CUSP FORMS OF WEIGHT 2 FOR THE GROUP $\Gamma_2(4, 8)$

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1. Introduction. This paper is devoted to the study of L-functions attached to certain Siegel modular forms. In the case of holomorphic modular forms of one variable, i.e. modular forms for congruence subgroups of finite index of $SL_2(\mathbf{Z})$, one has a very rich and extensive theory. Classical results by Hecke and others tell that the Dirichlet series associated to cusp forms which are common eigenforms for the so called Hecke operators, have analytic continuation and satisfy a functional equation. Eichler, Shimura, Kuga and more generally Deligne and Serre have shown that these L-series can also be obtained from (compatible systems of ℓ-adic) Galois representations. Deligne proved the 'Ramanujan conjecture', which asserts that the eigenvalues of Frobenius elements in these representations are algebraic integers, all having the same absolute value (depending of course on the prime at which one takes Frobenius elements) for any embedding into the complex numbers. There are still a lot of very interesting open problems in this theory. One has e.g. Serre's conjecture ([19], p. 196; compare [18]) and the celebrated Taniyama-Weil conjecture (compare [1], pp. 27–28).

The far reaching consequences of some of the results and conjectures in this theory motivate to search for generalizations. One (certainly not the only) way to do this, is to replace the usual upper half plane in the complex numbers by the Siegel upper half plane $\mathbf{H}_g$ of complex symmetric $g \times g$ matrices with positive definite imaginary part. This $\mathbf{H}_g$ plays the same role in the theory of abelian varieties over the complex numbers, as the usual upper half plane does in the theory of elliptic curves. On $\mathbf{H}_g$ the symplectic group $Sp_{2g}(\mathbf{Z})$ acts, and this is the starting point for a theory of modular forms, theta functions, classification problems concerning abelian varieties, etc. which was developed by Riemann, Siegel, Igusa, Mumford and many others. However, the appropriate generalizations of more arithmetic problems like the ones mentioned above for $g = 1$, evolved only quite recently, and in some instances, are still missing. We give a very brief sketch.

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In 1963, Shimura published the generalization to higher genus of the classical 'Eichler-Shimura relations' (see [9], Section 1 for an excellent readable account). From these one can guess what an Euler product of a Dirichlet series associated to cusp forms which are common eigenforms for the Hecke operators should look like. It was done only for the full group $\mathrm{Sp}_{2g}(\mathbf{Z})$. In the beginning of the seventies, Andrianov proved that in case $g = 2$ this L-function, (now called Andrianov L-function or Hasse-Weil-Andrianov L-function) has an analytic continuation and satisfies a functional equation. The generalization of Andrianov's results to congruence subgroups is due to Evdokimov (1976), and of Shimura's results to Faltings (1984–85).

The 'generalized Ramanujan conjecture' turns out to be false in general. More precisely, the polynomials of degree 2^g which yield the Euler factors of the Hasse-Weil-Andrianov L-function, have complex zeroes of different absolute value in certain examples. The first published counterexamples are from Kurokawa (1978; [16], (1.2), p. 151).

As far as we know, it is unknown whether the Andrianov L-functions arise as L-functions of (compatible systems of ℓ-adic) Galois representations. Special cases have been settled however: e.g. the Saito-Kurokawa lifts. In the general case, for forms of weight $k = g + 1 + ng$, and $n \geq 0$, one has a geometric candidate for this Galois representation, but nothing has been proven (compare [9], pp. 7–8). For other weights, something like an analogue of the Deligne-Serre theorem is still wide open. In his Bourbaki talk in June 1989 Guy Henniart mentions ideas of R. Taylor which may turn out to be of importance here.

In 1980, H. Yoshida (in [22], p. 243) stated a conjecture which is an analogue for $g = 2$ of the Taniyama-Weil conjecture. It asserts that the L-function $L(A, s)$ associated to the Tate module of an abelian surface A over $\mathbf{Q}$ is (upto finitely many Euler factors) the Andrianov L-function of a cusp form of weight 2 for some congruence subgroup of $\mathrm{Sp}_4(\mathbf{Z})$, provided that $L(A, s)$ has analytic continuation and satisfies a functional equation with sign +1.

This sign condition looks quite curious. Yoshida puts it in because he has a procedure to 'lift' two cusp forms of weight 2 for $\Gamma_0(N)$ to a Siegel modular form of weight 2 for $\Gamma_2(N)$. This 'lifting' is zero if the two forms he starts with have functional equations with opposite signs. If the lifting is nonzero, then the associated Andrianov L-function is the product of the two L-functions attached to the modular forms of one variable. In particular this implies that his lifting technique only provides examples of Andrianov L-functions with a sign +1 in the functional equation.

Using this lifting procedure he is able to prove his conjecture for simple abelian surfaces which are (upto isogeny) factors of the jacobian of a modular curve $X_0(p)$ where p is a prime number not congruent to 1 modulo 24 (see [23], p. 185 and Remark 6.9, pp. 216–217). For more information concerning these liftings, compare results of Böcherer and Schulze-Pillot [2,3].

The aim of this paper is to provide examples of cusp forms of weight 2 in the $g = 2$ case. It seems desirable to have such examples. One can multiply them by any modular form of weight at least 1 and thus obtain cusp forms of higher weight. One can also try to find more examples where the generalized Ramanujan conjecture does not hold, or examples which seem to correspond to abelian surfaces. Especially, we hoped to find an example where the corresponding abelian surface has a -1 in its functional equation. Such an example we did not find. What we did find is a complete description of the cusp forms of weight 2 for the group $\Gamma_2(4,8)$, and a basis for it consisting of eigenfunctions for the Hecke operators. Moreover, we have formulas for the eigenvalues, hence a precise description of the corresponding Andrianov L-functions. They turn out to be all equal and provide very simple counterexamples to the generalized Ramanujan conjecture.

In the next section some results concerning the commutator subgroup of the principal congruence subgroups of $Sp_{2g}(\mathbf{Z})$ are given. We also compute orbits and stabilizers of the action of $Sp_4(\mathbf{Z})$ on certain characters of quotients of congruence subgroups. These calculations will allow us to decompose the space of modular forms of weight 2 for $\Gamma_2(8)$ arising from theta constants into 'manageable' subspaces. This will be done in Sections 3 and 4. Section 5 contains the main results; in the remainder of the paper some examples of forms of higher level are given.

2. Computations with subgroups of $\mathbf{Sp_{2g}(Z)}$. Let Sp_{2g} be the group of $2g \times 2g$ matrices which preserve the symplectic form corresponding to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here 1 is the $g \times g$ identity matrix. Elements of Sp_{2g} are sometimes written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with A, B, C and D $g \times g$ matrices. The principal congruence subgroups of $\Gamma_g = Sp_{2g}(\mathbf{Z})$ are the groups

$$\Gamma_g(N) = \mathrm{Ker}(\Gamma_g \to Sp_{2g}(\mathbf{Z}/N\mathbf{Z})).$$

Here the homomorphism is the obvious reduction map. Some other subgroups of Γ_g are

$$\Gamma_g(2n, 4n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(2n); \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \bmod 4n \right\},$$

and

$$\Gamma_g(n,2n,4n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(2n,4n);\ \operatorname{trace}(A) \equiv g \bmod 4n \right\}.$$

All these groups are normal subgroups of Γ_g, except if g is odd and $\neq 1$ in which case $\Gamma_g(n,2n,4n)$ is not normal. Also, if g is even, by [9], Proposition 2.8, $\Gamma_g(n,2n,4n)$ is the unique normal subgroup of Γ_g lying between $\Gamma_g(4n)$ and $\Gamma_g(2n,4n)$.

The first result we want to prove is

PROPOSITION 2.1. *Take $g > 1$. The commutator subgroup of $\Gamma_g(n)$ is*

$$\begin{cases} \Gamma_g(n^2,2n^2) & \textit{in case } n \textit{ is even;} \\ \Gamma_g(n^2) & \textit{in case } n \textit{ is odd.} \end{cases}$$

Proof. Write H_n for the commutator subgroup of $\Gamma_g(n)$. This is a normal subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$. A direct verification shows that $\Gamma_g(n)/\Gamma_g(n^2)$, which can be regarded as subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z}/n^2\mathbf{Z})$, is abelian. Hence

$$H_n \subset \Gamma_g(n^2).$$

Consider the $g \times g$ matrix A which has 1's on the diagonal, n in the upper right corner and zeroes on all other places. Let B be the $g \times g$ matrix with n in the lower left and upper right corner, and for the rest zeroes. Then both

$$\begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

are elements of $\Gamma_g(n)$. Their commutator is

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix},$$

with C consisting of only zeroes except for the value $2n^2$ in the upper left corner. It is a result of Mennicke ([17], Satz 10, p. 128) that any normal subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ which contains this matrix, contains the whole $\Gamma_g(2n^2)$.

Let D be the $g \times g$ matrix with n in the upper left corner and zeroes everywhere else. The commutator of

$$\begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ D & 1 \end{pmatrix}$$

is not in $\Gamma_g(\frac{1}{2}n^2, n^2, 2n^2)$ in case n even, respectively not in $\Gamma_g(n^2, 2n^2)$ in case n is odd. (We did not define this group for odd n; see [11], last line of p. 220.) It is straightforward to verify that no normal subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ exists properly between $\Gamma_g(n^2, 2n^2)$ and $\Gamma_g(n^2)$. This can e.g. be done by checking that the conjugation action of Sp_{2g} on the set of matrices

$$\begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix},$$

with A a nonzero diagonal matrix, is transitive. This proves the proposition for odd n.

For even n, a computation in $\mathrm{Sp}_{2g}(\mathbf{Z}/2n^2\mathbf{Z})$ yields that $H_n \subset \Gamma_g(n^2, 2n^2)$. Since we know all normal subgroups of $\mathrm{Sp}_{2g}(\mathbf{Z})$ between $\Gamma_g(n^2, 2n^2)$ and $\Gamma_g(2n^2)$ by [9], Proposition 2.8, it follows that $H_n = \Gamma_g(n^2, 2n^2)$ in this case. □

The remainder of this section will be devoted to the study of the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$ on characters of $\Gamma(2n)/\Gamma(4n)$. Since we only deal with $g = 2$ the notation Γ instead of Γ_2 is used. We will omit the proof of the following lemma, which reviews some easy facts about quotients like $\Gamma(2n)/\Gamma(4n)$. Compare [11], pp. 222–223 for analogous and more general statements.

LEMMA 2.2. *The group $\Gamma(2n)/\Gamma(4n)$ is a 10-dimensional $\mathbf{F}_2$-vectorspace. A basis is given by*

$$e_1(n) = \begin{pmatrix} 1 & & & \\ 2n & 1 & & \\ & & 1 & -2n \\ & & & 1 \end{pmatrix}, \qquad e_2(n) = {}^t e_1,$$

$$e_3(n) = \begin{pmatrix} 1 & & & 2n \\ & 1 & 2n & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad e_4(n) = {}^te_3,$$

$$e_5(n) = \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}, \qquad e_6(n) = \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix},$$

$$e_7(n) = \begin{pmatrix} 1 & & 2n & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad e_8(n) = \begin{pmatrix} 1 & & & \\ & 1 & & 2n \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$e_9(n) = {}^te_7(n), \qquad e_{10}(n) = {}^te_8(n).$$

Here

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is some fixed matrix in $SL_2(\mathbf{Z})$ *which is congruent to*

$$\begin{pmatrix} 1+2n & 0 \\ 0 & 1+2n \end{pmatrix}$$

modulo $4n$.

