $j = \{1, 2, \cdots, k\}$, we add a total number of Y_{ij} arcs from some available nodes in C_j to ℓ_i if $Y_{ij} > 0$. Note that multiple arcs between two vertices is not allowed. This is always possible since $d_{r_i j} \leqslant |C_j|$, and hence $Y_{ij} \leqslant |C_j| - |N(\ell_i, C_j)|$. Similarly, if $Y_{ij} < 0$, we remove a total number of $|Y_{ij}|$ existing arcs which are from some nodes in C_j to ℓ_i. This is also always implementable, as $-Y_{ij} \leqslant |N(\ell, C_j)|$. Denoting the arc set obtained in this way by F, it is easy to observe that the partition π is an AEP of $H = (V, F)$ by construction.

It remains to show that the third statement implies either of the other two. Assume that there exists a simple graph $H = (V, F)$ obtained from $G = (V, E)$ by adding some non-existing or removing some existing arcs from some vertices in V to vertices in C such that π is an almost equitable partition of H. Let $L(H)$ denote the Laplacian matrix of H. Then, by Lemma 7.2.2, we have

$$L(H)P(\pi) = P(\pi)X \tag{7.28}$$

for some matrix X. Hence, $L^{\alpha}(H)P(\pi) = P^{\alpha}(\pi)X$ for $\alpha = V\backslash C$. Now, since the head of all arcs which are added or removed from G are all in C, we have $L^{\alpha}(H) = L^{\alpha}(G)$. Consequently, π is an AEP of G with respect to C by Lemma 7.2.4. ■

Example 7.2.6 As mentioned in Example 7.2.3, the partition π given by (7.17) is an almost equitable partition of G with respect to C, where $C = \{2\}$ and the graph G is given by Figure 7.2. Moreover, as pointed out in Example 7.2.1, π is an AEP of the graph H given by Figure 7.2. Now, observe that graph H is obtained from G by removing the arc from vertex 8 to 2, and adding an arc from vertex 3 to 2. This indeed corresponds to the equivalence between the first and the third statement of Theorem 7.2.5.

7.3 Disturbance decoupling of diffusively coupled multi-agent systems

In this section, we propose solutions for Problems 7.1.5 and 7.1.6. We assume without loss of generality that leaders are not affected by the disturbance signals, i.e. $V_L \cap V_D = \varnothing$. Indeed, it is easy to show that if $V_L \cap V_D$ is nonempty then one can redefine the leader set as $V_L' = V_L \setminus (V_L \cap V_D)$, and solve the DDP with respect to the leader set V_L'. Recall that $V_D = \{w_1, w_2, \dots, w_r\}$. Now, let the partition π_S of V be defined as

$$\pi_S = \{\{w_1\}, \{w_2\}, \dots, \{w_r\}, V \setminus V_D\}. \tag{7.29}$$

Obviously, we have

$$\operatorname{im} S \subseteq \operatorname{im} P(\pi_S). \tag{7.30}$$

Moreover, it is easy to observe that there exists a partition of G, say π_R such that

$$\operatorname{im} P(\pi_R) = \ker R^{\top}. \tag{7.31}$$

Then, the following result holds.

Lemma 7.3.1 *The multi-agent system* (7.12) *is disturbance decoupled only if* $\pi_R \leqslant \pi_S$*. Similarly, the disturbance decoupling problem for system* (7.13) *is solvable only if* $\pi_R \leqslant \pi_S$ *holds.*

Proof. Suppose that the system (7.12) is disturbanced decoupled, or the DDP for system (7.13) is solvable. Then by Lemmas 7.1.2 and 7.1.4, it follows that $\operatorname{im} S \subseteq \ker R^\top = \operatorname{im} P(\pi_R)$. Hence, by (7.29) and the structure of S in (7.6), we obtain that $\pi_R \leqslant \pi_S$. ∎

Next, we discuss the open loop and the closed loop disturbance decoupling, and establish sufficient conditions for Problems 7.1.5 and 7.1.6.

7.3.1 Open loop disturbance decoupling

The following theorem gives a sufficient (topological) condition for multi-agent system (7.12) to be disturbance decoupled.

Theorem 7.3.2 *Let* π_S *and* π_R *be given by* (7.29) *and* (7.31)*, respectively. Then the multi-agent system* (7.12) *is disturbance decoupled if there exists a partition* π *such that both of the following conditions hold:*

1. π *is an AEP of* G
2. $\pi_R \leqslant \pi \leqslant \pi_S$

Proof. Suppose that conditions 1 and 2 hold. Then, by Lemma 7.2.2 and (7.16), we obtain that $\operatorname{im} P(\pi)$ is L-invariant and $\operatorname{im} S \subseteq \operatorname{im} P(\pi) \subseteq \ker R^\top$. Hence, it follows from Lemma 7.1.2 that (7.12) is disturbance decoupled. ∎

Based on Theorem 7.3.2, the DDP for (7.12) is solvable if there exists an almost equitable partition which is finer than π_S and coarser than π_R. In principle, this requires searching for all almost equitable partitions of G to find one which satisfy the partial ordering constraint of Theorem 7.3.2. However, similar to the idea of largest/smallest invariant subspaces for ordinary linear systems (see Subsection 7.1.2), one may try to find a partition, say π^*, which is extremal in certain sense. Then, providing that such a partition exists and can be efficiently computed, disturbance decoupling of (7.12) can be guaranteed upon a satisfaction of a single and easily verifiable condition. This will be addressed in details in Subsection 7.4.1.

7.3.2 Closed loop disturbance decoupling

Now suppose that the DDP is not solvable for (7.12). Then, similar to the case of general linear systems in Subsection 7.1.2, one may try to make the system (7.12) disturbance decoupled by using a control input. Indeed this brings us to Problem 7.1.6.

Recall the notion of controlled invariant subspaces in Subsection 7.1.2. As we are dealing with graph topological conditions, we are not interested in all subspaces but only those which can be written as an image of a partition. As observed in the previous subsection, almost equitable partitions corresponds to L-invariant subspaces. Now, the following Lemma establishes the relationship between almost equitability with respect to a cell and controlled invariance of the pair (L, M).

Lemma 7.3.3 *For a given graph G, let V_L, M, and L be defined as before. Let π be a partition of G. Then $\operatorname{im} P(\pi)$ is controlled invariant for the pair (L, M) if and only if π is an almost equitable partition with respect to V_L.*

Proof. Recall that $V_L = \{\ell_1, \ell_2, \ldots, \ell_m\}$. Note that $\operatorname{im} M = \operatorname{im} P(\rho)$ where

$$\rho = \{\{\ell_1\}, \{\ell_2\}, \ldots, \{\ell_m\}\}.$$

The result now immediately follows by using (7.11) together with Theorem 7.2.5. ∎

Now we are at the position to apply the results of Section 7.2.2 to disturbance decoupling problem of multi-agent system (7.13). This is discussed in the following theorem.

Theorem 7.3.4 *Let V_L, π_R, and π_S be defined as before. Then the disturbance decoupling problem for multi-agent system (7.13) is solvable if there exists a partition π of G such that both of the following conditions hold:*

1. *π is almost equitable with respect to V_L*
2. *$\pi_R \leqslant \pi \leqslant \pi_S$*

Proof. Suppose that the conditions 1 and 2 hold. Then, by Lemma 7.3.3, $\operatorname{im} P(\pi)$ is controlled invariant for the pair (L, M). Moreover, we have $\operatorname{im} P(\pi_S) \subseteq \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi_R)$. Hence, by (7.30) and (7.31), we obtain that $\operatorname{im} S \subseteq \operatorname{im} P(\pi) \subseteq \ker R^\top$. Consequently, the DDP for system (7.13) is solvable by Lemma 7.1.4. ∎

Suppose that the conditions of Theorem 7.3.4 hold, and hence the DDP for system (7.13) is solvable. This means that there exists a state feedback $u(t) = Kx(t)$ such that the resulting closed loop system is disturbance decoupled. To see the structure of the controller, note that π is an AEP of $G = (V, E)$ with respect to the leader set V_L. Hence, π is an AEP of $H = (V, F)$ where H is obtained from G by adding or removing arcs from vertices in V to vertices in V_L, as discussed in the proof of Theorem 7.2.5. These adding and removing of the arcs are indeed associated with the state feedback controller which makes the system (7.13) disturbance decoupled. In particular, it is easy to observe that adding an arc from a vertex in V, say i, to a vertex in V_L, say ℓ_k, corresponds to the control signal $x_i(t) - x_{\ell_k}(t)$ which is to be applied to the leader vertex ℓ_k. Similarly, removing an arc from i to ℓ_k corresponds to the term $x_{\ell_k}(t) - x_i(t)$ in the control signal. Consequently, the controller can be expressed as

$$u_k(t) = \sum_{(j,\ell_k)\in F\setminus E} (x_j(t) - x_{\ell_k}(t)) - \sum_{(j,\ell_k)\in E\setminus F} (x_j(t) - x_{\ell_k}(t)) \tag{7.32}$$

for each $k = \{1, 2, \ldots, m\}$. This shows that, as demanded in the context of distributed control, the controller only uses the relative information of the states of the agents to achieve disturbance decoupling for system (7.13). Observe that, by applying the controller (7.32) to system (7.13), we obtain the following input/state/output system:

$$\dot{x}(t) = -L(H)x(t) + Sd(t) \tag{7.33a}$$

$$y(t) = R^\top x(t), \tag{7.33b}$$

where $L(H)$ denotes the Laplacian matrix of the graph $H = (V, F)$. The system (7.33) is indeed disturbance decoupled by Theorem 7.3.2, as π is an AEP of H and $\pi_R \leqslant \pi \leqslant \pi_S$. This is in accordance with the fact that the DDP for system (7.13) is solvable.

Note that Theorem 7.3.4 provides a sufficient condition in terms of graph topological conditions for the multi-agent system (7.13) to be disturbance decoupled. However, in principle, one needs to perform an exhaustive search to find a promising partition satisfying the required constraints. To avoid this, in Subsection 7.4.2, we propose an efficient algorithm to verify the conditions provided in Theorem 7.3.4.

Remark 7.3.5 It is worth mentioning that, in general, one should not expect necessary and sufficient conditions in terms of graph partitions either for the multi-agent system (7.12) to be disturbance decoupled or for the DDP of (7.13) to be solvable. The reason is that not all the subspaces can be written in terms of the image of the characteristic matrix of a partition. In fact, the lack of necessary and sufficient conditions here, are mainly associated with the gap between the image of the characteristic matrices of partitions and arbitrary subspaces.

7.4 Algorithms

7.4.1 An algorithm for open loop disturbance decoupling

Let Π denote the set of all partitions of V. With the partial order "$\leqslant$", the set Π becomes a complete lattice [11], meaning that every subset of Π has both its greatest lower bound and least upper bound within Π. We use $\vee\Pi'$ to denote the least upper bound of a subset $\Pi' \in \Pi$. By definition, $\vee\Pi'$ has the following property:

$$\vee\Pi' \geqslant \pi, \text{ for all } \pi \in \Pi' \tag{7.34a}$$

$$\exists\tilde{\pi} \in \Pi \text{ s.t. } \pi \leqslant \tilde{\pi} \text{ for all } \pi \in \Pi' \quad \Longrightarrow \quad \vee\Pi' \leqslant \tilde{\pi}. \tag{7.34b}$$

Let Π_{AEP} denote the set of all almost equitable partitions of G. For a given partition π_0 of G, we define

$$\Pi_{\text{AEP}}(\pi_0) = \{\pi \in \Pi_{\text{AEP}} : \pi \leqslant \pi_0\}. \tag{7.35}$$

Then, the following result holds.

Lemma 7.4.1 *Let π_R and π_S be given as before. Then the following two statements are equivalent.*

1. *There exists a partition $\pi \in \Pi_{\text{AEP}}$ such that $\pi_R \leqslant \pi \leqslant \pi_S$*
2. *$\pi_R \leqslant \vee\Pi_{\text{AEP}}(\pi_S)$*

Proof. Suppose that the first statement holds. Then, $\pi \in \Pi_{\text{AEP}}(\pi_S)$. Hence, by definition, $\pi \leqslant \vee\Pi_{\text{AEP}}(\pi_S)$ which yields $\pi_R \leqslant \vee\Pi_{\text{AEP}}(\pi_S)$.

Conversely, suppose that the second statement holds. Then, obviously $\pi_R \leqslant \vee\Pi_{\text{AEP}}(\pi_S) \leqslant \pi_S$. Besides, it is shown in [106] that $\vee\Pi_{\text{AEP}}(\pi_0) \in \Pi_{\text{AEP}}(\pi_0)$ for any given partition π_0. Therefore, $\vee\Pi_{\text{AEP}}(\pi_S)$ serves as a partition satisfying the conditions in Statement 1 of the lemma. ∎

By Lemma 7.4.1 and Theorem 7.3.2, we obtain the following result.

Corollary 7.4.2 *The multi-agent system* (7.12) *is disturbance decoupled if*

$$\pi_R \leqslant \vee\Pi_{\text{AEP}}(\pi_S). \tag{7.36}$$

As observed, verification of the conditions provided in Theorem 7.3.2 boils down to the computation of $\vee\Pi_{\text{AEP}}(\pi_S)$. An algorithm to compute this bound is provided in [106]. However, here we use a rather different terminology and techniques which will be extended in the next subsection to the case of almost equitability with respect to a cell. First, we define some new notions and notation.

Let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of V. Then we call i a *cellmate* of j in π, if i and j belong to the same cell of π. In this case, we say i and j are cellmates in π and we write as $i \overset{\pi}{\equiv} j$. Note that by definition, i is a cellmate of itself.

Next, for a given partition π, we define the notion of a *friend* as follows. Let $v, w \in V$. Then we say v is a *friend* of w in π, and we write as $v \overset{\pi}{\sim} w$ if both of the following hold:

1. $v \overset{\pi}{\equiv} w$; in particular let $v, w \in C_j$.
2. $|N(v, C_i)| = |N(w, C_i)|$ for each $i = 1, 2, \ldots, k$ with $i \neq j$.

We simply say v is a friend of w whenever the underlying partition is clear from the context. Obviously, v is a friend of w if and only if w is a friend of v in π. Hence, we sometimes say v and w are friends in π.