Similar $\mathbf{F}_2$*-vectorspaces with a basis are given in the following table.*

$\Gamma(2n,4n)/\Gamma(4n)$	$e_1(n),\dots,e_6(n)$
$\Gamma(n,2n,4n)/\Gamma(4n)$	$e_1(n),\dots,e_5(n)$
$\Gamma(2n,4n)/\Gamma(n,2n,4n)$	$e_6(n)$
$\Gamma(2n,4n)/\Gamma(4n,8n)$	$e_1(n),\dots,e_6(n),e_7(2n),\dots,e_{10}(2n)$

The action of $\mathrm{Sp}_4(\mathbf{Z})$ on groups $\Gamma(2n)/\Gamma(4n)$ we want to consider, arises from the usual conjugation action given by $\sigma\cdot x = \sigma x\sigma^{-1}$. This action is in fact linear on $\Gamma(2n)/\Gamma(4n)$ regarded as $\mathbf{F}_2$-vectorspace. Moreover, one verifies that $\Gamma(2)$ acts trivially, hence we work with a linear representation over $\mathbf{F}_2$ of the quotient group $\mathrm{Sp}_4(\mathbf{Z})/\Gamma(2) \cong \mathrm{Sp}_4(\mathbf{F}_2)$ which is actually isomorphic to S_6, the group of permutations on a set of 6 elements. On characters $\chi : \Gamma(2n)/\Gamma(4n) \to \{\pm 1\}$, an element $\sigma \in \mathrm{Sp}_4(\mathbf{Z})$ acts as

$$\sigma \cdot \chi = (x \mapsto \chi(\sigma x \sigma^{-1})).$$

Our main result needed about these actions is provided by the following proposition.

PROPOSITION 2.3. *The action by conjugation of* $\mathrm{Sp}_4(\mathbf{Z})$ *on the set of characters of* $\Gamma(2n)/\Gamma(4n)$, *which are nontrivial on* $\Gamma(n, 2n, 4n)/\Gamma(4n)$, *divides this set into 12 different orbits. For each of these orbits, a character in it is given in the following table.*

character:	*orbit length:*
$+1$ *on* $e_1, \dots, e_4, e_6, \dots, e_{10}$ -1 *on* e_5	*60*
$+1$ *on* $e_1, \dots, e_4, e_6, e_8, e_9, e_{10}$ -1 *on* e_5, e_7	*180*
$+1$ *on* $e_1, \dots, e_4, e_6, e_7, e_9$ -1 *on* e_5, e_8, e_{10}	*20*
$+1$ *on* $e_1, \dots, e_4, e_6, e_7$ -1 *on* e_5, e_8, e_9, e_{10}	*60*
$+1$ *on* $e_1, e_2, e_3, e_5, \dots, e_{10}$ -1 *on* e_4	*45*
$+1$ *on* $e_1, e_2, e_3, e_5, e_6, e_7, e_9, e_{10}$ -1 *on* e_4, e_8	*180*
$+1$ *on* $e_1, e_2, e_3, e_5, \dots, e_8$ -1 *on* e_4, e_9, e_{10}	*15*
$+1$ *on* $e_1, e_2, e_3, e_5, e_7, \dots, e_{10}$ -1 *on* e_4, e_6	*60*
$+1$ *on* $e_1, e_2, e_3, e_5, e_8, e_9, e_{10}$ -1 *on* e_4, e_6, e_7	*90*
$+1$ *on* e_1, e_2, e_3, e_5, e_7 -1 *on* $e_4, e_6, e_8, e_9, e_{10}$	*90*
$+1$ *on* $e_1, e_2, e_6, \dots, e_{10}$ -1 *on* e_3, e_4, e_5	*120*
$+1$ *on* e_1, e_2, e_6, e_7 -1 *on* $e_3, e_4, e_5, e_8, e_9, e_{10}$	*72*

(sketch of) Proof. The group $\mathrm{Sp}_4(\mathbf{Z})/\Gamma(2)$ is generated by e.g. the following 4 matrices (compare [6], Satz A5.4, p. 326):

$$\alpha = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

By a straightforward calculation one obtains the action of these generators on a basis of $\Gamma(2n)/\Gamma(4n)$. The result is given in the following table. Throughout, the matrix $e_i(n)$ defined above is written as e_i. Also, multiplication in the group $\Gamma(2n)/\Gamma(4n)$ is written as $+$, to stress the fact that we work in a vectorspace over $\mathbf{F}_2$.

	α	β	γ	δ
e_1	e_2	$e_1 + e_3$	e_1	e_1
e_2	e_1	e_2	$e_2 + e_3$	e_2
e_3	e_4	e_3	e_3	e_3
e_4	e_3	$e_2 + e_4$	$e_1 + e_4$	$e_3 + e_4 + e_5$
e_5	e_5	e_5	e_5	e_5
e_6	e_6	e_6	e_6	$e_3 + e_6$
e_7	e_9	e_7	e_7	e_7
e_8	e_{10}	e_8	e_8	e_8
e_9	e_7	$e_6 + e_7 + e_9$	e_9	$e_1 + e_8 + e_9$
e_{10}	e_8	e_{10}	$e_5 + e_6 + e_8 + e_{10}$	$e_2 + e_7 + e_{10}$

In order to find the orbits and stabilizers for the action of $\mathrm{Sp}_4(\mathbf{Z})$ on the characters of $\Gamma(2n)/\Gamma(4n)$, one may proceed as follows. First, it will turn out that for later use only characters which are nontrivial on the subgroup $\Gamma(n,2n,4n)/\Gamma(4n)$ are of importance. We restrict ourselves to those characters. Any character of a group which is a vectorspace over $\mathbf{F}_2$ is of course determined by its kernel. In our situation, starting from a nontrivial character of $\Gamma(n,2n,4n)/\Gamma(4n)$, or equivalently, a codimension 1 subspace of this group, it is easy to compute the orbit of this kernel. It turns out that there are three such orbits:

An orbit of length 10, coming from the character χ_1 which has as kernel $\langle e_1, e_2, e_3, e_4\rangle$;

One of length 15, coming from χ_2 with kernel $\langle e_1, e_2, e_3, e_5\rangle$;

The orbit of χ_3, the character with kernel $\langle e_1, e_2, e_3+e_5, e_4+e_5\rangle$; this one has length 6.

From this starting point, one can extend representatives of the orbits to characters of larger groups. The first extension is to the group $\Gamma(2n,4n)/\Gamma(4n)$. For each of the three characters χ_i one has to consider, there are two possible extensions: $\chi_{i,0}$, which sends the extra basis vector e_6 to 1, and $\chi_{i,1}$ which is -1 on e_6. One computes that the element $\delta\alpha\delta\alpha\delta$ of $\mathrm{Sp}_4(\mathbf{Z})$ maps $\chi_{1,0}$ to $\chi_{1,1}$ and δ sends $\chi_{3,0}$ to $\chi_{3,1}$. The two extensions of χ_2 are in different orbits.

It is not hard to compute the stabilizers $\mathrm{Stab}(\chi)$, as subgroups of $\mathrm{Sp}_4(\mathbf{F}_2)$, for the four relevant characters we now have.

- $\mathrm{Stab}(\chi_{1,0})$ is the group of order 36, generated by α, β and γ.
- $\mathrm{Stab}(\chi_{2,0} = \mathrm{Stab}(\chi_{2,1}) = \mathrm{Stab}(\chi_2)$ is the subgroup of order 48 in $\mathrm{Sp}_4(\mathbf{F}_2)$, consisting of all matrices of the form

$$\begin{pmatrix} A & B \\ & {}^t\!A^{-1} \end{pmatrix}.$$

It is generated e.g. by β, γ, δ and

$$\sigma = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

- The stabilizer of $\chi_{3,0}$ has order 60 and is generated by $\alpha, \epsilon = \beta\alpha\gamma$ and $\rho = \delta\alpha\delta$.

The remaining step is, to compute the action of these stabilizers on all possible extensions of the character they correspond to. This leads to Proposition 2.3. □

3. Siegel modular forms and Hecke operators. We briefly recall some basic facts about Siegel modular forms and the action of Hecke operators on them. Details can be found in [4], §1–3, pp. 432–442.

Let $k \geq 0$ and $g \geq 1$ be integers. For a function $f : \mathbf{H}_g \to \mathbf{C}$ and a matrix

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{R})$$

one defines the function $f|_k\sigma$ on $\mathbf{H}_g$ by

$$f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix}(\tau) = \det(C\tau + D)^{-k} f((A\tau + B)(C\tau + D)^{-1}).$$

One has the formula $f|_k(\sigma_1\sigma_2) = (f|_k\sigma_1)|_k\sigma_2$. For $g \geq 2$ and Γ a subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$, a (Siegel) modular form of genus (or degree) g and weight k for Γ is a holomorphic function f on $\mathbf{H}_g$ which satisfies $f|_k\sigma = f$ for all $\sigma \in \Gamma$. (In case $g = 1$, an additional boundedness condition is put into the definition; this is automatically fulfilled for $g > 1$.) The complex vectorspace of all modular forms of weight k for Γ is denoted $M_k(\Gamma)$. A modular form of level q is an element of $M_k(\Gamma_g(q))$.

To define cusp forms, one uses the notion of the Siegel operator Φ. This is a linear map from modular forms of genus g to modular forms of genus $g-1$, defined by

$$\Phi f(\tau) = \lim_{\lambda \to \infty} f\left(\begin{pmatrix} \tau & \\ & i\lambda \end{pmatrix}\right).$$

Here λ is (of course) a real parameter. The space of cusp forms of weight k for the group Γ is

$$S_k(\Gamma) = \{f \in M_k(\Gamma); \Phi(f|_k\sigma) = 0, \quad \forall \sigma \in \mathrm{Sp}_{2g}(\mathbf{Z})\}.$$

We will now define the Hecke operators $T_k(m)$ on the spaces of modular forms $M_k(\Gamma_g(q))$ for $\gcd(m, q) = 1$. One writes

$$\overline{S_g}(q) = \left\{\sigma \in M_{2g}(\mathbf{Z});\ \sigma \equiv \begin{pmatrix} 1_g & \\ & r(\sigma)\cdot 1_g \end{pmatrix} \bmod q \text{ for some } r(\sigma) \in \mathbf{Z},\right.$$

$$\left.\text{with } \gcd(r(\sigma), q) = 1 \text{ and } \sigma \begin{pmatrix} & -1_g \\ 1_g & \end{pmatrix} {}^t\sigma = r(\sigma)\cdot \begin{pmatrix} & -1_g \\ 1_g & \end{pmatrix}\right\}.$$

For $\sigma \in \overline{S}_g(q)$, the double coset $\Gamma_g(q)\sigma\Gamma_g(q)$ can be written as a finite union of left cosets:

$$\Gamma_g(q)\sigma\Gamma_g(q) = \cup_i \Gamma_g(q)\sigma_i.$$

The double coset yields an operator $T_k(\Gamma_g(q)\sigma\Gamma_g(q))$ on $M_k(\Gamma_g(q))$ defined by

$$T_k(\Gamma_g(q)\sigma\Gamma_g(q))f = r(\sigma)^{gk-\frac{1}{2}g(g+1)} \cdot \sum_i f|_k\sigma_i.$$

The Hecke operator $T_k(m)$, with m an integer such that $\gcd(m,q) = 1$, is defined as

$$T_k(m) = \sum T_k(\Gamma_g(q)\sigma\Gamma_g(q)).$$

This sum is taken over all double cosets $\Gamma_g(q)\sigma\Gamma_g(q)$ such that $r(\sigma) = m$.