Lemma 7.4.3 *Let $v, w \in V$ and π_0 be a partition of V. Then, the following implication holds:*

$$v \overset{\pi_0}{\sim} w \Longrightarrow v \overset{\pi}{\sim} w \textit{ for all } \pi \geqslant \pi_0$$

Proof. Let $\pi = \{C_1, C_2, \ldots, C_k\}$ and $\pi \geqslant \pi_0$. Also let π_0 be written as

$$\pi_0 = \{C_1^1, \ldots, C_1^{r_1}, C_2^1, \ldots, C_2^{r_2}, \ldots, C_k^1, \ldots, C_k^{r_k}\},$$

where $\sum_{i=1}^k r_i = |\pi_0|$ and $\cup_{j=1}^{r_i} C_i^j = C_i$ for each $i = 1, 2, \ldots, k$. Now suppose that $v \overset{\pi_0}{\sim} w$. Then clearly v and w are cellmates in π_0, and thus in $\pi \geqslant \pi_0$. Without loss of generality assume that $v, w \in C_1^1$, and hence $v, w \in C_1$. Since $v \overset{\pi_0}{\sim} w$, we have $|N(v, C_q^j)| = |N(w, C_q^j)|$ for each $q = 2, 3, \ldots, k$ and $j = 1, 2, \ldots, r_q$. Then, for each q we have

$$|N(v, C_q)| = \sum_{j=1}^{r_q} |N(v, C_q^j)| = \sum_{j=1}^{r_q} |N(w, C_q^j)| = |N(w, C_q)|.$$

Therefore, v and w are friends in π. ∎

It is easy to observe that *friendship* in π defines an equivalence relation. The corresponding equivalent classes are given by

$$[i] = \{j \in V : i \overset{\pi}{\sim} j\} \tag{7.37}$$

for each $i \in V$. Let $\mathfrak{F}(\pi)$ denote the set of all (friendship) equivalent classes for a given partition π, that is

$$\mathfrak{F}(\pi) = \{[i] : i \in V\}. \tag{7.38}$$

Note that obviously $\mathfrak{F}(\pi)$ constitutes a partition of V. Then we have the following Lemma.

Lemma 7.4.4 *Let π_1 and π_2 be two partitions of V. Then, we have:*

$$\pi_1 \leqslant \pi_2 \Longrightarrow \mathfrak{F}(\pi_1) \leqslant \mathfrak{F}(\pi_2).$$

Proof. Suppose that i and j are cellmates in $\mathfrak{F}(\pi_1)$. Hence, $i \overset{\pi_1}{\sim} j$. Then, by Lemma 7.4.3 we have $i \overset{\pi_2}{\sim} j$ which implies that i and j are cellmates in $\mathfrak{F}(\pi_2)$. Therefore, we conclude that $\mathfrak{F}(\pi_1) \leqslant \mathfrak{F}(\pi_2)$. ■

Note that clearly $\mathfrak{F}(\pi) \leqslant \pi$. Moreover, the equality $\mathfrak{F}(\pi) = \pi$ holds whenever π is an AEP of V, as stated in the following lemma.

Lemma 7.4.5 *Let π be a partition of V, and $\mathfrak{F}(\pi)$ be given by* (7.38)*. Then we have $\mathfrak{F}(\pi) = \pi$ if and only if π is an almost equitable partition of V.*

Proof. Suppose that $\pi = \{C_1, C_2, \dots, C_k\}$ is an AEP of V. Let r_i be an integer such that $i \in C_{r_i}$ for each $i \in V$. Then, by definition of almost equitability and (7.37), it is easy to observe that $[i] = C_{r_i}$ for each $i \in V$. Hence, we have $\mathfrak{F}(\pi) = \pi$. The "only if" part is also straightforward from the definition of almost equitability and the notion of a friend. ■

Observe that, based on Lemma 7.4.5, an almost equitable partition can be characterized as a partition in which cellmates vertices are friends. This will be used to develop a simple algorithm to compute $\vee\Pi_{\text{AEP}}(\pi_0)$. We first prove the following crucial result.

Lemma 7.4.6 *Let π_0 be a partition of V. Then $\vee\Pi_{\text{AEP}}(\pi_0) = \vee\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0))$.*

Proof. Clearly $\mathfrak{F}(\pi_0) \leqslant \pi_0$, and we have $\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0)) \subseteq \Pi_{\text{AEP}}(\pi_0)$. Hence, we obtain that

$$\vee\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0)) \leqslant \vee\Pi_{\text{AEP}}(\pi_0). \tag{7.39}$$

Now, let $\pi \in \Pi_{\text{AEP}}(\pi_0)$. Then π is an AEP of V and $\pi \leqslant \pi_0$. So, by Lemma 7.4.4 we obtain that $\mathfrak{F}(\pi) \leqslant \mathfrak{F}(\pi_0)$. This is simplified to $\pi \leqslant \mathfrak{F}(\pi_0)$ by Lemma 7.4.6. Therefore, $\pi \in \Pi_{\text{AEP}}(\mathfrak{F}(\pi_0))$. Hence, by definition, $\pi \leqslant \vee\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0))$. As this holds for any $\pi \in \Pi_{\text{AEP}}(\pi_0)$, the partition $\vee\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0))$ serves as an upper bound for the set $\Pi_{\text{AEP}}(\pi_0)$. Consequently, by (7.34b), we obtain that

$$\vee\Pi_{\text{AEP}}(\pi_0) \leqslant \vee\Pi_{\text{AEP}}(\mathfrak{F}(\pi_0)).$$

This together with (7.39) completes the proof. ■

Roughly speaking, Lemma 7.4.6 implies that $\vee\Pi_{\text{AEP}}(\pi_0)$ is invariant under the act of $\mathfrak{F}$. Thanks to this result, $\Pi_{\text{AEP}}(\pi_0)$ can be computed as follows. Given an initial partition π_0, construct a sequence of partitions as:

$$\pi_{i+1} = \mathfrak{F}(\pi_i) \tag{7.40}$$

where i takes its value from the set $\{0, 1, 2, \ldots\}$. This sequence converges to $\vee\Pi_{\text{AEP}}(\pi_0)$ as stated in the following theorem.

Theorem 7.4.7 *Let π_0 be a partition of V, and $n = |V|$. Consider the sequence given by* (7.40). *Then there exits $j \in \{0, 1, \ldots, n-1\}$ such that*

$$\pi_i = \vee\Pi_{AEP}(\pi_0)$$

for all $i \geqslant j$.

Proof. By definition, we have $\mathfrak{F}(\pi) \leqslant \pi$ for any given partition π of V. Clearly, if $\mathfrak{F}(\pi) \leqslant \pi$ but $\mathfrak{F}(\pi) \neq \pi$, then $|\mathfrak{F}(\pi)| > |\pi|$. Besides, we know that $|\mathfrak{F}(\pi)| \leqslant |V| = n$. Therefore, there exists $j \in \{0, 1, \ldots, n-1\}$ such that $\mathfrak{F}(\pi_j) = \pi_j$. Then, clearly, $\pi_i = \pi_j$ for all $i \geqslant j$. Hence, it remains to show that $\pi_j = \vee\Pi_{\text{AEP}}(\pi_0)$.

Since $\pi_j = \mathfrak{F}(\pi_j)$, the partition π_j is an AEP of V by Lemma 7.4.5. Thus, clearly $\pi_j = \vee\Pi_{\text{AEP}}(\pi_j)$. Then, by (7.40) and a repetitive use of Lemma 7.4.6, we obtain that $\vee\Pi_{\text{AEP}}(\pi_j) = \vee\Pi_{\text{AEP}}(\pi_0)$. This completes the proof. ∎

7.4.2 An algorithm for closed loop disturbance decoupling

In this subsection, we aim to extend the ideas provided in the previous subsection to almost equitability with respect to a cell. Our ultimate goal is to propose an efficient algorithm to verify the conditions provided in Theorem 7.3.4. Again an idea here would be to try to compute an extremal partition, and derive a similar result to that of Corollary 7.4.2. However, unfortunately, one can show that the semi-lattice structure of (7.35) is lost when ordinary almost equitability is replaced by almost equitability with respect to a cell. Therefore, additional treatments are needed for extending the result of the previous subsection to the case of almost equitability with respect to a cell.

We make one additional assumption here on the structure of π_R. In particular, we assume that each cell of π_R contains at least one follower, i.e. one vertex in V_F. Before proceeding any further, some new notions need to be defined.

We first extend the notion of partial ordering to partial partitions. Let $\pi_1 = \{C_1, C_2, \ldots, C_r\}$ and $\pi_2 = \{C'_1, C'_2, \ldots, C'_{r'}\}$ be partial partitions of V. Then we say

π_1 is finer than π_2, or equivalently π_2 is coarser than π_1, if each cell of π_1 is a subset of a cell of π_2, and $\cup_i C_i = \cup_i C_i'$. In this case, we write as $\pi_1 \leqslant \pi_2$.

Let $\pi = \{C_1, C_2, \dots, C_k\}$ be a partition of V, and $V_1 \subseteq V$. By $\pi|_{V_1}$, we denote the *restriction* of π to V_1 which is defined as

$$\pi|_{V_1} = \{C_1 \cap V_1, C_2 \cap V_1, \dots, C_k \cap V_1\}.$$

Note that $\pi|_{V_1}$ is a partial partition of V.

Now, let $\pi' = \{C_1, C_2, \dots, C_r\}$ with $\cup_i C_i = V' \subseteq V$ be a partial partition of V. Also let π be a partition of V. Then we call π an *extension* of π' to V if $\pi|_{V'} = \pi'$. For a given partial partition π' of V, we define the set Ω as

$$\Omega(\pi') = \{\pi \in \Pi : \pi|_{V'} = \pi'\}. \tag{7.41}$$

In fact, Ω collects all extensions of π' to V.

Recall the notion of a friend discussed in Subsection 7.4.1. It is easy to see that the result of Lemma 7.4.4 is extended as follows.

Lemma 7.4.8 *Let π_1 and π_2 be two partitions of V. Also let $V' \subseteq V$. Then, we have:*

$$\pi_1|_{V'} \leqslant \pi_2|_{V'} \Longrightarrow \mathfrak{F}(\pi_1)|_{V'} \leqslant \mathfrak{F}(\pi_2)|_{V'}.$$

Proof. The proof is analogous to that of Lemma 7.4.4. ■

We observed in the previous subsection that an AEP, say π, is a partition where vertices that are cellmates in π are also friends in π. By definition, it is easy to observe that an almost equitable partition with respect to a cell C can be characterized in a similar way by restricting this friendship requirement to the vertices that are cellmates in $\pi|_{(V \setminus C)}$. This brings us to the following lemma, which we state without a proof.

Lemma 7.4.9 *Let π be a partition of V and C be a cell of V. Then π is an AEP with respect to C if and only if $\mathfrak{F}(\pi)|_{V'} = \pi|_{V'}$ where $V' = V \setminus C$.*

Recall that $V_L = \{\ell_1, \ell_2, \dots, \ell_m\}$. For each $j = \{1, 2, \dots, m\}$, let r_j be an integer such that $\ell_j \in C_{r_j}$. In addition, assume that each cell of $\pi_R = \{C_1, C_2, \dots, C_k\}$ contains at least one follower, i.e. $C_i \cap V_F \neq \varnothing$ for each $i = \{1, 2, \dots, k\}$. Now, for each i, choose $q_i \in C_i \cap V_F$. Then, we construct a sequence of partitions as stated in Algorithm 1.

Algorithm 1

1: Set $\pi_0 = \pi_S$ and $i = 0$.
2: Compute $\mathfrak{F}(\pi_i)$
3: Compute $\pi' = \mathfrak{F}(\pi_i)|_{V_F}$
4: Choose $\pi_{i+1} \in \Omega(\pi')$ such that $\ell_j \overset{\pi_{i+1}}{\equiv} q_{(r_j)}$ for each $j = \{1, 2, \ldots, m\}$
5: Set $i = i + 1$, and return to Step 2

Some illustrations and clarifications are needed here. First note that, after choosing q_is, the partition π_{i+1} in Step 4 of the algorithm is obtained in a unique way. Moreover, the proposed sequence converges to a partition, say π^*, after a finite number of steps. Besides, π^* is almost equitable with respect to V_L, as stated in the following lemma.

Lemma 7.4.10 *Consider the sequence given by algorithm* (1)*. Then, there exists an integer $t < n = |V|$ such that $\pi_i = \pi_t$ for all $i \geqslant t$. Moreover, π_t is an almost equitable partition with respect to V_L.*

Proof. First, observe that

$$\pi_{i+1}|_{V_F} = \mathfrak{F}(\pi_i)|_{V_F} \leqslant \pi_i|_{V_F} \tag{7.42}$$

for each i in Algorithm 1. Now, as cardinality of $\pi_i|_{V_F}$ is no more than that of V, there exists a positive integer $t < n$ such that $\pi_{t+1}|_{V_F} = \mathfrak{F}(\pi_t)|_{V_F} = \pi_t|_{V_F}$. Then, since $\ell_j \overset{\pi_{i+1}}{\equiv} q_{(r_j)}$, for each i and j, we conclude that $\pi_{t+1} = \pi_t$, and consequently the proposed sequence converges to π_t. Moreover, by Lemma 7.4.9, $\mathfrak{F}(\pi_t)|_{V_F} = \pi_t|_{V_F}$ implies that π_t is almost equitable with respect to V_L. ■

Now, the following theorem states the main result of this subsection.

Theorem 7.4.11 *Let $\pi_R = \{C_1, C_2, \ldots, C_k\}$. Assume that $C_i \cap V_F \neq \varnothing$, for each $i \in \{1, 2, \ldots, k\}$. Let π^* denote the partition to which the sequence in Algorithm 1 converges. Then the following two statements are equivalent.*

1. *There exists a partition π of V such that π is almost equitable with respect to V_L and $\pi_R \leqslant \pi \leqslant \pi_S$.*
2. *$\pi_R \leqslant \pi^* \leqslant \pi_S$.*

Proof. Suppose that the first statement holds. By (7.42), we obtain that $\mathfrak{F}(\pi^*)|_{V_F} \leqslant \pi_S|_{V_F}$. Now, let $V_D = \{w_1, w_2, \ldots, w_r\}$ and π_S be given by (7.29), as before. Recall that $V_D \cap V_L = \varnothing$. Since $\pi_R \leqslant \pi_S$, we have $\{w_j\} \in \pi_R$ for each $j = \{1, 2, \ldots, r\}$.

Then, based on Algorithm 1, it is easy to observe that $\{w_j\} \in \pi_i$ for each i and j. Therefore, by the structure of π_S, it follows that $\pi^* \leqslant \pi_S$. Besides, π^* is an AEP of G with respect to V_L by Lemma 7.4.10. Hence, it remains to show that $\pi_R \leqslant \pi^*$.

Clearly, as $\pi_R \leqslant \pi$, it suffices to show that $\pi \leqslant \pi^*$. Since $\pi \leqslant \pi_0 = \pi_S$, we have $\pi|_{V_F} \leqslant \pi_0|_{V_F}$. As π is almost equitable with respect to V_L, by Lemma 7.4.9 and Lemma (7.4.6), it follows that $\pi|_{V_F} \leqslant \mathfrak{F}(\pi_0)|_{V_F}$. This yields $\pi|_{V_F} \leqslant \pi_1|_{V_F}$ by (7.42). By repeating this argument, we obtain that

$$\pi|_{V_F} \leqslant \pi^*|_{V_F}. \tag{7.43}$$

Now, suppose that $v, w \in V$ are cellmates in π. To prove $\pi \leqslant \pi^*$, we need to show that v and w are also cellmates in π^*. For that, we distinguish three cases. First, suppose that $v, w \in V_F$. Then, by (7.43) it follows that v and w are also cellmates in π^*. Next, suppose that $v \in V_F$ and $w = \ell_j \in V_L$. By Algorithm 1, ℓ_j and $q_{(r_j)}$ are cellmates in π_i for each i, where $q_{(r_j)}$ is defined as before. Thus,

$$\ell_j \overset{\pi^*}{\equiv} q_{(r_j)}. \tag{7.44}$$

By definition $\ell_j \overset{\pi_R}{\equiv} q_{(r_j)}$, and hence $\ell_j \overset{\pi}{\equiv} q_{(r_j)}$ as $\pi_R \leqslant \pi$. Then, since v and ℓ_j are cellmates in π, we obtain that $v \overset{\pi}{\equiv} q_{(r_j)}$. Therefore, as $v, q_{(r_j)} \in V_F$, by (7.43) it follows that $v \overset{\pi^*}{\equiv} q_{(r_j)}$. Hence, by (7.44) we have $v \overset{\pi^*}{\equiv} \ell_j$. For the third case, suppose that $i, j \in V_L$. Since $C_i \cap V_F \neq \varnothing$ and $\pi_R \leqslant \pi$, there exists $q \in V_F$ such that $v \overset{\pi}{\equiv} q \overset{\pi}{\equiv} w$. Then, by the result of the previous case it follows that $v \overset{\pi^*}{\equiv} q \overset{\pi^*}{\equiv} w$. Therefore, summarizing the results of the aforementioned three cases, we conclude that $\pi \leqslant \pi^*$.