Note that $T_k(m)$ does not depend on the level in an essential way: if a form of level q is regarded as a form of level qs, then the action of $T_k(m)$ on it remains the same. (To see this, write $\Gamma(q) = \cup_j \Gamma(qs) \cdot \alpha_j$. The contribution to $T_k(m)$ coming from a matrix

$$\sigma \in \Gamma_g(q) \begin{pmatrix} 1 & \\ & m \end{pmatrix}$$

can be written as $|_k \sum_{i,j} \alpha_j \sigma_i$. The matrices $\sigma\alpha_l$ yield contributions as well; all this together is just what one obtains from a coset $\Gamma_g(qs)\tau\Gamma_g(qs)$ for a τ which maps modulo q to σ.) In fact, one can prove more generally:

LEMMA 3.1. *Suppose $\Gamma_g(q) \subset \Gamma \subset Sp_{2g}(\mathbf{Z})$. Denote by $\overline{\Gamma}$ the image of Γ under $Sp_{2g}(\mathbf{Z}) \to Sp_{2g}(\mathbf{Z}/q\mathbf{Z})$. Let χ be any character: $\Gamma \to \mathbf{C}^*$.*

If for every $m \in (\mathbf{Z}/q\mathbf{Z})^*$ and every

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \overline{\Gamma}$$

one has that

$$\begin{pmatrix} A & m^{-1}B \\ mC & D \end{pmatrix} \in \overline{\Gamma},$$

then the Hecke operator $T_k(m)$ maps the space

$$M_k(\Gamma,\chi) = \{f \in M_k(\Gamma_g(q)); f|_k\sigma = \chi(\sigma)f \quad \forall\sigma \in \Gamma\}$$

to the space $M_k(\Gamma,\chi')$ where $\chi' : \overline{\Gamma} \to \mathbf{C}^*$ is given by

$$\chi'\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \chi\begin{pmatrix} A & m^{-1}B \\ mC & D \end{pmatrix}.$$

The condition mentioned here is e.g. satisfied for each of the inclusions $\Gamma_g(q) \subset \Gamma_g(n)$, $\Gamma_g(q) \subset \Gamma_g(2n,4n)$ and $\Gamma_g(q) \subset \Gamma_g(n,2n,4n)$.

In the special case $g = 2$, $q = 8$ and χ a homomorphism: $\Gamma(2)/\Gamma(4,8) \to \mathbf{C}^*$ the Hecke operators $T_k(m)$ for $m \equiv 1 \bmod 4$ leave $M_k(\Gamma(2),\chi)$ invariant, while those for $m \equiv 3 \bmod 4$ send $M_k(\Gamma(2),\chi)$ to $M_k(\Gamma(2),\overline{\chi})$. Here $\overline{\chi}$ is the complex conjugate of χ.

The proof of this lemma is a straightforward calculation using the definition of Hecke operators. It is indicated e.g. on the first part of p. 305 of [5]. Note however that there it is claimed that the Hecke operators leave the character spaces invariant, which is false in general.

The great advantage of Lemma 3.1 is, that it reduces the problem of finding eigenforms for the Hecke operators in the (usually) very big space $M_k(\Gamma)$ to the easier problem of studying the character spaces

$$M_k(\Gamma, \chi) = \{f \in M_k(\Gamma_g(q));\ f|_k\sigma = \chi(\sigma)f \quad \forall \sigma \in \Gamma\}.$$

With the Hecke operators defined above, one can introduce the Andrianov L-function. To do this, one starts by decomposing $M_k(\Gamma_g(q))$ into character spaces for the action of the 'torus group' of diagonal matrices in $\mathrm{Sp}_{2g}(\mathbf{Z}/q\mathbf{Z})$. Let $1 \leq i \leq g$ be an integer, and $a \in (\mathbf{Z}/q\mathbf{Z})^*$. Define $\sigma_i(a)$ to be any matrix in $\mathrm{Sp}_{2g}(\mathbf{Z})$ with $\sigma_i(a) \bmod q$ the diagonal matrix

$$\mathrm{diag}(\overbrace{a^{-1}, \ldots, a^{-1}}^{i}, 1, \ldots, \overbrace{a, \ldots, a}^{i}, 1, \ldots, 1).$$

For any g-tuple of characters $\chi_i : (\mathbf{Z}/q\mathbf{Z})^* \to \mathbf{C}^*$, write

$$M_k(\Gamma_g(q), \chi_1, \ldots, \chi_g) = \{f \in M_k(\Gamma_g(q));\ f|_k\sigma_i(a) = \chi_i(a) \cdot f \quad (1 \leq i \leq g)\}.$$

Note that this notation differs from the one introduced above. In $M_k(\Gamma, \chi)$ the character χ is defined on Γ. In the notation $M_k(\Gamma_g(q), \chi_1, \ldots, \chi_g)$, the χ_i are characters on $(\mathbf{Z}/q\mathbf{Z})^*$. We trust that this will not lead to any confusion.

Suppose $f \in M_k(\Gamma_g(q), \chi_1, \ldots, \chi_g)$ is eigenform for all the $T_k(m)$; write $T_k(m)f = \lambda(m)f$. Then it turns out that for prime numbers p which do not divide q, the Dirichlet series

$$\sum_{n=0}^{\infty} \lambda(p^n)p^{-ns}$$

is a rational function in p^{-s}. One can write it in the form $\frac{P_p(p^{-s})}{Q_p(p^{-s})}$, for polynomials P_p and Q_p of degree $2^g - 2$ and 2^g respectively. In particular,

$$g = 1 : Q_p(T) = 1 - \lambda(p)T + \chi_1(p)p^{k-1}T^2,$$

and $g = 2 : Q_p(T) =$

$$1 - \lambda(p)T + (\lambda(p)^2 - \lambda(p^2) - \chi_2(p)p^{2k-4})T^2 - \chi_2(p)p^{2k-3}\lambda(p)T^3 + \chi_2(p^2)p^{4k-6}T^4.$$

The Andrianov L-function is defined by the Euler product

$$L(f,s) = \prod_{p\nmid q} Q_p(p^{-s})^{-1}.$$

To compute these L-functions in specific examples, one can write modular forms for $\Gamma_g(q)$ as Fouries series and use the action of the Hecke operators on the Fourier coefficients. The only case we are interested in is $g = 2$ and the only Hecke operators we need there are $T_k(p)$ and $T_k(p^2)$. For this case, explicit formulas for the action on Fourier coefficients will be given. These are special cases of [4], Formula 3.7, p. 440.

By the so-called 'Koecher-Prinzip', a modular form $f \in M_k(\Gamma_g(q))$ can be written as a Fourier series

$$f(\tau) = \sum a_N \exp\left(\frac{2\pi i}{q}\operatorname{trace}(N\tau)\right).$$

This summation is taken over the set of all $g \times g$ matrices $N = (n_{i,j})$ which are symmetric, positive semi-definite and semi-integral, i.e. $n_{i,i} \in \mathbf{Z}$ and $2n_{i,j} \in \mathbf{Z}$. In case $g = 2$, we will write these Fourier coefficients as

$$a_N = a(n,r,m) \text{ for } N = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}.$$

The Fourier expansion of the image $T_k(m)f$ of the modular form f is denoted

$$T_k(m)\left(\sum a_N \exp\frac{2\pi i}{q}\operatorname{trace}(N\tau)\right) = \sum a(m;N)\exp\frac{2\pi i}{q}\operatorname{trace}(N\tau).$$

For $g = 2$, the Fourier coefficients of the image under $T_k(p^e)$ are written as $a(p^e;n,r,m)$.

Let $f \in M_k(\Gamma_2(q),\chi_1,\chi_2)$ be given, with Fourier coefficients $a(n,r,m)$. For prime numbers p which do not divide q, formulas for $a(p^e;n,r,m)$ in terms of Fourier coefficients of f can be found in Evdokimov's paper [4], p. 440; see also [20], Appendix A for a more explicit version of it. There is one very simple special case: if p and n,r,m are such that the polynomial nX^2+rX+m is irreducible and of degree 2 in $\mathbf{F}_p[X]$, then

$$a(p^i;n,r,m) = a(p^i n, p^i r, p^i m)$$

for all exponents i.

4. Theta functions. Let $\mathbf{m} = (m', m'') \in \mathbf{R}^{2g} = \mathbf{R}^g \oplus \mathbf{R}^g$ be a row vector. The theta function corresponding to the characteristic $\mathbf{m}$ is the function $\theta_{\mathbf{m}}$:

$\mathbf{H}_g \times \mathbf{C}^g \to \mathbf{C}$ defined by

$$\theta_{\mathbf{m}}(\tau, z) = \sum_{a \in \mathbf{Z}^g} \exp \pi i \left\{ (a + m')\tau^t(a + m') + 2(a + m')^t(z + m'') \right\}.$$

A theta constant is a function $\tau \mapsto \theta_{\mathbf{m}}(\tau, 0)$, for a characteristic $\mathbf{m} \in \mathbf{Q}^{2g}$. This function will be written as $\theta_{\mathbf{m}}(\tau)$. Our aim is to find Siegel modular forms which are products of theta constants. In this section the properties of theta constants we will need are given. The theta constant as defined here is the same as in Igusa's book [13], p. V and p. 49. In [6], p. 40 the function $\vartheta(\tau; 2m', 2m'')$ differs from this one by a factor $\exp \pi i (m''^t m''/2)$.

The following two properties are direct consequences of the definition.

$$\theta_{\mathbf{m}+\mathbf{a}}(\tau) = \exp 2\pi i m''^t a'' \cdot \theta_{\mathbf{m}}(\tau) \text{ for } \mathbf{a} \in \mathbf{Z}^g$$

and

$$\theta_{1-\mathbf{m}}(\tau) = \exp(-2\pi i m''^t 1)\theta_{\mathbf{m}}(\tau).$$

Here 1 denotes the row vector $(1, \ldots, 1)$.

From the first formula one deduces that the entries e occurring in a characteristic $\mathbf{m}$ may be chosen such that $0 \leq e < 1$. The second one diminishes the number of possible different theta constants by another factor 2.