Conversely, suppose that the second statement holds. Then, by Lemma 7.4.10, π^* is almost equitable with respect to V_L, and thus the first statement holds true. ■

Note that Theorem 7.4.11 obviates the need to search for a partition satisfying the conditions of Theorem 7.3.4. Indeed, by using the same notations and assumptions as in Theorem 7.4.11, we have the following corollary.

Corollary 7.4.12 *The disturbance decoupling problem for system* (7.13) *is solvable if* $\pi_R \leqslant \pi^* \leqslant \pi_S$.

7.5 Numerical example

To illustrate the proposed results, consider the multi-agent system (7.4) with the communication graph G as shown in Figure 7.3 (left). For this system, let black vertices denote the leaders, i.e. $V_{\mathrm{L}} = \{2\}$. Also let the square vertices correspond to the agents affected by disturbance signals, i.e. $V_D = \{3,5\}$. We are interested in decoupling the outputs $x_1(t) - x_2(t)$ and $x_4(t) - x_6(t)$ from the disturbance. Hence, the output variables in this case is given by $y = R^\top x$ where

$$R^\top = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and $x \in \mathbb{R}^8$. Then π_R and π_S are given by:

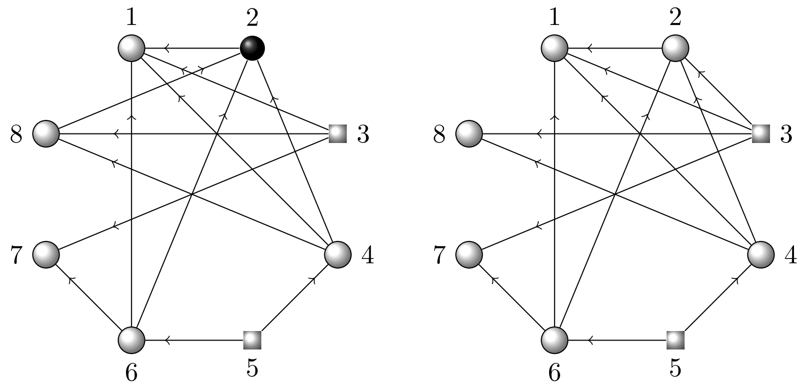

Figure 7.3: *The simple directed graph G (left) and H (right) of a diffusively coupled multi-agent system*

$$\pi_R = \{\{1,2\},\{3\},\{4,6\},\{5\},\{7\},\{8\}\}$$
$$\pi_S = \{\{1,2,4,6,7,8\},\{3\},\{5\}\}$$

Next, we compute $\vee\Pi_{\text{AEP}}(\pi_S)$ by Theorem 7.4.7. The sequence (7.40) is obtained as:

$$\pi_0 = \pi_S$$
$$\pi_1 = \{\{1,7,8\},\{4,6\},\{2\},\{3\},\{5\}\}$$
$$\pi_2 = \{\{1\},\{7,8\},\{4,6\},\{2\},\{3\},\{5\}\}$$

and $\pi_i = \pi_2$ for all $i \geqslant 2$. Hence, by Theorem 7.4.7 we obtain that $\vee\Pi_{\text{AEP}}(\pi_S) = \pi_2$. Now since $\pi_R \not\leqslant \pi_2$, the disturbance decoupling of the open loop system (7.12) is not guaranteed by Corollary 7.4.2. Next, we consider solvability of the disturbance decoupling problem for this system by verifying the proposed (topological) conditions.

We construct a sequence by following the steps of Algorithm 1. Let $\pi_0 = \pi_S$. Then, $\mathfrak{F}(\pi_0)$ is computed as $\mathfrak{F}(\pi_0) = \{\{1,7,8\},\{4,6\},\{2\},\{3\},\{5\}\}$, and hence

$$\mathfrak{F}(\pi_0)|_{V_F} = \{\{1,7,8\},\{4,6\},\{3\},\{5\}\}.$$

Therefore, π_1 is obtained as

$$\pi_1 = \{\{1,2,7,8\},\{4,6\},\{3\},\{5\}\}.$$

Similarly, π_2 is computed as

$$\pi_2 = \{\{1,2\},\{7,8\},\{4,6\},\{3\},\{5\}\}, \tag{7.45}$$

and it can be verified that $\pi_i = \pi_2$ for $i \geqslant 2$. Now, since $\pi_R \leqslant \pi_2 \leqslant \pi_S$, the disturbance decoupling problem in this case is solvable by Corollary 7.4.12.

Note that indeed π_2 given by (7.45) is an almost equitable partition of G with respect to V_L. In addition, based on Theorem 7.2.5, and as observed in Example (7.2.6), the partition π_2 becomes almost equitable in a graph H by removing the arc from vertex 8 to 2, and adding an arc from 3 to 2. The graph H is depicted in Figure 7.3 (right). Note that, except from the different symbols/coloring of the vertices, the graphs G and H in Figure 7.3 are the same as those depicted in Figures 7.2 and 7.1, respectively. Consequently, by (7.32), the state feedback which renders the system disturbance decoupled is given by

$$u(t) = (x_3(t) - x_2(t)) - (x_8(t) - x_2(t)) = x_3(t) - x_8(t).$$

7.6 Conclusions

We have studied the disturbance decoupling problem for multi-agent systems. By extending the notion of almost equitability to almost equitability with respect to a cell, an appropriate topological translation for controlled invariant subspaces is provided. We have considered disturbance decoupling of both the open loop and the closed loop system. In the open loop case, we have established sufficient conditions ensuring the system is disturbance decoupled with no input applied to the leaders. In the case of closed loop, we have derived sufficient conditions guaranteeing the solvability of the disturbance decoupling problem. In case the DDP is solvable, an admissible controller rendering the system disturbance decoupled has been provided. As desired, this controller uses the relative information of the states of the agents. The proposed sufficient conditions are in terms of existence of certain almost equitable partitions with respect to a cell containing leaders. Algorithm verifying these conditions have been developed in this chapter. It is worth mentioning that the lack of necessary and sufficient conditions here is associated with the gap between the image of characteristic matrices of partitions and arbitrary subspaces.

Chapter 8
Conclusions

8.1 Contributions

The main contributions of this thesis can be summarized as follows:

- Necessary and sufficient conditions have been derived for simultaneous balancing of the subsystems of a switched linear system. Consequently, a model reduction technique based on simultaneous balancing has been established. Sufficient conditions have been proposed under which global uniform exponential stability, passivity, or contractivity of the SLS is preserved in the reduced order models. In case simultaneous balancing is not possible, an extended balanced truncation scheme has been established that is based on balancing the average gramians of the system.

- A model reduction method has been established for networks of dynamical agents that aims at reducing the dynamic order of the agents. Preservation of stability as well as preservation of synchronization in the reduced order models has been addressed in the proposed method. The behaviors of the original and the reduced order network have been compared by establishing a priori model reduction error bounds.