The 'transformation formula' for the theta function yields the behavior of $\theta_{\mathbf{m}}(\tau)$ under the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$. It reads ([13], Corollary, p. 176 and Theorem 3, p. 182):

For $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{Z})$ and $\mathbf{m} \in \mathbf{R}^{2g}$, put

$$\sigma \cdot \mathbf{m} = \mathbf{m}\sigma^{-1} + \frac{1}{2}(\mathrm{diag}(\gamma^t\delta)\ \mathrm{diag}(\alpha^t\beta))$$

and

$$\begin{aligned} \phi_{\mathbf{m}}(\sigma) &= -\frac{1}{2}(m''^t\delta\beta^t m' - 2m''^t\beta\gamma^t m'' + m'''^t\gamma\alpha^t m'') \\ &\quad + \frac{1}{2}(m''^t\delta - m'''^t\gamma)^t(\mathrm{diag}(\alpha^t\beta)). \end{aligned}$$

Then

$$\theta_{\sigma\cdot\mathbf{m}}((\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}) = \kappa(\sigma)\exp(2\pi i \phi_{\mathbf{m}}(\sigma))\sqrt{\det(\gamma\tau + \delta)}\theta_{\mathbf{m}}(\tau),$$

in which $\kappa(\sigma)$ is a complex number of absolute value 1 which depends only on σ and the choice of the square root. In particular, it does not depend on the characteristic $\mathbf{m}$. Thus $\kappa(\sigma)^2$ is well defined; one has

$$\kappa(\sigma)^2 = (-1)^{\operatorname{trace}(\delta-1)/2}$$

if $\sigma \in \Gamma_g(2)$. This implies $\kappa(\sigma)^4 = 1$ whenever $\sigma \in \Gamma_2(2)$.

From now on we work with $g = 2$ and write Γ instead of Γ_2.

LEMMA 4.1. *Let k, n be positive integers and $\mathbf{m}_i \in \frac{1}{2n}\mathbf{Z}^4$; $i = 1, \dots, 2k$. The product*

$$f(\tau) = \theta_{\mathbf{m}_1}(\tau)\theta_{\mathbf{m}_2}(\tau) \cdot \dots \cdot \theta_{\mathbf{m}_{2k}}(\tau)$$

is a modular form of weight k for $\Gamma(4n^2, 8n^2)$. The group $\Gamma(2n)$ acts linearly on the space $\mathbf{C} \cdot f$.

In case $k = 2$, the action of the generators $e_1(n^2), \dots, e_{10}(n^2)$ of the group $\Gamma(2n^2)/\Gamma(4n^2, 8n^2)$ (notation from Lemma 2.2) is given in the following table. The characteristics are written as $\mathbf{m}_j = (a_j, b_j, c_j, d_j)$; the b, c are coefficients of e_6 as defined in Lemma 2.2.

e_1	$\exp(4\pi i n^2 \sum b_j c_j)$
e_2	$\exp(4\pi i n^2 \sum a_j d_j)$
e_3	$\exp(4\pi i n^2 \sum a_j b_j)$
e_4	$\exp(4\pi i n^2 \sum c_j d_j)$
e_5	$\exp(4\pi i n^2 \sum (a_j c_j + b_j d_j))$
e_6	$\exp(\pi i \sum (4n^2 a_j c_j + a_j^2 b + c_j^2 c))$
e_7	$\exp(2\pi i n^2 \sum a_j^2)$
e_8	$\exp(2\pi i n^2 \sum b_j^2)$
e_9	$\exp(2\pi i n^2 \sum c_j^2)$
e_{10}	$\exp(2\pi i n^2 \sum d_j^2)$

Proof. Let k, n be positive integers and $\mathbf{m} \in \frac{1}{2n}\mathbf{Z}^4$ a theta characteristic. For all $\sigma \in \mathrm{Sp}_4(\mathbf{Z})$, one checks that $\sigma \cdot \mathbf{m} \in \frac{1}{2n}\mathbf{Z}^4$ as well. (Note that this is not true for $\frac{1}{2n-1}\mathbf{Z}^4$.) Moreover, a trivial computation shows that if $\sigma \in \Gamma(2n)$, then $\sigma \cdot \mathbf{m} - \mathbf{m} \in \mathbf{Z}^4$. Take $f = \prod_{i=1}^{2k} \theta_{\mathbf{m}_i}$, for theta characteristics $\mathbf{m}_i \in \frac{1}{2n}\mathbf{Z}^4$. It follows that $f|_k\sigma = \chi(\sigma)f$ for $\sigma \in \Gamma(2n)$, where χ is a homomorphism: $\Gamma(2n) \to \mathbf{C}^*$. Since χ is trivial on the commutator subgroup of $\Gamma(2n)$, i.e. on $\Gamma(4n^2, 8n^2)$, the conclusion is that such products are modular forms of weight k for $\Gamma(4n^2, 8n^2)$.

Concerning the action of the quotient $\Gamma(2n^2)/\Gamma(4n^2, 8n^2)$ on products of four theta constants with characteristics in $\frac{1}{2n}\mathbf{Z}^4$, we know generators of such a quotient by Lemma 2.2. The transformation formula for the theta function is given above. Hence the remainder of the proof is just a straightforward calculation. $\square$

We will now consider the Siegel operator and investigate whether such products of theta constants can be cusp forms. The basic property one needs for that, is a precise criterion when a function $\theta_{\mathbf{m}}(\tau)$ is nontrivial. This is provided by [13], Theorem 1, p. 174. It turns out that a theta constant $\theta_{\mathbf{m}}(\tau)$ with characteristic $\mathbf{m} = (m', m'')$ is identically zero if and only if $2\mathbf{m} \in \mathbf{Z}^{2g}$ and $2m'\,{}^t\!m'' \notin \mathbf{Z}$.

If we want to find cusp forms, we have to consider all possible limits of the form

$$\lim_{\lambda\to\infty} f|_k\sigma\left(\begin{pmatrix} \tau & \\ & i\lambda \end{pmatrix}\right),$$

where σ runs over the group $\mathrm{Sp}_4(\mathbf{Z})$. Using the transformation formula, this comes down to the action of the Siegel operator on products of arbitrary theta constants. For that, one only needs the following easily computable limit ([6], Bemerkung 3.10, p. 46).

$$\lim_{\lambda\to\infty} \theta_{(a,b,c,d)}\begin{pmatrix} \tau & \\ & i\lambda \end{pmatrix} = \theta_{(a,c)}(\tau)\cdot\begin{cases} 0 & \text{if } b \notin \mathbf{Z}; \\ 1 & \text{if } b \in \mathbf{Z}. \end{cases}$$

We will return to the problem whether or not certain products of theta constants are cusp forms in the next section.

5. All cusp forms of weight 2 for $\Gamma_2(4,8)$. In this section we will compute explicit generators of the space of cusp forms of weight 2 for $\Gamma_2(4,8)$. It will turn out that these generators are in fact common eigenvectors of all Hecke operators $T_2(m)$ for odd m. Moreover, they are counterexamples to the so-called 'generalized Ramanujan conjecture'. Our first object is to prove:

THEOREM 5.1. *The space $M_2(\Gamma_2(4,8))$ of modular forms of weight 2 for $\Gamma_2(4,8)$ has dimension 695. The subspace $S_2(\Gamma_2(4,8))$ of cusp forms is 15-dimensional; a basis of it is given in the following table.*

$$\begin{array}{ll}
\theta_{(0,0,0,0)}\theta_{(0,0,\frac12,0)}\theta_{(\frac12,0,0,0)}\theta_{(\frac12,\frac12,\frac12,\frac12)} & \theta_{(0,0,0,0)}\theta_{(0,0,\frac12,0)}\theta_{(\frac12,\frac12,0,0)}\theta_{(\frac12,0,0,\frac12)} \\
\theta_{(0,0,0,0)}\theta_{(0,0,0,\frac12)}\theta_{(0,\frac12,0,0)}\theta_{(\frac12,\frac12,\frac12,\frac12)} & \theta_{(0,0,0,0)}\theta_{(0,0,0,\frac12)}\theta_{(\frac12,\frac12,0,0)}\theta_{(0,\frac12,\frac12,0)} \\
\theta_{(0,0,0,0)}\theta_{(0,0,\frac12,\frac12)}\theta_{(\frac12,0,0,0)}\theta_{(0,\frac12,\frac12,0)} & \theta_{(0,0,0,0)}\theta_{(0,0,\frac12,\frac12)}\theta_{(0,\frac12,0,0)}\theta_{(\frac12,0,0,\frac12)} \\
\theta_{(0,0,\frac12,0)}\theta_{(0,0,0,\frac12)}\theta_{(\frac12,0,0,0)}\theta_{(0,\frac12,0,0)} & \theta_{(0,0,\frac12,0)}\theta_{(0,0,0,\frac12)}\theta_{(\frac12,0,0,\frac12)}\theta_{(0,\frac12,\frac12,0)} \\
\theta_{(0,0,\frac12,0)}\theta_{(0,0,\frac12,\frac12)}\theta_{(0,\frac12,0,0)}\theta_{(\frac12,\frac12,0,0)} & \theta_{(0,0,\frac12,0)}\theta_{(0,0,\frac12,\frac12)}\theta_{(0,\frac12,\frac12,0)}\theta_{(\frac12,\frac12,\frac12,\frac12)} \\
\theta_{(0,0,0,\frac12)}\theta_{(0,0,\frac12,\frac12)}\theta_{(\frac12,0,0,0)}\theta_{(\frac12,\frac12,0,0)} & \theta_{(0,0,0,\frac12)}\theta_{(0,0,\frac12,\frac12)}\theta_{(\frac12,0,0,\frac12)}\theta_{(\frac12,\frac12,\frac12,\frac12)} \\
\theta_{(\frac12,0,0,0)}\theta_{(0,\frac12,0,0)}\theta_{(\frac12,0,0,\frac12)}\theta_{(0,\frac12,\frac12,0)} & \theta_{(\frac12,0,0,0)}\theta_{(\frac12,\frac12,0,0)}\theta_{(\frac12,0,0,\frac12)}\theta_{(\frac12,\frac12,\frac12,\frac12)} \\
\theta_{(0,\frac12,0,0)}\theta_{(\frac12,\frac12,0,0)}\theta_{(0,\frac12,\frac12,0)}\theta_{(\frac12,\frac12,\frac12,\frac12)} &
\end{array}$$

These 15 forms are eigenforms of the Hecke operators $T_2(m)$ for odd m.

The main ingredient in the proof is a result of Igusa, which states that all modular forms for $\Gamma_2(4,8)$ are products of an even number of theta constants $\theta_{\mathbf{m}_j}$, for $\mathbf{m}_j \in \frac{1}{2}\mathbf{Z}^4$ (see [12], Theorem 1, p. 396). This provides us with generators. To find all relations, cusp forms and eigenspaces for the Hecke operators, we decompose the space $M_2(\Gamma_2(4,8))$ into character spaces for the action of $\Gamma_2(2)$. By Lemma 3.1, we only have to look for Hecke eigenfunctions inside these character spaces. The same is true for cusp forms; more precisely,

LEMMA 5.2. *Suppose Γ is a normal subgroup of finite index in $\Gamma' \subset Sp_{2g}(\mathbf{Z})$. Let*

$$M_k(\Gamma) = \sum_{\chi} M_k(\Gamma', \chi)$$

denote the decomposition of $M_k(\Gamma)$ into character spaces for the action of Γ'. If $f = \sum f_\chi \in M_k(\Gamma)$ is a cusp form, then each f_χ is a cusp form.