- By means of suitable graph partitions, we have established a projection based model reduction method for multi-agent systems defined on a graph. The spatial structure of the network is preserved in the proposed reduction technique, and reduced order models are realized as multi-agent systems defined on a new graph of smaller size. For almost equitable partitions, an explicit formula for the $\mathcal{H}_2$-norm of the error system has been obtained. We have also shown that the error obtained by taking an arbitrary partition of the graph is bounded from below by the one obtained by using the maximal almost equitable partition finer than the given partition.

- Stability and synchronization of networks with general linear dynamics and arbitrary switching topologies have been studied. A small gain condition has been derived under which the stability of the overall network is guaranteed. By using an appropriate decomposition of the overall network dynamics, sufficient conditions guaranteeing synchronization of networks under arbitrary switching topologies have been established. The proposed conditions solely depend on the agents dynamics and the (nontrivial) extremal eigenvalues of the Laplacian matrices. The results established have also been extended to the case of observer-based protocols.

- We have considered the problem of controllability of the network for a family of matrices carrying the structure of an underlying directed graph. A one-to-one correspondence between the set of leaders rendering the network (structurally) controllable and zero forcing sets has been established. We have shown how our proposed results together with the existing results on zero forcing sets lead to a minimal leader selection scheme in particular cases.

- The disturbance decoupling problem for networks of dynamical agents has been studied from a topological perspective. A class of graph partitions, namely almost equitable partitions with respect to a cell, has been developed that provide a topological characterization for controlled invariant subspaces. We have established sufficient (topological) conditions under which the network is disturbance decoupled, as well as conditions guaranteeing the solvability of the disturbance decoupling problem for the network.

8.2 Further research topics

The following points can be considered for carrying forward the work of this thesis.

- It is of interest to establish a priori model reduction error bound to compare the input-output behavior of the original switched linear system to that of the reduced order models.

- For networks of dynamical agents, a relevant issue is to extend the proposed model reduction techniques to the case of directed graphs.

- We have proposed a model reduction technique based on clustering the agents by means of appropriate graph partitions. In view of the derived formula for the $\mathcal{H}_2$-norm of the error system, it is of interest to characterize an optimal partition, i.e. a partition leading to the least model reduction error compared to all partitions with the same number of cells.

- By exploiting the notion of zero forcing sets and the results established in this thesis, an open problem is to obtain a leader selection scheme with a minimum number of leaders to achieve (structural) controllability for more general subclasses of graphs than those already investigated in this thesis (path, cycle, complete, (di)trees).

- As we have dealt with structural controllability and disturbance decoupling of networks, we believe that it is an interesting line of research to study certain properties of networks from a topological prospective. This will lead to numerically stable methods, will provide a better understanding of the network behavior, and will facilitates the design.

Bibliography

[1] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM, Philadelphia, Pa, USA, 2005.

[2] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. S. Zhou. Synchronization in complex networks. *Physics Reports-Review Section of Physics Letters*, 469:93–153, 2008.

[3] A. Banerjee and J. Jost. On the spectrum of the normalized graph. *arXiv:0705.3772*, 2007.

[4] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. Van Den Driessche, and H. Van Der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433(2):401–411, 2010.

[5] G. Basile and G. Marro. *Controlled and conditioned invariants in linear system theory*. College of Engineering, University of California, 1968.

[6] G. Basile and G. Marro. Controlled and conditioned invariant subspaces in linear system theory. *Journal of Optimization Theory and Applications*, 3(5):306–315, 1969.

[7] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.

[8] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.

[9] A. Birouche, B. Mourllion, and M. Basset. Model order-reduction for discrete-time switched linear systems. *International Journal of Systems Science*, 43(9):1753–1763, 2012.

[10] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Transactions on Automatic Control*, 58(9), 2013.

[11] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer-Verlag, New York, 1981.

[12] M. K. Camlibel, S. Zhang, and M. Cao. Comments on 'Controllability analysis of multi-agent systems using relaxed equitable partitions'. *International Journal of Systems, Control and Communications*, 4(1/2):72–75, 2012.

[13] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47(4):575–600, 2008.

[14] D. M. Cardoso, C. Delorme, and P. Rama. Laplacian eigenvectors and eigenvalues and almost equitable partitions. *European Journal of Combinatorics*, 28(3):665–673, 2007.

[15] C. G. Cassandras, D. L. Pepyne, and Y. Wardi. Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, 46(3):398–415, 2001.

[16] A. Chapman and M. Mesbahi. Strong structural controllability of networked dynamics. Submitted for publication.

[17] Xin Chen and John T. Wen. Positive realness preserving model reduction with H_∞ norm error bounds. *IEEE Trans. Circuits Systems*, 42, no.1:23–29, 1995.

[18] D. Cheng, L. Guo, and J. Huang. On quadratic Lyapunov functions. *IEEE Transactions on Automatic Control*, 48(5):885–890, 2003.

[19] D. Cheng, L. Guo, Y. Lin, and Y. Wang. Stabilization of switched linear systems. *IEEE Transactions on Automatic Control*, 50(5):661–666, 2005.

[20] J. Cortes and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.

[21] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243–255, 2004.

[22] U. B. Desai and D. Pal. A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, 29(2):1097–1100, 1984.

[23] G. E. Dullerud and E. G. Paganini. *A Course in Robust Control Theory: A Convex Approach*. Springer, Newyork, 2000.

[24] M. Egerstedt, S. Martini, M. Cao, M. K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus network? *Control Systems Magazine*, 32(4):66–73, 2012.

[25] D. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *Proceedings of the 23rd IEEE conference on Decision and Control*, volume 23, pages 127–132, 1984.

[26] Z.Y. Fan, J. Xu, Y. Wang, and D. Liang. The Laplacian spread of a tree. *Discrete Math. Theor. Comput. Sci.*, 10(1):79–86, 2008.

[27] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[28] P. Feldman and R. W. Freund. Efficient linear circuit analysis by Pade approximation via a Lanczos method. *IEEE Transactions on Computer-Aided Design*, 14:639–649, 1995.

[29] H. Gao, J. Lam, and C. Wang. Model simplification for switched hybrid systems. *Systems and Control Letters*, 55(12):1015–1021, 2006.

[30] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[31] C. D. Godsil. Control by quantum dynamics on graphs. *Physical Review A*, 81(5):052316–1:5, 2010.

[32] E. J. Grimme. *Krylov Projection Methods for Model Reduction*. PhD thesis, ECE Dept., U. of Illinois, Urbana-Champaign, 1997.

[33] R. Grone and R. Merris. The Laplacian spectrum of a graph II. *SIAM J. Discrete Math.*, 7(2):221–229, 1994.

[34] S. Gugercin and A. C. Antoulas. A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, 77(8):748–766, 2004.

[35] J. P. Hespanha. *Linear Systems Theory*. Princeton University Press, 41 William St., Princeton, New Jersey 08540, 2009.

[36] J.W. Hoffmann and P.A. Fuhrmann. On balanced realizations of bounded real and positive real functions. *Linear Algebra App.*, 245:107–146, 1996.

[37] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications*, 432:1961–1974, 2010.

[38] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 1988.

[39] J. Imura. Clustered model reduction of large-scale complex networks. In proceeding of the 20^{th} International Symposium on Mathematical Theory of Networked and Systems, Melbourne, Australia, 2012.

[40] T. Ishizaki, K. Kashima, J. Imura, and K. Aihara. Model reduction of multi-input dynamical networks based on clusterwise controllability. In *Proc. of American Control Conference*, pages 2301–2306, 2012.