Proof. This is an immediate consequence of the well-known fact that each f_χ equals upto a scalar $\sum \chi(\sigma) f|_k \sigma$, the summation taken over the quotient Γ'/Γ.□

Write Γ instead of Γ_2. The proof of Theorem 5.1 uses of course the decomposition of the space of forms $\theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4}$ into character spaces for the action of $\Gamma(2)/\Gamma(4,8)$. This is easily done using Lemma 4.1. As a byproduct we will obtain

PROPOSITION 5.3. *All modular forms of weight and genus 2 and level 4 are described as follows.*

$$M_2(\Gamma(4)) = \left\langle \theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4}; \sum \mathbf{m}_j \equiv (0,0,0,0) \bmod \mathbf{Z}^4 \right\rangle,$$

$$M_2(\Gamma(2,4)) = M_2(\Gamma(1,2,4)) = \left\langle \theta^2_{\mathbf{m}_1}\theta^2_{\mathbf{m}_2} \right\rangle$$

and

$$M_2(\Gamma(2)) = \left\langle \theta^4_{\mathbf{m}} \right\rangle.$$

One has $\dim M_2(\Gamma(2)) = 5$, $\dim M_2(\Gamma(2,4)) = 35$ *and* $\dim M_2(\Gamma(4)) = 50$. *None of these spaces contains a nonzero cusp form.*

Concerning the Andrianov L-functions corresponding to the 15 cusp forms in $M_2(\Gamma(4,8))$, we have the following

PROPOSITION 5.4. *Denote by $\lambda(m)$ the eigenvalue at $T(m)$ of some common eigenfunction of all Hecke operators. The 15 cusp forms in $M_2(\Gamma(4,8))$ yield* *for all primes* $p \equiv 3 \bmod 4$ *eigenvalues* $\lambda(p) = 0$ *and* $\lambda(p^2) = p^2$. *It follows that these cusp forms provide a counterexample to the generalized Ramanujan conjecture.*

For primes $p \equiv 1 \bmod 4$, *one* $\lambda(p) = 2p + 2$ *and* $\lambda(p^2) = 3p^2 + 4p + 2$.

An immediate consequence of this is

THEOREM 5.5. *The 15 products of theta constants which are cusp forms of weight 2 for* $\Gamma(4, 8)$ *all have Andrianov L-function*

$$L(f, s) = \zeta_{\mathbf{Q}(i)}(s)\zeta_{\mathbf{Q}(i)}(s-1)$$

where $\zeta_{\mathbf{Q}(i)}(s)$ *is the Dedekind* ζ*-function of the field* $\mathbf{Q}(i)$.

Proof (of Theorem 5.1 and Proposition 5.3). First one decomposes the space of theta constants $\theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4}$ into character spaces for the action of $\Gamma(4)/\Gamma(4, 8)$. The 15 spaces attached to nontrivial characters of $\Gamma(4)/\Gamma(4, 8)$ are conjugate under the action of $\mathrm{Sp}_4(\mathbf{Z})$. One of them, corresponding to χ defined by $\chi(e_7(2)) = -1$ and $\chi(e_j(2)) = 1$ for $j = 8, 9, 10$, is

$$M_2(\Gamma(4), \chi) = \left\langle \theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4}; \sum \mathbf{m}_j \equiv \left(\frac{1}{2}, 0, 0, 0\right) \bmod \mathbf{Z}^4 \right\rangle.$$

The trivial character yields the description of $M_2(\Gamma(4))$ given in Proposition 5.3.

Using the same method this space $M_2(\Gamma(4))$ will be treated first. The action of $\Gamma(2, 4)$ yields $M_2(\Gamma(2, 4))$, plus 15 more forms on which $\Gamma(2, 4)/\Gamma(4)$ acts by nontrivial characters. One of these corresponds to the character given in the 7th line of the table in Proposition 2.3 for $n = 1$, namely $\theta_{(0,0,0,0)}\theta_{(0,0,0,\frac{1}{2})}\theta_{(0,0,\frac{1}{2},0)}\theta_{(0,0,\frac{1}{2},\frac{1}{2})}$. Since the orbit of that character has length 15, the 15 forms we have are in different 1-dimensional character spaces for the action of $\Gamma(2)/\Gamma(4)$. In particular, they are linearly independent. The form given above is obviously not a cusp form, hence none of these 15 is.

With the space $M_2(\Gamma(2, 4))$ one proceeds similarly. The result is the description of $M_2(\Gamma(2))$ provided by Proposition 5.3, and again one finds 15 conjugate spaces attached to the nontrivial characters of $\Gamma(2)/\Gamma(2, 4)$. One of these is generated by the 3 forms $\theta^2_{(0,0,0,0)}\theta^2_{(\frac{1}{2},0,0,0)}$, $\theta^2_{(0,0,0,\frac{1}{2})}\theta^2_{(\frac{1}{2},0,0,\frac{1}{2})}$ and $\theta^2_{(\frac{1}{2},\frac{1}{2},0,0)}\theta^2_{(0,\frac{1}{2},0,0)}$. The Riemann theta formula [13], Theorem 1, p. 137 provides the relation

$$\theta^2_{(0,0,0,0)}\theta^2_{(\frac{1}{2},0,0,0)} = \theta^2_{(0,0,0,\frac{1}{2})}\theta^2_{(\frac{1}{2},0,0,\frac{1}{2})} + \theta^2_{(\frac{1}{2},\frac{1}{2},0,0)}\theta^2_{(0,\frac{1}{2},0,0)}.$$

If a nontrivial linear combination of these forms would be a cusp form, then applying the Siegel operator yields that $\theta^2_{(\frac{1}{2},\frac{1}{2},0,0)}\theta^2_{(0,\frac{1}{2},0,0)}$ is also a cusp form. This is not the case, as can e.g. be seen by applying the matrix

$$\begin{pmatrix} & -1_2 \\ 1_2 & \end{pmatrix}.$$

We also conclude that such a space is 2-dimensional (compare [8], Proposition 4.15, p. 347).

For the space $M_2(\Gamma(2))$ one has 10 generators. The Riemann theta formula provides 5 relations. As above, one can check that 5 remaining forms are independent (see also [10], pp. 336–337 or [8], Corollary 4.10, p. 345) and that there are no cusp forms in this space.

The question whether a product $F = \prod \theta_{\mathbf{m}_i}$ with $\mathbf{m}_i \in 1/2\,\mathbf{Z}^4$ defines a cusp form for $\Gamma(4,8)$, can be answered as follows. The function F^4 is a modular form for $\Gamma(2)$, and F is a cusp form if and only if F^4 is a cusp form. This is the same as saying that F^4 vanishes on all 15 1-dimensional boundary components of a certain compactification of $\mathbf{H}_2/\Gamma(2)$. By [10], Proposition 1.1, p. 323 these boundary components correspond to the 15 nonzero vectors in $\mathbf{F}_2^4$. Under this bijection, the action of $\mathrm{Sp}_4(\mathbf{Z})/\Gamma(2)$ on the set of 1-dimensional boundary components corresponds to the linear action of $\mathrm{Sp}_4(\mathbf{F}_2)$ on $\mathbf{F}_2^4$.

Let $\mathbf{m} = (m', m'') \in \frac{1}{2}\mathbf{Z}^4$ be an even theta characteristic. To $\mathbf{m}$ we associate a 'quadratic form' $Q_{\mathbf{m}} : \mathbf{F}_2^4 \to \mathbf{F}_2$ defined by

$$Q_{\mathbf{m}}(a', a'') = (a' + 2m')\,{}^t(a'' + 2m'') \bmod 2.$$

The group $\mathrm{Sp}_4(\mathbf{F}_2)$ acts on the set of $Q_{\mathbf{m}}$'s by

$$\sigma \cdot Q_{\mathbf{m}} = Q_{\mathbf{m}} \circ \sigma^{-1}.$$

It turns out that $\sigma \cdot Q_{\mathbf{m}} = Q_{\sigma\cdot\mathbf{m}}$. By checking only the boundary component $(0,0,0,1)$ which corresponds to the usual Siegel operator, and using the actions described above plus the remarks made at the end of Section 4 one finds

PROPOSITION 5.6. *With the notations introduced above, a modular form $\theta^4_{\mathbf{m}}$ with even characteristic $\mathbf{m} \in \frac{1}{2}\mathbf{Z}^4$ vanishes at the boundary component corresponding to $a \in \mathbf{F}_2^4$ if and only if $Q_{\mathbf{m}}(a) = 1$.*

Using this criterion, one computes the table given below. Vectors in $\mathbf{F}_2^4$ are written as column vectors. A minus sign means that the characteristic given in the same row yields a theta function which vanishes at the 1-dimensional boundary *component given in the column. A plus sign means it does not vanish on the corresponding component.*

	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$
$(0,\frac{1}{2},0,0)$	−	+	−	+	+	+	+	−	+	+	+	−	−	+	−
$(0,\frac{1}{2},\frac{1}{2},0)$	−	+	−	+	−	−	+	+	+	−	+	+	+	−	+
$(\frac{1}{2},0,0,0)$	+	−	−	+	+	+	−	+	−	−	+	+	+	+	−
$(\frac{1}{2},0,0,\frac{1}{2})$	+	−	−	−	+	−	+	+	+	+	−	+	−	+	+
$(\frac{1}{2},\frac{1}{2},0,0)$	−	−	+	+	+	+	−	−	+	+	−	+	+	−	+
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	−	−	+	−	−	+	+	+	−	+	+	−	+	+	+
$(0,0,0,0)$	+	+	+	+	+	+	+	+	−	−	−	−	−	−	+
$(0,0,0,\frac{1}{2})$	+	+	+	−	+	−	−	+	+	+	+	−	+	−	−
$(0,0,\frac{1}{2},0)$	+	+	+	+	−	−	+	−	−	+	−	+	+	+	−
$(0,0,\frac{1}{2},\frac{1}{2})$	+	+	+	−	−	+	−	−	+	−	+	+	−	+	+

We continue with the proof of Theorem 5.1. The only space that remains to be investigated is $M_2(\Gamma(4),\chi)$, for the nontrivial character χ on $\Gamma(4)/\Gamma(4,8)$ introduced above. It is straightforward to check that this space is spanned by 43 forms. One finds 30 forms of the type $\theta^2_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}$ and 13 products of 4 pairwise unequal theta constants. It turns out that the action of $\Gamma(2)/\Gamma(4,8)$ completely decomposes this space into 1-dimensional spaces. In 42 cases one easily finds using the table above a σ such that $*|_2\sigma$ is not in the kernel of the Siegel operator. The 43d case is the form

$$\theta_{(0,0,\frac{1}{2},0)}\theta_{(0,0,\frac{1}{2},\frac{1}{2})}\theta_{(0,\frac{1}{2},\frac{1}{2},0)}\theta_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}$$

which occurs in the table given in Theorem 5.1. One checks that all 15 forms in the orbit of this one under $\mathrm{Sp}_4(\mathbf{Z})$ are in the kernel of the Siegel operator. Hence these are cusp forms. Of course this also follows using the criterion provided by Proposition 5.6.