[41] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules . *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[42] I. M. Jaimoukha and E. M. Kasenally. Implicitly restarted Krylov subspace methods for stable partial realizations. *SIAM Journal on Matrix Analysis and Applications*, 18:633–652, 1997.

[43] Y. Kuramoto. *Chemical Oscillators, Waves and Turbulence*. Springer Verlag, Berlin, 1984.

[44] S. Lall, P. Krysl, and J. E. Marsden. Structure-preserving model reduction for mechanical systems. *Physica D: Nonlinear Phenomena*, 184:304–318, 2003.

[45] A. Laub, M. Heath, C. Paige, and R. Ward. Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms. *IEEE Transactions on Automatic Control*, 32(2):115–122, 1987.

[46] R. C. Li and Z. Bai. Structure-preserving model reduction. *Applied Parallel Computing*, 3732:332–332, 2006.

[47] Z.-K. Li, Z.-S. Duan, G.-R. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems*, 57(1):213–224, 2010.

[48] D. Liberzon. *Switching in Systems and Control*. Birkhauser, Boston, 2003.

[49] C.-A. Lin and T.-Y. Chiu. Model reduction via frequency weighted balanced realization. *Control Theory and Advanced Technology*, 8:341–351, 1992.

[50] C. T. Lin. Structural controllability. *IEEE Transactions on Automatic Control*, 19(3):201–208, 1974.

[51] H. Lin and P. Anstalkis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.

[52] Y. Y. Liu, J. J. Slotine, and A. L. Barabasi. Controllability of complex networks. *Nature*, 473:167–173, 2011.

[53] C. Ma and J. Zhang. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 55(5):1262–1268, 2010.

[54] S. Martini, M. Egerstedt, and A. Bicchi. Controllability analysis of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control and Communications*, 2(1/2/3):100–121, 2010.

[55] H. Mayeda and T. Yamada. Strong structural controllability. *SIAM Journal on Control and Optimization*, 17(1):123138, 1979.

[56] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[57] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton and Oxford, 2010.

[58] R.E. Mirollo and S.H. Strogatz. Synchronization of pulse-coupled biological oscillators. *SIAM Journal of Applied Mathematics*, 50(6):1645–1662, 1990.

[59] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. A projection based approximation of multi-agent systems by using graph partitions. Submitted for publication.

[60] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Uniform synchronization in multi-agent systems with switching topologies. Submitted for publication.

[61] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Simultaneous balancing and model reduction of switched linear systems. In *Proceedings of the* 50^{th} *IEEE conference on Decision and Control*, Orlando, USA, 2011.

[62] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. A simultaneous balanced truncation approach to model reduction of switched linear systems. *IEEE Transactions on Automatic Control*, 57(12):3118–3131, 2012.

[63] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Synchronization analysis of multi-agent systems with switching topologies. In *Proceedings of the 20^{th} International Symposium on Mathematical Theory of Networks and Systems*, Melbourne, Australia, 2012.

[64] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Stability and synchronization preserving model reduction of multi-agent systems. *Systems and Control Letters*, 62(1):1–10, 2013.

[65] N. Monshizadeh, S. Zhang, and M.K. Camlibel. Disturbance decoupling problem for multi-agent systems: A graph topological approach. Submitted for publication.

[66] N. Monshizadeh, S. Zhang, and M.K. Camlibel. Zero forcing sets and controllability of systems defined on graphs. *IEEE Transactions on Automatic Control*. Conditionally accepted.

[67] B. C. Moore. Principle component analysis in linear system: controllability, observability and model reduction. *IEEE Transactions on Automatic Control*, AC-26, 1981.

[68] C. T. Mullis and R. A. Roberts. Synthesis of minimum roundoff noise fixed point digital filters. *IEEE Transactions on Circuits and Systems*, CAS-23:551–562, 1976.

[69] M. Nabi-Abdolyousefi and M. Mesbahi. On the controllability properties of circulant networks. *IEEE Transactions on Automatic Control*. To appear.

[70] R. Ober. Balanced parametrization of classes of linear systems. *SIAM Journal of Control and Optimization*, 29(6):1251–1287, 1991.

[71] R. Olfati-Saber. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3):401–420, 2006.

[72] R. Olfati-Saber, J. A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. In *Proceedings of the IEEE*, volume 95, pages 215–233, 2007.

[73] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays . *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[74] P. C. Opdenacker and E. A. Jonckheere. A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Transactions on Circuits and Systems*, CAS-35:184–189, 1988.

[75] G. Parlangeli and G. Notarstefano. On the reachability and observability of path and cycle graphs. *IEEE Transactions on Automatic Control*, 57(3):743–748, 2012.

[76] L. Pernebo and L.M. Silverman. Model reduction via balanced state space representation. *IEEE Transactions on Automatic Control*, AC-27:382–387, 1982.

[77] J.R. Philips, L. Daniel, and L.M. Silveira. Guaranteed passive balancing transformations for model order reduction. *IEEE Trans. on Computer-Aided Des. of. Integr. Circ. and Systems*, 22, no. 8:1–15, 2003.

[78] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[79] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 53(6):1503–1509, 2008.

[80] W. Ren and R.W. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[81] C. Reynolds. Flocks, birds, and schools: a distributed behavioral model. *Computer Graphics*, 21(4):25–34, 1987.

[82] H. Sandberg and M. Murray. Model reduction of interconnected linear systems. *Optimal Control Applications and Methods*, 30(3):225–245, 2009.

[83] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 14(11):2557–2562, 2009.

[84] J. H. Seo, H. Shim, and J. Back. Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica*, 45(11):2659–2664, 2009.

[85] H. Shaker and R. Wisniewski. Generalized gramian framework for model/controller order reduction of switched systems. *International Journal of Systems Science*, 42(8):1277–1291, 2011.

[86] R. Shorten, O. Mason, F. O. Cairbre, and P. Curran. A unifying framework for the SISO circle criterion and other quadratic stability criteria. *International Journal of Control*, 77(1):1–8, 2004.

[87] D.C. Sorensen. Implicit application of polynomial fillters in a k-step arnoldi method. *SIAM Journal on Matrix Analysis and Applications*, 13(1):357–385, 1992.

[88] G.B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256–270, 2007.

[89] D. J. Stilwell, E. M. Bollt, and D. G. Roberson. Sufficient conditions for fast switching synchronization in time-varying network topologies. *SIAM J. Applied Dynamical Systems*, 5(1):140–156, 2006.

[90] Z. Sun and S. S. Ge. *Switched Linear Systems*. Springer, London, 2005.

[91] T. Takamatsu, I. Hashimoto, and Y. Nakai. Decoupling and pole assignment in linear multivariable systems: a geometric approach. *Automatica*, 15(4):387–402, 1979.

[92] H. G. Tanner. On the controllability of nearest neighbor interconnections. In *Proceedings of the 43rd IEEE conference on Decision and Control*, pages 2467–2472, 2004.

[93] M. Trefois and J. C. Delvenne. Zero forcing sets, constrained matchings and minimum rank. *Linear and Multilinear Algebra*. Submitted for publication.

[94] H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus. *Control Theory for Linear Systems*. Springer, London, 2001.

[95] H. L. Trentelman, K. Takaba, and N. Monshizadeh. Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 58(6):1511–1523, 2013.

[96] S. E. Tuna. Synchronizing linear systems via partial-state coupling. *Automatica*, 44(8):2179–2184, 2008.

[97] T. Vicsek, A. Czirok, E. Ben Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.

[98] G. Wang, V. Sreeram, and W. Q. Liu. A new frequency weighted balanced truncation method and an error bound. *IEEE Transactions on Automatic Control*, 44:1734–1737, 1999.

[99] J. Wang, A. Cheng, and X. Hu. Consensus of multi-agent linear dynamic systems. *Asian Journal of Control*, 10(2):144–155, 2008.