Denote by χ_1 the character on $\Gamma(2)$ determined by the cusp form above. It is easy to check that the space corresponding to the complex conjugate character is generated by $\theta^2_{(\frac{1}{2},\frac{1}{2},0,0)}\theta_{(\frac{1}{2},0,0,\frac{1}{2})}\theta_{(0,0,0,\frac{1}{2})}$. Since this form is not a cusp form, it follows using Lemma 3.1, which asserts in our situation that the space $M_2(\Gamma(2),\chi_1)\oplus M_2(\Gamma(2),\overline{\chi_1})$ is stable under the action of the Hecke operators, that the cusp form above is an eigenform of all the Hecke operators. Moreover it is mapped to 0 by all Hecke operators $T(m)$ for $m\equiv 3 \bmod 4$. This completes the proof of Theorem 5.1. □

In order to say something about the Andrianov L-functions attached to the cusp forms we found, an alternative way to describe these forms can be used. This was pointed out by R. Weissauer.

Proof (of Proposition 5.4). Take a couple $\mathbf{m},\mathbf{n}$ of odd theta characteristics, i.e. such that $\theta_{\mathbf{m}}(\tau) = \theta_{\mathbf{n}}(\tau) = 0$. Consider the function

$$\Theta_{\mathbf{m},\mathbf{n}}(\tau) = \frac{1}{\pi^2} \det \left(\left| \begin{matrix} \frac{\partial}{\partial z_1}\theta_{\mathbf{m}}(\tau,z) & \frac{\partial}{\partial z_1}\theta_{\mathbf{n}}(\tau,z) \\ \frac{\partial}{\partial z_2}\theta_{\mathbf{m}}(\tau,z) & \frac{\partial}{\partial z_2}\theta_{\mathbf{n}}(\tau,z) \end{matrix} \right) \right|_{z=(0,0)}.$$

It is a classical result ([21], p. 182, formula (9) or [7], p. 246) that these are integral multiples of the forms given in Theorem 5.1 (see also [14], Theorem 3, p. 439). For any pair of odd theta characteristics $\mathbf{m} = (m_1, m_2, m_3, m_4)$ and $\mathbf{n} = (n_1, n_2, n_3, n_4)$ write

$$M' = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}$$

and

$$M'' = \begin{pmatrix} m_3 & n_3 \\ m_4 & n_4 \end{pmatrix}.$$

From the definition one easily computes

$$\Theta_{\mathbf{m},\mathbf{n}}(\tau) = -4 \sum_{X \in M_2(\mathbf{Z})} (-1)^{2\mathrm{trace}({}^tM''X)} \det(X+M') \exp \pi i(\mathrm{trace}({}^t(X+M')\tau(X+M'))).$$

This implies that upto a constant the Fourier coefficients a_N of such a form are given by

$$a_N = \sum_Y (-1)^{\mathrm{trace}({}^tM''Y)} \det(Y),$$

where one sums over all matrices $Y \in M_2(\mathbf{Z})$ satisfying $Y \equiv (2M') \bmod 2$ and $Y\,{}^tY = N$.

Using this description one can compute the action of most of the generators of $\Gamma(2)/\Gamma(4,8)$ on these forms directly. Of course, one could also do that using Weber's result [21] mentioned above. As a result one finds that a $\Theta_{\mathbf{m},\mathbf{n}}$ corresponds to that $\theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4}$ listed in Theorem 5.1 for which

$$\mathbf{m} + \mathbf{n} \equiv \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4 \bmod \mathbf{Z}^4.$$

We will use the characteristics $\mathbf{m} = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $\mathbf{n} = (0, \frac{1}{2}, 0, \frac{1}{2})$. In that case, the eigenform to be studied is

$$D(\tau) = \frac{1}{4} \sum_{X \in M_2(\mathbf{Z})} (-1)^{\mathrm{trace}(X)} \det(2X+1) \exp\left(\frac{2\pi i}{8}\mathrm{trace}({}^t(2X+1)\tau(2X+1))\right).$$

The Fourier coefficients $a(n,r,m)$ of $D(\tau)$ are given by the formula

$$a(n,r,m) = \frac{1}{4}\sum_A (-1)^{\mathrm{trace}(A-1)/2}\det(A),$$

where the sum is taken over all integral 2×2 matrices A which are congruent to 1 modulo 2 and satisfy

$$A\,{}^t\!A = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}.$$

It is immediate that $a(1,0,1) = 1$. Hence $\lambda(p^i) = a(p^i;1,0,1)$ which we will compute now for $i = 1,2$ using Evdokimov's formulas.

For primes $p \equiv 3 \bmod 4$ one has $\lambda(p^e) = a(p^e,0,p^e)$. A straightforward calculation reveals that $a(p,0,p) = 0$ and $a(p^2,0,p^2) = p^2$ in this case. It follows that the corresponding Euler factors come from the polynomials $Q_p(t) = (1-t^2)(1-p^2t^2)$. Obviously the zeroes of these $Q_p(t)$ do not have the same absolute value, hence we find a counterexample to the 'generalized Ramanujan conjecture'. This proves the first part of Proposition 5.4.

From now on we can assume p is a prime number congruent to 1 modulo 4. Let $0 < j < p/2$ be the unique integer such that $1 + 8^2j^2 \equiv 0 \bmod p$. Then

$$\lambda(p) = a(p;1,0,1) = a(p,0,p) + a\left(\frac{1+8^2j^2}{p}, 16j, p\right) + a\left(\frac{1+8^2j^2}{p}, -16j, p\right).$$

As mentioned above, computing $a(p,0,p)$ is the same as computing all integral matrices A which are $\equiv 1 \bmod 2$ and satisfy

$$A\,{}^t\!A = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.$$

Since p can be written in a unique way as the sum of two squares, one finds 8 such matrices A. It easily follows that $a(p,0,p) = 2p$.

Before computing the other contributions to $\lambda(p)$, note that $a(n,r,m) = a(n,-r,m)$ for this form $D(\tau)$. This follows from the fact that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

yields a contribution to the one then

$$B = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$$

yields a contribution to the other. One checks that $A \equiv 1 \bmod 2$ implies

$$(-1)^{\mathrm{trace}(A-1)/2}\det(A) = (-1)^{\mathrm{trace}(B-1)/2}\det(B).$$

An alternative way to see that $a(n,r,m) = a(n,-r,m)$ in our case is provided by the remark that the diagonal matrix with entries $(1,-1,1,-1)$, regarded as element of $\Gamma(2)$ acts trivially on our form. The matrices A which contribute to $a((1+64j^2)/p, 16j, p)$ are of the form

$$\begin{pmatrix} c & d \\ b & a \end{pmatrix}$$

with a odd and $a^2+b^2 = p$. Since A^tA has determinant 1 it follows that $ac-bd = \pm 1$. Furthermore one has the relation $bc+ad = 8j$. By Cramer's rule, for each choice of the sign of $\det(A)$ this system of linear equations has exactly one solution (c,d) in $\mathbf{Q}\times\mathbf{Q}$. If we consider the same system over $\mathbf{F}_p$, it follows that for precisely one choice of $\det(A)$ the solution (c,d) is in $\mathbf{Z}\times\mathbf{Z}$. Moreover, for 2 of the choices for (a,b) this happens when $\det(A) = 1$ and for the other 2 when $\det(A) = -1$. Using this it is not hard to verify that $(-1)^{\mathrm{trace}(A-1)/2}\det(A) = 1$ for each of the 4 possible A's. It follows that $a((1+64j^2)/p, 16j, p) = 1$, hence $\lambda(p) = 2p+2$.

Let j be as above and take $0 < \ell < p^2/2$ to be the unique integer such that $1+64\ell^2 \equiv 0 \bmod p^2$. Again using Evdokimov's formulas, plus the fact that $a(n,r,m) = a(n,-r,m)$ one finds

$$\lambda(p^2) = a(p^2,0,p^2) + 2a(1+64j^2, 16pj, p^2) + 2a((1+64\ell^2)/p^2, 16\ell, p^2).$$

There are 12 matrices which contribute to $a(p^2,0,p^2)$, namely the 4 matrices

$$\begin{pmatrix} \pm p & 0 \\ 0 & \pm p \end{pmatrix},$$

and the 8 matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

where $a \equiv 1 \bmod 2$ and b are nonzero integers satisfying $a^2+b^2 = p^2$. One finds $a(p^2,0,p^2) = 3p^2$.

The matrices contributing to $a(1+64j^2, 16jp, p^2)$ have determinant $\pm p$, and second row (b, a) with a odd and $a^2+b^2 = p^2$. One finds 4 possibilities with $b=0$, having first row of the form $(\pm 1, \pm 8j)$. Using an argument similar to the one given for $a((1+64j^2)/p, 16j, p)$ above one finds 4 possibilities with $b \neq 0$ as well. It follows that $a(1+64j^2, 16jp, p^2) = 2p$.

The remaining coefficient to compute is $a((1+64\ell^2)/p^2, 16\ell, p^2)$. Again the second row of a matrix contributing to it is of the form (b, a) with a odd and $a^2+b^2 = p^2$. In this case one verifies that $b \neq 0$, and as before there are 4 matrices. It turns out that $a((1+64\ell^2)/p^2, 16\ell, p^2) = 1$.

Adding all the contributions it follows that $\lambda(p^2) = 3p^2+4p+2$.

To finish the proof one could go through a similar calculation for the other 14 eigenforms. Alternatively one may use the fact that the forms are in the same orbit for the action of $\Gamma(1)$, plus the transformation formula, to show that the L-functions at worst differ by a character modulo 8. Using a machine it is easy to compute for small primes in each congruence class modulo 8 the eigenvalues for all the 15 forms. It turns out they are all equal, hence the L-functions are all the same. This finishes the proof of Proposition 5.4.

Remark. It is interesting to consider other eigenforms of the Hecke operators as well. For instance, the product

$$\theta_{(0,0,0,0)}\theta_{(\frac{1}{2},0,0,\frac{1}{2})}\theta_{(0,0,\frac{1}{2},0)}\theta_{(0,0,\frac{1}{2},\frac{1}{2})}$$

defines an eigenform which is not a cusp form. Computing the polynomials $Q_p(t)$ for small prime numbers p one obtains the squares of the polynomials occurring in the L-function of the elliptic curve $E/\mathbf{Q}$ given by $y^2 = x^3 - x$. Hence it seems that the Andrianov L-function is just $L(E \times E, s)$.

It turns out to be fairly easy to prove this phenomenon. Applying the Siegel operator, we find the cusp form (genus 1)

$$f(\tau) = \theta_{(0,0)}\theta_{(\frac{1}{2},0)}\theta^2_{(0,\frac{1}{2})}.$$

From the commutation rules between the Siegel operator and the Hecke operators one deduces that the Andrianov L-function we want to compute is the square of the L-function attached to this form ([24], Theorem 1, p. 115; compare [5], Theorem 4.1, p. 315 and [6], Kapitel IV, §4). Write $g(\tau) = f(8\tau)$. It is not hard to check that g defines a cusp form of weight 2 for $\Gamma_0(32)$. The genus of the corresponding modular curve $X_0(32)$ is 1, hence the space of cusp forms of weight 2 for $\Gamma_0(32)$ has dimension 1. A generator is $\eta(4\tau)^2\eta(8\tau)^2$ (with η the classical η-function). One has

$$\theta_{(0,0)}(8\tau)\theta_{(\frac{1}{2},0)}(8\tau)\theta^2_{(0,\frac{1}{2})}(8\tau) = 2\eta(4\tau)^2\eta(8\tau)^2.$$

6. Some cusp forms of weight 2 for $\Gamma_2(8)$. The next case to be considered is $M_2(\Gamma_2(8))$. Here one does not have an explicit description of all modular forms. Analogous to the situation above we will study subspaces (for $\Gamma \supset \Gamma_2(8)$)

$$\mathrm{Th}_2(\Gamma, \chi) = \left\langle \prod_{i=1}^{4} \theta_{\mathbf{m}_i} \in M_2(\Gamma, \chi);\ \mathbf{m}_i \in \frac{1}{4}\mathbf{Z}^4 \right\rangle.$$

Here χ is any character of Γ which is trivial on $\Gamma_2(8)$. It follows from a result of Igusa ([13], p. 224) that the quotient field of the ring of modular forms for $\Gamma_2(8)$ is the same as the quotient field of $\mathrm{Th}_*(\Gamma_2(8))$. However, the spaces $M_2(\Gamma_2(8))$ and $\mathrm{Th}_2(\Gamma_2(8))$ are not the same. We will see this by looking at certain products $\theta_{\mathbf{m}_1}(\tau)\theta_{\mathbf{m}_2}(\tau)\theta_{\mathbf{m}_3}(\tau)\theta_{\mathbf{m}_4}(2\tau)$, with $\mathbf{m}_i \in \frac{1}{2}\mathbf{Z}^4$. The first result needed is a description of $\mathrm{Th}_2(\Gamma_2(8))$. This is provided by

PROPOSITION 6.1. *Let $n > 0$ be an integer. A product*

$$\theta_{\mathbf{m}_1}(\tau)\theta_{\mathbf{m}_2}(\tau)\theta_{\mathbf{m}_3}(\tau)\theta_{\mathbf{m}_4}(\tau),$$

with $\mathbf{m}_i \in \frac{1}{2n}\mathbf{Z}^4$, defines a modular form for $\Gamma_2(2n^2)$ if and only if $\mathbf{m}_i \equiv \mathbf{m}_j \bmod \frac{1}{n}\mathbf{Z}^4$, for all i, j.

Proof. This follows by considering the action of $\Gamma_2(2n^2)/\Gamma_2(4n^2, 8n^2)$ on such a product. This is described in Lemma 4.1. From the action of $e_7, \ldots, e_{10}$ one concludes that the condition on the characteristics is necessary. It is easy to see that it is also sufficient. □

We want to decompose $\mathrm{Th}_2(\Gamma_2(8))$ into character spaces for the action of $\Gamma_2(4)/\Gamma_2(8)$.

LEMMA 6.2. *One has*

$$Th_2(\Gamma_2(8)) \cap M_2(\Gamma_2(4,8)) = Th_2(\Gamma_2(8)) \cap M_2(\Gamma_2(2,4,8)).$$

Proof. Let δ be the nontrivial character of $\Gamma_2(4,8)/\Gamma_2(2,4,8)$. We have to prove that the space $\mathrm{Th}_2(\Gamma_2(4,8), \delta)$ is trivial. The condition on the characteristics $\mathbf{m}_i = (\frac{a_i}{4}, \frac{b_i}{4}, \frac{c_i}{4}, \frac{d_i}{4})$ such that $\prod_{i=1}^{4} \theta_{\mathbf{m}_i}$ is in this space are easily determined using the theta transformation formulas. They arc as follows. Write $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and analogously $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Regard these as vectors in $(\mathbf{Z}/4\mathbf{Z})^4$ and let $\langle \cdot, \cdot \rangle$ denote the standard pairing $\langle \mathbf{a}, \mathbf{b} \rangle = \sum a_i b_i$. Working over $\mathbf{Z}/4\mathbf{Z}$ we must have

- The coordinates of $\mathbf{a}$ are equal modulo 2; the same for $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$;
- $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{c}, \mathbf{d} \rangle = \langle \mathbf{a}, \mathbf{d} \rangle = \langle \mathbf{b}, \mathbf{c} \rangle = 0$, and
- $\langle \mathbf{a}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{d} \rangle = 2$.

It is easy to check that this has no solutions. □

From this lemma one draws the conclusion that all products $\prod \theta_{\mathbf{m}_i}$ which are not modular forms for $\Gamma_2(4,8)$, are elements of spaces $\mathrm{Th}_2(\Gamma_2(4),\chi)$, for characters χ on $\Gamma_2(4)/\Gamma_2(8)$ which are nontrivial on $\Gamma_2(2,4,8)/\Gamma_2(8)$. Such characters are classified in Proposition 2.3. It follows that we only need to consider the 12 characters listed there (with $n = 2$). A computation in the same spirit as the one given in the proof of Lemma 6.2 yields

LEMMA 6.3.*The space $Th_2(\Gamma_2(4),\chi)$ is zero for all characters considered in Proposition 2.3 which are not in the orbit of one of the following two:*

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
χ_1	1	1	1	1	-1	1	1	-1	1	-1
χ_2	1	1	1	-1	1	1	1	-1	1	1

It remains to study $\mathrm{Th}_2(\Gamma_2(4),\chi_i)$ with χ_i (for $i = 1,2$) defined in the lemma above. One uses the relation between $\theta_{\mathbf{m}}$ and $\theta_{\mathbf{a}-\mathbf{m}}$ (for $\mathbf{a} \in \mathbf{Z}^4$) to compute that both $\mathrm{Th}_2(\Gamma_2(4),\chi_i)$ are spanned by 120 products of theta constants. It turns out that 40 of them are congruent to $(0,0,0,\frac{1}{4})$ modulo $\frac{1}{2}\mathbf{Z}^4$, another 40 congruent to $(0,0,\frac{1}{4},0)$ modulo $\frac{1}{2}\mathbf{Z}^4$, and the remaining ones to $(0,0,\frac{1}{4},\frac{1}{4})$ modulo $\frac{1}{2}\mathbf{Z}^4$. The Riemann theta relation writes any product in such a set of 40 forms as a sum of forms in the other two sets. Hence any such congruence class of 40 forms generates $\mathrm{Th}_2(\Gamma_2(4),\chi_i)$. We choose generators of the form $\prod \theta_{\mathbf{m}_i}$ with $\mathbf{m}_i = (a_i,b_i,c_i,\frac{1}{4})$ and $a_i,b_i,c_i \in \{0,\frac{1}{2}\}$. One verifies that if the characteristics $\mathbf{m}_i$ yield a form in one of our character spaces, then for any $\mathbf{a} \in \frac{1}{2}\mathbf{Z}^4$, the characteristics $\mathbf{m}_i + \mathbf{a}$ yield a form in the same character space. This provides a fairly efficient way to describe a set of generators for each of the two character spaces to be considered.

PROPOSITION 6.4. *The space $Th_2(\Gamma_2(4),\chi_1)$ is generated by the 40 products of theta constants*

$$\begin{gathered}
\theta_{[(0,0,0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta^3_{[(0,\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau),\\
\theta_{[(0,0,0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[0,\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta^2_{[(0,\frac{1}{2},\frac{1}{2},\frac{1}{4})+\mathbf{a}]}(\tau),\\
\theta_{[(0,0,0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[(0,\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta^2_{[(\frac{1}{2},\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau),\\
\theta_{[(0,0,0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[(0,\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta^2_{[(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4})+\mathbf{a}]}(\tau),\\
\theta_{[(0,0,0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[(\frac{1}{2},\frac{1}{2},0,\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[(0,\frac{1}{2},\frac{1}{2},\frac{1}{4})+\mathbf{a}]}(\tau)\theta_{[(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4})+\mathbf{a}]}(\tau).
\end{gathered}$$

Here $\mathbf{a}$ *runs through the elements of $\{0,\frac{1}{2}\}^3 \oplus \{0\}$. The Fourier coefficients of these generators are in $\sqrt{2}\cdot\mathbf{Z}$.*

Similarly, generators of $Th_2(\Gamma_2(4),\chi_2)$ can be obtained by taking the functions given above, and interchanging the second with the third coordinate in all characteristics. The resulting forms have Fourier coefficients in $\mathbf{Z}$.

Proof. The only assertion in this proposition for which no proof was sketched in the preceding discussion, is the one about the Fourier coefficients. From the definition of such a product of theta constants it follows that these coefficients can be written as a sum of primitive eighth roots of unity in the case of χ_1. One checks that for each term ζ, also a term $\overline{\zeta}$ occurs. For χ_2 the situation is similar, but with fourth roots of unity. □

We found a description of $\mathrm{Th}_2(\Gamma_2(8))$ as a direct sum of 35 character spaces, each with an explicit set of 40 generators which can be computed using Proposition 6.4. What remains to be computed is the subspace of cusp forms. The following is only a partial result in that direction.

LEMMA 6.5. *The 16 products $\prod \theta_{\mathbf{m}_i}$ with $\mathbf{m}_i \neq \mathbf{m}_j$ for $i \neq j$ which are given in Proposition 6.4, are cusp forms.*

Proof. If such a product were not a cusp form, then by the discussion at the end of Section 4 one can find $\sigma \in \mathrm{Sp}_4(\mathbf{Z})$ such that (upto translation by an element of $\mathbf{Z}^4$) the characteristics satisfy $\sigma \cdot \mathbf{m}_i = (0, b_i, c_i, d_i)$. All b_i or all c_i or all d_i can be chosen $\frac{1}{4}$. Since $(0, b_i, c_i, d_i) \equiv (0, b_j, c_j, d_j) \bmod \frac{1}{2}\mathbf{Z}^4$, there are essentially 4 possibilities for these $\sigma \cdot \mathbf{m}_i$. In particular, all 4 must occur and one concludes that $\sum \sigma \cdot \mathbf{m}_i \in \mathbf{Z}^4$. This contradicts the fact that $\sum \mathbf{m}_i \notin \mathbf{Z}^4$, because σ acts as a linear automorphism on sums of an even number of theta characteristics. □

By computing for odd primes $p \leq 41$ the action of the Hecke operators $T(p)$ on a large number of Fourier coefficients, and of $T(p^2)$ on some Fourier coefficients, one is led to the conjecture that the 8 cusp forms in $\mathrm{Th}_2(\Gamma_2(4), \chi_1)$ described above are common eigenforms for all Hecke operators. If this is true, then in all 8 cases the Euler factors at these primes correspond to the polynomials given in the following table.