[100] J.T. Wen. Time domain and frequency domain conditions for strict positive realness . *IEEE Transactions on Automatic Control*, 33(10):988–992, 1988.

[101] P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068–1074, 2011.

[102] W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, New York, 1985.

[103] W. M. Wonham and A. S. Morse. Decoupling and pole assignment in linear multivariable systems: a geometric approach. *SIAM Journal of Control*, 8(1):1–18, 1970.

[104] A. Y. Yazicioglu, W. Abbas, and M. Egerstedt. A tight lower bound on the controllability of networks with multiple leaders. In *Proceedings of the 51st IEEE conference on Decision and Control*, pages 1978–1983, 2012.

[105] M. Zhai, J. Shu, and Y. Hong. On the Laplacian spread of graphs. *Applied Mathematics Letters*, 24(12):2097–2101, 2011.

[106] S. Zhang, M. K. Camlibel, and M. Cao. Controllability of diffusively-coupled multi-agent systems with general and distance regular topologies. In *Proceedings of the 50th IEEE conference on Decision and Control and 2011 European Control Conference*, pages 759–764, 2011.

[107] S. Zhang, M. Cao, and M. K. Camlibel. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. *IEEE Transactions on Automatic Control*. To appear.

[108] J. Zhao, D. J. Hill, and T. Liu. Synchronization of complex dynamical networks with switching topology:A switched system point of view. *Automatica*, 45(11):2502–2511, 2009.

[109] K. Zhou. Frequency-Weighted L_∞ norm and optimal Hankel norm model reduction. *IEEE Transactions on Automatic Control*, 40:1687–1699, 1995.

[110] K. Zhou and J. C. Doyle. *Essentials of Robust Control*. Prentice Hall, New Jersey, 1998.

[111] K. Zhou and P. P. Khargonekar. An algebraic Riccati equation approach to H^∞ optimization. *Systems and Control Letters*, 11(2):85–91, 1988.

Summary

This thesis addresses several problems related to model reduction and control of complex systems. Two classes of complex systems considered in this thesis are switched systems and networks of dynamical agents. The problems studied in this thesis can be summarized as follows.

First, an extended balanced truncation method is established for model reduction of switched linear systems (SLS). The idea is to seek for conditions under which there exists a single state space transformation that brings all subsystems (modes) of a given SLS in balanced coordinates. Moreover, we derive sufficient conditions under which global uniform exponential stability of the SLS is preserved in the reduced order model. Similarly, we propose conditions for preservation of positive realness or bounded realness of the SLS.

Second, we propose a method to reduce the dynamic order of agents in a network, while stability or synchronization is preserved in the reduced order model. In this model reduction process, the interconnection structure of the network remains unchanged whereas the dynamic order of the agents is reduced. In addition, we establish a priori model reduction error bounds to compare the behavior of the original network to that of the reduced order model.

Third, in contrast to the previous model order reduction method, we aim at reducing the size of the underlying communication graph of the network. For this purpose, vertices (agents) of the network communication are clustered by means of suitable graph partitions. We also provide an explicit formula for the H_2-norm of the error system obtained by comparing the input-output behaviors of the original model and the reduced order model for the case that the clusters are chosen using the so-called almost equitable partitions of the graph.

As a forth problem, we carry out stability and synchronization analysis for networks where agents have general, yet identical, linear dynamics and the underlying

communication topology may switch arbitrarily within a finite set of admissible topologies. The primary conditions established are in terms of a pair of Lyapunov strict inequalities. Following these conditions, small gain and passivity types of conditions are proposed under which synchronization is guaranteed.

Fifth, we investigate the property of strong structural controllability for systems defined on a graph. In particular, it is shown that there is a one-to-one correspondence between the set of leaders rendering the network controllable and the so-called zero forcing sets. Special subclasses of graphs including path, cycle, and complete graphs are exploited to illustrate the proposed results.

Finally, the disturbance decoupling problem for networks of dynamical agents is studied from a graph topological perspective. A class of graph partitions is developed in this thesis that can be described as a "topological translation" of controlled invariant subspaces in the context of dynamical networks. Then, sufficient conditions are derived in terms of this class of partitions such that the network is disturbance decoupled. Likewise, topological conditions guaranteeing solvability of the disturbance decoupling problem are established.

Samenvatting

Dit proefschrift behandelt een aantal vraagstukken gerelateerd aan modelreductie en regeling van twee soorten complexe systemen: 'switched linear systems' en netwerken van dynamische agenten. De volgende problemen worden in dit proefschrift bestudeerd.

Eerst geven we een uitgebreide, gebalanceerde afkappingsmethode voor modelreductie van 'switched linear systems' (SLS). Hiervoor zoeken we naar voorwaarden waaronder er n transformatie van de toestandsruimte bestaat die alle subsystemen (modi) van een gegeven SLS in gebalanceerde coördinaten brengt. Verder leiden we voldoende voorwaarden af waaronder globale uniforme exponentile stabiliteit van een SLS behouden blijft in het gereduceerde orde model. Eveneens geven we voorwaarden voor het behoud van de eigenschappen van het positief reëel of begrensd reëel zijn van een SLS.

Vervolgens introduceren we een methode om de dynamische orde van agenten in een netwerk te reduceren, waarbij stabiliteit of synchronisatie behouden blijft in het gereduceerde orde model. In dit proces van modelreductie blijven de onderlinge verbindingen in het netwerk onveranderd, terwijl de dynamische orde van de agenten wordt gereduceerd. Tevens vinden we 'a priori' foutgrenzen voor de modelreductie, zodat we het gedrag van het originele netwerk kunnen vergelijken met dat van het gereduceerde model.

Het derde probleem richt zich, in tegenstelling tot de vorige modelreductiemethode, op het reduceren van de omvang van de onderliggende communicatiegraaf van het netwerk. Hiervoor worden de knopen van de communicatiegraaf, de agenten, geclusterd door middel van een geschikte graafpartitie. In het geval dat de clusters zijn gekozen met behulp van de zogenaamde bijna billijke partities van de graaf, geven we een expliciete formule voor de H_2-norm van het afwijkingssysteem, welke het ingang-uitgangsgedrag van het originele systeem vergelijkt met dat van het gereduceerde model.

Als vierde probleem analyseren we de stabiliteit en synchronisatie van netwerken waarbij de agenten een algemene, maar identieke, lineaire dynamica hebben en de onderliggende communicatietopologie willekeurig mag schakelen tussen een eindig aantal toegestane topologieën. Eerst vinden we voldoende voorwaarden in termen van een tweetal strikte Lyapunov ongelijkheden. Daarna leiden we hieruit voldoende voorwaarden voor synchronisatie af in termen van passiviteit en kleine versterkingsfactor.

De sterk structurele regelbaarheid van systemen die op een graaf gedefinieerd zijn, is het vijfde onderwerp van dit proefschrift. In het bijzonder tonen we aan dat er een een-op-eenrelatie is tussen de verzameling van leiders die ervoor zorgen dat het netwerk regelbaar is en de zogenaamde 'zero-forcing' verzamelingen. Enkele speciale deelverzamelingen van grafen, waaronder paden, cykels en de volledige graaf, worden gebruikt om de resultaten te illustreren.

Tenslotte bestuderen we het probleem van storingsontkoppeling voor netwerken van dynamische agenten vanuit een graaftopologisch perspectief. In dit proefschrift ontwikkelen we een klasse van graafpartities die gezien kan worden als een topologische vertaling van regelbare, invariante deelruimten in het kader van dynamische netwerken. Vervolgens vinden we voldoende voorwaarden, in termen van deze klasse van partities, waaronder het netwerk storingsontkoppeld is. Eveneens geven we topologische voorwaarden voor de oplosbaarheid van het storingsontkoppelingsprobleem.