3	$X^4 - 2X^2 + 9$	19	$X^4 - 34X^2 + 19^2$
5	$(X^2 - 5)^2$	23	$(X^2 + 23)^2$
7	$(X^2 + 7)^2$	29	$(X^2 - 29)^2$
11	$X^4 + 14X^2 + 11^2$	31	$(X^2 - 29)^2$
13	$(X^2 - 13)^2$	37	$(X^2 - 37)^2$
17	$(X^2 + 6X + 17)^2$	41	$(X^2 + 6X + 41)^2$

These are exactly the characteristic polynomials of Frobenius elements acting on the Tate module of the jacobian of $y^2 = x^5 - x$ over $\mathbf{Q}$ (compare [20], Section 2.5). We summarize this in

CONJECTURE 6.6. *The eight cusp forms $\theta_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\theta_{\mathbf{m}_3}\theta_{\mathbf{m}_4} \in Th_2(\Gamma_2(4), \chi_1)$ are common eigenforms of all Hecke operators. Upto an Euler factor at the prime 2,* *their Andrianov L-function equals the Hecke L-function corresponding to the jacobian of the curve over* $\mathbf{Q}$ *given by* $y^2 = x^5 - x$.

One can exhibit more modular forms for $\Gamma_2(8)$ by searching for products of theta series of the form $\theta_{\mathbf{m}_1}(\tau)\theta_{\mathbf{m}_2}(\tau)\theta_{\mathbf{m}_3}(\tau)\theta_{\mathbf{n}}(2\tau)$. Our aim is to prove

PROPOSITION 6.7. *Let* $\mathbf{m}_i, \mathbf{n} \in \frac{1}{2}\mathbf{Z}^4$. *The function*

$$\theta_{\mathbf{m}_1}(\tau)\theta_{\mathbf{m}_2}(\tau)\theta_{\mathbf{m}_3}(\tau)\theta_{\mathbf{n}}(2\tau)$$

is a modular form in $M_2(\Gamma_2(8,16))$.

It is a modular form for $\Gamma_2(2,4,8)$ *if and only if the last two coordinates of* $\mathbf{n}$ *are integers. If this is the case then this form is an element of* $M_2(\Gamma_2(4,8),\delta)$ *for* δ *the nontrivial character of* $\Gamma_2(4,8)/\Gamma_2(2,4,8)$.

From Lemma 6.2 and this proposition one deduces (compare [13], p. 224, Corollary)

COROLLARY 6.8. *The ring of modular forms for* $\Gamma_2(16,32)$ *is strictly larger than the ring generated over* $\mathbf{C}$ *by the products* $\theta_{\mathbf{m}}\theta_{\mathbf{n}}$ *for* $\mathbf{m}, \mathbf{n} \in \frac{1}{4}\mathbf{Z}^4$.

Proof. (of Proposition 6.7.) From the trivial relation

$$2 \cdot (A\tau + B)(C\tau + D)^{-1} = (A \cdot 2\tau + 2B)\left(\frac{1}{2}C \cdot 2\tau + D\right)^{-1}$$

it follows that we have to investigate the following problem. Suppose $\theta_{\mathbf{m}}\theta_{\mathbf{n}}$ is given; compute

$$\det(C\tau + D)^{-1}\theta_{\mathbf{m}}\left((A\tau + B)(C\tau + D)^{-1})\theta_{\mathbf{n}}((A \cdot 2\tau + 2B)\left(\frac{1}{2}C \cdot 2\tau + D\right)^{-1}\right)$$

for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2(4,8).$$

Using the transformation formula, this reduces to the computation of

$$\kappa\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \cdot \kappa\left(\begin{pmatrix} A & 2B \\ \frac{1}{2}C & D \end{pmatrix}\right).$$

It turns out to be useful to treat the case $g = 1$ first.

LEMMA 6.9. *Let* $\mathbf{m}, \mathbf{n} \in \frac{1}{2}\mathbf{Z}^2$ *be even theta characteristics. Then*

$$\theta_{\mathbf{m}}(\tau)\theta_{\mathbf{n}}(2\tau) \in M_1(\Gamma_1(8)) \Leftrightarrow \mathbf{n} \notin \mathbf{Z} \oplus \left(\frac{1}{2} + \mathbf{Z}\right).$$

None of these forms is modular for $\Gamma_1(4,8)$.

Proof. The numbers $\kappa(\sigma)$ do not depend on the theta characteristics. Using the formulas (4.3) and (4.4) from [15], pp. 148–149 this implies for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$$

that

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot \kappa\left(\begin{pmatrix} a & 2b \\ \frac{1}{2}c & d \end{pmatrix}\right) = 1$$

and

$$\kappa\left(\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}\right) \cdot \kappa\left(\begin{pmatrix} 5 & 16 \\ 4 & 13 \end{pmatrix}\right) = -1.$$

The lemma is a direct consequence of these formulas. □

We are now ready to deal with the case $g = 2$. The group $\Gamma_2(4)$ fixes the involved characteristics. This implies that $\Gamma_2(4)$ acts on such a product $f = \theta_{\mathbf{m}}(\tau)\theta_{\mathbf{n}}(2\tau)$ via a character. In particular, f is a modular form for the commutator subgroup $\Gamma_2(16,32)$. From the addition formula [13], Theorem 2, p. 139 one concludes that f^2 is a modular form for $\Gamma_2(4,8)$. Hence the squares of elements in $\Gamma_2(4,8)$ act trivially on f. It follows that f is modular for $\Gamma_2(4,8,16)$. A generator of $\Gamma_2(8,16)/\Gamma_2(4,8,16)$ is given by

$$e_6(4) = \begin{pmatrix} 9 & & 32 & \\ & 1 & & \\ 16 & & 57 & \\ & & & 1 \end{pmatrix}.$$

One can find the corresponding 'κ-contribution' in the action of $e_6(4)$ on f by computing it on the function

$$\theta_{(0,0,0,0)}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right) \cdot \theta_{(0,0,0,0)}\left(\begin{pmatrix} 2\tau_1 & \\ & 2\tau_2 \end{pmatrix}\right).$$

It equals

$$\kappa\left(\left(\begin{array}{cc} 9 & 32 \\ 16 & 57 \end{array}\right)\right) \cdot \kappa\left(\left(\begin{array}{cc} 9 & 64 \\ 8 & 57 \end{array}\right)\right) = 1.$$

Using this it is straightforward to check that f is even modular for $\Gamma_2(8,16)$. A similar computation involving the generators $e_7(4), \ldots, e_{10}(4)$ of $\Gamma_2(8)/\Gamma_2(8,16)$ yields that here one does find an obstruction. The matrices $e_9(4)$ and $e_{10}(4)$ act on $f = \theta_{\mathbf{m}}(\tau)\theta_{(a,b,c,d)}(2\tau)$ by multiplication by $\exp 2\pi i\phi_{(a,b,c,d)}(e_{9,10}(4))$. This implies that f is modular for $\Gamma_2(8)$ if and only if $c, d \in \mathbf{Z}$. The remaining statements in the proposition follow from analogous computations. □

7. Examples with higher level. It turns out that some of the modular forms for $\Gamma_2(8,16)$ described in Proposition 6.7 are actually cusp forms. To find such examples, notice that by the addition formula [13], Theorem 2, p. 139 the square of such a form is a sum of products of 'usual' theta constants. Hence the square is modular (of weight 4) for $\Gamma_2(4,8)$. If we require each summand to be a cusp form, this certainly implies that the form we started with is a cusp form. We give 2 examples.

PROPOSITION 7.1. *Consider the two functions on* $\mathbf{H}_2$ *given by*

$$\begin{aligned} F_1(\tau) &:= \theta_{(\frac{1}{2},0,0,0)}(\tau)\theta_{(0,\frac{1}{2},0,0)}(\tau)\theta_{(0,\frac{1}{2},\frac{1}{2},0)}(\tau)\theta_{(\frac{1}{2},0,0,\frac{1}{2})}(2\tau); \\ F_2(\tau) &:= \theta_{(0,\frac{1}{2},0,0)}(\tau)\theta_{(0,\frac{1}{2},\frac{1}{2},0)}(\tau)\theta_{(\frac{1}{2},\frac{1}{2},0,0)}(\tau)\theta_{(0,0,0,\frac{1}{2})}(2\tau). \end{aligned}$$

These are cusp forms for the group $\Gamma_2(8,16)$.

Based on computer calculations of the action of Hecke operators on lots of Fourier coefficients one may state

CONJECTURE 7.2. *The cusp forms* F_i *introduced in Proposition 7.1 are eigenforms for the Hecke operators.*

The corresponding Andrianov L-functions are

$$\begin{aligned} L(F_1,s) &= L(E_1 \times E_{2,s}); \\ L(F_2,s) &= \zeta_{\mathbf{Q}(\sqrt{-2})}(s) \cdot \zeta_{\mathbf{Q}(\sqrt{-2})}(s-1). \end{aligned}$$

Here the E_i *are elliptic curves over* $\mathbf{Q}$ *given by the equations*

$$\begin{aligned} E_1 &\quad y^2 = x^3 - x; \\ E_2 &\quad y^2 = x^3 + x. \end{aligned}$$

Remark 1. It seems quite likely that at least part of the statement about F_2 can be proved analogously to the proof of Proposition 5.4. To be more specific, consider for a (this time ordered) pair $\mathbf{m}, \mathbf{n}$ of odd theta characteristics the function

$$G_{\mathbf{m},\mathbf{n}}(\tau) = \frac{1}{\pi^2} \det \begin{pmatrix} \frac{\partial}{\partial z_1}\theta_{\mathbf{m}}(\tau, z) & \frac{\partial}{\partial z_1}\theta_{\mathbf{n}}(2\tau, z) \\ \frac{\partial}{\partial z_2}\theta_{\mathbf{m}}(\tau, z) & \frac{\partial}{\partial z_2}\theta_{\mathbf{n}}(2\tau, z) \end{pmatrix}\Bigg|_{z=(0,0)}.$$

One can try to relate such a $G_{\mathbf{m},\mathbf{n}}$ to the function F_2 just as the $\Theta_{\mathbf{m},\mathbf{n}}$ correspond to the cusp forms of weight 2 for $\Gamma_2(4,8)$. If this can be achieved, then at least for primes $p \equiv 5$, 7 mod 8 it follows that the corresponding Euler factors are as desired. However, we did not push this method through.

Remark 2. One can also find examples by replacing in a product of theta constants one of the τ's by e.g. 4τ. An example of a cusp form (for the group $\Gamma_2(16,32)$) obtained in this way is

$$F_3(\tau) := \theta_{(0,\frac{1}{2},0,0)}(\tau)\theta_{(\frac{1}{2},\frac{1}{2},0,0)}(\tau)\theta_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}(\tau)\theta_{(0,0,\frac{1}{2},0)}(4\tau).$$

This function seems to be an eigenform of the Hecke operators as well. For the primes $p \leq 47$ which are congruent to 1, 3 or 7 modulo 8 the Euler factor one finds corresponds to a product of two elliptic curves of conductor 128. However, for primes $p \equiv 5$ mod 8 we do not know what should be the case. The computer indicates that the eigenvalues $\lambda(p^i)$ of $T(p^i)$ are as follows.

p	$\lambda(p)$	$\lambda(p^2)$
5	0	1
13	0	-15
29	0	49
37	0	225

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